A Toolbox for the Numerical Study of Linear Dynamic Rational Expectations Models

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Abstract

By simplifying the computational tasks and by providing step-by-step explanations of the procedures required to study a linear dynamic rational expectations (LDRE) model, this paper and the accompanying "LDRE Toolbox" of Matalb functions guide a researcher with almost no experience in computational work to resolve and study his own model. After coding the model following specific guidelines, a single function call is all that is needed to log-linearize the model; simulate it under exogenous sequences of shocks; compute sample and population moment conditions; and obtain impulse-response functions. Three classical models in the Real-Business-Cycles literature are solved and studied throughout to give detailed examples of the steps involved in solving and studying LDRE models using the LDRE Toolbox. Namely, the economies in Brock and Mirman (Optimal Growth and Uncertainty: the Discounted Case, *Journal of Economic Theory* 4(3): 479-513; 1972); King, Plosser, and Rebelo (Production, Growth and Business Cycles I: The Basic Neoclassical Model, *Journal of Monetary Economics* 21: 195-232; 1988); and Mendoza (Real Business Cycles in a Small Open Economy, *American Economic Review* 81(4): 797-818; 1991).

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1   Introduction

The use of numerical methods to study dynamic economic systems has pervaded all fields of economics in the last three decades. In macroeconomics, the pioneering studies by Kydland and Prescott (1982) and Long and Plosser (1983) sparked off a new approach to study the overall performance of an economy. The approach, called real business cycle (RBC) theory, gained adoption from a large number of macroeconomists because RBC models were able to reproduce several features of the economywide fluctuations that characterize market economies. Agents in these models face uncertainty about future economic developments and have to solve dynamic stochastic problems forming expectations about the future rationally. Except for some special cases, dynamic stochastic models lack a close form solution and they have to be solved using numerical methods. A numerical technique widely used to solve these models involves the log-linearization of their equations. The log-linearized model is an approximation to the original model that is called a linear dynamic rational expectations (LDRE) model.

Some of the reluctance to use numerical methods in economics can be linked to the difficulty of translating economic models into a language that can be read by a computer. This paper and the accompanying set of Matlab functions which we call the LDRE Toolbox are designed to help a researcher with almost no experience in computational work to resolve and study his own dynamic stochastic model.[1] The simplification of the computational tasks required to study a dynamic model is one distinguishing characteristic of the Toolbox, vis-à-vis other computational algorithms available elsewhere.[2] The provision of step-by-step explanations of the procedures a researcher should follow to study his own model is the second distinguishing characteristic and goal of this work. Briefly, once a macroeconomic model fitting in the class of models of section 2 is outlined, the Toolbox accompanying this paper log-linearizes the model equations; returns impulse-response functions; sample and population moments conditions of macroeconomic aggregates and ratios; and simulations of the model under a given sequence of shocks. Although non macroeconomic models could also be studied using the Toolbox, the characteristics of its output make it more suitable for the study of macroeconomic models.

The most important reason to work with the linear (or log-linear) version of a model instead of its original non-linear formulation is the minimization of the computational costs, measured in terms of computer time and programming effort, involved in solving a

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[1] The Toolbox is available at www.econ.iastate.edu/faculty/oviedo.

[2] Some of the functions in the Toolbox were borrow from these other sources although their documentation has been changed here. See for example http://www.ssc.uwo.ca/economics/faculty/klein

model. Furthermore, whereas most alternative solution techniques are subject to the 'curse of dimensionality', the dimension of the state space does not represent a problem for linear approximation techniques.[3] This advantage could be shadowed by a lower accuracy of the results. However, it has been shown that linear models produce highly accurate results when the variance of the shocks hitting the system is not too large, something that we illustrate later in section six.[4]

The organization of the paper is as follows. In the second section, we show the type of models than can be studied using the LDRE Toolbox. In the third section, we consider an open and two closed-economy versions of the standard RBC model to serve as examples for the application of the solution technique discussed in section four. In section five we review the Matlab functions in the Toolbox and in section six we illustrate the output of the Toolbox for the three RBC models used as examples. We summarize the paper in section seven

2   LDRE Models and their Variables

To establish a convention on how to classify the variables in a LDRE model, we distinguish four types of variables. First, predetermined or backward-looking variables. Second, non-predetermined or forward-looking variables. Third, innovations to the backward-looking variables. Fourth, flow or additional variables.

The value of a predetermined variable at time t is given by optimizing decisions taken at $t-1$, the evolution of the system between $t-1$ and t, and the innovations observed at t. Commonly, backward-looking variables correspond to the state variables in a dynamic programming framework. Throughout the paper as well as in the Toolbox, we use $x_1(t)$ to denote the values of the predetermined variables at time t.[5]

Within the set of predetermined variables, we distinguish between *endogenous* and *exogenous* variables. While the values of the endogenous predetermined variables are affected by economic optimizing decisions, the values of the exogenous predetermined variables are independent of these decisions and only rely on their exogenous driving processes. The stock of capital and the amount of foreign assets are typical examples of

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[3]In the numerical methods literature, the course of dimensionality refers to the impossibility of solving models endowed with a relatively large state space. What is a "relatively large state space" cannot be stated more precisely without referring to specific models, but it can be said that difficulties arise, in general, when models have five or more state variables.

[4]On the accuracy of the log-linear solutions, see for example Dotsey and Mao (1992) and Danthine and Mehra (1989).

[5]For the sake of clarifying the exposition, we depart from the convention of dating discrete-time variables with a subindex.

endogenous backward-looking variables in an open economy, while a productivity shock is an example of an exogenous backward-looking variable. The values of both types of predetermined variables at time t are included in $x_1(t)$.

The value of a time-t forward-looking or non-predetermined variable depends on actions taken at time t. We group all forward-looking variables in the vector $x_2(t)$. Appealing to the parallelism with the dynamic programming framework, policy variables like consumption and the accumulation of assets are examples of non-predetermined variables.

The time-t values of some predetermined variables depend on the innovations to specific stochastic processes. For example, in an open-economy model, the international interest rate and the economywide productivity shock might be modeled as exogenous autoregressive processes subject to period-by-period innovations. Let $\epsilon(t)$ denote the vector of innovations to the variables in the model. At a high level of generality, both forward and backward-looking variables could be subject to innovations; in this case, the length of $\epsilon(t)$ is equal to the number variables in the model. In most models, however, most of the components of $\epsilon(t)$ are equal to zero because no shock hits some of the predetermined variables (overall, the endogenous ones) and most, if not all, non-predetermined variables.

Finally, we call flow or additional variables to the variables of interest to the researcher whose value can be inferred from the model, although they are not specifically part of the model. For example, the trade balance may not be a variable in an open-economy model but its dynamics might be of interest to the researcher and can be computed from the dynamics of output, consumption and investment. We use $x_3(t)$ to refer to the value of the vector of flow variables at time t.

The specification of a macroeconomic model usually involves three sets of equations, namely: a set of optimality conditions; a set of resource constraints and market clearing conditions; and a set of equations specifying the dynamics of the exogenous predetermined variables. Examples of these equations are, respectively, the equalization of the marginal rate of substitution of labor for consumption to the real wage rate; the equalization of output to the sum of investment and consumption; and an autoregressive process describing the dynamics of the productivity shock. After log-linearizing the described sets of equations and using a "hat" over a variable to express percentage deviations with respect to its steady-state value, we can write the log-linear version of a model as follows:

$$AE_t\hat{x}(t+1) = B\hat{x}(t) \tag{1}$$

where $\hat{x}(t) \equiv (\hat{x}_1(t), \hat{x}_2(t))'$. Without loss of generality, we assume throughout that the

models have n_1 backward-looking variables, n_2 forward-looking variables, and n_3 flow variables. Consequently, $\hat{x}_1(t)$ is a n_1-vector, $\hat{x}_2(t)$ is an n_2-vector, and $\hat{x}(t)$ is an n-vector, where $n = n_1 + n_2$. Thus, the matrices of coefficients A and B in eq. (1) are matrices of size $n \times n$.

The actual evolution of the system is also affected by the innovations to the n variables in $\hat{x}$:[6]

$$A\hat{x}(t+1) = B\hat{x}(t) + \epsilon(t+1)$$

If only the backward-looking variables are subject to innovations, a partition of the vector $\epsilon(t+1)$ permits writing the system as:

$$A \left(\begin{array}{c} \hat{x}_1(t+1) \\ \hat{x}_2(t+1) \end{array} \right) = B \left(\begin{array}{c} \hat{x}_1(t) \\ \hat{x}_2(t) \end{array} \right) + \left(\begin{array}{c} \epsilon_1(t+1) \\ 0_{n_2 \times 1} \end{array} \right)$$

where $\epsilon_1(t+1)$ is an n_1-vector. Furthermore, when only the exogenous backward-looking variables are subject to innovations, some components of $\epsilon_1(t+1)$ are also equal to zero.

Let $x_3(t)$ be the vector of additional or flow variables at time t whose value depends on the value of x_1 and x_2, at times t and $t+1$. By log-linearizing the equations that define the flow variables, we can write:

$$\hat{x}_3(t) = C\,\hat{x}(t+1) + D\,\hat{x}(t) \tag{2}$$

where $\hat{x}_3$ is an n_3-vector and C and D are matrices of dimension $n_3 \times n$.

3 Examples: Three Neoclassical-Growth Models

In this section we present three versions of the one-sector neoclassical growth model of optimal capital accumulation augmented by productivity shocks that fit into the class of models discussed above. The three models are purely real in the sense that money does not play any role in the determination of the optimal allocation of resources. This characteristic of the models led Long and Plosser (1983) to call them *real business cycles* (RBC) models. The first model is the stochastic growth example of Brock and Mirman (1972) which became the workhorse in the macroeconomic literature. Since a closed form solution to this model exists when preferences are logarithmic and the capital depreciates completely after production in every period, we will be able to compare numerically the

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[6]Observe that $A\hat{x}(t+1) = AE_t\,\hat{x}(t+1) + \epsilon(t+1)$.

closed form solution with the solution to the log-linear approximation of the model. The second model is an extended version of the Brock and Mirman economy that lacks a closed-form solution. In this extension, only a fraction of the capital stock depreciates in every period, labor supply responds to economic conditions, and the productivity of the economy grows exogenously over time. We call this model the KPR model because we are going to replicate the results in King, Plosser, and Rebelo (1988a) who calibrate it to the US economy. The third model discussed below is the extension of the RBC framework to a small open economy (SOE), extension developed by Mendoza (1991) who calibrated the model to the Canadian economy.

The three artificial RBC economies are consistent with the conditions under which the second welfare theorem holds. Consequently, the allocations optimally chosen by a central planner are the same allocations that would arise in a decentralized competitive economy where households supply the factors of production employed by atomistic firms to produce the final output. Although we are ultimately interested in the dynamics of the decentralized economies, we exploit the equivalence above and focus on the less involved central planner problems. We can then back out the competitive equilibrium prices from the optimal allocations chosen by the planner.

Below, we start specifying the models and characterizing their first-order optimality conditions; we then show how to sort their variables according to the criteria discussed in section 2; next we calibrate the models and obtain their log-linear approximations.

3.1 Models Specification and Optimality Conditions

3.1.1 Model 1: The Brock and Mirman Stochastic Growth Model

In the Brock and Mirman problem, the central planner of a closed economy populated by a large number of identical households is endowed with a production technology that transforms capital and labor services into consumption and investment goods. Households have a time endowment measured in hours and normalized to equal 1. The planner seeks to maximize the expected lifetime utility of the representative household by choosing a sequence of consumption, hours of labor supply, and capital stock, $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$.[7] The planner's problem is summarized as follows:

$$\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \log(c_t) \right] \tag{3}$$

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[7]Variables are in per capita terms.

subject to the following constraints for $t = 0,...,\infty$:

$$z_t \, k_t^{\alpha} h_t^{1-\alpha} \geq c_t + i_t \tag{4a}$$

$$i_t = k_{t+1} - k_t(1 - \delta) \tag{4b}$$

$$z_{t+1} = (1 - \rho)z + \rho z_t + \epsilon_{t+1} \tag{4c}$$

$$k_0 \quad \text{and} \quad z_0 \qquad \text{given} \tag{4d}$$

The lifetime utility function in eq. (3) indicates that the representative household only derives utility from the consumption of goods in each period, c_t. Households discount future utility at the rate implicit in the discount factor β. The flow budget constraint in eq. (4a) says that at any time $t \geq 0$, the sum of consumption and investment, $c_t + i_t$, must not exceed the total amount of output arising from exploiting the Cobb-Douglas technology. For a given pair of labor and capital, (h_t, k_t), the total amount of output depends on the capital share in output α, and the productivity shock z_t. The investment equation (4b) depicts the law of motion of the capital stock which shows that the capital stock depreciates exponentially at the rate δ. Eq. (4c) is the forcing process governing the dynamics of the productivity shock; there, ρ is an autocorrelation parameter, z is the stationary value of z_t, and ϵ_t is an i.i.d. shock with $E[\epsilon_t] = 0$ and $E[\epsilon_t, \epsilon_{t+j}] = \sigma_z^2$ if $j = 0$, and it is equal to 0 otherwise. The central planner finds optimal to set $h_t = 1$ at any $t > 0$ because leisure is not an argument of the household utility function. Furthermore, the non-satiation assumption implicit in the functional form of the objective function implies that the budget constraint must hold as a strict equality at the optimum. After substituting i_t from eq. (4b) into the budget constraint, the first order conditions for optimality in the problem at hand are:

$$\frac{1}{c_t} = \beta \, E\left[\frac{1}{c_{t+1}}(1 + \alpha z_{t+1} k_{t+1}^{\alpha-1} - \delta)\right]; \qquad t \geq 0 \tag{5a}$$

$$z_t \, k_t^{\alpha} + k_t(1 - \delta) = c_t + k_{t+1}; \qquad t \geq 0 \tag{5b}$$

Eq. (5a) shows that the economy must accumulate capital until the marginal cost in utility terms (on the left-hand side of the equation), is equal to the marginal expected benefit, in utility terms (on the right-hand side), and eq. (5b) imposes the technological constraint on the allocation of resources.

The equilibrium and solution of the Brock and Mirman economy is a sequence $\{z_{t+1}, k_{t+1}, c_t\}_{t=0}^{\infty}$ that satisfies eq. (4c) and the optimality conditions (5) for $t \geq 0$, given the

initial conditions specified in (4d).[8] To completely characterize the efficiency allocations of the Brock and Mirman economy, the solution must satisfy the following limiting condition:

$$\lim_{t\to\infty} E\left[\beta^t(1/c_t)k_{t+1}\right] = 0$$

that (loosely speaking) indicates that it would be suboptimal for the planner to accumulate capital forever. Notice that eq. (4c) along with eqs. (5) represent a system of three equations for each $t \geq 0$ that can be solved for the optimal sequence $\{z_t, k_{t+1}, c_t\}_{t=0}^{\infty}$.

A researcher studying the Brock and Mirman model might be interested in studying the dynamics of output, y_t, the wage rate, w_t, and the rental rate of capital, r_t^k, which are three possible flow variables of this model. The following equations allow to back out these variables:

$$y_t = z_t k_t^{\alpha} \tag{6a}$$

$$w_t = (1-\alpha)z_t k^{\alpha} \tag{6b}$$

$$r_t^k = \alpha z_t k^{\alpha-1} \tag{6c}$$

The outlined model is known to have a closed form solution when $\delta = 1$. In this case, investment is equal to k_{t+1} and output is split between c_t and k_{t+1}. It is out of the scope of this paper to show that the closed form solution to this model involves:[9]

$$k_{t+1} = \beta\alpha z_t k_t^{\alpha} \tag{7a}$$

$$c_t = (1-\beta\alpha)z_t k_t^{\alpha} \tag{7b}$$

3.1.2 Model 2: The King-Plosser-Rebelo Economy

The King-Plosser-Rebelo (KPR) economy is similar to the economy in the preceding model except that in the KPR economy leisure is an argument of the utility index. Consequently, the household labor supply is no longer a constant equal to its time endowment. Furthermore, labor-augmenting productivity in the KPR economy grows over time at the (net) exogenous rate γ, which is the rate at which all variables except labor grow over time. In order to work with a stationary version of the model where the variables are constant in the non-stochastic steady state, we have to transform the variables and express them in units of effective labor. We do not show here how to perform this transformation and we

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[8]If the problem above were expressed in recursive form, it can be said that for given values of the predetermined variables at t, the model solution returns a value for z_{t+1}, k_{t+1}, and c_t.

[9]See, for instance, Sargent (1988).

do not discuss its implications nor the restrictions that a balanced growth path imposes on the functional form of the production and utility function.[10]

Here, the central planner faces the following problem:

$$\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \log(c_t) + \omega \log(1 - h_t) \right] \tag{8}$$

subject to the following constraints:

$$z_t \, k_t^{\alpha} h_t^{1-\alpha} \geq c_t + (1+\gamma)k_{t+1} - k_t(1-\delta) \tag{9a}$$

$$z_{t+1} = (1-\rho)z + \rho z_t + \epsilon_{t+1} \tag{9b}$$

$$k_0 \quad \text{and} \quad z_0 \quad \text{given} \tag{9c}$$

$$E[\epsilon_t] = 0; \quad E[\epsilon_t, \epsilon_{t+j}] = \sigma_z^2, \text{ if j=1; and } = 0 \text{ otherwise}$$

In this economy, all variables except h_t and ϵ_t represent "de-trended" variables, that is stationary versions of the growing variables. Except for ω, which is a parameter of the utility function that determines the fraction of time allocated to leisure, the nomenclature is the same as in the foregoing model. The transformation required to make the variables stationary modifies the investment equation multiplying k_{t+1} by the gross growth rate $(1+\gamma)$.

The equilibrium and solution of the model can be stated as a contingent sequence of four variables $\{z_{t+1}, k_{t+1}, c_t, h_t\}_{t=0}^{\infty}$ that, for some initial conditions (9c), satisfies the following four equations for $t \geq 0$:[11]

$$z_{t+1} = (1-\rho)z + \rho z_t + \epsilon_{t+1} \tag{10a}$$

$$(1+\gamma)\frac{1}{c_t} = \beta \, E \left[\frac{1}{c_{t+1}}(1 + \alpha z_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} - \delta) \right]; \tag{10b}$$

$$z_t \, k_t^{\alpha} h_t^{1-\alpha} + k_t(1-\delta) = c_t + (1+\gamma)k_{t+1}; \tag{10c}$$

$$\frac{\omega c_t}{(1 - h_t)} = (1-\alpha)z_t k_t^{\alpha} h_t^{-\alpha} \tag{10d}$$

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[10]References for the interested reader include Barro and Sala-i-Martin (1995) and King, Plosser, and Rebelo (1988b).

[11]The ordering of the equations is consistent with the ordering followed in the numerical part to be discussed later. Notice that we start writing the equation for the forcing process, then we write all other equations involving variables dated both at t and $t+1$, and finally we write the "intratemporal" equations, that is equations whose variables belong to only one period.

The optimal allocations should also be consistent with the limiting condition imposed on the Brock and Mirman economy. Different from that economy where it was optimal to set $h_t=1$, in the KPR economy eq. (10d) is used to find the number of hours that equates the marginal rate of substitution of labor for consumption to the marginal productivity of labor.

The flow variables that we (arbitrarily) choose to study are output, y_t; investment, i_t; the wage rate, w_t; and the rental return on capital, r_t^k. The following equations define these variables:

$$y_t = z_t k_t^\alpha h_t^{1-\alpha} \tag{11a}$$

$$i_t = (1+\gamma)k_{t+1} - k_t(1-\delta) \tag{11b}$$

$$w_t = (1-\alpha)z_t k^\alpha h_t^{-\alpha} \tag{11c}$$

$$r_t^k = \alpha z_t k^{\alpha-1} h_t^{1-\alpha} \tag{11d}$$

3.1.3 Model 3: The Small-Open-Economy RBC Model

In the small-open-economy (SOE) version of the RBC model, the economy borrows and lends in international capital markets so as to smooth out consumption over time. This requires incorporating the asset position of the economy into the analysis, taking into consideration that only domestic households own the domestic capital stock.

To produce fluctuations in the relative price of investment to consumption goods, the model includes capital adjustment costs; this means that changing the stock of capital generates costs that increase with the speed of the desired adjustment and that agents find optimal to undertake investment changes gradually (Mendoza (1991)). The inclusion of capital-adjustment costs has been motivated by the otherwise extremely high investment volatility produced by cross-country variations in the real return on capital. Specifically, we are going to assume that the cost of increasing the capital stock from k_t at t to k_{t+1} at $t+1$, involves a cost equal to $(\phi/2)(k_{t+1} - k_t)^2$.

As explained by Arellano and Mendoza (2003), and Schmitt-Grohé and Uribe (2003), when domestic agents have only access to a risk-free bond whose rate of return is exogenously determined abroad, the equilibrium dynamics posses a random walk component. This is a major problem when we want to study the model dynamics around a stationary point because the random walk component essentially means that the stationary point does not exist. Several techniques have been proposed in the literature to overcome this problem (see Schmitt-Grohé and Uribe (2003)). Following Mendoza (1991), we choose to make the discount factor dependent on the history of consumption and labor supply. Un-

der this formulation, the central planner seeks to maximize an Epstein's (1983) Stationary Cardinal Utility:

$$\max_{\{c_t,h_t,b_{t+1},k_{t+1}\}_{t=0}^{\infty}} E \left\{ \sum_{t=0}^{\infty} \exp\left[-\sum_{\tau=0}^{t-1} \psi \log\left(1 + c_\tau - \frac{h_\tau^\omega}{\omega} \right) \right] \frac{\left(c_t - \frac{h_t^\omega}{\omega} \right)^{1-\sigma}}{1-\sigma} \right\} \quad (12)$$

where the endogenous discount factor $\sum_{t=0}^{\infty} \exp\left[-\sum_{\tau=0}^{t-1} \psi \log\left(1 + c_\tau - \frac{h_\tau^\omega}{\omega} \right) \right]$ introduces an impatient effect because the rate of time preference increases with past consumption and labor supply. However, since this effect has shown negligible in quantitative applications (see Mendoza (1991)), we follow Schmitt-Grohé and Uribe (2003) in assuming that atomistic households do not internalize the effects of their consumption plans on the rate of time preference. In the formulation above, σ is the coefficient of relative risk aversion of the household; ψ is the elasticity of the rate of time preference with respect to $(1 + c_t + h_t^\omega/\omega)$; $1/(1-\omega)$ is the elasticity of the labor supply; and b_t is the stock of international assets.

The central planner maximizes the lifetime expected utility subject to the following flow budget constraint for $t \geq 0$:

$$z_t k_t^\alpha h_t^{1-\alpha} - \frac{\phi}{2}(k_{t+1} - k_t)^2 + k_t(1-\delta) + (1+r)b_t \geq c_t + k_{t+1} + b_{t+1}$$

where the left hand side contains the source of resources and the right hand side contains the uses of resources. Among the sources, the planner counts on: the net-of-adjustment-costs output arising from exploiting the production technology given the available stock of capital and labor supply; the after-depreciation capital inherited from the precedent period; and the inherited stock of international assets along with its accruing interest payments. The value of b_t determines the economy's asset position (with negative values representing net indebtedness); furthermore, an increment of b_t, i.e $b_{t+1} - b_t > 0$, implies a current account surplus of the same magnitude, and viceversa. International assets are assumed to have a fixed net return equal to r and the process for the productivity shock is the same as in the precedent two models.

The solution to this SOE model is a contingent sequence of five variables $\{z_{t+1}, b_{t+1}, k_{t+1}, c_t, h_t\}_{t=0}^{\infty}$, that for some initial conditions (z_0, k_0, b_0), satisfies the following system of five equations for $t \geq 0$:

$$z_{t+1} = (1-\rho)z + \rho z_t + \epsilon_t \quad (13a)$$

$$(c_t - h_t^\omega/\omega)^{-\sigma} [1 + \phi(k_{t+1} - k_t)] =$$

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$$\beta_t E\left[\left(c_{t+1} - h_{t+1}^\omega/\omega\right)^{-\sigma}\left(1 + \alpha z_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha} - \delta + \phi(k_{t+2} - k_{t+1}))\right)\right] \quad (13b)$$

$$\left(c_t - h_t^\omega/\omega\right)^{-\sigma} = \beta_t E\left[\left(c_{t+1} - h_{t+1}^\omega/\omega\right)^{-\sigma}\left(1 + r\right)\right] \quad (13c)$$

$$z_t k_t^\alpha h_t^{1-\alpha} - \frac{\phi}{2}(k_{t+1} - k_t)^2 + k_t(1-\delta) + (1+r)b_t = c_t + k_{t+1} + b_{t+1} \quad (13d)$$

$$h_t^{\omega-1} = z_t(1-\alpha)k_t^\alpha h_t^{-\alpha} \quad (13e)$$

where we have used $\beta_t \equiv \left(1 + c_t + \frac{h_t^\omega}{\omega}\right)^{-\psi}$ in eqs. (13b) and (13c). Except for the functional form of the utility function, the capital adjustment costs, and the discount factor, eq. (13b) is the same as eq. (5a) in the Brock and Mirman economy and eq. (10b) in the KPR economy; eq. (13c) is the dynamic efficiency condition for the accumulation of debt; it equates the utility cost of borrowing to the expected discounted utility of repaying the amount borrowed at the international interest rate; eq. (13d) is the flow budget constraint, and eq. (13e) equates the marginal rate of substitution of labor for consumption to the marginal productivity of labor.

Strictly speaking, the SOE model does not fit in the class of models of section 2 because k_{t+2} in eq. (13b) is a variable dated at time $t+2$. To make the model conformable with the class of models of section 2, we will add an equation to system (13) to define the auxiliary variable k_t^a as $k_t^a = k_{t+1}$. This allows us to write k_{t+1}^a instead of k_{t+2} in eq. (13b) and all model variables are then dated either at t or $t+1$.

The sequence $\{z_{t+1}, b_{t+1}, k_{t+1}, c_t, h_t\}_{t=0}^\infty$, that satisfies conditions (13), given a sequence of innovations to the productivity shock, $\{\epsilon_t\}_{t=1}^\infty$, and given initial conditions (b_0, k_0, z_0), is a solution if it does not violate the limiting condition

$$\lim_{t\to\infty} E\left\{\sum_{t=0}^\infty \exp\left[-\sum_{\tau=0}^{t-1}\psi\log\left(1 + c_\tau - \frac{n_\tau^\omega}{\omega}\right)\right]\left(c_t - \frac{h_t^\omega}{\omega}\right)^{-\sigma}(k_{t+1} + b_{t+1})\right\} = 0$$

This limiting condition rules out solutions where assets $(k_t + b_t)$ grow indefinitely for the same reasons we ruled them out in the precedent examples. The condition also rules out indefinite increments in borrowing to avoid situations where the household indefinitely borrows and then repays interest and principal by borrowing more.

Since we want to obtain the results in Table 6 (pp. 812) of Mendoza (1991), we consider the following flow variables:

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GDP:	$y_t = z_t k_t^\alpha h_t^{1-\alpha} - (\phi/2)(k_{t+1} - k_t)^2$	(14a)
GNP:	$\widetilde{y}_t = y_t - r\, b_t$	(14b)
Savings:	$s_t = y_t - c$	(14c)
Investment:	$i_t = k_{t+1} - k_t$	(14d)
Productivity:	$pr_t = y_t / h_t$	(14e)
Interest Payments:	$ip_t = r\, b_t$	(14f)
Trade-balance-output ratio:	$tb_t = (y_t - c_t - i_t)/y_t$	(14g)

3.2 Variable and Equation Sorting and a Recursive Interpretation of the Models

Tables 1 and 2 classify the variables and equations of the three RBC models following the criteria of section 2. Having a recursive interpretation of the dynamics of these models will be useful to understand the theoretical solution of the next section. Starting with the Brock and Mirman economy, at $t = 0$, the central planner wakes up with a given stock of capital and a value of productivity, i.e. a pair (k_0, z_0), and he has to decide how much output to allocate to current consumption, c_0. Given the current output y_0 (a quantity dependent on k_0 and z_0) and the chosen level of consumption, c_0, then he finds how much capital will be available for production at $t = 1$, i.e. k_1. At time $t = 1$, the planner faces the same problem he faced at $t = 0$: he knows (k_1, z_1) and has to choose c_1; then, he finds how much capital he will have at $t = 2$, i.e. k_2, and so on.

The recursive interpretation of the planner's decision problem indicates that at any t, he observes the state variables (z_t, k_t), and the optimal policy function $c_t = \mathbf{c}(k_t, z_t)$ tells him the optimal level of consumption.[12] Then, the state-transition eqs. (4c) and (5b) determine the value of k_{t+1} and z_{t+1} for a given innovation to the productivity shock. In the recursive form of the KPR economy we have the same vector of state variables (k_t, z_t); two policy functions, one for consumption, $c_t = \mathbf{c}(k_t, z_t)$, and one for labor supply $h_t = \mathbf{h}(k_t, z_t)$; and two state transition eqns., i.e. eqns. (10a) and (10c). In the SOE model, at any t, the planner knows the value of the state variables (b_t, k_t, z_t), and he follows an optimal policy rule for consumption, $c_t = \mathbf{c}(b_t, k_t, z_t)$; an optimal policy rule for saving in capital, $s_t^k = \mathbf{s}^k(b_t, k_t, z_t)$; and an optimal policy rule for saving in international assets, $s_t^b = \mathbf{s}^b(b_t, k_t, z_t)$. Then, a state transition function along with the policy functions and the

---

[12]The policy functions are part of the solution to the models.

innovation to the productivity shock determine the value of next period states.

To link the precedent recursive interpretation of our three models with the general numerical solution to LDRE models discussed in the next section, it will be useful to call **f** to the set of policy functions; to call **p** to the set of transition equations; and to think that: a) given the value of the state variables at time t, the policy functions in **f** indicate the optimal value of the control variables at t; b) given the value of the state variables at t, the transition equations in **p** indicate the value of the state variables at $t + 1$.

3.3 Calibration

To calibrate the three foregoing RBC models, we focus on their non-stochastic steady states. These states are the rest points where the economies would stay in the long run if we shut out all innovations to the exogenous backward-looking variables. To find the steady states of our models, we have to impose $\epsilon_t = 0$, for $t \geq 0$, and solve the respective systems of equations: (5) for the Brock and Mirman economy; (10) for the KPR economy; and (13) for the SOE.

The system of steady state equations is solved to find the value of some model parameters and variables, given the information available on some macroeconomic aggregates and parameters. In each case, deciding upon which parameter and variable values are taken from the data and which ones are arising from the equilibrium conditions of the model depends, among other things, on the availability of statistical information of the modeled economy. We calibrate the Brock and Mirman and the KPR model to the US economy following King et al. (1988a), and the SOE model to the Canadian economy following Mendoza (1991). For the first two models, the time interval is a quarter and for the SOE model, the time interval is a year.

Using the definition of output in (6a), the following three equations characterize the non-stochastic steady state of the Brock and Mirman economy:

$$1 = \beta(1 + \alpha \frac{y}{k} - \delta) \tag{15a}$$

$$y = c + k\delta \tag{15b}$$

$$y = zk^{\alpha} \tag{15c}$$

We normalize the calibration setting $k = 10$ and we impose $\delta = 1$ to obtain a closed form solution to this model. We set the capital share in output α=0.40 and choose a value of β consistent with an annual net-of-depreciation return to capital, $1 + \alpha\, y/k - \delta$,

equal to 6.5%, which gives $\beta = 0.9844$ on quarterly basis. When $\delta = 1$ (15a) implies that the output-to-capital ratio $y/k = 1.065^{0.25}/\alpha = 2.54$ and equation (15b) implies that $c/y = 0.6062$; additionally, from the equality $z\alpha k^{\alpha-1} = \alpha y/k$, we obtain $z = 10.11$;

In the KPR economy the system of steady-state equations is:

$$(1 + \gamma) = \beta(1 + \alpha \frac{y}{k} - \delta) \tag{16a}$$

$$y = c + k(\delta + \gamma) \tag{16b}$$

$$\frac{\omega c}{(1 - h)} = (1 - \alpha)\frac{y}{h} \tag{16c}$$

$$y = zk^\alpha h^{1-\alpha} \tag{16d}$$

We normalize the calibration setting $k = 10$; consistent with the fact that the fraction of time allocated to work is equal to 20% in the US economy, we follow King et al. (1988a) to set $h = 0.2$. We set the capital share in output α equal to 0.42; $\delta = 0.025$; and $\gamma = 0.004$; and we find the value of β consistent with a net annual return on capital of 6.5% in equation (16a): $\beta = (1 + 0.004)/(1 + 0.065)^{0.25} = 0.988$; the same equation then implies the ratio $y/k = 0.0973$ and by using the investment eq. (11b) and eq. (10c), we obtain $i/k = 0.29$ and $c/y = 0.702$. Eq. (16c) then implies $\omega = 3.305$ and (16d) $z = 0.9409$.

In the SOE version of the RBC model, the calibration of the model to the Canadian economy in Mendoza (1991) starts with the following parameter values: $\alpha = 0.320$; $\delta = 0.100$; $\omega = 1.455$; $\phi = 0.019$; $\rho = 0.42$; $\sigma = 2.000$; $r = 0.040$; and $\psi = 0.11$.[13] We normalize the calibration setting $k = 3.398$, which approximates the middle point of the capital-stock grid in Mendoza (1991). The system of equations characterizing the steady state of the model is:

$$1 = (1 + c - h^\omega/\omega)^{-\psi}(1 + \alpha \frac{y}{k} - \delta) \tag{17a}$$

$$1 = (1 + c - h^\omega/\omega)^{-\psi}(1 + r) \tag{17b}$$

$$y + rb = c + k\delta \tag{17c}$$

$$h^{\omega-1} = (1 - \alpha)\frac{y}{h} \tag{17d}$$

$$y = zk^\alpha h^{1-\alpha} \tag{17e}$$

Thinking of input prices in the decentralized equilibrium, notice that $h^{\omega-1} = w = (1 - \alpha)(y/h)$ and that $r^k = r + \delta = \alpha(y/k)$ (see eqs. (17a), (17b), and (17d)). From eq.

---

[13]To reproduce the quantitative results in Mendoza (1991), we have lightly deviated from the original parameterization in which $\phi = 0.028$.

(17d) and the equalization of the relative input prices to the ratio of marginal products, $h = [r^k k (1 - \alpha)/\alpha]^{1/\omega} = 1.007$. Eq. (17b) then shows that $c = 1.123$, which then implies that the discount factor is equal 0.962. By using equation (17e) and the right hand side of (17a), we obtain $z = 1.00$ and $y = 1.486$.

Table 3 summarizes the value of the parameters in the three RBC models and Table 4 summarizes the value of some macroeconomic variables and ratios.

3.4   Log-Linearization

The LDRE Toolbox accompanying this paper carries out the numerical log-linearization of the equations of any dynamic model fitting in the class of models of section 2. To show the equivalence between the paper-and-pencil and the numerical log-linearization of a model we next work out the log-linear approximation to the KPR model.

Log-linearizing a model implies finding the matrices of coefficients A and B in eq. (1) and C and D in eq. (2). For the KPR economy, we start totally differentiating eqs. (10) around the steady state:

$$\rho dz_t - dz_{t+1} = 0 \tag{18a}$$

$$-\frac{1+\gamma}{c^2} dc_t - \beta E \left[-\frac{1}{c^2} \left(1 + \alpha z k^{\alpha-1} h^{1-\alpha} - \delta \right) dc_{t+1} + \frac{1}{c} \alpha k^{\alpha-1} h^{1-\alpha} dz_{t+1} \right.$$

$$\left. +\frac{1}{c} \alpha z (\alpha - 1) k^{\alpha-2} h^{1-\alpha} dk_{t+1} + \frac{1}{c} \alpha z (1 - \alpha) k^{\alpha-1} h^{-\alpha} dh_{t+1} \right] = 0 \tag{18b}$$

$$k^\alpha h^{1-\alpha} dz_t + [\alpha z k^{\alpha-1} h^{1-\alpha} + (1 - \delta)] dk_t +$$

$$(1 - \alpha) z k^\alpha h^{-\alpha} dh_t - dc_t - (1 + \gamma) dk_{t+1} = 0 \tag{18c}$$

$$\frac{\omega}{(1-h)} dc_t - (1 - \alpha) k^\alpha h^{-\alpha} dz_t - z(1 - \alpha) \alpha k^{\alpha-1} h^{-\alpha} dk_t -$$

$$[\frac{\omega c}{(1-h)^2} dh_t + z(1 - \alpha)(-\alpha) k^\alpha h^{-\alpha-1}] dh_t = 0 \tag{18d}$$

Let define $\hat{x}_t \equiv dx_t/x$ as the percentage deviation of x_t from its steady-state value x. This permits writing system (18) as:

$$\rho z \hat{z}_t - z \hat{z}_{t+1} = 0 \tag{19a}$$

$$-\frac{1+\gamma}{c} \hat{c}_t - \beta E \left[-\frac{1}{c} \left(1 + \alpha z k^{\alpha-1} h^{1-\alpha} - \delta \right) \hat{c}_{t+1} + \right.$$

$$\left. \left(\frac{1}{c} z \alpha k^{\alpha-1} h^{1-\alpha} \right) \left(\hat{z}_{t+1} + (\alpha - 1) \hat{k}_{t+1} + (1 - \alpha) \hat{h}_{t+1} \right) \right] = 0 \tag{19b}$$

$$z k^\alpha h^{1-\alpha} \hat{z}_t + [\alpha z k^\alpha h^{1-\alpha} + (1 - \delta) k] \hat{k}_t +$$

$$(1 - \alpha)zk^{\alpha}h^{1-\alpha}\hat{h}_t + c\hat{c}_t - (1 + \gamma)k\hat{k}_{t+1} = 0 \tag{19c}$$

$$\left(z(1 - \alpha)k^{\alpha}h^{-\alpha}\right)\left(\hat{z}_t + \alpha\hat{k}_t\right) + \left[\frac{\omega c\, h}{(1 - h)^2} - z(1 - \alpha)(-\alpha)k^{\alpha}h^{-\alpha}\right]\hat{h}_t = 0 \tag{19d}$$

Since all but the "hat" variables above have numerical values, the log-linear approximation to the KPR model arises from the evaluation of the system (19) at the steady state using the numerical information in Tables 3 and 4:

$$0.847\hat{z}_t - 0.941\hat{z}_{t+1} = 0 \tag{20a}$$

$$-1.470\hat{c}_t + 1.470E[\hat{c}_{t+1}] - 0.059E[\hat{z}_{t+1}] + 0.034E[\hat{k}_{t+1}] - 0.034E[\hat{h}_{t+1}] = 0 \tag{20b}$$

$$0.973\hat{z}_t + 10.159\hat{k}_t + 0.564\hat{h}_t - 0.683\hat{c}_t - 10.04\hat{k}_{t+1} = 0 \tag{20c}$$

$$2.822\hat{c}_t + 1.891\hat{h}_t - 2.822\hat{z}_t - 1.185\hat{k}_t = 0 \tag{20d}$$

The coefficients multiplying the variables dated at $t + 1$ are the elements of matrix A and the coefficients multiplying the variables dated at t are the elements of matrix B. When we use the LDRE Toolbox to obtain the log-linear version of the KPR model we obtain the following matrices:

$$
\begin{pmatrix}
0.941 & 0 & 0 & 0 \\
0.059 & -0.034 & -1.470 & 0.034 \\
0 & 10.040 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
E\,\hat{z}_{t+1} \\
E\,\hat{k}_{t+1} \\
E\,\hat{c}_{t+1} \\
E\,\hat{h}_{t+1}
\end{pmatrix}
=
\begin{pmatrix}
0.847 & 0 & 0 & 0 \\
0 & 0 & -1.470 & 0 \\
0.973 & 10.159 & -0.683 & 0.564 \\
-2.822 & -1.185 & 2.822 & 1.891
\end{pmatrix}
\begin{pmatrix}
\hat{z}_t \\
\hat{k}_t \\
\hat{c}_t \\
\hat{h}_t
\end{pmatrix}
$$

This equation is equivalent to system (20); furthermore, the matrices of coefficients on the left- and right-hand side correspond to matrices A and B in eq. (1). Following the same steps above, we can log-linearize the flow-variables in (11) to obtain:

$$
C = 0_{4\times 4} \qquad D =
\begin{pmatrix}
1.000 & 0.420 & 0 & 0.580 \\
3.355 & 1.409 & -2.355 & 1.9461 \\
2.822 & 1.185 & 0 & -0.185 \\
0.041 & -0.024 & 0 & 0.024
\end{pmatrix}
$$

which are examples of the matrices in eq. (2). In general, whereas A and B have a row per equation and a column per variable, C and D have a row per flow variable and a column per model variable.

The matrices of log-linear coefficients A to D of the Brock and Mirman model are:

$$A = \begin{pmatrix} 10.111 & 0 & 0 \\ 0.065 & -0.039 & -0.065 \\ 0 & 10.000 & 0 \end{pmatrix} ; \quad B = \begin{pmatrix} 9.100 & 0 & 0 \\ 0 & 0 & -0.065 \\ 25.397 & 10.159 & -15.397 \end{pmatrix}$$

$$C = 0_{3\times3} \quad D = \begin{pmatrix} 1.000 & 0.400 & 0 \\ 1.000 & 0.400 & 0 \\ 1.000 & -0.600 & 0 \end{pmatrix}$$

And the matrices of log-linear coefficients A to D of the SOE model are:

$$A = \begin{pmatrix} 1.000 & 0 & 0 & 0 & 0 & 0 \\ 0.734 & 0 & -1.207 & -28.570 & 26.213 & 0.347 \\ 0 & 0 & 0 & -28.570 & 25.714 & 0 \\ 0 & -0.590 & 3.398 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.398 & 0 & 0 & 0 \end{pmatrix} ;$$

$$B = \begin{pmatrix} 0.402 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.361 & -28.100 & 25.290 & 0.327 \\ 0 & 0 & 0 & -28.100 & 25.290 & 0 \\ -1.486 & -0.614 & 3.534 & -1.123 & 1.0108 & 0 \\ -1.003 & 0 & -0.321 & 0 & 0.777 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.398 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.286 & 0 & 0 & 0 \end{pmatrix} ; \quad D = \begin{pmatrix} 1.000 & 0 & 0.320 & 0 & 0.680 & 0 \\ 1.016 & -0.016 & 0.325 & 0 & 0.691 & 0 \\ 4.091 & 0 & 1.309 & -3.090 & 2.782 & 0 \\ 0 & 0 & -9.000 & 0 & 0 & 0 \\ 1.000 & 0 & 0.320 & 0 & -0.320 & 0 \\ 0 & -0.024 & 0 & 0 & 0 & 0 \\ 0.984 & 0 & 2.372 & -0.755 & 0.669 & 0 \end{pmatrix}$$

4 A Theoretical Solution to LDRE Models

Klein (2000) proposes a theoretical solution to LDRE models that is summarized by a pair of functions, one representing the law of motion of the state (or backward-looking) variables, and the other representing the optimal policy rule (the optimal decision on forward-looking variables). The first maps the state space into itself and the second maps the state space into the set of optimal policy functions. These functions correspond to **f**

and **p** introduced in section 3. The solution is, therefore, the recursive representation of the stable solution to a system of log-linear difference equations. Here, we follow Soderling (1999) to show how to obtain this pair of functions from the matrices of coefficients A and B.

Klein's method is based on the Schur decomposition of the matrices A and B in eq. (1). The decomposition gives the square complex matrices Q, S, T, and Z such that:[14]

$$A = QSZ^H \quad \text{and} \quad B = QTZ^H$$

where Z^H denotes the transpose of the complex conjugate of Z. Q and Z are unitary matrices, that is $Q^H Q = Z^H Z = I$, and S and T are upper triangular.

The generalized eigenvalues of the matrices A and B are equal to the i-th diagonal element of T divided by the i-th diagonal element of S. When the matrix A is singular, some of the generalized eigenvalues are infinite. Let us define as stable generalized eigenvalues those that are less than one. The unstable are those larger or equal to one in absolute value, including infinite values.

The decomposition can be reordered without altering the result. The reordering made here is such that the block of stable generalized eigenvalues come first.

Starting with the vector of variables $\hat{x}$, define the auxiliary variable $\hat{y}$ as $\hat{y} = Z^H \hat{x}$, so

$$\hat{y}(t) = \begin{pmatrix} \hat{y}^s(t) \\ y^u(t) \end{pmatrix} = Z^H \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{pmatrix} \tag{21}$$

By virtue of the Schur decomposition, pre-multiplying both sides of (1) by Q^H gives:

$$SE_t Z^H \hat{x}(t+1) = T\, Z^H \hat{x}(t)$$

which, employing the auxiliary variables defined in (21), becomes:

$$S\, E_t \hat{y}(t+1) = T\, \hat{y}(t)$$

---

[14]If the reader does not feel comfortable with the statement above, notice that it takes just a Matlab call to get the Schur decomposition of two matrices. Given two square matrices A and B, through the sentence *[S,T,Q,Z]=qz(A,B)*, Matlab returns the matrices S, T, Q, and Z. In fact, the decomposition that Matlab makes is such that $A = Q^H S T^H$.

A partition of the matrices S and T conformably with y^s and y^u, is:

$$\begin{pmatrix} S_{x_1x_1} & S_{x_1x_2} \\ 0 & S_{x_2x_2} \end{pmatrix} E_t \begin{pmatrix} \hat{y}^s(t+1) \\ \hat{y}^u(t+1) \end{pmatrix} = \begin{pmatrix} T_{x_1x_1} & T_{x_1x_2} \\ 0 & T_{x_2x_2} \end{pmatrix} \begin{pmatrix} \hat{y}^s(t) \\ \hat{y}^u(t) \end{pmatrix} \tag{22}$$

Notice that the second difference equation contains the unstable roots due to the reordering of the eigenvalues. Therefore, a stable solution for that equation requires $\hat{y}^u(t)=0$ for all t, which implies the existence of a linear dependency among the components of the vector $\hat{y}^u(t)$. The remaining equations in (22) can be written as:

$$E_t\,\hat{y}^s(t+1) = S_{x_1x_1}^{-1} T_{x_1x_1} \hat{y}^s(t) \tag{23}$$

where it can be shown that $S_{x_1x_1}^{-1}$ exists. This is because S is a triangular matrix that in turn was reordered in such a way that none of its elements in the diagonal are zero (otherwise an unstable generalized eigenvalue would have arisen and its correspondent block could not be part of the upper right block of S).[15]

Multiplying eq. (21) by Z gives $Z\,Z^H x(t) = \hat{x}(t) = Z\,\hat{y}(t)$ because $Z\,Z^H = I$. Making a partition of Z conformably with the partition of $\hat{x}$ and $\hat{y}$ in eq. (21), we can write:

$$\begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{pmatrix} = \begin{pmatrix} Z_{x_1y_s} & Z_{x_1y_u} \\ Z_{x_2y_s} & Z_{x_2y_u} \end{pmatrix} \begin{pmatrix} \hat{y}^s(t) \\ \hat{y}^u(t) \end{pmatrix} = \begin{pmatrix} Z_{x_1y_s} \\ Z_{x_2y_s} \end{pmatrix} \hat{y}^s(t) \tag{24}$$

where the second equality arises because $\hat{y}^u(t) = 0$ and the first equation in (24) says that $\hat{x}_1(t) = Z_{x_1y_s}\hat{y}^s(t)$. Recalling that the value of $\hat{x}_1(0)$ arises from the initial conditions, one can solve for $\hat{y}^s(0)$:

$$\hat{y}^s(0) = Z_{x_1y_s}^{-1} \hat{x}_1(0)$$

provided that $Z_{x_1y_s}$ is invertible. It can be shown that the invertibility is assured when the number of predetermined variables (rows in $Z_{x_1y_s}$) equals the number of stable roots (columns in $Z_{x_1y_s}$).

Some of the backward-looking variables are affected by innovations to the exogenous processes and some others are not. For instance, while the next period stock of financial assets may be affected by changes in the interest rate, the stock of capital will be equal to the existent stock plus the net investment in the current period. Hence, we may write $\hat{x}_1(t+1) = E_t\,\hat{x}_1(t+1) + \epsilon_1(t+1)$, with some components of $\epsilon_1(t+1)$ equal to zero. On the other hand, recalling that the first equation in (24) implies that $\hat{x}_1(t) = Z_{x_1y_s}\hat{y}^s(t)$, the

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[15]The determinant of a triangular matrix is equal to the product of the diagonal elements, which are all non zero in matrix S.

expression $\hat{x}_1(t+1) = E_t\ \hat{x}_1(t+1) + \epsilon_1(t+1)$ can be written as:

$$Z_{x_1y_s}\left[\hat{y}^s(t+1) - E_t\ \hat{y}^s(t+1)\right] = \epsilon_1(t+1)$$

So:

$$\hat{y}^s(t+1) = E_t\ \hat{y}^s(t+1) + Z_{x_1y_s}^{-1}\ \epsilon_1(t+1)$$

This is not an explicit solution for $\hat{y}^s(t+1)$ because its expected value as of time t is also on the right hand side of the equation. However, expression (23) may then be used to give:

$$\hat{y}^s(t+1) = S_{x_1x_1}^{-1}T_{x_1x_1}\hat{y}^s(t) + Z_{x_1y_t^s}^{-1}\ \epsilon_1(t+1) \qquad (25)$$

It remains to get the solution in terms of the original variables in $\hat{x}_1(t)$, something that is straightforward if we appeal to the definition of the auxiliary variable $\hat{y}(t)$ and the partition of Z made above (see equations (21) and (24)):

$$\hat{x}_1(t+1) = Z_{x_1y_s}\hat{y}^s(t+1) = Z_{x_1y_s}S_{x_1x_1}^{-1}T_{x_1x_1}Z_{x_1y_s}^{-1}\hat{x}_1(t) + \epsilon_1(t+1) \qquad (26)$$

Now, in order to solve for $\hat{x}_2(t)$, notice that (24) implies that $\hat{x}_2(t) = Z_{x_2y_s}\hat{y}^s(t)$, and again, since $\hat{x}_1(t) = Z_{x_1y_s}\hat{y}^s(t)$,

$$\hat{x}_2(t) = Z_{x_2y_s}Z_{x_1y_s}^{-1}\hat{x}_1(t) \qquad (27)$$

To establish a link between these results, the recursive interpretation of the models in section 3, and the Matlab functions in the Toolbox, define

$$\mathbf{p} \equiv Z_{x_1y_s}S_{x_1x_1}^{-1}T_{x_1x_1}Z_{x_1y_s}^{-1} \qquad (28)$$

$$\mathbf{f} \equiv Z_{x_2y_s}Z_{x_1y_s}^{-1} \qquad (29)$$

This notation permits writing (26) as $\hat{x}_1(t+1) = \mathbf{p}\hat{x}_1(t) + \epsilon_1(t+1)$, and (27) as $\hat{x}_2(t) = \mathbf{f}\hat{x}_1(t)$. Thus, (28) and (29) completely describe the evolution of the system once the initial conditions and the shocks hitting the economy are specified. Particularly, $\mathbf{p}$ represents the state transition function governing the evolution of the state variables. Likewise $\mathbf{f}$ represents the policy function or decision rule and it maps the state of the economy into the decisions about the forward-looking variables. To see the dynamics implied by $\mathbf{p}$ and $\mathbf{f}$, remember that the initial conditions are specified in $\hat{x}_1(0)$. Then $\mathbf{f}$ gives the value of $\hat{x}_2(0)$ and $\mathbf{p}$ gives the value of the backward-looking variables at $t = 1$ for a particular shock $\epsilon_1(1)$. At $t=1$, $\hat{x}_2(1) = \mathbf{f}\hat{x}_1(2)$, and $\hat{x}_1(2) = \mathbf{p}\hat{x}_1(1) + \epsilon_1(2)$, and son on.

The value of the additional variables is straightforward to get at this point. Making partitions of the matrices C and D and the vector x at times t and $t+1$ in eq. (2),

$$\hat{x}_3(t) = \left(\begin{array}{c|c} C_1 & C_2 \end{array} \right) \left(\begin{array}{c} \hat{x}_1(t+1) \\ \hat{x}_2(t+1) \end{array} \right) + \left(\begin{array}{c|c} D_1 & D_2 \end{array} \right) \left(\begin{array}{c} \hat{x}_1(t) \\ \hat{x}_2(t) \end{array} \right) \tag{30}$$

where both C_1 and D_1 are of dimension $n_3 \times n_1$ while both C_2 and D_2 are of dimension $n_3 \times n_2$. Notice that the elements of C_2 are all zeros in most of the cases. Also recalling that $\hat{x}_1(t+1) = \mathbf{p}\,\hat{x}_1(t) + \epsilon_1(t+1)$ and that $\hat{x}_2(t) = \mathbf{f}\,\hat{x}_1(t)$, (30) can be re-written as:

$$\hat{x}_3(t) = (C_1\,\mathbf{p} + C_2\,\mathbf{f}\,\mathbf{p} + D_1 + D_2\,\mathbf{f})\hat{x}_1(t) + (C_1 + C_2\,\mathbf{f})\,\epsilon_1(t+1) \tag{31}$$

The following notation is used in the Toolbox:

$$\mathbf{g} \equiv C_1\,\mathbf{p} + C_2\,\mathbf{f}\,\mathbf{p} + D_1 + D_2\,\mathbf{f} \tag{32}$$

$$\mathbf{h} \equiv C_1 + C_2\,\mathbf{f} \tag{33}$$

which simplifies (30) to:

$$\hat{x}_3(t) = \mathbf{g}\,x_1(t) + \mathbf{h}\,\epsilon_1(t+1) \tag{34}$$

5   The Matlab Functions in the LDRE Toolbox

This section describes the role of the Matlab functions of the LDRE Toolbox and shows how a researcher should adapt the model-specific functions to his own model. The functions in the Toolbox can be grouped in three subsets and they are all coordinated by the function `CONTROL.m`. The first subset carries out the log-linearization; the second obtains the fundamental equations (28) and (29), as well as the set of coefficients in (32) and (33); and the third subset plots impulse response functions, computes business cycles statistics, and simulates the economy under a user-provided sequence of innovations to the exogenous predetermined variables. Table 5 lists the functions in each subset and uses a star to identify the model-specific functions.

In designing the LDRE Toolbox, we have minimized the number of function calls to just one. After providing his model-specific information, the researcher just has to call the function `CONTROL.m` to solve his model and obtain his results. To provide the model specific information, the researcher has to edit 5 of the 16 functions in the Toolbox. First, he has to pass some in-file information within `CONTROL.m`. To serve as an example, a

simplified version of this function follows:

```
function [X,f,p]=CONTROL

global n1 n2 n3 ne nv n
%--- Model specific information goes here: ----------------------------
    n1=2; n2=2; n3=4; ne=1;
    proc = [1 0 0];
    SIGMA = diag([0.0099818^2]);
    shocks=[1];     lengthIR = 60;
    directory = 'C:\Your directory\';
%-----------------------------------------------------------
 nv=n1+n2+n3; n=n1+n2;
 [g,h,f,p]=recrepres;
 X=procedure(proc,shocks,lengthIR,SIGMA,GDP,g,h,f,p,directory);
```

The first piece of model-specific information corresponds to the number of each type of variable in the model. `n1` is the number of backward-looking variables (including both the endogenous and the exogenous); `n2`, the number of forward-looking variables; `ne`, the number of exogenous variables among the backward-looking ones; and, `n3`, the number of additional or flow variables. As indicated in Table 1, to solve the Brock and Mirman economy we set `n1=2`, `n2=1`, `n3=3`, and `ne=1`; to solve the KPR economy, we set `n1=2`, `n2=2`, `n3=4`, and `ne=1`; and to solve the SOE economy, we set `n1=3`, `n2=3`, `n3=7`, and `ne=1`.

The researcher must indicate the required output by filling in the binary vector `proc` which of size 1×3. The first component of the vector corresponds to population moment conditions; the second to impulse-response functions; and the third to simulations with a researcher-provided sequence of innovations to the exogenous predetermined variables. Setting `proc=[1,0,0]`, the Toolbox returns the population moment conditions; setting `proc=[0,1,0]`, returns the impulse response functions; and setting `proc=[0,0,1]`, returns the results of the simulations with the researcher-provided innovations to the exogenous predetermined variables.

The statistical properties of the innovations to the exogenous predetermined variables is specified through the variance-covariance matrix `SIGMA`, a square matrix of size `ne`.

When the researcher seeks to obtain impulse-response functions, the vector `shocks` specifies the predetermined variables receiving impulses. The size of `shocks` is $1 \times n_e$ and its ordering matters: its first element corresponds to the first exogenous predetermined variable, the second element to the second exogenous predetermined variable, and so on. `shocks` is a binary vector and a 1 in the j-th position indicates that the j-th exogenous predetermined variable is receiving an impulse equal to a 1% deviation with respect to its steady-state value. Furthermore, the researcher must specify the length of the responses

by providing a value to the scalar variable `lengthIR`.

The last piece of information that has to be passed within `CONTROL.m` is the name of a directory where the tables of moment conditions will be saved. To simulate a model under a particular sequence of shocks, the file containing the innovations to the shocks must also be saved in this directory following the guidelines given below.

Once `CONTROL.m` has collected the above in-file information, it first calls `recrepres.m` to solve for the recursive representation of the model, and then calls `procedure.m` to obtain the desired results. The function `recrepres.m`, in turns, calls `lincoef.m` to log-linearize the model equations as it is explained next.

5.1 The Log-linearization Funtions

To log-linearize the model, `lincoef.m` calls `stst.m` once and `eqns.m` and `flows.m` twice. The researcher has to adapt these functions to his own model as we illustrate next by adopting the KPR model as example.

`stst.m` has to return the values of the variables and parameters of the model and this requires solving the system of steady-state equations. An adaptation of the function `stst.m` to the KPR economy follows:

```
function [X,Xf,p] = stst    %STEADY STATE OF THE KPR ECONOMY

%=======Parameter values and steady state values of k and h ==============
 alpha = 0.420; gamma = 0.004;  delta = 0.025;  rho = 0.900;
 h = 0.200;     k = 10.00;      r = 1.065^.25-1;

%======Calibration========================================================
 beta       =  (1+gamma)/(1+r);
 ratioyk    = ( (1+gamma)/beta - 1 + delta)/alpha;
 ratioxy    = (1/rayk)*(delta+gamma);
 ratiocy    = 1 - ratioxy;
 z          = ratioyk*k^(1-alpha)*h^(alpha-1);
 w          = (1-alpha)*z*k^alpha*h^-alpha;
 rk         = r + delta;
 y          = z*k^alpha*h^(1-alpha);
 c          = y - (gamma+delta)*k;
 i          = k*(1+gamma) - k*(1-delta);
 omega      = (c/y)*(h/(1-h))/(1-alpha);
 ratiock    = c/k;
 ratioiy    = i/y;

%=====Model and flow variables============================================
 X   = [z k c h]';
 Xf  = [y i w rk]';

 p   = struct('zss',z,'alpha',alpha,'beta',beta,'gamma',gamma,'delta',...
```

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```
            delta,'omega',omega,'rho',rho,'xi',xi);
```

After writing the parameter values and setting the values of k and h, the function solves the system of equations (16a) to find the values of all variables and ratios shown in Table 4, as well as the value of ω. Then it sets the values of the vectors X and Xf, which contain the steady-state values of the variables in the model and the values of the flow variables. The last two lines construct a structure p containing the values of the parameters. It is important to keep unchanged the input-output syntaxis of stst.m because other functions in the Toolbox have been programmed to work with variables called X, Xf, and p.

The system of equations defining the model has to be written within the function eqns.m. A simplified version of eqns.m adapted to the KPR economy follows:

```
function coeff=eqns(X,flag,p)        %EQUATIONS IN THE KPR ECONOMY

s=X';                          %used to calculate percentage deviations
X=repmat(X,1,size(X,1));       %matrix X is constructed repeating vector X
SmId=1e-6*eye(size(X));        %finite differences: 1st step

switch flag1                   %coefficients, A or B
case 'A'
    Xp=X+SmId;                 %deviation on variables dated at t+1
case 'B'
    Xp=X; X=X+SmId;            %deviation on variables dated at t
end

%======== Only the following lines are model specific ===========================
 z=X(1,:);  k=X(2,:);        c=X(3,:);        h=X(4,:);
 zp=Xp(1,:);kp=Xp(2,:);      cp=Xp(3,:);      hp=Xp(4,:);

 uc  = 1./c;     uh=-p.xi./(1-h); ucp = 1./cp;

 coeff(1,:)  = (1-p.rho)*p.zss + p.rho*z - zp;
 coeff(2,:)  = (1+p.gamma)*uc - p.beta * ucp .* ...
              (1 + p.alpha*zp.*kp.^(p.alpha-1).*hp.^(1-p.alpha) - p.delta);
 coeff(3,:)  = z.*k.^p.alpha.*h.^(1-p.alpha) + ...
              (1-p.delta)*k - c - kp*(1+p.gamma);
 coeff(4,:)  = -uh./uc - (1-p.alpha)*z.*k.^p.alpha.*h.^(-p.alpha);
%===============================================================================
coeff=coeff/1e-6;                   %finite diff.: 2nd step: (f(x+Dx)-f(x))/Dx
s=repmat(s,size(s,2),1);
coeff=coeff.*s;
```

The inputs of eqns.m are the vector of variables X created in stst.m, the variable flag which indicates whether the matrix of coefficients A or B is requested, and the structure of parameters p that was defined within stst.m. The model-specific part of the function starts creating two vectors for each variable that permits obtaining the t- and $t+1$-values

of the variables. For example, for the productivity shock, which is the first component of X, we wrote `z=X(1,:)` and `zp=Xp(1,:)` to obtain the values of z_t and z_{t+1}. The user should be aware that the position of the variables in X is the position indicated in `stst.m` and that the order of the variables remains unchanged across the functions in the Toolbox.

In the next step we have written the system (10). We use `.*` to indicate a product and `.^` to indicate exponentiation because this is the way Matlab recognizes these operations with matrices. For each equation $i = 1, ..., 4$, we first wrote `coeff(i,:)=` and then the equation as if they were equated to zero. Parameters are indicated anteceding the expression `p.` to the nomenclature introduced in `stst.m`.

The equations defining the flow variables should be written in the function `flows.m`. A simplified version of the function adapted to the KPR economy (see eqs. (11)) follows:

```
function coeff=flows(X,Xf,flag,p)    %FLOWS OF THE KPR ECONOMY

s=X';                       %used for percentage deviations
X=repmat(X,1,size(X,1));     %X is made repeating the vector X
SmId=1e-6*eye(size(X));      %creates finite differences

switch flag                 %coefficients, C or D
case 'C'
    Xp=X+SmId;              %deviation on variables dated at t+1
case 'D'
    Xp=X; X=X+SmId;         %deviation on variables dated at t
end

%======== Only the following lines are model specific =======================
 z=X(1,:);   k=X(2,:);       c=X(3,:);        h=X(4,:); zp=Xp(1,:);
 kp=Xp(2,:); cp=Xp(3,:);     hp=Xp(4,:);

 coeff(1,:) = - Xf(1) + z.*k.^p.alpha.*h.^(1-p.alpha);
 coeff(2,:) = - Xf(2) + z.*k.^p.alpha.*h.^(1-p.alpha) - c;
 coeff(3,:) = - Xf(2) + (1-p.alpha)*z.*k.^p.alpha.*h.^-p.alpha;
 coeff(4,:) = - Xf(4) + p.alpha)*z.*k.^(p.alpha-1).*h.^(1-p.alpha);
%============================================================================
 coeff=coeff/1e-6;              %: (f(x+Dx)-f(x))/Dx
 s=repmat(s,size(coeff,1),1);
 Xf = repmat(Xf,1,size(coeff,2));
 coef_flow=coeff.*s./Xf;
```

The format of `flows.m` is similar to the format of `eqns.m` although `flows.m` has an extra input, the vector Xf which was created in `stst.m` and which contains the value of the flow variables. The input variable `flag` now indicates whether the matrix C or D is requested. Each equation in `flows.m` is preceded by the expression `coeff(i,:) = - Xf(i)`, where $i = 1, ..., 4$.

5.2 The Core Functions

To solve the model, the function `CONTROL.m` calls `recrepres.m` to compute **p** and **f** (see eqs. (28) and (29)) as well as **g** and **h** (see eqs. (32) and (33)). `recrepres.m` calls `reorder.m`, `qzswitch.m` and the Matlab built-in function `qz.m`.[16] `reorder.m` checks, at pairs and starting from the top, whether or not the generalized eigenvalues are arranged in an ascending order. If they are, then it checks the next pair and so on. If they are not, it then calls `qzswitch.m`, which in turns completes the desired ordering. None of the core functions are model specific.

5.3 The Output Functions

Once the function `CONTROL.m` has obtained the solution to the recursive representation of the LDRE model, it calls `procedure.m` to obtain the requested results. `procedure.m`, in turns, calls other functions which are described next.

In an almost completely model-specific function `nomeclature.m`, the researcher has to write the names of the variables in his model. These names are used in the headings of graphs and tables of results. The nomenclature must be ordered according to the positions of the variables in the vectors `X` and `Xf` of the file `stst.m`. The model variables must precede the flow variables, and among the flow variables, it is assumed that the first variable is the GDP. A simplified version of nomenclature, adapted to the KPR model follows:

```
function [Names,Notation] = nomenclature

 A=1;    k=2;    c=3;      h=4;  y=5;    i=6;    w=7;      r=8;

 Names{A} = 'Productivity';     Names{k} = 'Capital';
 Names{c} = 'Consumptions';     Names{h} = 'Hours';
 Names{y} = 'GDP';              Names{i} = 'Investment';
 Names{w} = 'Wage rate';        Names{r} = 'Ret. on Cap."

Notation       =    {'A';'k';'c';'n';'y';'i';'w';'r'};
```

Among the remaining output functions, `impres.m` computes the impulse response functions; `moments1.m`, the population moment conditions; and `simushocks.m` carries out the simulations of the model with the user-provided sequence of innovations to the exogenous predetermined variables. When the researcher requests the simulations with actual shocks, the function `moments2,m` computes the sample moment conditions.

---

[16]The function *recrepres.m* is Paul Klein's function *solab.m* to which we have added the lines necessary to obtain the coefficients of the flow variables. *qzswitch* is a function written by Christopher Sims and the version in the Toolbox's has minor notation and documentations changes. See Sims (1995).

`impres.m` computes the impulse response functions assuming that the impulse is equal to 1% with respect to the steady state value of the variable being shocked. Briefly, for a given impulse $\hat{x}_1(1)$, `impres` iterates $\hat{x}_1(t+1) = \mathbf{p}\hat{x}_1(t)$ for the sate variables; $\hat{x}_2(t) = \mathbf{f}\hat{x}_1(t)$ for the control variables; and $\hat{x}_3(t) = \mathbf{g}\hat{x}_1(t)$ for the additional variables.

`moments1.m` computes the population moment conditions reading the variance-covariance matrix specified in `CONTROL.m` along with the matrices $\mathbf{p}$ to $\mathbf{h}$. Its output contains the standard deviation (or absolute volatility) of the variables, along with the relative volatility (measured with respect to the volatility of GDP) and several correlations. The results are written in four tables that are saved in the file `tables.out`. This file, which can be opened with the Matlab editor, is overwritten each time the researcher requests to recalculate the moment conditions.

Finally, `simushocks.m` reads the user-provided innovations to the exogenous predetermined variables and computes the same statistics as `moments1.m`, except that they are sample and not population moments. The statistics are also reported in the file `tables.out`.

6   The Toolbox at Work

This section discusses the output of the Toolbox by showing some results related to the RBC models of Section 3. Starting, with the Brock and Mirman model, the following matrices summarize the solution to its recursive representation:

$$\mathbf{p} = \begin{pmatrix} 0{,}90 & 0 \\ 1.00 & 0.40 \end{pmatrix}; \qquad \mathbf{f} = \begin{pmatrix} 1.00 & 0.4 \end{pmatrix};$$

As for the coefficients of the flow variables, $\mathbf{g}$ and $\mathbf{h}$ are both of 3×1. $\mathbf{h}$ is a null matrix and $\mathbf{g}$ has a unitary vector in its first column and the vector $(0.4, 0.4, -0.6)'$ in its second column. To understand better how the matrices $\mathbf{p}$ and $\mathbf{f}$ interact with other elements of the model, assume that starting at the steady state at time $t = 0$ (i.e. $\hat{z}_0 = \hat{k}_0 = \hat{c}_0 = 0$), there is a one-time 1% innovation to the productivity at time $t = 1$, i.e. $\epsilon_1 = 1$. At $t = 1$,

$$\begin{pmatrix} \hat{z}_1 \\ \hat{k}_1 \end{pmatrix} = \mathbf{p} \begin{pmatrix} \hat{z}_0 \\ \hat{k}_0 \end{pmatrix} + \epsilon_1 = \begin{pmatrix} 0{,}90 & 0 \\ 1.00 & 0.40 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which shows that $\hat{z}_1 = 1$, and $\hat{k}_1 = 0$. The optimal response of consumption is

$$\hat{c}_1 = \mathbf{f} \begin{pmatrix} \hat{z}_1 \\ \hat{k}_1 \end{pmatrix} = \begin{pmatrix} 1.0 & 0.4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

that is to say, consumption rises by 1%. At time $t = 2$, $(\hat{z}_2, \hat{k}_2)' = \mathbf{p} \times (\hat{z}_1, \hat{k}_1)'$, which gives $\hat{z}_2 = 0.9\hat{z}_1$ and $\hat{k}_2 = \hat{z}_1$. Thus, the percentage deviation of the capital stock at $t = 2$ is equal to the percentage deviation of consumption at $t = 1$. The result extends to any two contiguous periods so that in the Brock and Mirman economy $\hat{k}_{t+1} = \hat{c}_t$ for any t. This is not a coincidence for what we have discussed here is the log-linear approximation to the exact policy functions $k_{t+1} = \beta\alpha z_t k_t^\alpha$ and $c_t = (1 - \beta\alpha)z_t k_t^\alpha$ of eqs. (7). If we use paper and pencil to log-linearize these policies following the guidelines given in section 3.4, we obtain precisely that the approximate policy rules for $\hat{k}_{t+1}$ and $\hat{c}_t$ are equal, i.e. $\hat{k}_{t+1} = \hat{c}_t = \hat{z}_t + \alpha\hat{k}_t$.

The equality $\hat{k}_{t+1} = \hat{c}_t$ is not a general result but a particular result valid for the Brock and Mirman economy. Notwithstanding, the equality is useful to illustrate the accuracy of the log-linear approximations in this particular model for which we know the exact solution. Figure 1 shows the consumption policy function $\mathbf{c}(k_t, z_t)$ for particular values of (k_t, z_t); in the left panel, z_t is equal to its steady-state value and k_t deviates between -10% and +10% from its steady-state value. The solid line shows the exact policy function $c_t = (1 - \beta\alpha)z_t k_t^\alpha$, and the dashed line shows the linear approximation to this policy, i.e. $c_t = c[1 + (\alpha\hat{k}_t + \hat{z}_t)]$, where c is the steady-state value of consumption. When the deviation of k_t from its steady-state value is equal to 10% (i.e. $k_t = 11$), the approximate policy rule returns a consumption value that is 6% higher than the value indicated by the exact policy function. Observing that it is highly unlikely to find that k_t is 10% above its steady state value, we can conclude that the approximation is relatively accurate in the Brock and Mirman economy. In the right panel of Figure 1, $\hat{k}_t = 0$, and z_t deviates between -10% and +10% from its steady-state value. The approximation is equal to the exact consumption policy function, and this happens because the exact policy function is linear in z_t.

The other two models do not have known exact solutions and their approximate solutions are summarized by their corresponding matrices of coefficients $\mathbf{f}$ to $\mathbf{h}$. In the KPR economy, these matrices are:

$$
\mathbf{f} = \begin{pmatrix} 0.296 & 0.617 \\ 1.051 & -0.294 \end{pmatrix}; \quad
\mathbf{p} = \begin{pmatrix} 0.900 & 0.000 \\ 0.136 & 0.953 \end{pmatrix}; \quad
\mathbf{g} = \begin{pmatrix} 1.609 & 0.250 \\ 4.702 & -0.615 \\ 1.577 & 1.533 \\ 0.066 & -0.031 \end{pmatrix}; \quad
\mathbf{h} = 0_{4\times 2};
$$

and in the SOE economy the matrices **f** to **h** are:

$$
\mathbf{f} = \begin{pmatrix} 1.282 & -0.030 & 0.543 \\ 1.290 & 0 & 0.413 \\ 0.816 & 0 & 0.473 \end{pmatrix} ; \qquad \mathbf{p} = \begin{pmatrix} 0.402 & 0 & 0 \\ 2.411 & 0.984 & -3.229 \\ 0.816 & 0 & 0.423 \end{pmatrix} ;
$$

$$
\mathbf{g} = \begin{pmatrix} 1.877 & 0 & 0.601 \\ 1.908 & -0.016 & 0.610 \\ 3.718 & 0.092 & 0.781 \\ 8.164 & 0 & -4.774 \\ 0.587 & 0 & 0.188 \\ 0 & -0.024 & 0 \\ -0.987 & 0.022 & 1.272 \end{pmatrix} ; \qquad \mathbf{h} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10.000 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2.286 \end{pmatrix} ;
$$

Turning our attention toward the output of the LDRE Toolbox, we start calculating the population moment conditions of the variables in the three RBC models by setting `proc=[1,0,0]` in the file `CONTROL.m`. After executing `[X,f,p]=CONTROL` at the Matlab command window, the population moment conditions are summarized in the structure `X`. Additionally, the file `tables.out` will display the four tables showing these moment conditions. We can now write any of the following at the Matlab command window: `X.sdx` to obtain percentage standard deviations; `X.sdxsdGDP` to obtain the ratios between the percentage deviation of each variable and the percentage deviation of GDP; `X.rhoxlagx`, autocorrelations; `X.rhoxGDP`, contemporaneous correlations with GDP; `X.rholagxGDP`, correlation between (four) lags of each of the variables and the GDP at time t; and `X.rhoxlagGDP`, correlation between (four) lags of GDP and the remaining variables at time t. In Table 6 we summarize the population moment conditions of the three RBC models that we obtain using the LDRE Toolbox. The population moment conditions of the KPR model reproduce the results in Table 4 (pg. 222) of King et al. (1988a) (see the fourth row and last six columns of that table) and the population moment conditions of the SOE model correspond to the central columns of Table 6 (pg. 812) of Mendoza (1991).

Setting `proc=[0,1,0]` and `lengthIR = 15` in `CONTROL.m`, and executing `[X,f,p]= CONTROL` at the Matlab command window, we obtain impulse response functions for 15 periods. `X` is now a matrix containing a row per time period and a column per variable. A graph per variable is displayed on the screen after pressing any key. For the sake of illustration, three response functions following a 1% productivity impulse in the SOE are shown in Figure 2.

Setting `proc=[0,0,1]` in `CONTROL.m` allows us to simulate the models under a user-specified sequence of shocks. For this end, we have created a vector of innovations to the

productivity shocks called `innovations`, which is of size 30,000, using a (quasi)-random number generator. More in general, to simulate a model under a particular sequence of innovations to the exogenous predetermined variables, the researcher has to write the innovations in a matrix `innovations` which has to be saved writing `save innovations innovations`. The matrix will be of size $T \times n_e$, where T is the number of simulation periods and n_e is the number of exogenous variables in the model. In our example, we have set the variance of our innovations equal to the variance specified in `SIGMA` within the function `CONTROL.m` of the Brock and Mirman economy because we want to show that the population moment conditions are very similar to the sample moment conditions when the length of the sample is long enough. When `[X,f,p]=CONTROL` is executed, the output structure `X` has the same format as the structure obtained setting `proc=[1,0,0]`, except that a new field `X.Y` will contain a matrix `Y` which has a row per simulation period and a column per variable. Table 7 shows the sample moment conditions of the Brock and Mirman economy when the economy is simulated with the shocks described above. The results are similar to those shown in Table 6.

7 Summary

We have provided step-by-step guidance on the procedures required to approximate a model with a log-linear approximation and study its economic implications. The Toolbox of Matlab functions accompanying this paper and the explanations in the paper can help a researcher with almost no experience in computational economics to approximate and study his own dynamic rational expectations model. He must edit 5 out of the 16 functions in the Toolbox to provide his model-specific information. Part of this information is the values of parameters and variables when the model rests in its non-stochastic steady state. A calibration exercise is required for this end and we have shown how three standard RBC models have been calibrated in the literature.

Once the researcher has supplied his model specific information, a set of functions in the Toolbox carries out the numerical log-linearization, and we have shown the equivalence between the numerical and analytical log-linearization of one RBC model. A second set of functions solves for the recursive representation of the approximated model. These functions implement the solution proposed by Klein (2000), solution that has been discussed in detail in the paper. The solution to the approximated model is summarized in four matrices that equip the third set of Matalb functions with the inputs necessary to deliver three types of results. Namely, population moment conditions, impulse response

functions, and sample moment conditions when the model is simulated under a sequence of user-provided shocks. Three RBC examples have been worked out thoroughly to show how the functions in the Toolbox are adapted to particular cases of log-linear dynamic rational expectations models. In the process, we replicated some results published in the RBC literature.

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Table 1: Variable Sorting According to the Criteria of Section 2

Model	Predetermined $x_1(t)$ Endo-genous	Predetermined $x_1(t)$ Exo-genous	Non-Predetermined $x_2(t)$	Inno-vations $\epsilon_1(t)$	Flows $x_3(t)$
Model 1	k	z	c	ϵ	$y,\ w,\ r^k$
Model 2	k	z	$c,\ h$	ϵ	$y,\ w, r^k$
Model 3	$b,\ k$	z	$c,\ h\ k^a$	ϵ	$y,\ \widetilde{y},\ s,\ i,\ pr,\ ip,\ tb$

Notation: $k \equiv$ capital stock; $z \equiv$ productivity shock; $c \equiv$ consumption; $h \equiv$ hours; $\epsilon \equiv$ innovation to the productivity shock; $y \equiv$ GDP; $w \equiv$ wage rate; $r^k \equiv$ return on capital; $b \equiv$ international debt; $k^a \equiv$ auxiliary capital variable; $\widetilde{y} \equiv$ GNP; $s \equiv$ savings; $i \equiv$ investment; $pr \equiv$ productivity; $ip \equiv$ interest payments; $tb \equiv$ trade-balance-output ratio.

Table 2: Equation Sorting According to the Criteria of Section 2

Model	Optimality Conditions	Budget Constraint & Market Clearing	Forcing Processes
Model 1	(5a)	(5b)	(4c)
Model 2	(10b), (10d)	(10c)	(9b)
Model 3	(13b), (13c), (13e)	(13d)	(13a)

Note: The equation defining the auxiliary variable k^a completes the system of equations in Model 3 and that equation can be understood as a budget constraint.

Table 3: Parameters Values used in the Calibration

Model	α	β	γ	δ	ω	ϕ	ψ	ρ	σ
Model 1	0.40	0.984	-	1.000	-	-	-	0.90	-
Model 2	0.42	0.988	0.004	0.025	3.305	-	-	0.90	-
Model 3	0.32	0.961	-	0.100	1.455	0.019	0.11	0.42	2.00

Note: The time period in Models 1 and 2 is one quarter and in Model 3 is one year.

Table 4: Macroeconomic Aggregates and Ratios at Steady State

Macroeconomic aggregates						
Model	h	c	y	i	k	z
Model 1	1.000	15.40	25.40	10.00	10.00	10.11
Model 2	0.200	0.683	0.973	0.290	10.00	0.941
Model 3	1.007	1.123	1.486	0.340	3.398	1.000
Macroeconomic ratios						
Model	c/y	i/y	k/y	b/y	tb/y	-
Model 1	0.606	0.394	0.394	-	-	-
Model 2	0.702	0.298	10.28	-	-	-
Model 3	0.755	0.229	2.286	-0.397	0.016	-

Note: See Table 1 for nomenclature details.

Table 5: Matlab functions in the Toolbox

Log Linearization	Core Functions	Output Functions
stst.m*	recrepres.m	procedure.m
lincoeff.m	qzswitch.m	nomenclature.m*
eqns.m*	reorder.m	impres.m
flows.m*		simushocks.m
		tables.m
		tables.out
		graphs.m
		moments1.m
		moments2.m
		innovations.mat*
CONTROL.m*		

Notes: a) The function `CONTROL.m` coordinates the execution of all other functions to produce the required results. b) The functions that have to be adapted to each model are starred. c) The files `tables.out` and `innovations.mat` are not Matlab functions but a text file to show business cycle statistics and a variable containing innovations to exogenous predetermined variables, respectively.

Table 6: Population Moments in the Example RBC Models

| Variable	Standard deviation (percent)	Relative standard deviation[a]	Autocorrelation	Cross Correlations with GDP								
					GDP at t and variables at				GDP at t and variables at			
				t	$t+1$	$t+2$	$t+3$	$t+4$	$t-1$	$t-2$	$t-3$	$t-4$
Brock and Mirman (1972)[b]												
GDP, (y)	3.642	1.000	0.956	1.000	0.956	0.883	0.803	0.727	0.956	0.883	0.803	0.727
Consumption, (c)	3.642	1.000	0.956	1.000	0.956	0.883	0.803	0.727	0.956	0.883	0.803	0.727
Capital, (k)	3.642	1.000	0.956	0.956	1.000	0.956	0.883	0.803	0.883	0.803	0.727	0.655
Wages, (w)	3.642	1.000	0.956	1.000	0.956	0.883	0.803	0.727	0.956	0.883	0.803	0.727
Return on k, (r^k)	1.082	0.297	0.330	0.149	-0.149	-0.247	-0.267	-0.258	0.247	0.267	0.258	0.240
Prod. Shock, (z)	2.290	0.629	0.900	0.982	0.884	0.796	0.716	0.645	0.959	0.893	0.815	0.739
King, Plosser, and Rebelo (1988)[c]												
GDP, (y)	4.251	1.000	0.926	1.000	0.926	0.858	0.796	0.738	0.926	0.858	0.796	0.738
Consumption, (c)	2.719	0.640	0.993	0.818	0.836	0.847	0.853	0.855	0.760	0.707	0.658	0.613
Investment, (i)	9.751	2.294	0.876	0.926	0.805	0.698	0.603	0.519	0.855	0.791	0.732	0.677
Hours, (h)	2.049	0.482	0.858	0.792	0.649	0.524	0.414	0.318	0.730	0.673	0.622	0.574
Capital, (k)	3.727	0.877	0.997	0.677	0.728	0.768	0.798	0.821	0.631	0.588	0.549	0.512
Wages, (w)	2.912	0.685	0.981	0.903	0.895	0.884	0.870	0.854	0.838	0.779	0.724	0.674
Return on k, (r^k)	3.240	0.762	0.863	0.533	0.378	0.243	0.126	0.025	0.489	0.449	0.413	0.380
Prod. shock, (z)	2.290	0.539	0.900	0.983	0.884	0.796	0.716	0.645	0.909	0.841	0.779	0.723
Mendoza (1991)[d]												
GDP, (y)	2.813	1.000	0.634	1.000	0.634	0.357	0.188	0.095	0.634	0.357	0.188	0.095
GNP, ($\hat{y}$)	2.868	1.020	0.637	0.991	0.628	0.354	0.186	0.094	0.625	0.367	0.214	0.130
Consumption, (c)	2.130	0.757	0.712	0.941	0.592	0.332	0.175	0.088	0.640	0.411	0.271	0.193
Savings, (s)	5.766	2.050	0.646	0.922	0.589	0.333	0.176	0.089	0.535	0.243	0.066	-0.031
Investment, (i)	9.909	3.523	0.024	0.599	0.427	0.254	0.138	0.071	-0.232	-0.301	-0.213	-0.126
Capital, (k)	1.372	0.488	0.716	0.622	0.348	0.183	0.092	0.045	0.992	0.726	0.435	0.238
Hours, (h)	1.933	0.687	0.634	1.000	0.634	0.357	0.188	0.095	0.634	0.357	0.188	0.095
Productivity, (y/h)	0.880	0.313	0.634	1.000	0.634	0.357	0.188	0.095	0.634	0.357	0.188	0.095
Int. Payments, (ip)	23.926	8.506	0.984	0.042	0.027	-0.015	0.08	0.004	0.050	-0.083	-0.194	-0.262
Tde. Bce. ratio, (tb/y)	1.792	0.637	0.190	-0.057	-0.093	-0.068	-0.041	-0.022	0.698	0.563	0.316	0.132
Prod. shock, (z)	1.273	0.0452	0.420	0.963	0.627	0.357	0.190	0.096	0.404	0.170	0.071	0.030

Notes: a) Relative to the standard deviation of GDP. b) The standard deviation of the productivity shock z has been set equal to 2.29%, to make results comparable with the results of the KPR model; c) Replicates results in the 4th row and last six columns of Table 4, pg. 222, of King et al. (1988a). d) Replicates results in the three central columns of Table 6, pg. 812, of Mendoza (1991).

Table 7: Sample Moment Conditions in the Brock and Mirman Economy

| Variable	Standard deviation (percent)	Autoco- rrelation	Cross Correlations with GDP at time t						
			variables at				variables at		
			t	$t+1$	$t+2$	$t+3$	$t-1$	$t-2$	$t-3$
GDP	3.653	0.956	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Consumption	3.653	0.956	1.000	0.956	0.884	0.804	0.956	0.884	0.804
Capital	3.653	0.956	0.956	1.000	0.956	0.884	0.884	0.804	0.725
Wages	3.653	0.956	1.000	0.956	0.884	0.804	0.956	0.884	0.804
Return on k	1.078	0.333	0.148	-0.148	-0.246	-0.270	0.246	0.270	0.268
Prod. shock	2.295	0.901	0.983	0.886	0.798	0.717	0.959	0.895	0.818

Notes: Simulations setting the standard deviations of innovations to the productivity-shock process equal to 0.9981%. Results are comparable to the first block of Table 6. Notice that the sample productivity shock is lightly more volatile than the theoretical shock.

Figure 1: Consumption Policy Functions in the Brock and Mirman Economy: Exact and Linear Approximate Solutions
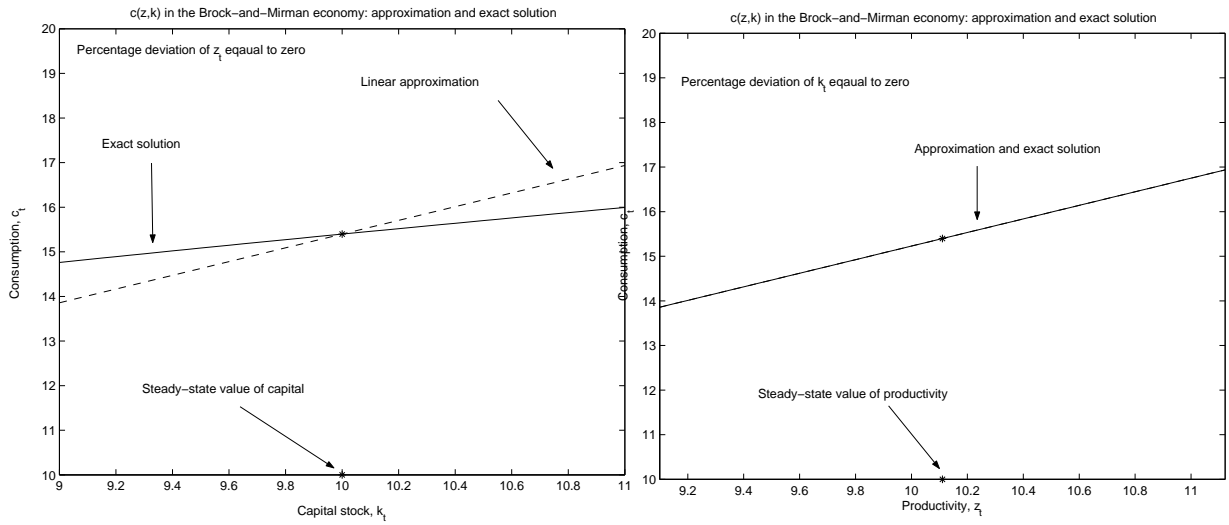
Note: In the left panel of the figure, the solid line shows the exact consumption optimal policy rule for different values of k when $\hat{z}_t = 0$; the dashed line shows the approximate policy function arising from the solution method discussed in the paper. In the right panel the comparison is made for different values of z_t when $\hat{k}_t = 0$.

Figure 2: Impulse-Response Functions in the SOE
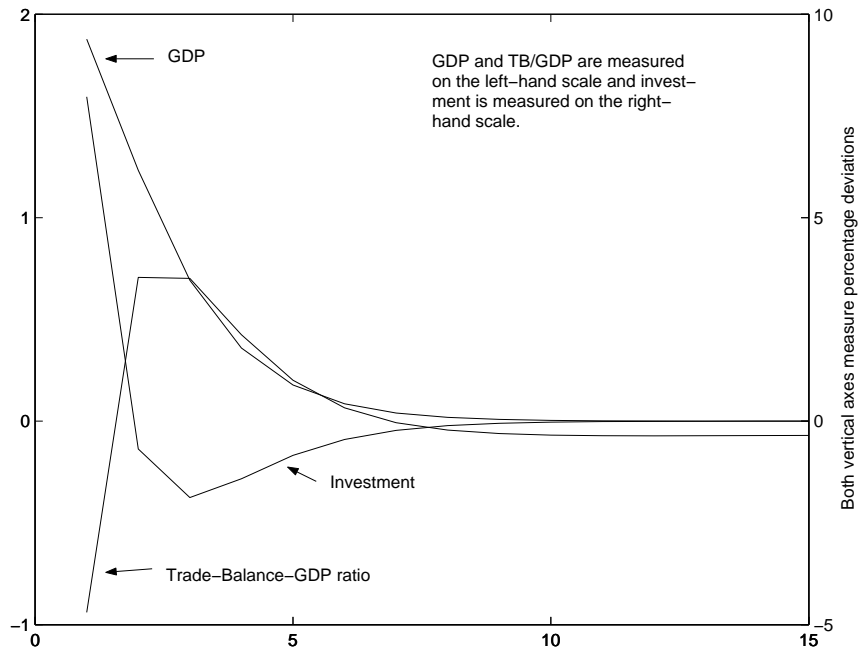
Note: Response functions of GDP, investment, and the trade balance-GDP ratio to a 1% innovation to the productivity shock.