Asset prices and capital accumulation in a monetary economy with incomplete markets

Sunanda Roy *

College of Business and Public Administration, Drake University

email: sunanda.roy@drake.edu, Tel: (515) 271 1983

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Abstract

The paper studies asset prices and capital accumulation in a monetary economy with non-diversifiable idiosyncratic risks (incomplete markets). A government issued unbacked currency is introduced into agent's preferences in a dynamic GEI (General Equilibrium with Incomplete market) model with CARA preferences and normal disturbances. Closed form expressions for equlibrium allocations and prices are derived under finite and infinite horizons. The paper addresses several monetary issues. In particular, money is shown to be neutral but not superneutral at the steady state. The rate of inflation is shown to adversely affect the steady state capital stock under some situations. Finally the Friedman rule is shown to be non-optimal for some economies.

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1 Introduction

The paper studies a heterogenous agent monetary economy in which agents are exposed to uninsurable idiosyncratic risks (markets are incomplete) and proves the following results. First, the rate of money growth affects the steady state real riskfree rate and the capital stock - that is to say money is neutral but not superneutral at the steady state. Second, since the real riskfree rate is affected by the rate of money growth or inflation, the Fisher relationship between the rate of inflation and the nominal interest rate is not one on one. In the general case in which the agents are affected by uninsurable production risks it is hard to specify whether the real riskfree rate will rise or fall - because of two opposing effects on the real riskfree rate and capital stock. In a special case, however, in which production is assumed to be riskfree but agents are exposed to an exogenous uninsurable endowment shock, a rise in the inflation rate is shown to increase the real riskfree rate and hence cause a greater than proportional increase in the nominal rate. Finally, the Friedman rule which sets the nominal interest rate to zero is shown to be suboptimal for some economies, as a result of this monetary non-superneutrality.

To prove the above results we introduce a government issued unbacked currency into an agent's preferences in a dynamic general equilibrium model with incomplete markets with CARA preferences and normal disturbances, thus combining a standard Sidrauski model (Sidrauski 1967) of money with an existing GEI (General Equilibrium with Incomplete markets) set up. Closed form expressions for equlibrium allocations and prices are derived under finite and infinite horizons. The paper then proceeds to address some of the standard issues in monetary economics mentioned above in the new set up.

In this new set up, money remains superneutral at the steady state when the variance of uninsurable risks is zero - that is markets are complete - as is to be expected in a standard Sidrauski model. However, when this variance is positive - that is markets are incomplete - there exist two distinct channels through which the rate of money growth is shown to influence the real riskfree rate and the capital stock.

First, households resort to precautionary savings to reduce the variance of future consumption which in this model is proportional to the variance of uninsured risks with the factor of proportionality given by the square of the marginal propensity to consume. The marginal propensity to consume is shown to be inversely related to the rate of inflation, (equal to the money growth rate at the steady state). Consequently, the real risk free rate which is determined by the amount of precautionary savings (proportional to the variance of future consumption) is affected by changes in the rate of inflation (rate of money growth). Changes in the real risk free rate through this first channel - which we describe as the ”precautionary channel” - in turn affects the steady state capital stock.

Secondly, when the return on physical capital is subject to uninsurable production risks, the price of capital in equilibrium is not equal to its marginal product (as it would be if markets were complete) but to the sum of the marginal product and a risk premium which is once again shown to be proportional to the variance of uninsurable risks with the factor of proportionality depending on the marginal propensity to consume. Given a real riskfree rate, the price of capital and hence the steady state capital stock is therefore affected by changes in the marginal propensity to consume as a result of changes in the rate of inflation (rate of money growth). We describe this second channel as the ”risk premium” channel.

Because of these two effects discussed above, the relationship between the rate of inflation and the nominal interest rate (the Fisher eqiation) is not one on one in general, as the real riskfree rate changes with changes in the rate of inflation. This is in direct contrast with the situation when markets are complete. When markets are complete, a rise in the inflation rate brings about an equal percentage rise in the nominal rate of interest. It is difficult to specify in general in this model whether a rise in the rate of inflation brings about a rise or a fall in the real riskfree rate - hence whether the relationship between the inflation rate and the nominal interest rate is more than or less than one on one - as the two effects described above have opposite signs. It is however possible to be more specific in the special case in which production is assumed to be riskfree and the uninsurable risks are to an exogenous stochastic endowment. In this case we can see that only the first channel is operative whereas the second channel is not. The rate of inflation can be shown to positively affect the riskfree rate (negatively affect the steady state capital stock) in this special case.

Finally the paper shows by an example that even with standard preferences which are separable in consumption and money, the Friedman rule - setting the money growth rate to the negative of the real rate of interest, such that the nominal rate of interest is zero - is not optimal for some economies when markets are incomplete. (The rule remains optimal when markets are complete) In fact for any specific parameterization of the model an optimal money growth rate - defined as the money growth rate which maximizes average steady state consumption - may not exist, may uniquely exist or there may be multiple such rates.

The question of how the rate of money growth influences capital and output is an old classic one and dates back to Tobin (Tobin 1965) at least, culminating in the more recent works of Barro (Barro 1996) and many others (Bruno and Easterly 1998, Andres and Hernando 1999). The present paper supplements that long literature in its attempt to establish a relationship based on certain key features of market incompleteness. According to the Tobin effect, a rise in the rate of inflation reduces the return on cash holdings and induces households to invest more of its wealth in an alternative asset, physical capital. Although such a tradeoff exists by construction in the present model too, the steady state relationships between the rate of inflation, the capital stock and the real interest rate are driven by the effect of the first on precautionary savings and the risk premium rather than by the Tobin effect. The present paper also partly vindicates Barro's conjecture (discussed later by many others) that the rate of inflation and output growth are inversely related. The discussion in the previous paragraph shows that theoretically such an inverse relationship exists albeit in the special case of uninsurable (exogenous) endowment but no production risks.

Criticisms of the Friedman rule as the optimal monetary policy also has a long history, such as in the work of Phelps (1973), Chari, Christiano and Kehoe (1991, 1996) and others. Most of these works assume complete markets and the essence of the criticism is usually that the government may have to resort to other welfare reducing, distortionary taxes to make up for the loss of seigniorage revenue under the Friedman rule. The current paper supplements this literature too, by showing that the Friedman rule may be suboptimal even when the government does not have a specific objective which requires it to impose compensatory taxes. Rather, the suboptimality comes from the feature that the steady state capital stock and consumption are affected by the money growth rate.

Dynamic general equlibrium models with uninsurable production risks which yield closed form solutions for equilibrium allocations (and are hence computable) are few in number.[1] In a series

[1] Some of the papers which discusses dynamic GEI models which have closed form solutions in various contexts are Magill and Quinzii, (2000), Krebs (2003a, 2003b).

of recent papers, Willen (1999), Calvet(2001), Calvet and Angeletos (2001, 2003) and Athanasoulis (2005) amongst others have studied issues related to asset prices, capital accumulation and endogenous cycles assuming CARA preferences and normal shocks. Very recently, Angeletos (2005) has studied capital accumulation and cycles in a model with the more mainstream CRRA preferences. To the best of my knowledge however, there has been no study of these specific monetary issues within a dynamic GEI model.[2] The present paper seeks to fill that gap and demonstrates that money can have different steady state effects when markets are incomplete compared to when they are not.

The CARA assumption on preferences has certain known drawbacks, chief of which is that unlike its more mainstream CRRA counterpart, it does not take into account the effect of wealth on precautionary savings behavior.[3] This specification is nonetheless used here for a first study of monetary issues within an incomplete markets framework, because of its great analytical convenience compared to the CRRA specification. The equilibrium conditions are simpler and easier to interpret than they are under CRRA assumption. The extension of the current analysis to address wealth effects using CRRA preferences is left for the future.

In addition to assuming that agents have CARA preferences, to keep the model simple, we also assume that they maximize expected utility. This has the drawback that the model does not distinguish between risk aversion and the intertemporal elasticity of substitution. Such distinctions are less important in this present context given the nature of the comparative static questions we ask and are also consequently left for future work.[4]

[2] There are several papers however which deal with money in two period incomplete market models - such as Gottardi (1994), Magill and Quinzii (1992), to mention some.

[3] Other important criticisms are that the CARA specification does not allow for balanced growth (King, Plosser and Rebelo, (2002) and that it can lead to negative consumption (see Ljungqvist and Sargent (2004).

[4] A usual method in the GEI literature to distinguish between the two effects is to assume that agents have non-expected utility of the Kreps-Porteus/Epstein-Zin type (see Kimball and Weil (2003); Angeletos (2005)).

Section 2 describes the model economy. Section 3 derives the dynamic equilibrium path assuming a finite horizon and discusses some of the interesting features of the monetary economy compared to the non-monetary one. Finally Section 4 derives the infinite horizon steady state and discusses the issues of monetary neutrality and super-neutrality, the Fisher relation and the Friedman rule.

2 The Model

The economy consists of a continuum of households, indexed by $h \in H = [0,1]$. Each household lives for T periods where T may be finite or infinite. Each household has access to a specific and risky production technology which uses capital as the only input. The same good is used for both consumption and investment. Capital depreciates by a constant fraction δ every period and the consumption good can be costlessly converted into physical capital. The hth household's production function is given by, $y_t^h = \eta_t^h f(k_t^h)$, where η_t^h is a household specific productivity shock, f is the production function and k_t^h the undepreciated capital stock at date t. The production function is assumed to satisfy the usual neoclassical assumptions of concavity and Inada conditions.

Households derive utility from consumption and from holding an unbacked government liability which does not yield any return and which we call currency or cash. Currency or cash can be purchased at date t for $\frac{1}{p_t}$ units of the consumption good where p_t is the price of the consumption good in units of currency. Cash holdings provide utility to the households by reducing the transactions costs of exchange (for example buying bonds) for households.

Besides being allowed to invest in physical capital, households are also free to trade in a real, one-period, riskfree bond whose payoff at date t is one unit of the consumption good or p_t units of money and whose price is π_t units of money.[5]

At the beginning of each period t, the households have a capital stock k_t^h and a cash stock m_t^h carried over from the previous period and an amount of one-period real bonds θ_{t-1}^h purchased in the previous period. At date t, it chooses its current consumption c_t^h, the stock of capital k_{t+1}^h to carry over to the next period, an amount of cash m_{t+1}^h and an amount of real bonds θ_t^h. The real balances $\frac{m_{t+1}^h}{p_t}$ purchased at date t provides current utility but the household is forced to carry it over to the next period. History begins at date 0 with a given capital k_0^h and cash stock m_0^h for each household and an initial price p_0, normalized to 1.

The household budget constraints at date 0 and date t are thus respectively given by,[6]

$$
\begin{aligned}
\frac{m_0^h}{p_0} + \eta_0^h f(k_0^h) + (1-\delta)k_0^h &= c_0^h + k_1^h + \frac{\pi_0}{p_0}\theta_0^h + \frac{m_1^h}{p_o}, \text{ where } p_0 = 1 \\
\frac{m_t^h}{p_t} + \eta_t^h f(k_t^h) + (1-\delta)k_t^h + \theta_{t-1}^h &= c_t^h + k_{t+1}^h + \frac{\pi_t}{p_t}\theta_t^h + \frac{m_{t+1}^h}{p_t} \qquad (1)
\end{aligned}
$$

[5] Assets may be assumed to be short lived without loss of generality in a CARA set up. Security prices turn out to be non-stochastic in equilibrium under this set up which implies that the introduction of long lived assets does not affect the span of the assets (i.e. the market subspace) at any date - a potential source of complication in any dynamic analysis of financial markets. Since the model doesn't change qualitatively by having long-lived assets we keep matters simple by assuming that the bond in question is a one-period bond.

[6] To keep the number of symbols used to a minimum we assume that agents in this model have no source of income other than production using physical capital. In the body of the paper, however, on a couple of occasions we compare the steady state of the present economy with risky production with the steady state of an economy in which production is riskfree but agents have exogenous endowments which are subject to uninsurable risks. Note that with an additional exogenous (risky) endowment term the household's current budget constraint becomes,

$$\frac{m_t^h}{p_t} + e_t^h + \eta_t^h f(k_t^h) + (1-\delta)k_t^h + \theta_{t-1}^h = c_t^h + k_{t+1}^h + \frac{\pi_t}{p_t}\theta_t^h + \frac{m_{t+1}^h}{p_t}$$

where e_t^h is the endowment. Introducing exogenous endowment to the model merely adds another term to the defintion of the current wealth of a household and does not change the model or the results in any way

All households have identical preferences,additively separable in consumption and real cash balances. The state independent utility functions are CARA for both consumption and cash balances. The lifetime utility of the hth household is

$$E_0 \sum_{t=0}^{T} \beta^t \left(-\frac{1}{A}\exp(-Ac_t^h) + \gamma\left(-\frac{1}{A}\exp\left(-A\frac{m_{t+1}^h}{p_t}\right)\right)\right) \tag{2}$$

where A is the degree of absolute risk aversion (assumed same for consumption and cash without loss of generality), β the discount factor and γ a preference parameter.

The government issues an amount $(M_{t+1} - M_t)$ of new currency at date t. It consumes the seigniorage $(M_{t+1} - M_t)/p_t$ from the new currency at date t. M_t is exogenously given and deterministic. Government consumption G_t at date t is thus given by

$$G_t = \frac{M_{t+1} - M_t}{p_t} \tag{3}$$

Assumption 1 *η_t^h is normal with mean η and variance σ_p^2. η_t^h is identically and independently distributed over time and across agents.*

In the present set up with a single riskfree asset, the variance of η_t^h, denoted σ_p^2, measures the non-diversifiable risk for a household and hence may be used as a measure of the extent of market incompleteness.[7]

We assume that idiosyncratic shocks cancel across households in the aggregate, that is

[7] In a model with multiple assets the variance of non-diversifiable risks and hence a measure of market incompleteness is given by the variance of the OLS residual $\tilde{\eta}_t^h$ under an OLS decomposition of the productivity shocks on the asset returns. Thus with J risky assets and the return of the jth asset given by $d_{j,t}$, we can express

$$\eta_t^h = \eta + \sum_{j=1}^{J} \kappa_j^h d_{j,t} + \tilde{\eta}_t^h$$

where $\eta = E(\eta_t^h)$, $\kappa_j^h = \mathrm{Cov}(\eta_t^h, d_{j,t})/\mathrm{Var}(d_{j,t})$.

Assumption 2 $\int_H \eta_t^h = \eta$.

This is a crucial assumption which keeps the model mathematically tractable because it removes aggregative shocks and causing asset prices and total output to be deterministic.

3 The dynamic equilibrium path

In this section we derive the dynamic equilibrium of the above economy, assuming a finite horizon first. Some of the interesting features of the monetary economy are identified and contrasted with those of the non-monetary economy. In the next section we extend the analysis to the infinite horizon and the steady state of the economy.

We begin by defining a competitive equilibrium for the economy.

Definition 1 *A competitive equilibrium is a set of individual allocations* $(\{c_t^h\}_{t=0}^T, \{k_{t+1}^h, m_{t+1}^h, \theta_t^h\}_{t=0}^{T-1})$ *and a set of market prices* $\{p_{t+1}, \pi_t\}_{t=0}^{T-1}$ *such that*

(i) each household takes prices to be given and maximizes (2) subject to (1) for each t

(ii) the markets for the currency, the final good and the bond clear,

$$\int_H m_{t+1}^h = M_{t+1} \tag{4}$$

$$\int_H (c_t^h + k_{t+1}^h) + g_t = \int_H (\eta_t^h f(k_t^h) + (1-\delta)k_t^h) \tag{5}$$

$$\int_H (\theta_t^h) = 0 \tag{6}$$

In the next three subsections we show that a competitive equilibrium exists in which prices and aggregate output are deterministic and consumption is affine in current wealth.

3.1 Individual decisions

The optimal choices of θ_t^h, m_{t+1}^h and k_{t+1}^h must satisfy the Euler equations for $t = 0 \ldots T-1$,

$$
\begin{aligned}
\frac{\pi_t}{p_t} u_c^h(c_t^h) &= \beta E_t(u_c^h(c_{t+1}^h) && (7)\\
\frac{1}{p_t} u_c^h(c_t^h) &= \gamma(u_m^h(\frac{m_{t+1}^h}{p_t})\frac{1}{p_t}) + \beta E_t(u_c^h(c_{t+1}^h \frac{1}{p_{t+1}}) && (8)\\
u_c^h(c_t^h) &= \beta E_t(u_c^h(c_{t+1}^h)(\eta_{t+1}^h f'(k_{t+1}^h) + (1-\delta))) && (9)
\end{aligned}
$$

where u_c^h, u_m^h represents the partial derivatives of the utility function with respect to consumption and cash balances.

When T is finite, households do not invest in physical capital or bonds in the last period. Neither do they have any demand for cash (no transactions). Therefore optimal consumption at date T is given by

$$c_T^h = \eta_T^h f(k_T^h) + (1-\delta)k_T^h + \frac{m_T^h}{p_T} + \theta_{T-1}^h$$

The difference between the non-monetary GEI and the present model is the presence of the Euler equation (8) characterizing the households optimal choice of cash holdings. The equation can be interpreted in the standard way. The left hand side represents the cost in terms of current utility forgone of one unit of cash. The right hand side is the sum of the current utility from holding cash and the expected return next period, as the cash carried over to date $t+1$ is added to the date $t+1$ wealth. The cost and benefit must be equal for the optimal choice of cash holdings.

To derive the equilibrium solutions using CARA specification, we begin by assuming that c^h_{t+1} is normally distributed. We use this assumption to derive the individual demand functions for the riskfree asset, physical capital and cash holdings, at date t from the Euler's equations.

Proposition 1 *Under CARA assumption and assuming c^h_{t+1} is normal, the hth household's demand for the financial, cash and physical assets, θ^h_t, m^h_{t+1} and k^h_{t+1} are given by*

$$\log(\frac{\pi_t}{p_t}) = A(c^h_t - E_t(c^h_{t+1})) + \frac{A^2}{2} Var(c^h_{t+1}) + \log\beta \tag{10}$$

$$\frac{m^h_{t+1}}{p_t} = \frac{1}{A}\log(\frac{\gamma}{\beta}) + E_t(c^h_{t+1}) - \frac{A}{2} Var(c^h_{t+1}) - \frac{1}{A}\log(\frac{p_t}{\pi_t} - \frac{p_t}{p_{t+1}}) \tag{11}$$

$$(\frac{p_t}{\pi_t} - (1-\delta)) = f'(k^h_{t+1})(\eta - ACov(c^h_{t+1}, \eta^h_{t+1})) \tag{12}$$

Proof: See Appendix

Equations (10) and (12) characterizing the demand for the real bond and the physical capital have the same forms and features here as in the CARA model without money (Calvet and Angeletos, 2003). Equation (10) reflects that household's demand for the riskfree asset is affected by three factors - (i) pure time preference (ii) a desire to smoothen fluctuations in future expected consumption, reflected in the term $A(c^h_t - E_t(c^h_{t+1})$ and (iii) precautionary (prudence) motives demonstrated by the fact that the demand for the bond increases if $\mathrm{Var}(c^h_{t+1})$ increases.

Equation (12) is the familiar CAPM formula requiring that the optimal capital stock have a return (its marginal product) equal to the real riskfree rate plus a risk premium (alternatively, as in the equation, the riskfree rate must equal the marginal product minus the risk premium). The risk premium is given by the covariance of the asset return with the consumption of the household in equilibrium.

Equation (11), the distinguishing feature of this model says that the demand for real balances depends on four major factors - (i) direct utility derived (ii) the level of future expected consumption (iii) the variance of future consumption and (iv) the difference in the rates of return on bond and money. The first dependence is straightforward - the higher the direct utility derived (the higher the γ), the higher the demand for cash. The other three relationships are more interesting.

In a standard money-in-the-utility-function (MIU) model, currency is a substitute for (similar to) current consumption as both provides current utility. It is also a substitute for (similar to) bonds in that both are ways to transfer income intertemporally. When the level of expected future consumption is high, the demand for both current consumption and real balances is high and the demand for bonds is low. This is because agents have less need to transfer income from the present to the future in order to smoothen consumption over time. This accounts for the positive relationship between demand for cash and $E_t(c^h_{t+1})$.

Similarly when $\mathrm{Var}(c^h_{t+1})$ is high, households demand both current consumption and cash balances less and bonds more because agents need to transfer more income from the present to the future to smoothen consumption across future states. Hence the demand for cash balances is negatively related to the variance of next period's consumption. The connection between the demand for real balances and consumption variability is a novel feature of this model. We come back to it again during the discussion of the steady state.

Equation (11) further reveals that the demand for cash balances is negatively related to the difference between the gross risk free rate $\frac{p_t}{\pi_t}$ and the gross rate of inflation/deflation $\frac{p_t}{p_{t+1}}$, the return on money. This is expected since money and bonds are substitute assets.

3.2 Equilibrium asset prices, policy functions and output

Asset prices

We now use the asset demand functions derived above together with the market clearing conditions and the assumption of no aggregative shocks to prove that the equilibrium asset prices are non-stochastic.

Denote aggregate output $\int_h \eta_t^h f(k_t^h) = Y_t$ and aggregate capital stock $\int_h k_t^h = K_t$. We begin by assuming that K_t and hence Y_t are deterministic and derive the equilibrium asset prices. Later on in the section we prove that in equilibrium aggregate output is indeed deterministic.

Proposition 2 *Under CARA assumption and assuming K_t, and Y_t to be deterministic, bond and currency prices are given by,*

$$\begin{aligned}
\log(\frac{\pi_t}{p_t}) &= A(Y_t + (1-\delta)K_t - K_{t+1} - \frac{M_{t+1} - M_t}{p_t}) + \log\beta \\
&\quad - A(Y_{t+1} + (1-\delta)K_{t+1} - K_{t+2} - \frac{M_{t+2} - M_{t+1}}{p_{t+1}}) + \frac{A^2}{2}\int_h Var(c_{t+1}^h) \qquad (13) \\
\frac{M_{t+1}}{p_t} &= \frac{1}{A}\log(\frac{\gamma}{\beta}) + (Y_{t+1} + (1-\delta)K_{t+1} - K_{t+2} - \frac{M_{t+2} - M_{t+1}}{p_{t+1}}) \\
&\quad - \frac{A}{2}\int_h Var(c_{t+1}^h) - \frac{1}{A}\log((\frac{p_t}{\pi_t})(\frac{p_{t+1}}{p_t}) - 1) + \frac{1}{A}\log(\frac{p_{t+1}}{p_t}) \qquad (14)
\end{aligned}$$

Proof:Aggregating equations (10) and (11) over households, and noting that since there are no aggregate risks in equilibrium the covariance term becomes zero, we get the required expressions.

Proposition (2) shows that so long as there are no aggregate risks, i.e Y_t and M_t are deterministic, the asset price and the price of the consumption good are also deterministic. This is a special feature of CARA preferences under which asset prices are independent of the income distribution and which keeps the analysis tractable and yields closed form characterization of the dynamic equilibrium.

Policy functions

For the finite horizon case, since date T consumption is known, the individual equilibrium policy functions are solved for by using backward recursion from date T.[8] The next proposition shows that under the CARA specification, the household's equilibrium consumption at date t is a simple affine function of its wealth at date t and that both are normally distributed.

We begin by defining the hth household's current "earnings" as $i_t^h = \frac{m_t^h}{p_t} + \eta_t^h f(k_t^h) + (1-\delta)k_t^h + \theta_{t-1}^h$. This includes output produced, interest income from the riskfree bond and endowment of real cash balances. We also denote by $\tilde{i}_t^h = \eta_t^h f(k_t^h) + (1-\delta)k_t^h$, the household's income from the risky asset - in this model, production only. A household's wealth at date t, denoted W_t^h, is defined as current earnings from all sources and the present value of all future income from risky assets. In symbols,

$$W_t^h = i_t^h + \frac{\pi_t}{p_t} E_t(\tilde{W}_{t+1}^h)$$

where $\tilde{W}_t^h$ is given by the recursive relationship,

$$\tilde{W}_t^h = \tilde{i}_t^h + \frac{\pi_t}{p_t} E_t(\tilde{W}_{t+1}^h), \quad \text{and} \quad \tilde{W}_{T-1}^h = \tilde{i}_{T-1}^h + \frac{\pi_{T-1}}{p_{T-1}} E_{T-1}(\tilde{i}_T^h)$$

[8]One can alternatively solve for it by using the Bellman operator.

Proposition 3 *Under CARA specification, c_t^h is normally distributed at each t and is of the form,*

$$c_t^h = a_t W_t^h - b_t^h \tag{15}$$

where

$$\begin{aligned}
a_t &= \frac{1}{(1+\frac{1}{a_{t+1}}\frac{\pi_t}{p_t})+(1-\frac{\pi_t}{p_t}\frac{p_t}{p_{t+1}})}, \quad \textit{and} \quad a_T = 1 \\
b_t^h &= a_t(b_{t+1}^h \frac{1}{a_{t+1}}\frac{\pi_t}{p_t} + \frac{1}{a_{t+1}}\frac{\pi_t}{p_t}\frac{A}{2}Var(c_{t+1}^h) + k_{t+1}^h \\
&\quad +(1-\frac{\pi_t}{p_t}\frac{p_t}{p_{t+1}})(\frac{1}{A}\log(\frac{\gamma}{\beta}) - \frac{1}{A}\log((\frac{p_t}{\pi_t})(\frac{p_{t+1}}{p_t})-1) + \frac{1}{A}\log(\frac{p_{t+1}}{p_t}))) \\
&\quad +(1-a_t)((\frac{1}{A}\log\beta + \frac{1}{A}\log(\frac{\pi_t}{p_t})), \quad \textit{and} \quad b_T^h = 0
\end{aligned}$$

and a_t and b_t^h are non-stochastic, a_t is uniform across households and W_t^h is normally distributed.

Proof: see Appendix.

Equation 15 asserts that consumption is linear in wealth, at every date. The marginal propensity to consume (as a proportion of wealth) is given by a_t and is uniform across all households. The constant b_t^h, on the other hand, is household specific and depends on a complex of factors including very importantly the variance of consumption at date t, $Var(c_{t+1}^h)$.

The linear form of the consumption function is a special feature of the general HARA class of utility functions (see Gollier (2001)) of which the CARA is a special case. Under CARA specification the marginal propensity to consume (henceforth, mpc) at date t has a relatively (relative to CRRA for example) simple form. In the present set up, a_t is a function of a_{t+1} and in particular of the current rates of return on the two riskfree assets, bonds and money. Given a_{t+1}, a_t is positively related to each of the two rates of return - $\frac{p_t}{\pi_t}$ for the bond and $\frac{p_t}{p_{t+1}}$ for money (the substitution effect of an increase in an asset return is weaker than its income effect). This positive relationship is more stark in the infinite horizon model as we are able to eliminate a_{t+1} from the expression through recursive substitution. The difference between the model without and the model with money is the additional inclusion of $\frac{p_t}{p_{t+1}}$ in the expression for a_t. We also see below that it is this dependence that drives most of the results discussed in section 4 causes this model to be different from the standard complete markets Sidrauski model.

Since the equilibrium policy rules for θ_t^h and m_{t+1}^h are not immediately relevant they are relegated to the Appendix.[9]

Aggregate output

With CARA specifications the demand for risky assets is independent of current wealth of households, a further simplifying feature which provides us with the desired property of the model - the absence of aggregate risks in equilibrium.

Proposition 4 *Investment in physical capital by the hth household at date t, is given by*

$$(\frac{p_t}{\pi_t} - (1-\delta)) = f'(k_{t+1}^h)(\eta - Aa_{t+1}f(k_{t+1}^h)\sigma_p^2) \tag{16}$$

Further, investment is uniform across households.

Proof: Substitute $c_{t+1}^h = a_{t+1}W_{t+1}^h - b_{t+1}^h$ into the equation (12) and simplify to get the required

[9] A slight manipulation of the policy rule for bonds reveal thet everything else constant, a rise in the inflation rate (fall in money return) increases the proportion spent of current wealth on bonds - a feature similar to the Tobin effect. As we see later, however, other effects present in this model drive this one out and can cause the steady state riskfree rate to rise (and capital stock to fall) when the rate of inflation increases.

expression. It is also obvious from the expression that k_t^h is the same for all households.[10]

The demand for capital (risky assets in general) depends on the covariance between the asset returns and the household's idiosyncratic risks. In the given context - productivity shocks are the sole source of such risks - this covariance reduces to the variance of the non-diversifiable productivity shocks. Further, as the mpc is uniform and the idiosyncratic risks identically distributed across households, the risk premium (for a given k) is uniform across households implying $k_t^h = k_t$ for all t, for all h, in equilibrium. The demand for the bond and money (riskfree assets in general), in contrast, varies across households, being dependent on current income realizations.

The following corollary is a direct consequence of Proposition (4) and Assumption(2).

Corollary 1 *Aggregate ouput is deterministic along the equilibrium path.*

Proof: $Y_t = \int_h \eta_t^h f(k_t^h) = f(k_t) \int_h \eta_t^h = \eta f(k_t)$. And $K_t = \int_h k_t^h = \int_h k_t = k_t$. Note that because we assume the set of households is a continuum along $[0,1]$, the aggregate and the per capita output (capital stock) are the same.

Finally, the variance of consumption of the hth household at date t, has a rather simple form in equilibrium in this model.

[10]The right hand side of (16) can be non-monotonic even with strictly concave $f(k)$ when $\sigma_p^2 > 0$ and consequently the equation may have multiple solutions. Calvet and Angeletos (2001) shows however that under reasonable conditions, the minimum solution to the equation is the optimal capital stock.

Proposition 5 *Along the equilibrium path, $Var(c_t^h)$*[11] *is given by,*

$$Var(c_t^h) = (a_t)^2\sigma_p^2(f(k_t^h))^2 \tag{17}$$

and is uniform across households.

Proof: Substitute for W_t^h, into the expression, $\mathrm{Var}(c_t^h) = \mathrm{Var}(a_t W_t^h - b_t^h)$ and simplify.

An important implication of equation (17) is that the higher the marginal propensity to consume, the higher the variance of consumption. In particular, everything else remaining constant, a rise in the rate of return on money (a fall in the rate of inflation) increases a_t and increases consumption variance.

3.3 The reduced form dynamic system

The equations (13), (14), (16) and the expression for a_t make up a computable reduced form system of recursive dynamic equations in the variables p_t, π_t, k_t and a_t from which the values of the remaining variables can be recursively computed. The reduced form of the dynamic equilibrium is thus given by[12],

$$\begin{aligned}
\left(\frac{p_t}{\pi_t} - (1-\delta)\right) &= f'(k_{t+1})(\eta - A a_{t+1} f(k_{t+1})\sigma_p^2) \qquad (18)\\
\frac{M_{t+1}}{p_t} &= \frac{1}{A}\log\left(\frac{\gamma}{\beta}\right) + \left(\eta f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2} - \frac{M_{t+2}-M_{t+1}}{p_{t+1}}\right)
\end{aligned}$$

[11]If agents have exogenous risky endowments in addition to output produced as income, the expression for $\mathrm{Var}(c_t^h)$ is given by,

$$\mathrm{Var}(c_t^h) = (a_t)^2(\sigma_e^2 + \sigma_p^2(f(k_t^h))^2)$$

where σ_e^2 is the variance of the non-diversifiable endowment shocks (see Calvet and Angeletos (2001).

[12]The expressions k, $\eta f(k)$ and $\frac{M}{p}$ in the reduced form system actually represents averages across households. Since we have assumed the set of households to be a continuum on $[0,1]$, the average and the aggregate magnitudes are the same.

$$-\frac{A}{2}(a_{t+1})^2\sigma_p^2(f(k_{t+1}))^2-\frac{1}{A}\log(\frac{p_t}{\pi_t}-\frac{p_t}{p_{t+1}}) \tag{19}$$

$$a_t = \frac{1}{(1+\frac{1}{a_{t+1}}\frac{\pi_t}{p_t})+(1-\frac{\pi_t}{p_t}\frac{p_t}{p_{t+1}})} \tag{20}$$

$$\begin{aligned}\log(\frac{\pi_{0,t}}{p_t}) &= A(\eta f(k_t)+(1-\delta)k_t-k_{t+1}-(\frac{M_{t+1}-M_t}{p_t}))\\ &\quad -A(\eta f(k_{t+1})+(1-\delta)k_{t+1}-k_{t+2}-(\frac{M_{t+2}-M_{t+1}}{p_{t+1}}))+\log\beta\\ &\quad +\frac{A^2}{2}(a_{t+1})^2(f(k_{t+1}))^2\sigma_p^2\end{aligned} \tag{21}$$

The dynamic equilibrium paths of p_t, π_t, k_t and a_t can be recursively computed in the following way, given a monetary process $\{M_t\}_{t=0}^T$, the initial capital stock k_0 and p_0 normalized to one.

We know that at the last date T, $\frac{p_T}{\pi_{0,T}}=0$, $a_T=1$ and $k_{T+1}=0$. Now consider any given pair of values of k_T, p_T. From equation (18) we solve for $\frac{\pi_{0,T-1}}{p_{T-1}}$. Substituting into (19), we solve for $\frac{p_{T-1}}{p_T}$. Substituting into equation (20), we solve for a_{T-1} and from equation (21) we solve for k_{T-1}. We repeat the process till we find k_0 and p_0. If this k_0 and p_0 are equal to the given initial k_0 and 1, the computed path is the equilibrium path. If not we start with a different k_T and p_T. [13]

The dynamic equilibrium path in the present set up is characterized by complex feedbacks between the riskfree rate, capital accumulation, the rate of inflation and the marginal propensity to consume. Equation (19) adds an extra dimension to the present set of equations compared to the corresponding set in the model without money and the exogenously given monetary policy $\{M_t\}_{t=0}^T$ presents an extra parameter. Although an analysis of the transitional properties of the system (18)-(21) is of considerable interest, we relegate such exercises for future work and focus on the steady state instead.

[13]The dynamic equilibrium path is always unique in the finite horizon case (for explanations, see Lucas and Stokey ( ).

4 Infinite horizon and the steady state

As in the real CARA-normal set up, the optimal decision rule of the household-investor when $T = \infty$ can be calculated by taking the pointwise limit of the finite horizon optimal policy function (see Calvet (2001), Calvet and Angeletos (2001, 2003)). To do this, we need the assumption of a bounded sequence of goods prices in addition to the asumption of a bounded sequence of asset prices required for a non-monetary economy. We denote the price at date t of a perpetual stream of one unit of the consumption good (perpetuity) by $\pi_L(t) = \sum_{j=0}^{\infty} \frac{\pi_t}{p_t} \cdots \frac{\pi_{t+j}}{p_{t+j}}$. Also note that the price of the perpetual stream of gross returns on a unit of cash starting at date t is given by $\sum_{j=0}^{\infty} \frac{\pi_t}{p_t} \cdots \frac{\pi_{t+j}}{p_{t+j}} \frac{p_{t+j}}{p_{t+j+1}}$ and denote this by $\pi_{ML}(t)$. Using forward recursion it follows, $a_t = \frac{1}{(1+\pi_L(t))+(1-\pi_{ML}(t))}$ under infinite horizon. For a_t and b_t^h to be well defined for all t we need the following assumption.

Assumption 3 *The sequences*

$\{\pi_t\}_{t=0}^{\infty}$, $\{\frac{\pi_t}{p_t}\}_{t=0}^{\infty}$, $\{R_t\}_{t=0}^{\infty}$, $\{p_t\}_{t=0}^{\infty}$, $\{\frac{p_t}{p_{t+1}}\}_{t=0}^{\infty}$, $\{\pi_L(t)\}_{t=0}^{\infty}$ *and* $\{\pi_{ML}(t)\}_{t=0}^{\infty}$

are bounded.

It can be easily checked (see Calvet and Angeletos (2001, 2003)) that in the infinite horizon case, under assumption (3), the consumption rule (3) is optimal.

Steady state

The remaining part of the paper focuses on the steady state of the economy assuming a constant exogenously given rate of money growth g and an implied rate of inflation equal to it (since M and p must grow at the same rate in the steady state).

Let $m = \frac{M}{p}$ denote the real balances, $R = \frac{p}{\pi}$ the gross real interest rate, k the capital stock and a the mpc, at the steady state. The steady state values of R, m, a and k are given by the solution of,

$$\begin{aligned}
\log(\frac{1}{R}) &= \log\beta + \frac{A^2}{2}(a^2(f(k))^2\sigma_p^2) && (22)\\
m &= \frac{1}{1+g}(\frac{1}{A}\log(\frac{\gamma}{\beta}) - \frac{1}{A}\log(\frac{R(1+g)-1}{(1+g)}) + \eta f(k) - \delta k \\
&\quad -\frac{A}{2}a^2\sigma_p^2(f(k))^2 && (23)\\
a &= \frac{(R-1)(1+g)}{2R(1+g)-1} && (24)\\
R-(1-\delta) &= f'(k)(\eta - Aaf(k)\sigma_p^2) && (25)
\end{aligned}$$

Note that equations (22), (24) and (25) are independent of steady state real balances m and constitute a reduced form system of equations which determine the steady state values of k, R and a, given an exogenous money growth rate g. Further we can use equation (24) to eliminate a from equations (22) and (25). The steady state capital stock and riskfree rate are thus given by,

$$\begin{aligned}
\log(\frac{1}{R}) &= \log\beta + \frac{A^2}{2}((\frac{(R-1)(1+g)}{2R(1+g)-1})^2(f(k))^2\sigma_p^2) && (26)\\
R-(1-\delta) &= f'(k)(\eta - A(\frac{(R-1)(1+g)}{2R(1+g)-1})f(k)\sigma_p^2) && (27)
\end{aligned}$$

The next theorem shows that the monetary economy always has a steady state for any non-negative value of g.

Theorem 1 *There exists a steady state for every economy for every $g \geq 0$.*

Proof: see Appendix.

Note that $g \geq 0$ ensures that $a < 1$ at the steady state. Solutions to the above equations for which $a < 1$ are not guaranteed to exist for any $g < 0$ although such solutions may exist for small negative values of g (see Appendix for details). We revisit this issue below in section 4.3 again.

Geometrically, equations (26) and (27) implicitly define the capital stock k as two functions, $K^1(R)$ and $K^2(R)$ of the real riskfree rate (see figure 1). The intersection of these two determine the steady state values of k and R. It can be easily checked that both $K^1(R)$ and $K^2(R)$ are decreasing. This introduces the possibility of multiple steady states - a result carried over from the economy without money (see Calvet and Angeletos (2001, 2003)). In the economy without money some of these steady states are shown to be unstable, locally indeterminate and capable of generating endogenous fluctuations. It is a natural question to ask whether and how these conditions and features generalize to the present model with money. We leave this exploration for the future however and for the present focus on steady states which are unique and stable, in order to study some of the standard issues in monetary economics in this new framework. Stability is ensured if $K^1(R)$ is steeper than $K^2(R)$ at the point of intersection.

4.1 Monetary neutrality and super-neutrality

Given the steady state values of k and R, equation (23) determines the steady state value of m. Since k, R and a are independent of m at the steady state, money is neutral.

Equation (23) further shows that the steady state value of m and hence utility from real balances

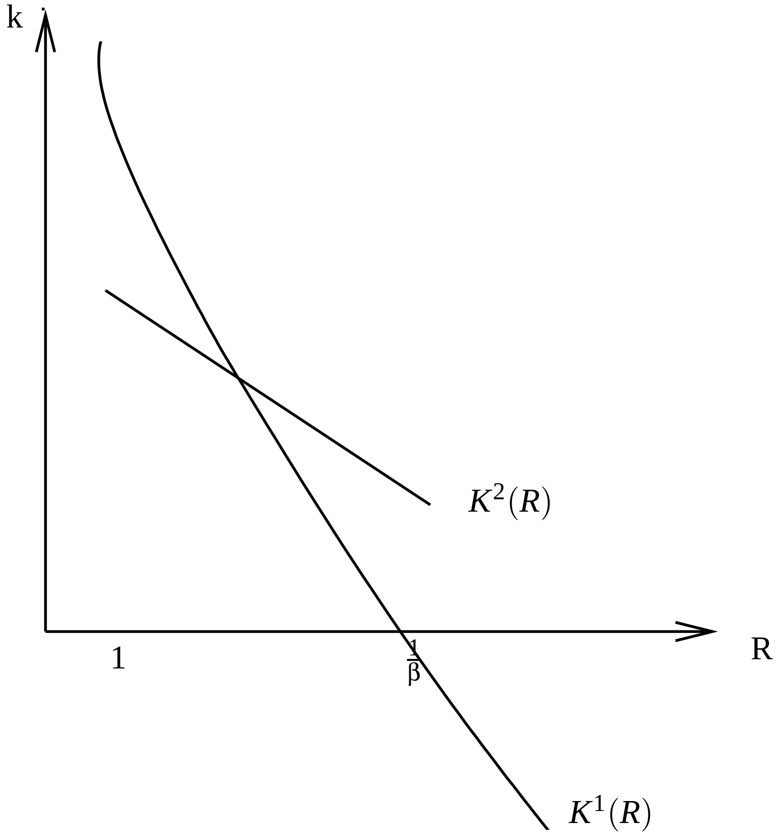

Figure 1:

are influenced by the extent of uninsurable risks σ_p^2 - implications to be explored in future research.

Although money is neutral it is clearly not superneutral at the steady state because k, R and a depend on the rate of inflation or money growth g, as the equations show. The next theorem and the following discussion makes clear that money is non-superneutral in this model because markets are incomplete.

Theorem 2 *Money is superneutral if markets are complete, i.e. if* $\sigma_p^2 = 0$.

Proof: When $\sigma_p^2 = 0$, the steady state capital stock k and riskfree rate r are given by,

$$
\begin{aligned}
\frac{1}{R} &= \beta \\
R-(1-\delta) &= \eta f'(k)
\end{aligned}
$$

The steady state capital stock and the riskfree rate are independent of the rate of inflation and money growth when markets are complete. Money is not only neutral but also superneutral at the steady state as in the standard Sidrauski set-up.

When some risks are non-diversifiable - that is when markets are incomplete - there are two distinct factors which destroy the super-neutrality property at the steady state. First a rise in the rate of inflation or money growth decreases the marginal propensity to consume a, as can be checked from equation (24). This is just a continuation into the steady state of the relationship between the marginal propensity to consume and the rate of return on money (alternatively the rate of inflation) discussed in Section 3.2. A fall in a reduces consumption variance and precautionary savings. This in turn has a positive effect on the steady state real riskfree rate and consequently a negative effect on the steady state capital stock (as the risk-adjusted rate of return on capital has to be equal to the riskfree rate in equlibrium). We described this channel of influence of the rate of inflation or money growth on the riskfree rate and capital stock as the precautionary channel in the introduction.

The second factor is that capital in this set-up is not priced according to the marginal product of capital only as in the case of complete markets. Instead the optimal capital stock is given by the equality of the riskfree rate and the marginal product adjusted for the risk that capital entails -

determined by the covariance between consumption and productivity shocks, which reduces to the term $Aa\sigma_p^2 f'(k)f(k)$ in equilibrium. The risk premium on capital at the steady state falls when the marginal propensity to consume a falls with a rise in the rate of inflation or money growth. The decrease in the risk premium has a positive effect (described as the ”risk premium” effect in the introduction) on the steady state capital stock which is quite opposite of the first precautionary effect.

The following thought experiment helps us understand how the two channels are distinct. Suppose idiosyncratic risks are assumed to come from some exogenous endowment (see footnote 5, section 2) rather than production sources - that is assume that there are no productivity shocks (capital is riskfree) but that households have some other source of income which is subject to non-diversifiable idiosyncratic shocks. It is easy to check that the steady state equations would then be given by,

$$\log(\frac{1}{R}) = \log\beta + \frac{A^2}{2}((\frac{(R-1)(1+g)}{2R(1+g)-1})^2\sigma_e^2) \tag{28}$$

$$R-(1-\delta) = f'(k)\eta \tag{29}$$

where σ_e^2 represents the variance of the exogenous (non-diversiafiable) endowment shocks. The riskfree rate would clearly be affected by the rate of inflation/money growth and so would the capital stock because of its dependence on R, even if there are no non-diversifiable production risks. Non-superneutrality breaks down because of the first but not the second factor in this case.

4.2 The Fisher equation

It is clear from the discussion in the previous section that the relationship between the nominal interest rate and the rate of inflation is generally not one-on-one as the riskfree real rate is influenced by changes in the rate of inflation. We explain below why it is not possible to further specify how the real riskfree rate will change because of a rise in the inflation rate, in the general model with production risks. But first we show that in the special simple case in which there are endowment but no productivity shocks the direction of this change is unambiguous.

Theorem 3 *In an economy with endowment risk but no production risk, the real riskfree rate is positively related and the capital stock negatively related to the rate of inflation.*

Proof: Equation (28) determines the steady state real riskfree rate R as independent of the capital stock but an implicitly increasing function of the rate of inflation g. Equation (29) determines the steady state capital stock as an inverse function of the real riskfree rate but without direct dependence on g. The steady state is thus unique. Geometrically, on the (R,k) plane the function $K^1(R)$ is a vertical straight line and the function $K^2(R)$ is more gently downward sloping (see Figure 2). An increase in g shifts $K^1(R)$ outwards, doesn't change $K^2(R)$ and the result follows.[14]

In the more general case in which productivity shocks are present the comparative static exercise of an increase in g yields ambiguous results. $K^1(R)$ and $K^2(R)$ are decreasing functions as in Figure 1. Both $K^1(R)$ and $K^2(R)$ are increasing in g. Hence as a result of a rise in the rate of inflation/money growth both $K^1(R)$ and $K^2(R)$ shift outwards. The net effect on the steady state values is thus ambiguous even when the steady state is unique.

[14]The theorem vindicates Barro's conjecture that the rate of inflation and capital stock are inversely related, albeit in the special case of endowment but no production risks.

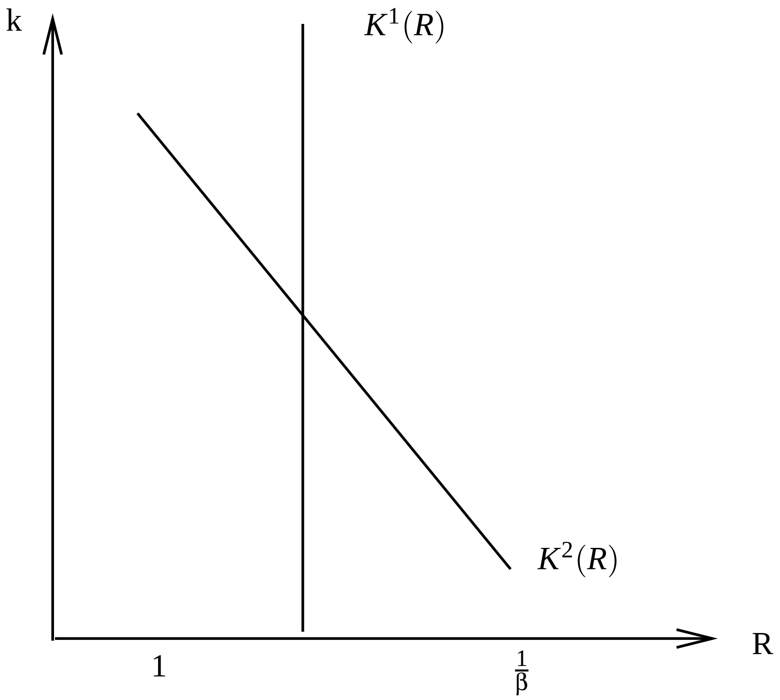

Figure 2:

Intuitively, a rise in the rate of inflation/money growth reduces the marginal propensity to consume which in turn reduces the precautionary savings and the risk premium. But a reduction in the risk premium in turn has positive effects on the capital stock, ouput which in turn has a positive feedback on consumption variance and precautionary savings. Thus when we take into account production risks, there are all these positive and negative feedbacks on the precautionary savings and the risk premium which render the net effect of a rise in the inflation rate ambiguous.

4.3 The optimal monetary policy

Since g is exogenously given, in this section we address the issue of its optimal value.

We begin by noting that in the present set up, although individual household consumption along the steady state path is stochastic, the average consumption across all households (or aggregate consumption for that matter) is not, since there are no aggregate risks in the economy. By optimal g we therefore denote the value of g that maximizes the steady state utility of the "average" consumer. The average consumption along the steady state is given by,

$$c = \eta f(k) - \delta k \tag{30}$$

From the previous sections, it is also clear that when markets are incomplete, the steady state real riskfree rate, capital stock and hence consumption depend on the rate of money growth. We are therefore in essence looking for a 3-tuple $(g, R(g), k(g))$ which maximizes the utility of the average consumer along a steady state path.

In addition to the two steady state equations (26) and (27), the optimal rate of money growth must satisfy

$$u_c \frac{\partial c}{\partial g} + \gamma u_m \frac{\partial m}{\partial g} = 0 \tag{31}$$

Note that from the Euler equation (8) at the steady state

$$u_c = \gamma u_m + u_c\left(\frac{1}{R(1+g)}\right)$$

Substituting for the partial derivatives and u_m in (31), using (30), (23) and the above expression and after simple manipulation, condition (31) reduces to,

$$u_c((\eta f'(k) - \delta)\frac{\partial k}{\partial g} + (\eta f'(k) - \delta)(1 - \frac{1}{R(1+g)}) - \frac{1}{A}(\frac{1}{R(1+g)^2})) = 0$$

Note that at the steady state $u_c \neq 0$, since k is finite. Hence the optimal money growth rate is given by the solution of (g, R, k) in the equations (26), (27) and

$$(\eta f'(k) - \delta)\frac{\partial k}{\partial g} + (\eta f'(k) - \delta)(1 - \frac{1}{R(1+g)}) - \frac{1}{A}(\frac{1}{R(1+g)^2}) = 0 \tag{32}$$

Two observations are in order. Firstly, given a certain parametric specification of the economy, equations (26) (27) and (32) together may have no solutions, or unique or multiple solutions which are meaningful. In particular a solution (if it exists) may involve a positive or a negative g. Secondly, at this point, given the complex form of the equations it is difficult to check (if not impossible) whether a solution(s) can be found in the form of a "rule" (such as the Friedman rule) which is invariant to parametric specifications.

The next theorem shows that the Friedman rule which sets the gross nominal rate of interest $R(1+g) = 1$ or $g \approx -r$ (the net real riskfree rate) may be suboptimal under the present set up.

Theorem 4 *The Friedman rule is suboptimal for some economies, when markets are incomplete.*

Proof: Note that the Friedman rule is optimal when markets are complete, since the riskfree rate (and hence capital stock) is invariant with respect to g.

For the incomplete market case, it is sufficient to give an example in which markets are incomplete and a meaningful steady state exists with $R(1+g) = 1$, $g = -r$ but for which (32) is violated. We assume the simple case once again under which endowment but no productivity shocks are present ($\sigma_p^2 = 0$, $\sigma_e^2 > 0$). Under this situation, it is easy to check that the condition (32) reduces to,

$$(R-1)\frac{1}{\eta f''(k)}\frac{\partial R}{\partial g} + (1 - \frac{1}{R(1+g)})(R-1) - \frac{1}{A}(\frac{1}{R(1+g)^2}) = 0 \tag{33}$$

We now assume $R(1+g)=1$ and $g=-r$. Note that under these conditions, a steady state in which $r \leq \frac{1}{2}$ can always be found for $\beta = 1/2$ and $A\sigma_e^2 > 8\log(4/3)$ (see Appendix).

The expression on the left of equation (33), evaluated at $R(1+g)=1$ and $g=-r$ becomes,

$$r(\frac{1}{\eta f''(k)})(\frac{A^2r^2}{(\frac{1}{1+r})+A^2\sigma_e^2 r(1-r)(1-2r)} - \frac{1}{A(1-r)}$$

which is always negative for $r \leq \frac{1}{2}$. Hence the optimality condition is violated for economies with $\beta = 1/2$ and $A\sigma_e^2 > 8\log(4/3)$ for $R(1+g)=1$ or $g \approx -r$.

5 Appendix

Proposition 1

To derive the demand function for the riskfree asset, we first evaluate the definite integral $E(u_c^h(c_{t+1}^h))$ on the right hand side of the first Euler equation. Assuming that $c_t^h \sim N(\bar{c}, \sigma_c^2)$

$$\begin{aligned} E_t(u_c^h(c_t^h)) &= \int_{\infty}^{\infty} \mathrm{Exp}(-Ac_t^h)\frac{1}{\sigma_c\sqrt{(2\pi)}}\mathrm{Exp}(-\frac{(c_t^h-\bar{c})^2}{2\sigma_c^2})d(c_t^h) \\ &= \mathrm{Exp}(-AE_t(c_t^h)+\frac{A^2}{2}\mathrm{Var}_t(c_t^h)) \end{aligned}$$

Substituting for $u_c^h(c_t^h) = e^{-Ac_t^h}$ on the left hand side and for $E_t(u_c^h(c_t^h))$ on the right hand side and simplifying we have,

$$\log(\frac{\pi_{0,t}}{p_t}) = Ac_t^h - AE(c_{t+1}^h) + A^2\mathrm{Var}(c_{t+1}^h) + \log\beta$$

The demand for real balances can be derived by dividing both sides of (8) by $E(u_c^h(c_{t+1}^h))\frac{1}{p_{t+1}}$ and simplifying.

To derive the demand for physical capital not that the Euler equation (9) can be written as,

$$\begin{aligned} u_c^h(c_t^h) &= \beta E_t(u_c^h(c_{t+1}^h))(1-\delta)+\beta E_t(u_c^h(c_{t+1}^h))E_t(\eta_{t+1}^h f'(k_{t+1}^h)) \\ &\quad +\mathrm{Cov}_t(u_c^h(c_{t+1}^h),\eta_{t+1}^h f'(k_{t+1}^h)) \end{aligned}$$

Since c_t^h is normal, applying Stein's lemma to the above expression, we have

$$\begin{aligned} u_c^h(c_t^h) &= \beta E_t(u_c^h(c_{t+1}^h))(1-\delta)+\beta E_t(u_c^h(c_{t+1}^h))E_t(\eta_{t+1}^h f'(k_{t+1}^h)) \\ &\quad +E(u_{cc}^h(c_{t+1}^h)\mathrm{Cov}(c_{t+1}^h),\eta_{t+1}^h f'(k_{t+1}^h) \end{aligned}$$

where $u_{cc}^h(.)$ represents the derivative of $u_c^h(.)$. Dividing the above expression by (7), and noting that $\frac{E(u_{cc}^h(c_{t+1}^h)}{E(u_c^h(c_{t+1}^h))}=-A$ and simplifying yields the required expression.

Proposition 3

To prove the normality of c_t^h and derive the functional form we start by solving for c_h^t backwards from date T,

$$c_T^h=\eta_T^h f(k_T^h)+(1-\delta)k_T^h+\theta_{T-1}^h+\frac{m_T^h}{p_T}$$

Since η_T^h is normal and prices are deterministic, c_T^h is normal. Hence equation (15) is true for date $T-1$ with $a_T = 1$, $b_T^h = 0$, and $i_T^h = W_T^h$ given by the right hand side.

To solve for c_{T-1}^h, we first derive the demand for cash and the risk free asset at date $T-1$. From equation (11), demand for real balances at $T-1$ must satisfy

$$\begin{aligned}\frac{m_T^h}{p_{T-1}} &= \frac{1}{A}\log(\frac{\gamma}{\beta}) + E_{T-1}(\eta_T^h f(k_T^h) + (1-\delta)k_T^h + \theta_{T-1}^h + \frac{m_T^h}{p_T} \\ &\quad -\frac{A}{2}\mathrm{Var}(c_T^h) - \frac{1}{A}\log(\frac{p_{T-1}}{\pi_{T-1}} - \frac{p_{T-1}}{p_T})\end{aligned}$$

Simplifying, we get

$$\begin{aligned}\frac{m_T^h}{p_{T-1}} &= \frac{1}{1-\frac{p_{T-1}}{p_T}}(\frac{1}{A}\log(\frac{\gamma}{\beta}) + E_{T-1}(\eta_T^h f(k_T^h) + (1-\delta)k_T^h) - \frac{A}{2}\mathrm{Var}(c_T^h) \\ &\quad -\frac{1}{A}\log(\frac{p_{T-1}}{\pi_{T-1}} - \frac{p_{T-1}}{p_T})) + \frac{1}{1-\frac{p_{T-1}}{p_T}}\theta_{T-1}^h\end{aligned}$$

From equation (10), bemand for the riskfree bond at date $T-1$ must satisfy

$$\log(\frac{\pi_{T-1}}{p_T - 1}) = Ac_{T-1}^h - AE(c_T^h) + A^2\mathrm{Var}(c_T^h) + \log\beta$$

Noting that $c_{T-1}^h = i_{T-1}^h - k_T^h - \frac{m_T^h}{p_{T-1}} - \frac{\pi_{T-1}}{p_{T-1}}\theta_{T-1}^h$ and substituting for $\frac{m_T^h}{p_{T-1}}$ from above and simplifying we have,

$$
\begin{aligned}
\theta^h_{T-1} &= \frac{1-\frac{p_{T-1}}{p_T}}{2+\frac{\pi_{T-1}}{p_{T-1}}(1-\frac{p_{T-1}}{p_T})}[i^h_{T-1}-\frac{2}{(1-\frac{p_{T-1}}{p_T})}E(\tilde{W}^h_T) \\
&+\frac{2}{(1-\frac{p_{T-1}}{p_T})}\frac{A}{2}\mathrm{Var}(c^h_T)+\frac{1+\frac{p_{T-1}}{p_T}}{1-\frac{p_{T-1}}{p_T}}\frac{1}{A}\log(\frac{p_{T-1}}{\pi_{T-1}}-\frac{p_{T-1}}{p_T}))-\frac{1}{A}\frac{\pi_{T-1}}{p_{T-1}} \\
&+\log\beta-\frac{1+\frac{p_{T-1}}{p_T}}{1-\frac{p_{T-1}}{p_T}}\frac{1}{A}\log(\frac{\gamma}{\beta})-k^h_T
\end{aligned}
$$

Finally, substitute for θ^h_{T-1} and $\frac{m^h_T}{p_{T-1}}$ into $c^h_{T-1}=i^h_{T-1}-k^h_T-\frac{m^h_T}{p_{T-1}}-\frac{\pi_{T-1}}{p_{T-1}}\theta^h_{T-1}$ and simplify to get,

$$c^h_{T-1}=a_{T-1}W^h_{T-1}-b^h_{T-1}$$

where

$$
\begin{aligned}
W^h_{T-1} &= i^h_{t-1}+\frac{\pi_{T-1}}{p_{T-1}}E_{T-1}(\tilde{i}^h_T) \\
a_{T-1} &= \frac{1}{(1+\frac{\pi_{T-1}}{p_{T-1}})+(1-\frac{\pi_{T-1}}{p_{T-1}}\frac{p_{T-1}}{p_T})} \\
b^h_{T-1} &= a_{T-1}(\frac{\pi_{T-1}}{p_{T-1}}\frac{A}{2}\mathrm{Var}(c^h_T)+k^h_T+(1-\frac{\pi_{T-1}}{p_{T-1}}\frac{p_{T-1}}{p_T}). \\
&(\frac{1}{A}\log(\frac{\gamma}{\beta})-\frac{1}{A}\log((\frac{p_{T-1}}{\pi_{T-1}})(\frac{p_T}{p_{T-1}})-1)+\frac{1}{A}\log(\frac{p_T}{p_{T-1}}))) \\
&+(1-a_{T-1})(\frac{1}{A}\log\beta+\frac{1}{A}\log(\frac{\pi_{T-1}}{p_{T-1}}),
\end{aligned}
$$

Thus c^h_{T-1} has the required form. Also since a_{T-1} and b^h_{t-1} are non-stochastic and W^h_{T-1} is normal, c^h_{T-1} is normal.

We repeat the above steps to solve for θ^h_{T-2}, $\frac{m^h_{T-1}}{p_{T-2}}$ and c^h_{T-2} and generalize to get the required forms for consumption. It can also be checked in the process that the household's equilibrium policy rules for bond and cash balances are given by,

$$\begin{aligned}
\theta_t^h &= a_t(i_t^h(\frac{1}{a_{t+1}} - \frac{p_t}{p_{t+1}}) - 2E_t(\tilde{W}_{t+1}^h) + \frac{A}{a_{t+1}}\mathrm{Var}(c_{t+1}^h) + 2\frac{b_{t+1}^h}{a_{t+1}} \\
&\quad (\frac{1}{a_{t+1}} + \frac{p_t}{p_{t+1}})(\frac{1}{A}\log((\frac{p_t}{\pi_t})(\frac{p_{t+1}}{p_t}) - 1) + \frac{1}{A}\log(\frac{p_{t+1}}{p_t}) \\
&\quad + (\frac{1}{a_{t+1}} - \frac{p_t}{p_{t+1}})\frac{1}{A}(\log\beta - \log(\frac{\pi_t}{p_t})) - \frac{1}{A}\log(\frac{\gamma}{\beta}) - k_{t+1}^h)) \qquad (34) \\
\frac{m_{t+1}^h}{p_1} &= \frac{1}{(\frac{1}{a_{t+1}} - \frac{p_t}{p_{t+1}})}(\theta_t^h + E_t(\tilde{W}_{t+1}^h) - \frac{A}{2a_{t+1}}\mathrm{Var}(c_{t+1}^h) \\
&\quad \frac{b_{t+1}^h}{a_{t+1}} + \frac{1}{a_{t+1}}(\frac{1}{A}\log((\frac{p_t}{\pi_t})(\frac{p_{t+1}}{p_t}) - 1) + \frac{1}{A}\log(\frac{p_{t+1}}{p_t}) - \log(\frac{\gamma}{\beta}))) \qquad (35)
\end{aligned}$$

Theorem 1

Note first that $g \geq 0$ is sufficient to guarrantee $a < 1$ at the steady state (assuming we are able to find one in which $R \geq 1$) since in this case

$$\begin{aligned}
(R-1)(1+g) &< 2(R-1)(1+g) \\
&= 2R(1+g) - 2(1+g) \\
&< 2R(1+g) - 1 \to a < 1
\end{aligned}$$

Also note from (26), that when $\sigma_p^2 = 0$, $R = 1/\beta$ and when $\sigma_p^2 > 0$, $R < 1/\beta$. We shall therefore look for solutions to (26) and (27) for $R \in [1, 1/\beta]$.

To prove existence of steady state we manipulate the equations and rewrite them as,

$$
\begin{aligned}
f(k) &= \left(\frac{2\log(\frac{1}{\beta R})}{\sigma_p^2}\right)^{1/2} A^{-1}\left(\frac{(R-1)(1+g)}{2R(1+g)-1}\right)^{-1} && (36)\\
f'(k) &= \frac{R-(1-\delta)}{\eta-\sigma_p(2\log(\frac{1}{\beta R}))^{1/2}} && (37)
\end{aligned}
$$

We get equation (37) by substituting

$$A\sigma_p\left(\frac{(R-1)(1+g)}{2R(1+g)-1}f(k) = (2\log(\frac{1}{\beta R}))^{1/2}$$

from equation (36). The equations (36) and (37) define the steady state capital stock k as two implicit functions $G^1(R)$ and $G^2(R)$ of the real riskfree rate. It can be easily checked that $G^1(1)=\infty$ and $G^1(1/\beta)=0$. Also $G^2(1)=(f')^{-1}(\frac{\delta}{\eta-\sigma_p(2\log(1/\beta)})=\overline{k}>0$ and $G^2(1/\beta)=(f')^{-1}(\frac{\delta+(1/\beta-1)}{\eta})=\underline{k}>0$. Thus, $G^1(1)-G^2(1)>0$ and $G^1(1)-G^2(1)<0$. Hence a zero exists in $[1,1/\beta]$.

Theorem 4

When $R(1+g)=1$ and $g=-r$, at the steady state $a=r$. We therefore need to look for a steady state in which $r\in[0,1]$. In particular note that for $\beta=(1/2)$, $\sigma_p^2=0$, and $\sigma_e^2>0$, the steady state equations are,

$$
\begin{aligned}
\frac{\log(\frac{2}{(1+r)})}{r^2} &= \frac{A^2\sigma_e^2}{2}\\
r &= \eta f'(k)-\delta
\end{aligned}
$$

The LHS of the first equation $\to \infty$ as $r \to 0$. At $r = 1/2$, LHS $= 4\log(4/3)$. Hence a solution to the first equation (and also the second) exists in the region $[0, 1/2]$ if $A^2\sigma_e^2 > 8\log(4/3)$.

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