Preference Representation for Multicriteria Decision Making

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Abstract: In this note we consider a multicriteria decision problem where the decision maker know the the state of the world but the set of consequences is multidimensional. We suppose that a value function is specified over the attribute of the decision problem and we analyze some classes of non additive functions that can represent interaction between criteria.

Keywords: Multicriteria decision making, value function, Choquet signed integral, Schur-decreasing functions.

M.S.C. CLASSIFICATION: 62C05, 91B08 .
J.E.L. CLASSIFICATION: C60.

1 Introduction

In multicriteria decision making we aim at ordering multidimensional alternatives. We suppose that a decision can be made using a value function which represents the preference structure of the decision maker. In this framework the critical point of solving multi-attribute decision problem is to determine the value function.
A traditional approach is to use a function that is a simple weight sum where each weight represents the importance given by the decision maker to a particular attribute. It should be noticed that the additive model implies independence between attributes so despite its simplicity this approach suffer a major drawback of not being able to take into account "inter-attribute" relations that are present in many situations.
The problem of modelling such an interaction is a difficult question because there are different types of dependence quite different from each other such as correlation, complementarity and preferential dependence.
In this note we study some particular classes of non additive value functions that are usually considered in the area of decision theory under uncertainty. So this note points out a similarity between decision under uncertainty and

multicriteria decision making two areas which have been investigate separately. However as shown by some recent papers under quite general hypothesis decision under uncertainty and multicriteria decision making are formally equivalent. A multicriteria decision problem is written in a decision under uncertainty framework by identifying the criteria with the states of the world and the consequences with the acts.
In the first part of the paper we will introduce some classes of multivariate real functions and we briefly review Choquet nonadditive integration theory while in the last part we prove our results concerning the preference relation of the decision maker.

2 Preliminaries

Let us consider a multicriteria decision problem where there is a set A of alternatives and a finite set of criteria $N = \{1, \ldots, n\}$ with respect to which the choice between alternatives is done. Each alternative $a \in A$ is represented by a n-dimensional vector $x^a = (x_1^a, \ldots, x_n^a) \in \mathbb{R}^n$, where for any $i \in N$, x_n^a represents the partial score of a related to criterion i. We assume that the partial scores are defined according the same interval scale.
Suppose that the preferences over A of the decision maker are known and expressed by a relation $\succeq$. We also assume that the preference relation is represented by a real value function defined $f$on $\mathbb{R}^n$ so that if $a, b \in A$

$$a \succeq b \Leftrightarrow f(x^a) \geq f(x^b)$$

In this note we study some particular classes of multidimensional value functions defined on $\mathbb{R}^n$.

We also consider the concept of majorization arising as a measure of diversity of the components of a n-dimensional vector. Majorization has been comprehensively treated by[1] and [5].
We aim to formalize the idea that the components of a vector x are less"spread out" or "more nearly equal" than the components of y.
For a vector $x \in \mathbb{R}^n$ we denote its elements ranked in descending order as

$$x_{\sigma(1)} \geq x_{\sigma(2)} \geq \ldots\ldots \geq x_{\sigma(n)} \tag{2.1}$$

where $\sigma$is a permutation defined on N.

Definition 1. *The vector y is said to majorize the vector x, which is denoted as $x \preceq y$, if*

$$\sum_{i=1}^{k} x_{\sigma(i)} \leq \sum_{i=1}^{k} y_{\sigma(i)} \quad k = 1, 2 \ldots, n-1 \quad \textit{and} \quad \sum_{i=1}^{n} x_{\sigma(i)} = \sum_{i=1}^{n} y_{\sigma(i)} \tag{2.2}$$

.

Majorization is a partial ordering among vectors, which applies only to vectors having the same sum. It is a measure of the degree to which the vector elements differ. For example it can be easily shown that all vectors of sum s majorize the uniform vector $u = (\frac{s}{n}, \ldots \frac{s}{n})$. Intuitively, the uniform vector is the vector with minimal differences between elements, so all vectors majorize it.

In order to introduce a nonadditive approach to multicriteria decision making we introduce a non- additive integral operator. As is well known the Choquet integral has been extensively applied in the context of decision under uncertainty.
In this context the Choquet integral may be viewed as a way of aggregating utility across different states in order to arrive to a decision criterion while in multicriteria decision making the nonadditive integral operator is a tool for aggregating over different criteria. The use of variants of the Choquet integral allows some flexibility in the way criteria are combined. In particular in this paper we consider a signed Choquet integral as in
For the sake of our application we restrict ourselves to the finite case and for the properties of nonadditive integration we refer to [2]. We define a nonmonotonic Choquet measure on 2^N and a signed Choquet integral for a n-dimensional vector.

Definition 2. *A function* $\upsilon : 2^N \rightarrow I\!R$ *is called a signed capacity if* $\upsilon(\emptyset) = 0$ *and* $\upsilon(N) = 1$.

We will assume here $\upsilon(N) = 1$ as usual although this is not necessary.
We note that if $S \subseteq N$, $\upsilon(S)$ can be viewed as the importance of the set of elements S.

Definition 3. *Let* υ *a signed capacity* $\upsilon : 2^N \rightarrow I\!R$, $x \in I\!R^n$ *and*

$$\pi_{\sigma(j)} = \upsilon(\{\sigma(1), \ldots\ldots, \sigma(j)\}) - \upsilon(\{\sigma(1), \ldots\ldots, \sigma(j-1)\}) \tag{2.3}$$

where σ *is defined by (2.2). The signed Choquet integral of* x *is*

$$\int x d\sigma = \sum_{j=1}^{n} \pi_j x_j \tag{2.4}$$

.

We are now able to present some aggregation operators as appropriate extensions to the weighted arithmetic mean for the aggregation of criteria.

3 Some class of multivariate value functions

Let us now recall some properties for real functions defined on $\mathbb{R}^n$.

Definition 4. *A function f, $\mathbb{R}^n \to \mathbb{R}$ is called Schur-increasing if $f(x) \leq f(y)$ when $x \preceq y$. f is Schur-decreasing if $-f$ is Schur-increasing.*

Schur increasing functions thus preserve majorization. We note also that a Schur increasing or decreasing function must be a symmetric function. Moreover a symmetric convex function is Schur increasing [5] .
A Schur decreasing value function consider the attribute as symmetric and prefer attributes that are less"spread out". In fact if x and y are interpreted as random variables with all state equally probable the condition $x \preceq y$ is equivalent to the statement that x is less risky than y in the sense of Rothschild and Stiglitz. Hence in this context Schur decreasing is a property of risk aversion.

The following result characterize the preference relations on $\mathbb{R}^n_+$ that can be represented by a Schur-decreasing function defined on $\mathbb{R}^n_+$.

Theorem 1. *Let $\geq$ a preference relation defined on $\mathbb{R}^n_+$. The preference relation is represented by a function f $\mathbb{R}^n_+ \to \mathbb{R}$ that is Schur-decreasing if and only if*

$$(x_1 \ldots x_i + c \ldots x_j - c \ldots x_n) \leq (x_1 \ldots x_i \ldots x_j \ldots x_n) \tag{3.1}$$

for all x_i, x_j with $x_i \geq x_j$ and $0 < c < x_j$.

Proof By the proof of theorem 2 of [4] if x and y are elements of $\mathbb{R}^n_+$ and $x \succeq y$ then x can be derived from y via a finite series of transformation of type (3.1). Hence if we assume that(3.1)is satisfied $x \succeq y$ if and only if $x \geq y$.

A nonadditive integral is a sort of weighted mean taking into account the importance of every subset of criteria and so considering the relative importance of the criteria. It should be noticed that we consider signed capacity that are nonmonotonic and may take negative values. We apply the signed integral operator to multicriteria decision making modelling interactions between criteria that are so strong that monotonicity is violated.

The following result characterize the continuos preference relations on $\mathbb{R}^n$ that can be represented by a signed integrals with respect to a signed capacities .

Theorem 2. *Let $\geq$ a preference relation defined on $\mathbb{R}^n_+$. We suppose that the preference relation is represented by a continuos function f $\mathbb{R}^n \to \mathbb{R}$ that is a signed Choquet integral if and only if*

i) if $a, b \in \mathbb{R}$, $a \geq b$ *then* $(a, \ldots, a) \geq (b, \ldots, b)$.

ii) if x, y, z are elements of $\mathbb{R}^n$ such that there exists a permutation σ defined on N.such that x, y, z are solution of the equation :

$$t_{\sigma(1)} \geq t_{\sigma(2)} \geq \ldots\ldots \geq t_{\sigma(n)} \tag{3.2}$$

if $x \geq y$ *then* $x + z \geq y + z$.

Proof The preference relation $\geq$ is continuous, constant-monotonic according to the definitions in [3]. Moreover if it is verified condition ii), the preference order satisfies comonotonic additivity (see [3]).Then by corollary 1 of [3] there exists a signed Choquet integral that represents the preference order.

4 Concluding remarks

In this note we have examined multicriteria decision problem where a value function is specified over the attributes of a deterministic decision problem. We have studied particular classes of value functions that are usually considered in the area of decision theory under uncertainty. In particular the Schur-increasing functions are used to compare situations according to their level of heterogeneity while Choquet integral operator can model violations of additivity and monotonicity. So this note points out a similarity between decision under uncertainty and multi-attribute decision making problems, two areas which have been developed in an almost completely independent way. This is a potentially fruitful area for future work.

References

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