The road to extinction: commons with capital markets[1]

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Abstract

We study extinction in a commons problem in which agents have access to capital markets. When the commons grows more quickly than the interest rate, multiple equilibria are found for intermediate commons endowments. In one of these, extinction is hastened and welfare decreases in the endowment, a *resource curse*. An extraction tax reduces welfare in this 'cursed' equilibrium, increases it in the other equilibrium in which the commons is eventually depleted, and expands the set of commons stocks that are never depleted. Capital market access harms societies with low commons endowments. In the limit, as marginal extraction costs become constant, 'jump extinctions' occur. Finally, when the commons grows less quickly than the interest rate, there is a unique extinction date for each endowment level.

Key words: commons, capital markets, extinction, resource curse, storage, multiple equilibria, rational expectations equilibrium

JEL classification numbers: C73, D91, O17, Q21

1 Introduction

In almost any contemporary common access problem, those drawing on the resource also have access to capital markets. Thus, the proceeds of their extraction need not be immediately consumed; they may be saved, or even borrowed against. In spite of this, relatively little attention has been paid to the effects of capital market access on commons problems. This paper seeks to redress this. A particular interest of the present paper is effect of access to capital markets on the 'extinction' of the commons.

The simplest commons analyses, dating to Gordon (1954), are static: free entry allows the rents associated with a natural resource to be competed away - the 'tragedy of the commons'. In the static framework, question of access to capital markets do not arise.

Much of the literature building on Gordon (1954) has been dynamic, and thus concerned with intertemporal issues. To date, the bulk of this has not allowed capital market access, forcing each agent into intertemporal autarky. See Mirman (1979); Levhari and Mirman (1980); Benhabib and Radner (1992); Dutta and Sundaram (1993); Dockner and Sorger (1996); Sorger (1998) for examples of this literature. Thus, given that the commons produces necessary consumption, it is never in anybody's interest to ransack to the point of extinction. Of course, the commons are often overfished or overgrazed relative to what a benevolent planner would determine.[1]

More recently, the possibility of relating consumption and extraction through a intertemporal constraint, rather than constraints in each period, has been explored. As we outline, this has been done in a number of ways, from simple storage technologies to access to capital markets. The present paper focusses on this latter possibility. We argue that the possibility of private storage can weaken the necessity of maintaining the commons, thereby allowing extinction, and possibly even greater inefficiency.

One of the earliest examples of this literature is Sinn (1984). In this, oligopolistic firms extract oil from underground reserves, either to sell immediately or store above ground. As firms own private oil fields, which only interact through seepage, is a problem of oligopolistic competition in the product market rather than extraction from a commons.

[1]Exceptions to this result exist. Dutta and Sundaram (1993) present a discrete time example in which under-exploitation of the commons can occur when trigger strategies are defined on the state variable; they note that this idea is also found in Fudenberg and Tirole (1983). Their equilibria remain inefficient. In continuous time, Benhabib and Radner (1992) find ranges of initial conditions that allow trigger strategies yielding efficient equilibria. Dockner and Sorger (1996) and Sorger (1998) also derive conditions under which equilibria are efficient.

Kremer and Morcom (2000) consider a model with genuine storage.[2] Competitive poachers may kill elephants (an open access resource) and store their ivory tusks at the opportunity cost. In contrast to the preceding literature that they cite, they argue that there may be

> multiple equilibria for open-access renewable resources used in the production of storable goods, because if others poach, the animal will become scarce, and this will increase the price of the good, making poaching more attractive.

In our model, there is an interval of initial commons stock levels within which multiple equilibria are found. In particular, there are three, two involving extinction in finite time; extinction does not occur in the remaining one.

Homans and Wilen (2001) also explore competitive equilibria. A fixed cost of fishing restricts the number of agents to be finite. Further, access to the commons is regulated, a framework that they regard as more consistent with existing fisheries than open-access. Their attention is focussed on the market for caught fish: fish caught during the fishing season must satisfy a year's consumer demand; that sold immediately is sold as fresh, and therefore at a higher price than that sold after being frozen. They argue that increased rents in the fishing industry both induce entry and shorten the fishing season, thus causing more fish to be sold on the inferior market.[3]

The most recent paper in this literature is Gaudet, Moreaux, and Salant (2002), which also considers competitive equilibria in a commons environment with private storage.[4] Its motivating examples are powerful: very rapid depletion of underground oil reserves, annual fishing quotas and groundwater. As average extraction costs become constant, their model captures these 'jump extinctions', which are similar to speculative attacks. We also obtain this result as a limit case of our model: generically, the road to extinction is smooth in our model.

Two earlier, related papers address considered the possibility not of costly storage, but of full access to capital markets: yield-bearing storage and costly borrowing. Tornell and Velasco (1992) ask why capital flows from poor countries to rich countries. They interpret poor countries as having not just high rates of return, but weak property rights so that domestic returns can be

[2]See Bulte, Horan, and Shogren (2003) and Kremer and Morcom (2003) for further discussion.

[3]Fish raised in farms are often fed fish pellets (Weiss, 2002). Such farms may therefore be a way of converting cheaper (frozen) fish into more expensive fresh fish.

[4]Their suggestion that the earlier literature neglects storage uses an unfortunate example given that cows are ruminants.

appropriated by anyone with equal facility; thus, domestic investment is investment in a commons. Capital flight corresponds to private storage of pillage from common property in a wealthy country, thus at lower rates of return. With perfect international capital markets, borrowing and lending may occur at the same rate. Their solution concept is a stationary Markov perfect equilibrium.

A sequel, Tornell and Lane (1999), again imagines a country with weak institutions. Now, the interest groups compete for control over government revenues, raised by taxing the formal sector. Thus, the formal (high return) sector becomes a commons; the informal (low return) sector is the medium for private storage.

These papers suggest a very different interpretation of the commons problem than the standard one, in which the tragedy is that of the rents' dissipation. In their view, the commons is associated with poorly defined property rights, weak institutions, and poverty.[5] From this point of view, the tragedy of these commons may be their persistence, not their extinction. By contrast, richer countries have likely enclosed their commons at some point in the past. Thus, to the extent that extinction corresponds to enhanced property rights, it may be a desirable outcome. The standard interpretation of the tragedy reminds us, though, that this is not assured: the costs of enforcing property rights may dissipate rents.

Equilibrium extinction may therefore be seen as voluntary privatization: a self-enforcing move from common to private property. Enclosures, gold rushes or, for that matter, the acquisition of Mesopotamian antiquities by private collections are all pertinent examples: are such equilibrium allocations of property rights likely to lead to efficiency? If so, might this provide a mechanism for a 'resource curse', whereby societies better endowed with resources may take longer to move to private property?

A weakness of Tornell and Velasco (1992) and Tornell and Lane (1999) in this context is that, by restricting extraction to shares of stock, they rule out the possibility of full extinction or enclosure. As extraction is costless in their model, an extreme version of the limit case of Gaudet et al. (2002), one might expect this to occur instantly.

To explore these questions further, this paper looks at perhaps the simplest possible representation of the problem, close to the industry standard on tragedy of the commons, but with capital market access. A model is presented in Section 2. This closely follows the structure of the models presented in Tornell and Velasco (1992) and Tornell and Lane (1999), generalising in two ways. First, extraction may be costly. Second, strategies are not required

[5]In the popular debate, this view has been argued forcefully by de Soto (2000).

to be shares of existing stocks. Without this latter generalisation, extinction is not technically possible.

While our model is consistent with the interpretations given to high and low rates of return, it is also consistent with a third variant: there is only one rate of return, but the costs of enclosing resources to protect them against expropriation reduce the net rate of return on privately held resources to the lower rate. Thus, the difference between the high and low rates may be seen as reflecting the weakness of property rights. This interpretation is very general, encompassing situations of both economic and biological growth.

Following Gaudet et al. (2002), Kremer and Morcom (2000) and Homans and Wilen (2001), the solution concept is a rational expectations equilibrium: there are no barriers to entry; individuals are small and do not take account of the impact of their actions on the evolution of aggregate capital stock. This is introduced in Section 3. (In a companion paper, Dutta and Rowat (2004), we evaluate the extent to which strategic, subgame perfect equilibria inherit these extinction properties. We find that they do, even for very small numbers of agents.)

As noted, we find the Gaudet et al. (2002) 'jump extinction' as a special case. More generally, low initial commons stocks correspond to unique solutions, with extinction in finite time; an intermediate range of initial commons stock produces the multiple equilibria already mentioned; initial stocks above this level are never exhausted.

Section 4 compares competitive outcomes with capital market access to autarkic ones in which competitive agents do not have access to capital markets. In the example analysed, welfare under autarky is higher for low initial stock levels. Once commons stocks are sufficient to support multiple extinction dates, superior welfare can be obtained with capital market access.

Section 5 considers the consequences of an extraction tax; it has the effect of shrinking the interval of commons stock levels that lead to multiple extinction dates. This shrinks the domain over which the 'cursed' equilibrium can arise and expands the set of commons stocks that are never extinguished. Its effect may be reversed, however, when the government imposing the tax is strong.

Section 6 concludes.

2 The model

Time, indexed by t, passes continuously toward an infinite horizon. At every point in time, a continuum of individuals, indexed by i and distributed on the unit interval with cumulative distribution F, decides on extraction, $x_i =$

$\{x_i(t)\}$, and consumption, $c_i = \{c_i(t)\}$.

There is a single consumption good, whose stock, $k = \{k(t)\}$, is common property. It grows at rate a. At each point in time, individuals extract, in total, $x(t) = \int_i x_i(t)dF(i)$ from the commons, storing it as their private property.[6] Thus, $x_i(t)$ is the (finite) extraction rate at time t of infinitesimal agents $dF(i)$.

The extraction path, $x = \{x(t)\}$, defines the initial value problem

$$\dot{k}(t) = ak(t) - x(t), k(0) > 0 \tag{1}$$

whose solution is the path of the capital stock whenever $k(t) > 0$.

An *extinction date* is the earliest $T \geq 0$, such that $k(T) = 0$. If $\lim_{t\to\infty} k(t) > 0$, then $T = \infty$, which corresponds to non-extinction. Since $k(T) = 0 \Rightarrow x(T) = 0$, the capital stock is absorbed at 0 and

$$k(t) = \max\left\{0, k(0)e^{at} - \int_0^t e^{a(t-\tau)}x(\tau)\,d\tau\right\}. \tag{2}$$

Equation 2 describes the unique solution to initial value problem whenever $x(t)$ is continuous over $[0, T)$ (Walter, 1998, p.28). Thus, aggregate extraction is *admissible* if $x(t)$ is continuous over $[0, T)$.[7]

Goods extracted from the commons are presented to capital markets. If saved, they earn a return of $r \leq a$. Equally, future extraction may be borrowed against, smoothing consumption, at the same rate.

Individuals choose extraction and consumption paths to maximize utility subject to their budget constraint, 5, and to a feasibility (or non-negativity) constraint

$$k(t) = 0 \Rightarrow x_i(t) = 0. \tag{3}$$

Individuals differ in their costs of extraction or access to common property. This difference is captured by an extraction cost parameter, θ, such that $\theta_i > \theta_j$ implies i has easier access.

Their utilities are

$$V_i(c_i, x_i) = \int_0^\infty e^{-\rho t}\left(U(c_i(t)) - \frac{C(x_i(t))}{\theta_i}\right)dt; \tag{4}$$

where utility and cost functions are

$$U(c) = \frac{c^{1-\alpha}}{1-\alpha}, \quad \alpha > 0 \qquad \text{and} \qquad C(x) = \frac{x^{1+\gamma}}{1+\gamma}, \quad \gamma > 0.$$

[6] This general specification is not generally taken advantage of. We often replace $dF(i)$ with the uniform di.

[7] Dockner, Jørgenson, Long, and Sorger (2000, p. 40, Definition 3.1) use 'feasible' in place of 'admissible'.

Utility from consumption is concave while extraction costs are convex: $U' > 0, C' > 0, U'' \leq 0, C'' \geq 0$. Values of $\alpha > 1$ yield utility function more concave than the log, which corresponds to $\alpha = 1$. As $\gamma = 0$ has an appealing interpretation, we consider it later in the paper.

As instantaneous utility is unbounded, integral 4 may may converge as $t \to \infty$. To avoid comparisons of infinite valuations, we impose Uzawa integrability conditions that ensure finite valuations:

$$(1-\alpha)r < \rho < (1+\gamma)r. \tag{U}$$

These ensure that, respectively, the utility of consumption and the disutility of extraction are finite.[8]

Without capital markets, agents are forced to set $x_i(t) = c_i(t)$: consumption must match extraction in every instant. Thus, agents maximise objective function 4 by choice of a single variable.

Capital market access, however, enables them to decompose their optimization problem into two separate problems. It does so by replacing the instantaneous feasibility constraints on consumption with a single intertemporal budget constraint:

$$\int_0^\infty e^{-rt} \left(c_i(t) - x_i(t)\right) dt \leq 0. \tag{5}$$

Thus, $x_i(t) < c_i(t)$ implies borrowing against future extraction. The possibility of default is not considered.

It may happen, for r high enough, that individuals' chosen paths satisfy $x_i(t) > c_i(t)$ up to some T. We do not impose this as a constraint on individuals. We do, in Section 4, address the commons problem without capital markets.

Finally, a *rational expectations equilibrium* is described by sequences $\{k \geq 0, c_i \geq 0, x_i \geq 0\}$ and an extinction time, T, such that $k(t) = 0$ for all $t \geq T$, where

1. individuals choose consumption and extraction paths to maximize utilities subject to their budget and feasibility constraints;

2. the evolution of the capital stock is determined by the aggregate extraction path.

The equilibrium is competitive in the standard sense: although individuals' aggregate behaviour influences the economic environment, T in this case,

[8]See Dockner et al. (2000, pp. 62 - 63) for a discussion of optimality criteria when valuations are infinite.

they disregard their individual effects on it. Thus, our agents are extinction date takers.

In the following, we solve the model for equilibrium, and, in particular, for extinction dates.

3 Rational expectations equilibria

The assumption that individuals can borrow and lend at rate r allows the decomposition of their problem of determining extraction and consumption into two separate problems:

1. the consumption-smoothing problem: choose $c_i(t)$ given total wealth $X_i(r) = \int_0^\infty e^{-rt} x_i(t) dt$ subject to constraint 5; and
2. the effort-smoothing problem: choose $x_i(t)$, given the extinction date T and feasibility constraint, 3.

3.1 The consumption smoothing problem

Agents are viewed as first solving the consumption problem:

$$\max_{c_i(t)} \int_0^\infty e^{-\rho t} U\left(c_i\left(t\right)\right) dt \text{ subject to equation 5.} \tag{6}$$

Defining

$$X_i(r) \equiv \int_0^\infty e^{-rt} x_i\left(t\right) dt;$$

facilitates writing the Lagrangian:

$$\mathcal{L} \equiv \int_0^\infty e^{-\rho t} \frac{c_i\left(t\right)^{1-\alpha}}{1-\alpha} dt - \lambda \left[\int_0^\infty e^{-rt} c_i\left(t\right) dt - X_i(r)\right].$$

The ensuing Euler equation is standard:

$$c_i\left(t\right) = \frac{\rho - r\left(1-\alpha\right)}{\alpha} X_i(r) e^{\frac{r-\rho}{\alpha} t}. \tag{7}$$

This is independent of a, γ and θ_i. Its constant term is determined by the constraint. Substitution into the objective function of equation 6 therefore produces the maximized present value of utility from consumption

$$U_B\left(X_i(r)\right) \equiv \nu \frac{\left(X_i(r)\right)^{1-\alpha}}{1-\alpha}; \tag{8}$$

where

$$\nu \equiv \left(\frac{\alpha}{\rho - (1-\alpha) r} \right)^{\alpha} = \frac{1}{(r - g_c)^{\alpha}}.$$

The Uzawa finite valuation condition for consumption is $\rho > r(1-\alpha)$. This is trivially satisfied if $\alpha \geq 1$ ($U(c)$ is more concave than $\log(c)$). We note that

$$c_i(t) = c_i(0) e^{g_c t};$$

where

$$g_c \equiv \frac{r - \rho}{\alpha};$$

is the growth rate of consumption and $c_i(0)$ is chosen to satisfy (5):

$$c_i(0) = (r - g_c) X_i(r). \tag{9}$$

The Uzawa condition may therefore be expressed as $g_c < r$: consumption growth is lower than the interest rate.

3.2 The effort smoothing problem

The agent's problem is now to

$$\max_{x_i(t) \geq 0} U_B \left(\int_0^{\infty} e^{-rt} x_i(t)\, dt \right) - \frac{1}{\theta_i} \int_0^{\infty} e^{-\rho t} C(x_i(t))\, dt; \tag{10}$$

subject to feasibility constraint 3.

When $t < T$, the ensuing Euler equation yields

$$\begin{aligned} x_i(t) &= e^{gt} \frac{[\theta_i \nu]^{\frac{1}{\gamma}}}{X_i(r)^{\frac{\alpha}{\gamma}}} \\ &= e^{gt} \frac{[\theta_i \nu]^{\frac{1}{\gamma}}}{\left[\int_0^T e^{-rt} x_i(t)\, dt \right]^{\frac{\alpha}{\gamma}}} \\ &= e^{gt} \kappa; \end{aligned}$$

where $g = \frac{\rho - r}{\gamma}$ and κ is a positive constant. Evaluating this at $t = 0$ produces $x_i(0) = \kappa$ so that

$$x_i(t) = \left\{ \begin{array}{lll} x_i(0) e^{gt} & \text{for} & t < T \\ 0 & \text{for} & t \geq T \end{array} \right\}. \tag{11}$$

This allows the term in $x_i(t)$ to be removed from the integral for

$$x_i(0)^{\alpha+\gamma} = \frac{\theta_i}{(\int_0^T \exp(-(r-g)t) dt)^{\alpha}} \nu. \tag{12}$$

Thus, extraction is smooth until the extinction date, T. More significantly, the problem of choosing an extraction path is reduced to a choice of $x_i(0)$. Note also that the extraction plan is a function of α: thus, full Fisher separation of extraction (production) and consumptions plans does not occur. This is a consequence of $\gamma > 0$: extraction costs are borne as non-transferable disutility.[9]

The Uzawa condition for extraction is now $g < r$. For now, we assume that $a > r$ as these situations might be expected to produce the most interesting economic behaviour. (When $a \leq r$ maintaining the commons offers no benefits.) Thus, for expositional purposes, we concentrate on $a > g$ at present. Theorem 2 treats the complementary cases as well.

Notice that $g = -\frac{\alpha}{\gamma} g_c$; thus, for $r > \rho$, individuals extract early but consume late. The $r = \rho$ case yields $g = g_c = 0$. This is a potentially important special case (and 'interest rate equals subjective rate of discount' is well justified along equilibrium growth paths).

Notice also that an expression for extraction as a function of capital stock may now be written. By equations 2 and 11,

$$k(t) = e^{at} \left[k(0) - \frac{x(0)}{g-a} \left(e^{(g-a)t} - 1 \right) \right].$$

Thus, when $g \neq a$, extraction cannot be expressed as a linear function of capital stock.

3.3 Characterising equilibrium

Define the function

$$Q_n(T) = \int_0^T \exp(-n\tau)\, d\tau = \frac{1 - \exp(-nT)}{n}$$

for $n \neq 0$ and $T \geq 0$.[10] Notice that $Q_n(T)$ increases with T and decreases with n. This allows simplification of equation 12 to

$$x_i(0) = \frac{(\theta_i \nu)^{\frac{1}{\alpha+\gamma}}}{(Q_{r-g}(T))^{\frac{\alpha}{\alpha+\gamma}}}. \tag{13}$$

Integrating over agents then produces **the first fundamental equation: the effect of anticipated extinction** on extraction:

$$x(0) = \frac{\mu}{(Q_{r-g}(T))^{\frac{\alpha}{\alpha+\gamma}}}; \tag{A}$$

[9]In the limit, as $\gamma \to 0, x_i(0)$ ceases to depend on α by exploding to infinity. This will also be demonstrated in Theorem 3.

[10]When $n = 0, Q_n(T) = T$.

where $\mu \equiv \nu^{\frac{1}{\alpha+\gamma}} \int \theta_i^{\frac{1}{\alpha+\gamma}} dF(i)$. This equation gives us a map $\mathbf{A} : T \to x(0)$, monotone decreasing.

We know that $x(0)$ determines the entire path of extractions. We now obtain the **second fundamental equation: the impact of extraction** on the possible extinction date of common property. As extinction occurs at the lowest T such that $k(T) = 0$, it represents a zero of equation 2:

$$k(0)\, e^{aT} = \int_0^T e^{a(T-\tau)} x(\tau)\, d\tau.$$

Thus, by equation 11 and integration over agents,

$$k(0) = x(0) \int_0^T e^{-(a-g)\tau} d\tau. \tag{14}$$

The Uzawa extraction condition ensures that $r > g$; thus $a > g$ follows. Therefore

$$\frac{k(0)}{x(0)} = \frac{1 - e^{-(a-g)T}}{a-g} = Q_{a-g}(T) \leq \frac{1}{a-g}.$$

This expression reaches its upper bound as $T \to \infty$. Therefore T implicitly solves,

$$Q_{a-g}(T) = \min\left\{ \frac{k(0)}{x(0)}, \frac{1}{a-g} \right\}. \tag{I}$$

This equation gives us a map $\mathbf{I} : x(0) \to T$, also monotone non-increasing. Here a low level of initial $x(0)$ guarantees the perpetuation of common property but a level higher than $(a-g)k(0)$ results in extinction in finite time.

The discussion above tells us the likely source of multiple equilibrium in extinction times. Both maps are decreasing: individuals choose to extract more if they believe that the commons will disappear soon; and higher extraction rates speed up extinction.

To formalise the preceding discussion, define

$$\Psi(t) \equiv \frac{Q_{a-g}(t)}{[Q_{r-g}(t)]^{\frac{\alpha}{\alpha+\gamma}}}; \tag{15}$$

and $\psi_* \equiv \lim_{t\to\infty} \Psi(t), \psi^* \equiv \max_t \Psi(t)$.

Theorem 1. *There exist* $0 < k_L \leq k_H \leq \infty$ *such that the following statements hold when agents have access to capital markets. Given intervals* $I_L = [0, k_L), I_M = (k_L, k_H)$, *and* $I_H = (k_H, \infty)$*:*

1. $k(0) \in I_L$ *implies unique equilibrium with finite extinction;*
2. $k(0) \in I_M$ *implies multiple equilibria, one with non-extinction and two with finite extinction;*
3. $k(0) \in I_H$ *implies unique equilibrium without extinction;*
4. $k(0) = k_L = k_H$ *implies a unique equilibrium with non-extinction;*
5. $k(0)$ *= either* k_L *or* k_H*, distinct, implies a unique equilibrium with extinction, and a unique equilibrium with non-extinction.*

The following lemmas are used to prove the theorem:

Lemma 1. $\lim_{t\to 0} \Psi(t) = 0$.

Proof. As $Q_n(0) = 0$, assessing $\Psi(0)$ requires use of l'Hôpital's rule: differentiating the numerator produces 1, while doing so to the denominator produces

$$\frac{\alpha}{\alpha+\gamma}\left[Q_{r-g}(0)\right]^{-\frac{\gamma}{\alpha+\gamma}} e^{-(r-g)0} = \frac{1}{0} = \infty.$$

□

Lemma 2. *When the Uzawa extraction condition holds and* $a > g, 0 < \psi_* < \infty$.

Proof. By definition, $\lim_{t\to\infty} Q_n(t) = \frac{1}{n}$ when $n > 0$.[11] Under the conditions of the lemma,

$$\psi_* = \frac{(r-g)^{\frac{\alpha}{\alpha+\gamma}}}{a-g}. \tag{16}$$

The Uzawa extraction condition ensures that the numerator is strictly positive. When $a > g$, the denominator is as well, ensuring the results. □

Lemma 3. *An equilibrium with finite extinction time* T *satisfies* $\Psi(T) = \frac{k(0)}{\mu}$.

Proof. Equations A and I, with $Q_{a-g}(T) = \frac{k(0)}{x(0)}$, are satisfied in equilibrium. The result follows by definition 15. □

Lemma 4. *An equilibrium with non-extinction exists iff*

$$\psi_* \leq \frac{k(0)}{\mu}.$$

[11]When $n \leq 0$ the limit is infinite.

Proof. Assume that $\psi_* \leq \frac{k(0)}{\mu}$ corresponds to a $T = \infty$ equilibrium. Therefore, by definition,

$$\begin{aligned}\frac{k(0)}{\mu} &\geq \frac{Q_{a-g}(\infty)}{[Q_{r-g}(\infty)]^{\frac{\alpha}{\alpha+\gamma}}} \\ &= Q_{a-g}(\infty)\frac{x(0)}{\mu} \text{ by equation A;}\end{aligned}$$

so that rearrangement produces

$$\frac{k(0)}{x(0)} \geq Q_{a-g}(\infty) = \frac{1}{a-g};$$

which satisfies equation I. Thus, the conditions for equilibrium are satisfied.

By contrast, if $\psi_* > \frac{k(0)}{\mu}$, the final inequality above does not satisfy equation I. □

Lemma 5. *For $\Psi(t)$ to be strictly quasiconcave, either of the following are sufficient:*

1. $a \neq g$;
2. $\alpha > 0$ *and the Uzawa extraction condition holds.*

Proof. Define

$$D(t) \equiv \frac{d\ln\Psi(t)}{dt} = \frac{a-g}{e^{(a-g)t}-1} - \frac{\alpha}{\alpha+\gamma}\frac{r-g}{e^{(r-g)t}-1}. \tag{17}$$

Therefore

$$D'(t) = -\frac{(a-g)^2}{\left[e^{(a-g)t}-1\right]^2} - \frac{\alpha}{\alpha+\gamma}\frac{(r-g)^2}{\left[e^{(r-g)t}-1\right]^2}.$$

By l'Hôpital's rule, when $a = g$ or $r = g$, the whole term in which it is contained is zero. Thus, the stated conditions of the lemma suffice to ensure that either the first or second term of $D'(t)$ is negative.

If either term is non-zero, the whole expression is strictly negative. This suffices for $\ln\Psi(t)$ to be strictly concave and, thus, for $\Psi(t)$ to be strictly quasiconcave. □

To prove the Theorem, define $k_L \equiv \mu\psi_*$ and $k_H \equiv \mu\psi^*$.

Proof. When $k(0) \in I_L$, $k(0) \geq 0$ and $\psi_* > \frac{k(0)}{\mu}$. Lemmas 1 and 2 and the continuity of $\Psi(t)$ ensure that there is a single finite T such that $\Psi(T) = \frac{k(0)}{\mu}$. By Lemma 3, this implies a unique equilibrium with finite extinction date.

As the inequality in ψ_* is the reverse of the necessary and sufficient condition in Lemma 4, there are no equilibria with infinite extinction dates.

Now consider $k(0) \in I_M \Rightarrow \psi_* < \frac{k(0)}{\mu} < \psi^*$. The first of these ensures, by Lemma 4, the existence of an equilibrium with an infinite extinction date. For the second inequality to hold, it must be that $\psi_* < \psi^*$. By the continuity of $\Psi(t)$ and the definition of ψ^*, there are two finite T such that $\Psi(T) = \frac{k(0)}{\mu} < \psi^*$. By Lemma 3, these are equilibria with finite extinction times.

When $k(0) \in I_H$, $\frac{k(0)}{\mu} > \psi^*$. Thus, by Lemma 3, there are no equilibria with finite extinction times; by Lemma 4, there is one without extinction.

Now consider the degenerate cases. First, $k(0) = k_L = k_H \rightarrow \frac{k(0)}{\mu} = \psi_* = \psi^*$. By Lemma 3, there is no equilibrium with finite extinction as ψ_* is only reached as $T \rightarrow \infty$. Lemma 4 is satisfied with equality, producing an equilibrium without extinction.

Finally, when $k_L \neq k_H$, Lemma 4 is satisfied. Now a single finite T satisfies Lemma 3, tangentially when $k(0) = k_H$. □

Figure 1 shows an example. Multiplying the horizontal axis by μ allows $\Psi(t)$ to be replaced by $k(0)$, easing interpretation.

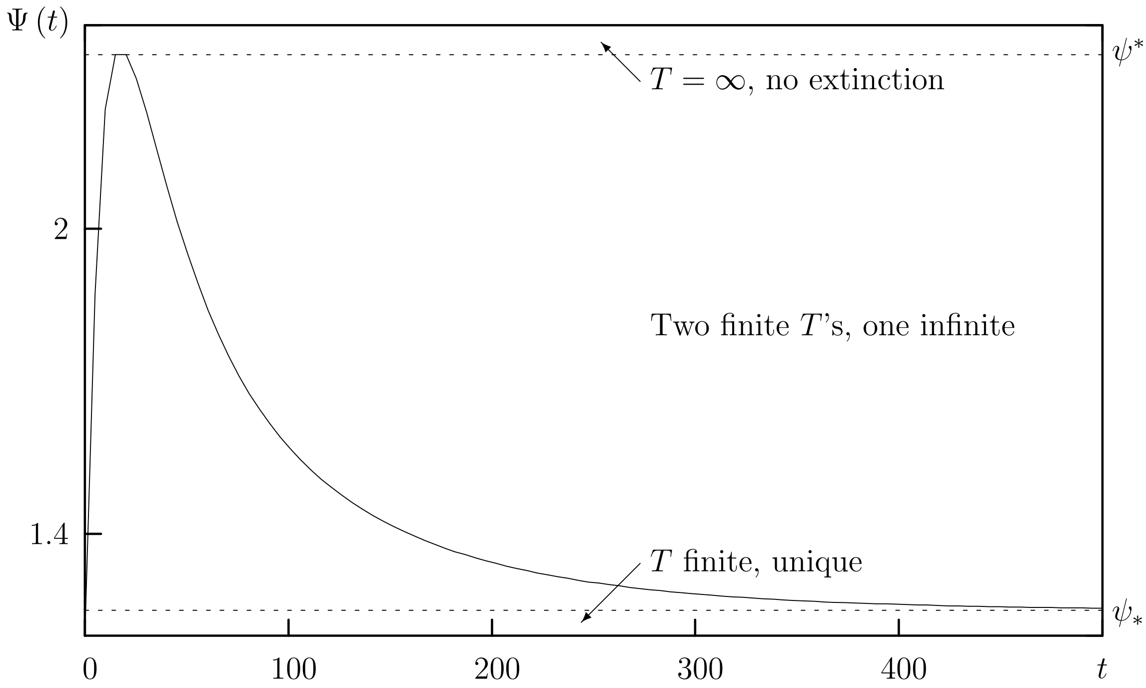

Figure 1: Extinction dates when $\alpha = \gamma = 1, \rho = 0.05, a = 0.1, r = 0.03$

To this point, it has been assumed that $a > r$. While this presents the most interesting class of cases, its complement contains some canonical cases. Non-renewable resources, for which $a = 0$ are the most obvious. These cases may be analysed using the objects already developed; in some cases, particular terms will be modified if their arguments are negative instead of positive.

Theorem 2. *When the Uzawa extraction condition holds:*

1. *I_L is always non-empty.*
2. *I_M is non-empty iff $a > r$.*
3. *I_H is empty when $a \leq g$.*

Proof. 1. the continuity of $\Psi(t)$ and $\Psi(0) = 0$ ensure the result if $\psi_* > 0$. When $a > g$, this has already been demonstrated in Lemma 2. When $a = g$ and $r > g$,

$$\Psi(t) = t\left[\frac{r-g}{1-e^{-(r-g)t}}\right]^{\frac{\alpha}{\alpha+\gamma}}. \tag{18}$$

Thus, the Uzawa extraction condition ensures that $\psi_* = \infty$. Finally, when $a < g, \lim_{t\to\infty} Q_{a-g}(t) = \infty$; as $\lim_{t\to\infty} Q_{r-g}(t), \psi_*$ is again infinite.

2. Sufficient conditions for the existence of I_M are that $\Psi'(t) = 0$ for a finite t and that $\Psi(t)$ be strictly quasiconcave. Consider all possible cases.

Under the Uzawa extraction condition, $a = g$ sets

$$\Psi'(t) = \frac{1}{[Q_{r-g}(t)]^{\frac{\alpha}{\alpha+\gamma}}}\left[1 + \frac{\alpha}{\alpha+\gamma}\frac{te^{-(r-g)t}}{Q_{r-g}(t)}\right] > 0 \forall t > 0.$$

Thus, $r > a = g$ suffices for an empty I_M.

Now consider $a \neq g$. Here

$$\Psi'(t) = \frac{e^{gt}}{Q_{r-g}(t)^{\frac{\alpha}{\alpha+\gamma}}}\left[e^{-at} - \frac{\alpha}{\alpha+\gamma}\frac{Q_{a-g}(t)}{Q_{r-g}(t)}e^{-rt}\right]. \tag{19}$$

Thus, a stationary point sets the square bracketed term to zero. Equivalently, it solves

$$\frac{e^{rt} - e^{gt}}{e^{at} - e^{gt}} = \frac{\alpha}{\alpha+\gamma}\frac{r-g}{a-g}. \tag{20}$$

To simplify analysis, define

$$\xi(t) \equiv \frac{e^{rt} - e^{gt}}{e^{at} - e^{gt}}.$$

Thus, $\xi(t)$ is continuous for all $t \geq 0$ and, by l'Hôpital's rule, $\xi(0) = \frac{r-g}{a-g}$. As this is greater in absolute value than the right hand side of equation 20 for all $\gamma > 0$, a sufficient condition for an empty I_M is that $(a-g)\,\xi'(t) \geq 0 \forall t$.

Calculation yields

$$\xi'(t) = \frac{(r-a)\,e^{(a+r)t} - (r-g)\,e^{(r+g)t} + (a-g)\,e^{(a+g)t}}{(e^{at} - e^{gt})^2}. \tag{21}$$

When $r > a > g$, this is positive.

Now consider $r > g > a$. By Lemma 5, $\Psi(t)$ is strictly quasiconcave. As $t \to \infty$, its denominator tends to $\left(\frac{1}{r-g}\right)^{\frac{\alpha}{\alpha+\gamma}}$, a positive finite number. Its numerator, however, tends to infinity. This, by strict quasiconcavity, precludes a maximum in finite t. Thus, I_M is empty under these conditions.

Now consider $a = r > g$. In this case, the square bracketed term in equation 21 is identically zero, so that $\xi'(t) = 0 \forall t$. This suffices, from above, for an empty I_M.

Finally, consider $a > r > g$, the case considered above. In this case, the denominator of $\xi(t)$ grows more quickly than the numerator, so that $\xi(t)$ asymptotes to zero as $t \to \infty$.

3. from the first steps in the proof, $\psi_* = \infty$ when $a \leq g$. With the Uzawa extraction condition, this suffices for an infinite ψ^*.

□

The non-extinction equilibrium may be explicitly eliminated by noting that allowing for $a < g$ replaces equation I with

$$Q_{a-g}(T) = \min\left\{\frac{k(0)}{x(0)}, \max\left\{\frac{1}{a-g}, 0\right\}\right\}.$$

Thus, the final inequality in Lemma 4 requires $x(0) = 0$. Equation A, in turn, then requires $\mu = 0$, a contradiction by definition of μ and ν.

In closing, consider the limit case of $\gamma = 0$, linear extraction costs. This may be interpreted as a situation in which there is a competitive market for the inputs (e.g. a labour market) into a CRS extraction function. Further, this corresponds to the case studied in Gaudet et al. (2002).

When $\gamma = 0$, extraction costs reflect only total extraction, rather than the rate of extraction. Thus, pulse extraction is no more costly than smooth extraction. Thus, "the extraction contest is so fierce that the common is drained in the instant storage is initiated" (Gaudet et al., 2002).

Theorem 3 (Gaudet et al. (2002) 'jump extinction'). *The extinction date, T, goes to zero with γ.*

The first condition includes $a = r$, the costless storage of Gaudet et al. (2002).

Proof. Assume that T is finite. Then, under the stated conditions, equations A and I combine to yield

$$\Psi(T) = \frac{k(0)}{\mu} = \frac{Q_{a-g}(T)}{Q_{r-g}(T)^{\frac{\alpha}{\alpha+\gamma}}}.$$

This may be rearranged and rewritten in terms of primitives for

$$\frac{k(0)}{\mu}\frac{r + a\gamma - \rho}{\left[(1+\gamma)r - \rho\right]^{\frac{\alpha}{\alpha+\gamma}}}\gamma^{-\frac{\gamma}{\alpha+\gamma}} = \left[1 - e^{-\frac{r+a\gamma-\rho}{\gamma}T}\right]\left[1 - e^{-\frac{(1+\gamma)r-\rho}{\gamma}T}\right]^{-\frac{\alpha}{\alpha+\gamma}};$$

so that, as $\lim_{\gamma\to 0}\gamma^{-\frac{\gamma}{\alpha+\gamma}}$,

$$\frac{k(0)}{\mu} = \lim_{\gamma\to 0}\left[1 - e^{-\frac{r+a\gamma-\rho}{\gamma}T}\right]\left[1 - e^{-\frac{(1+\gamma)r-\rho}{\gamma}T}\right]^{-\frac{\alpha}{\alpha+\gamma}}.$$

If T remained positive as $\gamma \to 0$, then the right hand side of the equation would converge to unity, a contradiction. Thus, $T \to 0$ as γ does. □

Thus, jump extinctions only require that extraction costs become linear. Unlike Gaudet et al. (2002), there is no condition on the cost of storage. The difference between these results does not seem reflect the difference between storage alone and full capital market access: agents are not taking advantage of their ability to borrow against future income here. Instead, they are banking and saving it all initially.

4 The commons without capital markets

This section compares the RE equilibria with capital market access to those without such access.

When individuals do not have access to capital markets, their consumption and effort smoothing problems are addressed as a unit. Intertemporal budget constraint 5 is replaced by $c_i(t) = x_i(t)$; feasibility constraint 3 remains the same. Thus, agents maximise

$$\int_0^{\infty} e^{-\rho t} \left[\frac{x_i(t)^{1-\alpha}}{1-\alpha} - \frac{1}{\theta_i} \frac{x_i(t)^{1+\gamma}}{1+\gamma} \right] dt \text{ subject to constraint 3.}$$

Maximising produces

$$\tilde{x}_i(t) = \left\{ \begin{array}{lll} \theta_i^{\frac{1}{\alpha+\gamma}} & \text{for} & t < T \\ 0 & \text{for} & t \geq T \end{array} \right\}. \quad (22)$$

Integrating over all agents yields aggregate extraction

$$\tilde{x}(t) = \left\{ \begin{array}{lll} \mu \nu^{-\frac{1}{\alpha+\gamma}} & \text{for} & t < T \\ 0 & \text{for} & t \geq T \end{array} \right\}. \quad (23)$$

The tilde distinguishes this solution from that with capital market access. As agents are unable to intertemporally smooth, they maximise myopically, extracting at the instantaneously optimal rate without consideration of the stock consequences. Thus, extraction is also independent of the initial capital stock.

Extinction dates without capital market access may be compared to the results derived in Theorem 1 for capital market access:

Theorem 4. *When agents do not have access to capital markets, there is a unique, finite extinction date iff*

$$k(0) < \frac{\mu}{a\nu^{\frac{1}{\alpha+\gamma}}}.$$

Otherwise, the unique equilibrium has no extinction.

Proof. Substitution of equation 23 into equation of motion 2 yields

$$e^{-a\tilde{T}} = 1 - \frac{a\nu^{\frac{1}{\alpha+\gamma}}}{\mu} k(0)$$

when $k(t)$ is set to zero. Under the conditions of the theorem, this has the unique, finite solution

$$\tilde{T} = \frac{1}{a} \left\{ \ln \mu - \ln \left[\mu - a\nu^{\frac{1}{\alpha+\gamma}} k(0) \right] \right\}. \quad (24)$$

Thus, as $k(0) \nearrow \frac{\mu}{a\nu^{\frac{1}{\alpha+\gamma}}}$, $\tilde{T}$ approaches infinity asymptotically. □

Thus, $\tilde{T}$ is convex in $k(0)$. Again, the Uzawa conditions derived earlier are assumed to hold.

Lemma 6. *A necessary condition for initial extraction without capital markets to exceed that with capital markets is*

$$\frac{\gamma}{(1+\gamma)r-\rho} > \left[\frac{\alpha}{\rho-(1-\alpha)r}\right]^{\alpha}.$$

Given any parameter values, initial extraction without capital markets will be less than that with capital markets for at least some commons stocks.

Proof. By equations A and 23, $\tilde{x}(0) > x(0)$ requires

$$[Q_{r-g}(T)]^{\alpha} > \nu > 0.$$

The right hand side is constant in $k(0)$; it is positive by the Uzawa consumption condition and the assumption that $\alpha > 0$. The left hand side increases in T. Thus, a necessary condition for the inequality to hold is that it hold at

$$T = \infty \Rightarrow Q_{r-g}(\infty) = \frac{1}{r-g}.$$

The condition follows from the definitions of g and ν.

The second part of the lemma follows from noting that

$$k(0) = 0 \Rightarrow T = 0 \rightarrow Q_{r-g}(0) = 0 < \nu;$$

when the Uzawa consumption condition holds. □

Figure 2 displays an example of the effect of capital market access on extinction dates.[12] The curve referring to capital market access is that in Figure 1. Here, low levels of commons stock are preserved for longer by individuals without access to capital markets. Above $k(0) = \mu\psi_*$, an intermediate zone is entered. In this, the extinction date without capital market access lies between the two finite extinction dates with capital market access. At the same time, there is an equilibrium with no extinction when individuals have access to capital markets.

Finally, above a higher level of $k(0)$, the extinction date without capital market access is greater than both of the finite dates with access. Again, though, there is a non-extinction equilibrium with capital market access.

Extinction dates are poor proxies for welfare: late extinction dates are obtained by low extraction rates. Consider a situation in which the commons

[12]Maple code available from the authors upon request.

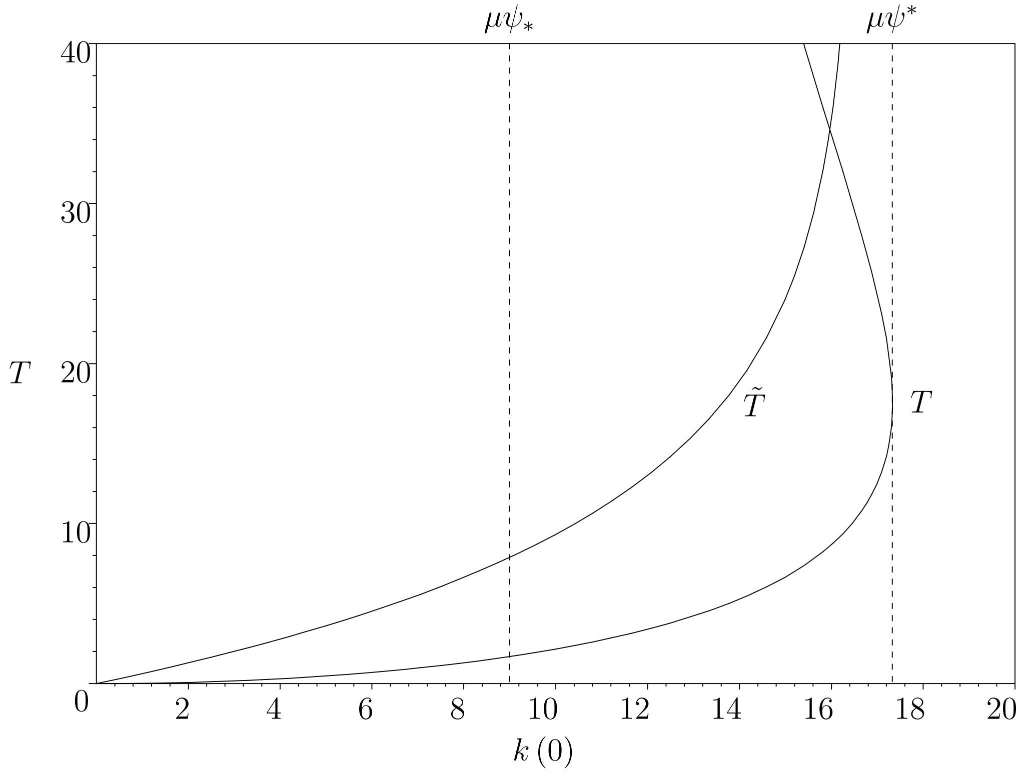

Figure 2: Extinction dates varying in $k(0)$ when $\alpha = \gamma = 1, \rho = 0.05, \theta_i = e \forall i, a = 0.1, r = 0.03$

is exhausted one period earlier under capital market access than it is without it. In this case, one period of returns at rate a are lost, but the privately stored extraction in that period then grows at rate r. By contrast, in the situation without capital market access, the extra period of a growth is not balanced by future r growth. Thus, we now compare welfare directly.

The equilibrium welfare obtained by the infinitesimal agents di with access to capital markets may be expressed in terms of initial extraction, $x_i(0)$ by substitution of equations 8, 11 and 12 into equation 4:

$$W_i = \frac{\alpha + \gamma}{(1-\alpha)(1+\gamma)} \theta_i^{\frac{1-\alpha}{\alpha+\gamma}} \nu^{\frac{1+\gamma}{\alpha+\gamma}} \left[Q_{r-g}(T)\right]^{\frac{1-\alpha}{\alpha+\gamma}\gamma}. \tag{25}$$

As T is not generally a function of $k(0)$, this cannot be expressed as a function of $k(0)$ directly.

The welfare obtained by individuals di without access to capital markets may also be calculated:

$$\tilde{W}_i = \frac{1}{\rho} \frac{\alpha + \gamma}{(1-\alpha)(1+\gamma)} \theta_i^{\frac{1-\alpha}{\alpha+\gamma}} \left[1 - e^{-\rho \tilde{T}}\right]. \tag{26}$$

In the special case of $\alpha = 1$, this becomes

$$\tilde{W}_i = \frac{\ln \theta_i - 1}{1+\gamma} \frac{1 - e^{-\rho \tilde{T}}}{\rho}. \tag{27}$$

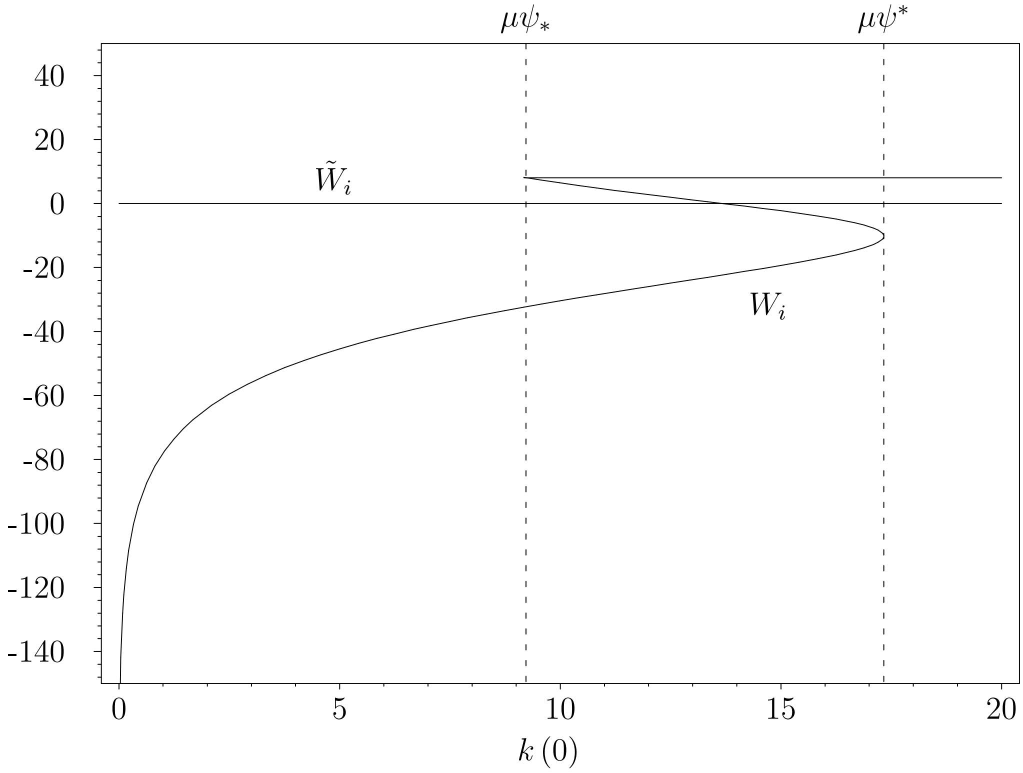

Figure 3: Welfare varying in $k(0)$ when $\alpha = \gamma = 1, \rho = 0.05, \theta_i = e \forall i, a = 0.1, r = 0.03$

Figure 3 displays the results for the same parameter values as those used above. These have been selected to 'normalise' welfare under autarky to zero for all commons endowments. This has a nice interpretation: all rents are dissipated regardless of initial commons stock.

When $\alpha = 1$, equation 27 shows that welfare may increase or decrease in $k(0)$ under finite extinction. We identify this latter case as a form of 'resource curse': higher levels of endowment support equilibria in which agents correctly expect that they will deplete the commons more energetically, extinguishing it too quickly. These cursed equilibria correspond to high extraction costs: $\theta < e$.

The rising curve, becoming a horizontal line, represents welfare with access to capital markets. It also exhibits a cursed equilibrium. Nevertheless, this still yields higher welfare than does the more intuitive upward sloping segment.

Comparing the levels of welfare under autarky and with capital market access, it may be seen that the autarkic equilibrium dominates for all $k(0) < \mu\psi_*$. Below that point, agents with capital market access expect such a rapid extinction that they undertake very rapid, thus expensive, extraction themselves, banking the proceeds.

Beyond that point, there are equilibria with capital market access that dominate the autarkic. At the point at which the equilibrium with most rapid extinction comes to dominate the autarkic, its extinction date is still much shorter than the autarkic's, justifying our earlier conjecture.

5 Taxation

Suppose now that the government imposes an extraction tax, δ, so that infinitesimal agents di retain $x_i^\delta = (1-\delta) x_i$ after having extracted quantity x_i.

In the case of literal commons, tax revenue earned might be spent providing public goods. We, however, follow the Tornell and Velasco (1992) interpretation, and consider institutional commons. Tax revenue, δx, is returned to the commons. Thus, taxes both reduce agents' productivity from θ_i and replenish the commons.

The equations in Section 3 may largely be rewritten in terms of x_i^δ instead of x_i. The consumption smoothing problem is subject to the feasibility constraint

$$X_i^\delta(r) = \int_0^\infty e^{-rt} x_i^\delta(t)\, dt;$$

which replaces X_i in subsequent calculations. The consumption calculations are otherwise unchanged.

As to extraction, first order condition 12 becomes

$$x_i^\delta(0)^{\alpha+\gamma} = \frac{\theta_i \nu}{\left[Q_{r-g}\left(T^\delta\right)\right]^\alpha} (1-\delta)^{1-\alpha};$$

while the first fundamental equation, A, is now

$$x^\delta(0) = \frac{\mu}{\left[Q_{r-g}\left(T^\delta\right)\right]^{\frac{\alpha}{\alpha+\gamma}}} (1-\delta)^{\frac{1-\alpha}{\alpha+\gamma}}.$$

The extraction rate relative to the situation without a tax therefore depends on two factors: a direct effect, and an indirect effect through the extinction date.

The second fundamental equation, I, becomes

$$Q_{a-g}\left(T^{\delta}\right) = \min\left\{\frac{1}{1-\delta}\frac{k(0)}{x(0)}, \frac{1}{a-g}\right\}.$$

Thus, for finite T^{δ}, the effect of an extraction tax is to 'inflate' $k(0)$. Whether this increases or decreases the extinction date depends on the equilibrium selected.

Theorem 1 goes through when k_L and k_H are replaced by $k_L^{\delta} \equiv \mu\psi_*(1-\delta)^{\frac{1+\gamma}{\alpha+\gamma}}$ and $k_H^{\delta} \equiv \mu\psi^*(1-\delta)^{\frac{1+\gamma}{\alpha+\gamma}}$. This, in turn, alters the conditions in Lemmas 3 and 4. Otherwise, the remaining results go through unchanged.

As $(1-\delta)^{\frac{1+\gamma}{\alpha+\gamma}} < 1$ for all $\delta > 0$, the effect of an extraction tax is to reduce toward zero the boundaries of the intervals defined in Theorem 1. Some initial capital stocks that, without taxation, were in I_L (*resp.* I_M) are, with taxation, in I_M (*resp.* I_H). Thus, taxation increases the set of $k(0)$ over which non-extinction is possible.

The welfare calculations in equation 25 may be altered for

$$W_i^{\delta} = \theta_i^{\frac{1-\alpha}{\alpha+\gamma}}\nu^{\frac{1+\gamma}{\alpha+\gamma}} \left\{\frac{1}{1-\alpha}(1-\delta)^{\frac{(1-\alpha)^2}{\alpha+\gamma}}\left[Q_{r-g}\left(T^{\delta}\right)\right]^{-\frac{1-\alpha}{\alpha+\gamma}\alpha} - \frac{1}{1+\gamma}(1-\delta)^{\frac{(1-\alpha)(1+\gamma)}{\alpha+\gamma}}\left[Q_{r-g}\left(T^{\delta}\right)\right]^{\frac{1-\alpha}{\alpha+\gamma}\gamma}\right\}. \quad (28)$$

The effect of taxation appears in both the consumption and extraction terms.

Figure 4 displays the welfare consequences of an extraction tax. This is set at $\delta = .5$ for illustrative purposes; the other parameters are as they were in earlier figures.

In the equilibria without extinction, the tax does not alter welfare in this example. This is a consequence of $\alpha = 1$, which reduces the difference between equations 28 and 25 to one of extinction dates. Without extinction, these are the same when $\alpha = 1$.

Taxation is welfare improving in the example: for any $k(0)$, the worst equilibrium dominates the best corresponding one without taxation. It can be seen to do this through two channels. First, it makes the non-extinction equilibrium feasible for lower levels of $k(0)$. Second, the finite extinction equilibrium in which welfare increases in $k(0)$ dominates that without an extraction tax. In this case, the replenishment effect of the extraction tax seems to dominate its reduced productivity effect.

Finally, note that the domain over which the cursed equilibrium is possible is reduced by taxation. In the extreme, at $\delta = 1$, the I_M interval disappears, eliminating the cursed equilibrium.

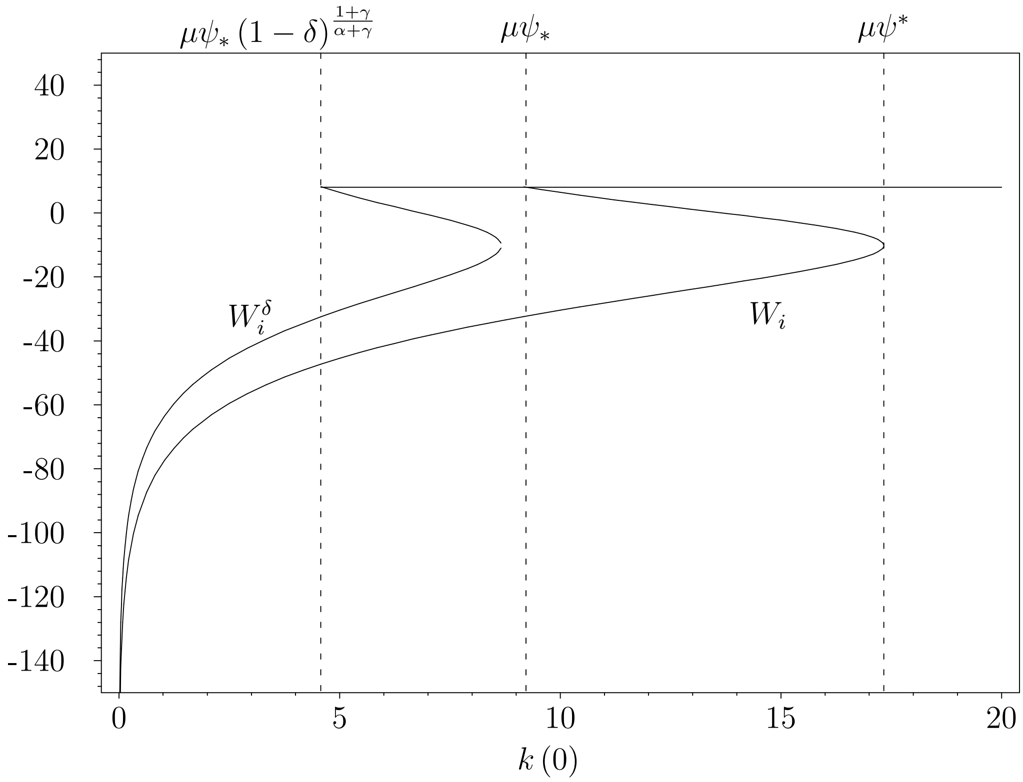

Figure 4: Welfare consequences of $\delta = .5$; other parameters as before

A consumption tax may also be considered using similar techniques. Supposing that the tax that reduces consumption to $c_i^\varepsilon = (1-\varepsilon)\, c_i$ modifies equation 1 for

$$\dot{k}(t) = ak(t) - x(t) + \varepsilon c(t), k(0) > 0.$$

All the steps taken above may be repeated. As the equation of motion is now more complicated, so is the new version of equation I.

Savings taxes are more difficult to consider. These would reduce $x_i - c_i$ when this difference was positive, but not otherwise, introducing a kink into agents' problems. Such a tax corresponds most closely to capital controls.

6 Discussion

We have analysed extraction from a commons when agents have access to capital markets. Comparison to the standard in the literature, in which agents do not have such access, shows that the results can differ significantly. Qualitatively, the difference appears to be largest when the resource in the commons grows quickly. A further comparison that may be of interest, given the existing literature, would be to the situation with storage.

We do not attempt to test these results empirically. Instead, we merely note some of the challenges that doing so presents. First, some proxy for the presented discounted value of utility is necessary. A measure like GDP may be a reasonable proxy. Finding a measure of $k(0)$ is more difficult, for at least two reasons. First, data refer to $t > 0$: savings and borrowing have begun. Second, communally owned resources do not seem to correspond to a clear category: measures of natural capital, for example, may overestimate (if some natural capital is privately owned) or underestimate them (if other resources are also weakly owned).

The extraction tax considered above may be thought of as decreasing individuals' θ_i, increasing their extraction costs. At the outset of the taxation discussion, it was noted that a literal understanding of the commons might not return tax revenue raised to the commons. Under this interpretation, governments may spend it on public goods of various sorts. Within the framework of this model, the one parameter that might be influenced by such expenditures is individuals' productivity, θ_i. If tax revenue increased θ_i, then the welfare consequences of the tax might be the opposite of those developed above.

Five directions for future research seem fairly natural. First, the specification of the constant a is, in general, an over-simplification. Biological models typically allow the growth rates to be functions of the stock (q.v. Dockner and Sorger, 1996; Kremer and Morcom, 2000), $a(k)$ in this case. This generalisation leaves equation A unchanged. Equation 2, however, becomes

$$k(t) = \max\left\{0, e^{\int_0^t a(k(\tau))d\tau}\left[k(0) - \int_0^t e^{-\int_0^\tau a(k(\sigma))d\sigma} x(\tau)\, d\tau\right]\right\}.$$

Thus, equation 14 becomes

$$k(0) = x(0)\int_0^T e^{gt - \int_0^t a(k(\tau))d\tau} dt.$$

Integrability requires a new Uzawa extraction condition; the equivalent of equation I will no longer be a clear expression of $Q_{a-g}(T)$.

Second, the present analysis has not considered strategic agents, as in Tornell and Velasco (1992) and Tornell and Lane (1999). Strategic analysis requires calculating optimal strategies for every combination of private savings and commons stock that can be reached from the problem's initial conditions. Thus, the Hamilton-Jacobi-Bellman equation is a PDE in private savings and commons stock.

Technically, this seems related to how relaxation of the commons assumption could be approached. In an environment with weak property rights, it may be more costly nonetheless to access the surplus generated by others than it is one's own. Again, this suggests differentiated commons stocks, and PDEs.

Fourth, the capital stock is not an argument in the utility function in this model. This seems more consistent with an interpretation of k as physical capital than as natural capital. In the latter case, k might provide ecosystem services directly. Allowing k to enter directly into the utility function would allow re-analysis of the problem as the marginal rate of substitution between k and c_i varied. This might contribute to the 'weak' and 'strong' sustainability debate.

Finally, the absence of default provisions is an obvious limitation of the present analysis.

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