Optimal growth path in an OLG economy without time-preference assumptions: main results

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This paper gives the main results of "Optimal growth path in an OLG economy without time-preference assumptions " (same author). Proofs and mathematical aspects are avoided in order to lighten the text and emphasize economic aspects. Full text is available at LAMSIN (ENIT) or on request to the author.

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1-Introduction

The aim of this work is to characterize optimal growth path in an OLG economy without using the assumptions of time preference theory on a social level, because such assumptions introduce necessarily inequality between the different generations of the society.

Section2 expounds the economic motivations of the problem and optimality concepts.

Section3 presents main formal definitions and assumptions.

Section4 sets up the first order necessary condition for Pareto optima and, with additional assumptions on U, particularly concavity and differentiability on the border, it studies the sufficiency of the first order condition to guarantee the Pareto-optimality of a bequests plan.

Section5 analyses examples of Pareto-optimal bequests plans particularly steady states ones.

The concept of consensual optimality, properties of consensual optimality criterions and relation between consensual optima and Pareto optima are worked out in section5. Section6 shows that, under assumption of intra-life time neutrality, golden rule is an asymptotical property of optimal growth paths. Section7 determines to what extent optimal growth paths are consistent with egoist spontaneous equilibrium.

2-Motivation

2.1-Infinite horizon sum

Optimization mathematics allowed to build economic models where decisive economic parameters like the rate of savings or the rate of technical change, were no longer data of the models, but results explained and computed by the models.

This increases the explanatory power of growth models but make them reckon on the assumption of infinite horizon sum.

Indeed, if we denote u the instantaneous utility function and *c(t)* the consumption at date *t*, the total utility of the society is supposed to be:

$$\int_0^{+\infty} u(c(t))e^{-\rho t}\,dt$$

Thus, to find the best growth path of economic growth, one maximizes the above criterion over *c(t)* under an evolution constraint like:

$$\dot{k} + c(t) = f(k(t)) - ak(t) \qquad \text{with } c \in [0, f(k)] \text{and } k(0) = k_0$$

where a is the rate of capital depreciation, f the production function, k the capital and c the consumption.

The discrete version of this kind of optimality criterion is a discounted sum of utility levels, depending on time-periods or generations in OLG models:

$$\sum_{t=1}^{+\infty} \frac{u(c(t))}{(1+\rho)^t}$$

Thus, the economy would behave like an infinite-lived individual with a time discount rate equivalent to the intergenerational discount rate.

As shown by Barro (1974), with operative intergenerational transfers, this is acceptable in the context of the usual assumptions of time-preference theory and with a positive rate of own generation preference.

On the personal level, time-preference theory assumes that one always prefers present goods to future goods. Although this can be criticized in absence of uncertainty and irrationality, it is not the purpose of this work.

However, the concept of "best growth path" is mainly concerned about social optimality which relation to time-preference theory is somewhat questionable: is it natural for social optimality to prefer present time over future? or present generations over distant ones?

Therefore, we can relevantly ask if we should keep on using the same time-preference assumptions when looking for the best growth path and if we should keep on using an infinite horizon sum criterion which inevitably favours present generations to the detriment of distant ones.

If we should not favour present generations to the detriment of distant ones, as pointed out by Allais (1946) or Mankiw (2001), how can we then compute a social optimum and test its economic efficiency without using the infinite horizon sum?

This is the object of the present work.

First of all, we have to specify the concepts of Pareto optimality and consensual optimality that form social optimality and will be used to define the optimal growth path.

2.2-Pareto optimality

To understand optimality problems in OLG models, we need first to look for Pareto-optimal consumption allocations c(t). This issue has been first studied by theorists like Cass(1972), Balasko-Shell(1980) and Wilson(1980), but with different mathematical tools and in a somewhat different context, probably more general because of the focus of this work on capital accumulation. However, this work permits (I hope) to give some interesting results in an issue as central as capital accumulation in growth theory.

Inter-generation Pareto-optimality of a consumption allocation means that it is not possible to find a better way to distribute the consumption so as to strictly enhance the utility of one generation without diminishing utilities of one or more other generations.

In the beginning of its economic life, a given generation receives a capital k_h as heritage. It consumes and invests during it's life and disappears bequeathing a capital k_l.

During its life, given k_h and k_l, the generation chooses the consumption *c(t)* that maximizes its individual intra-life utility.

Let k_0 be the capital inherited by the first generation g_1, k_1 the capital bequeathed by g_1 to g_2, k_2 the capital bequeathed by g_2 to g_3 and so on...

Given the vector bequests plan $K = (k_1, k_2, ...)$, each generation g_i maximizes its individual intra-life utility and determines its consumption $c_i(t)$ and its life-utility U_i.

Thus, we can see that the allocation of consumption and the distribution of utility between generations depends only on the vector K. So, we can speak of inter-generation Pareto-optimality of the vector K: a bequests plan K is Pareto-optimal if there is no other bequests plan that enhances strictly the utility of one generation without diminishing the utilities of one or more other generations.

If we exclude technical changes, the utility level reached with a heritage k_{i-1} and a bequest k_i depends only on k_{i-1}, k_i. So, we can write:

$$U_i = U_i(K) = U(k_{i-1}, k_i)$$

We can immediately see that if *K* is Pareto-optimal, it is a solution to the program $P_i(K)$ for all *i≥1*:

$$\max_B U(b_{i-1}, b_i) \quad \text{subject to } : U(b_{j-1}, b_j) \geq U(k_{j-1}, k_j) \forall j \geq 1, j \neq i$$

2.3-Consensual optimality

The concept of Pareto-optimality corresponds to the idea of efficient use of resources, but it does not take into account the social consensus underlying social stability and durability. For example, a situation where a unique individual owns all the wealth can be Pareto-optimal, but it is clearly not a socially stable situation and a social optimum.

Consequently, an optimal growth path has not only to be a Pareto-optimum, but it has also to respect a consensual criterion Ψ reflecting the social consensus. Consensual optimality is then given by the program S(Ψ):

$$\max_B \Psi\left[\left(U\left(b_{i-1}, b_i\right)\right)_{i\geq 1}\right]$$

The form of Ψ depends on the political system, social values...It is just as if Ψ expresses the preferences of an "out of the society and time" planner who incarnates the values and has widely agreed moral authority. We can think about this criterion as a kind of intergenerational GDP.

3-Formalism

To concentrate on optimality problems, consider an OLG economy without demographic growth and where individuals of each generation are exactly similar. Moreover, exclude intra-generational exchanges to eliminate wealth-distribution and prices questions. Exclude also, as a first approach, technical change. Capital accumulation is achieved through bequests from one generation to the next one.

3.1-Definitions

Denote $B = (b_i)_{i\geq 1}$ the sequence of bequests from one generation to the next one and $U(b_{i-1}, b_i)$ the level of utility a generation reaches with a heritage b_{i-1} and a bequest b_i, where *U* is a functional defined on a subset D_u strictly included in R_+^2.

$U(b_{i-1}, b_i)$ is then the life-utility. It is distinct from the instantaneous utility *u(c(t))* one achieves at the instant *t* with a consumption *c(t)*.

Name the sequence $B=(b_i)_{i\geq 1}$: a bequests plan.

Let $l_{\infty+}$ be the set of real positive and bounded sequences, k_0 a real positive number.

Denote:

$$D=\{K=(k_1,k_2,...)\in l_{\infty+} / \forall i\geq 1:(k_{i-1},k_i)\in D_u\}$$

Suppose that D_u is closed, and that it is such as the interior of *D* is not empty. Let Ψ be a Frechet-differentiable functional on l_∞.

For $K\in D$ and *i* an integer ≥1, denote respectively $P_i(K)$ and *S*(Ψ) the following programs:

$P_i(K)=$

$$\max_{B\in D} U(b_{i-1},b_i) \qquad \text{subject to : } U(b_{j-1},b_j)\geq U(k_{j-1},k_j)\forall j\geq 1, j\neq i$$

where b_{i-1}, b_i are the $i-1^{th}$ and the i^{th} components of *B*.

S(Ψ)=

$$\max_{B\in D} \Psi\left[\left(U\left(b_{i-1},b_i\right)\right)_{i\geq 1}\right]$$

A bequests plan *K* is a Pareto-optimal bequests plan if and only if it is solution to $P_i(K)$ for all i≥1 **and *K* is a consensual optimum if and only if it is solution to *S*(Ψ).**

The aim of this work is to characterize Pareto-optimal bequests plans and consensual optima. This amounts to characterize solutions of $P_i(K)$ and *S*(Ψ).

3.2-Assumptions

The following assumptions will be adopted when necessary:

- D_u is strictly included in R^2, closed and with a non-empty interior
- The interior of *D* is not empty
- *U* is of class C1 on the interior of D_u, and continuous on D_u
- $D_1U\succ 0$ (D_1U is the derivative of *U* with respect to its first variable)
- D_u Convex, *U* concave. One can then show easily that *D* is also convex.

The condition that *U* is C1 is a condition of preferences regularity.

The concavity of *U* means that every mixing between 2 bequests plans is preferred to the worst of them, which is a usual and acceptable assumption.

The condition $D_1U\succ 0$ means that life utility increases when one gets more heritage, which seems also quite reasonable.

When the utility $U(b_{i-1},b_i)$ of generation g_i comes from an optimization program like:

$$\max_{c} \int_0^T u(c(t))\delta(t)dt \quad \text{subject to : } k+c(t)=f(k(t))-a \cdot k(t);$$
$$k(0) = k_h \text{ ; } k(T)= k_l \text{ and } c \in [0,f(k)]$$

where u is the instantaneous utility function, δ is a function weighing instantaneous utilities of consumption during the life, a the rate of capital depreciation, f the production function and T the life period, we assume that u and f are **concave** and **increasing** and that f' **decreases below the parameter** a.[1]

We can then verify that the sequence of bequests is bounded.

We see also that for all $k_h \geq 0$ there is $k_{l\max}$ and $k_{l\grave{u}in}$ such that:

$$k_l \succ k_{l\max} \Rightarrow U(k_h, k_l) \text{ is not defined}$$

and

$$k_l \prec k_{l\max} \Rightarrow U(k_h, k_l) \text{ is not defined}$$

So, D_u is indeed strictly included in R^2. This means that, with a heritage k_h, whatever be the consumption sacrifice consented, one cannot bequeath more than $k_{l\max}$, and whatever be the consumption abuse, one cannot bequeath less than $k_{l\min}$.

Hence, generally, it comes out that if a bequests plan K is at the frontier of D then: either the bequest of at least one generation is extreme, or there is a tendency, even episodically, to this behaviour when time goes to infinity.

4-Pareto-optimality of a bequests plan

4.1-Necessary condition

Assumptions on U:

- D_u is strictly included in R^2, closed and with a non-empty interior
- The interior of D is not empty
- U is of class C1 on the interior of D_u, and continuous on D_u
- $D_1 U \succ 0$

Assumptions on K:

- K is an interior point to D and such that $(k_0, k_1) \in \overset{o}{D}_u$.
- $K \in D_l = \{B \in D / D_2 U(b_{j-1}, b_j) \leq 0 \forall j \geq 1\}$

[1] Since the object of this work is not the critique of the decreasing returns hypothesis, but that of infinite horizon optimization, I choose to stay in the framework of neoclassical concave production function.

· There is $i \geq 2$ such that $\prod_{j=1}^{i-1} D_2U(k_{j-1}, k_j) \neq 0$.

· Let

$$L_- = \left\{ B = (b_1, b_2, ...) \in D_l \,/\, \limsup \frac{-D_2U(b_{j-1}, b_j)}{D_1U(b_{j-1}, b_j)} \prec 1 \right\} \text{ and}$$

$$L_+ = \left\{ B = (b_1, b_2, ...) \in D_l \,/\, \liminf \frac{-D_2U(b_{j-1}, b_j)}{D_1U(b_{j-1}, b_j)} \succ 1 \right\}$$

Suppose that $K \in L_- \cup L_+$.

Under these assumptions, we have:

Proposition2 *If K is a Pareto-optimal bequests plan then for all i≥1 we have:*

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{D_2U(k_{i+n-1}, k_{i+n})}{D_1U(k_{i+n}, k_{i+n+1})} \right| \prec +\infty \;\blacksquare$$

4.2-Sufficiency

Assumptions on U:

· D_u is strictly included in R^2, closed, convex and with a non-empty interior

· U is of class C1 on the interior of D_u , and continuous on D_u.

· D_1U and D_2U are extendable by continuity on $\partial D_u = D_u - \overset{o}{D}_u$.

· $D_1U \succ 0$

· The interior of D is not empty

· U concave

U is then Frechet-differentiable on the border of D_u and D is closed and convex.

Proposition3 *Under the previous assumptions on U, let* $K \in D_l$ *and i such that if i≻1 we have:* $\prod_{j=1}^{i-1} D_2U(k_{j-1}, k_j) \neq 0$.

If

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{D_2U(k_{i+n-1}, k_{i+n})}{D_1U(k_{i+n}, k_{i+n+1})} \right| \prec +\infty$$

then K is solution of $P_i(K)$ ■

Proposition4 *Under the previous assumptions on U, let* $K \in D$ *such that for all i≥1 we have* $D_2U(k_{i-1}, k_i) \prec 0$.

If

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{D_2U(k_n, k_{n+1})}{D_1U(k_{n+1}, k_{n+2})} \right| \prec +\infty$$

then K is a Pareto-optimal bequests plan.■

Theorem1 *Under the previous assumptions on U, let* $K \in \overset{o}{D} \cap (L_- \cup L_+)$ *such that* $(k_0, k_1) \in \overset{o}{D}_u$ *and such that for all i≥1 we have* $D_2U(k_{i-1}, k_i) \prec 0$ *. Then K is a Pareto-optimal bequests plan if and only if*

$$\sum_{p=0}^{+\infty} \prod_{n=0}^{p} \left| \frac{D_2U(k_n, k_{n+1})}{D_1U(k_{n+1}, k_{n+2})} \right| \prec +\infty \ \blacksquare$$

5-Examples of Pareto optimal plans

5.1-A region of Pareto-optima

We adopt in all this section the same assumptions than in paragraph3-2 and 4-2.

Define the open region (supposed not empty) l_- between the two lines :

$L = \left\{ (h,l) \in D_u / \frac{-D_2U(h,l)}{D_1U(h,l)} = 1 \right\}$ and $L_0 = \{(h,l) \in D_u / D_2U(h,l) = 0\}$ as represented in figure1.

If *K* is in L_- (as defined in §4-1), this means graphically that from a given index n, *K*'s components are in l_-, under the line L without approaching it.

We then, apply theorem1:

Proposition5 *Under the assumptions of paragraph3-2 and 4-2 on U, all bequests plans which components are, from a given index in* l_- *without approaching the line L , are Pareto-optima.*■

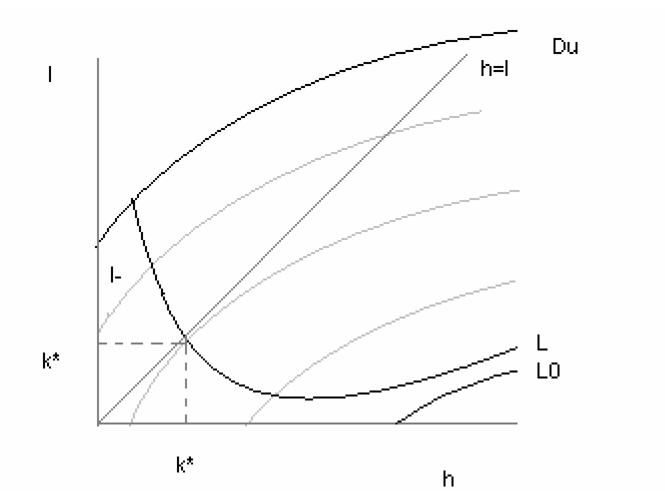

Figure 1: A region of Pareto-optima in a heritage-bequest diagram

This condition implies that, for "most" generations we have:

$$-D_2U(k_{n-1},k_n) \prec D_1U(k_n,k_{n+1})$$

which means that **if generation** g_n **decreases its bequest by one unit, it wins less than what is lost by generation** g_{n+1}. The condition

$$D_2U(k_{n-1},k_n) \prec 0$$

can be taken as an interiority condition with respect to the set of relevant bequests plans.

5.2-Steady state Pareto-optima

At the point (k^*,k^*) , the line *h=l* is tangent to a line *U(h,l)=U**, so we have

$$-D_2U(k^*,k^*) = D_1U(k^*,k^*)$$

Such a point does not necessarily exist. But if it does[2], a steady state bequests plan tending to k^* is an optimal steady state bequests plan with respect to the criterion

$$\max_K \lim U(k_{j-1},k_j)$$

We will suppose henceforth that k^* exists and that $D_2U(k^*,k^*) \neq 0$.

None of the steady state bequests plans which limits are over k^* is Pareto- optimum. All steady state bequests plans under k^* are Pareto-optima.

Let *B** be a steady state bequests plan which components are equal to k^* from a given index.

Proposition6 *B* is Pareto-optimal.*■

The criterion $\max_K \lim U(k_{j-1},k_j)$ gives best plans for remote generations. Thus *B** is the best steady state plan for remote generations.

If the economy begins with $k_0 \succ k^*$**, generations have to bequeath less than they inherit until they reach** k^*. If they do not, not only their bequests plan will not be good for remote generations but also it will not even be Pareto-optimal.

If the economy begins with $k_0 \leq k^*$, either it can tend to a steady state limit $k_\infty \prec k^*$ with a Pareto-optimal bequests plan, or it can tend to the limit k^* which is better for remote generations but requires from immediate generations to bequeath more. Consequently, if $k_0 \prec k^*$, **one cannot enhance utility level of remote generations without decreasing immediate generations ones.**

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[2] The question of existence of *k** is not a difficult problem. To lighten the text, it has been avoided. Nevertheless, it is addressed in a forthcoming work.

The condition $k_\infty \le k^*$ implies that

$$-D_2U(k_\infty,k_\infty) \le D_1U(k_\infty,k_\infty)$$

which means that, asymptotically, one unit decrease of bequest adds less utility than what is lost by the simultaneous and equal decrease of heritage.

If we consider long run interest, each generation should increase bequest until its marginal utility for next generation comes to heritage marginal utility. Then we get to the value k^* where

$$-D_2U(k^*,k^*) = D_1U(k^*,k^*)$$

6-Consensual optimum

6.1-First order condition

We adopt in all this section the same assumptions than in last section. Denote $G(B) = [(U(b_{i-1},b_i))_{i\ge1}]$. Denote *c* the set of real converging sequences.

As defined in section2, a consensual optimum is a bequests plan maximizing an inter-generations criterion Ψ(*G(B)*), where Ψ(*x*) is a Frechet-differentiable function from l_∞ to *R*. Suppose that $G(\overset{0}{D}) \ne \varnothing$. Suppose also that Ψ is increasing and concave[3]. This means that an increase in the utility of generation g_i without change for other generations, increases the value of Ψ. For example, we can take a linear criterion $\Psi(G(B)) = y \bullet G(B)$ where $y \in {l_{\infty+}}^*$.

Suppose that *K* is an interior solution of

$$\max_{B\in D} \Psi[G(B)]$$

The necessary first order condition is

$$\delta[\Psi G](K) \bullet \Delta B = 0 \forall \Delta B \in l_\infty$$

Denote the vector $\delta\Psi_1(G(K)) = \left[\frac{\partial\Psi}{\partial x_1}(G(K)), \frac{\partial\Psi}{\partial x_2}(G(K)),...\right]$ by $[\Psi'_1, \Psi'_2,...]$ and $\frac{\partial\Psi}{\partial\infty}(G(K))$ by Ψ'_∞.[4] Denote $D_1U(k_{j-1},k_j)$ by u'_{hj} and $D_2U(k_{j-1},k_j)$ by u'_{lj}.

The first order condition gives:

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[3] For the same reason than the concavity of U

[4] See proposition1 in appendix for the definition of $\frac{\partial\Psi}{\partial\infty}$

$$\Psi'_{i+1} = \Psi'_1 \prod_{j=1}^{i} \frac{-u'_{lj}}{u'_{hj+1}}$$

for all $i \geq 1$.
and:

Proposition7 *Under the assumptions of paragraph3-2 and 4-2, all interior maximizers of the criterion Ψ(G(B)) verifying* $\frac{\partial \Psi}{\partial x_1}(G(K)) \neq 0$ *and* $D_2U(k_{i-1}, k_i) \prec 0$ *for all i≥1, where Ψ is a Frechet-differentiable function on* l_∞*, are Pareto-optima.*■

6.2-Egalitarianism

First, we define some interesting properties for Ψ.

Definition1 *Ψ is non-saturable at infinity if and only if* $\Psi'_\infty(G(B)) \succ 0$ *for all* $B \in D$.

This property means that the consensual criterion always increases strictly when utility of remote generations increases strictly. Thus this property warrants that the criterion is sensitive to long run interest.

Definition2 *Ψ is "locally egalitarian" at a point G if and only if* $\Psi'_i(G) = 0$ *for all i≥1.*

As seen in section2, one of the essential reasons to work in an OLG context is the preoccupation about equality between generations. Even if equality is not a demanded condition, we need at least to compare it to the analysed situation[5]
The following proposition clarifies definition2:

Proposition8 *Let s be a one-to-one mapping on N* and define* $\hat{s}$ *the transformation on* l_∞ *such that, for* $G = (g_i) \in l_\infty$, $\hat{s}(G) = (g_{s(i)})$. *Then, Ψ is locally egalitarian at a point G if and only if:*

$$\delta\Psi(G) \bullet \Delta G = \delta\Psi(G) \bullet \hat{s}(\Delta G)$$

for all s and for all ΔG in c, in a given neighbourhood of G.■

The above condition means that if we change components order in Δ*G*, it does not change the consensual value. Thus, it expresses the idea of an equal importance of wealth increase for

[5] For much of the ideas developed here, I am indebted to M. Allais's "Economie et interet". I found also a more accessible exposé of some important issues in Macroeconomics of G.Mankiw. For example, about intergeneration equality we can read p116: "We then see that optimal capital accumulation is essentially function of the importance that we give to present and future generations. If we put them (generations) on the same level, (optimal path) will have to reach the golden rule's capital level." This is exactly what is shown in present and next section.
Notice that original G.Mankiw text is certainly somewhat different from what I quoted because I translated back to English the French translation available to me. Nevertheless, I hope that the meaning is preserved.

all generations in the eyes of what we called social consensus or "out of society and time" planner in section2.

6.3-Egalitarianism versus efficiency in steady states case

Suppose now that the solution K is in c. It is equivalent to say that K is a steady state plan. Take ΔB in c. Then $\delta G(K) \bullet \Delta B \in c$. Denote $u'_l = \lim D_2U(k_{n-1}, k_n)$ and $u'_h = \lim D_1U(k_{n-1}, k_n)$ and $\Delta b = \lim \Delta b_n$. We then have

$$\delta\Psi_2(G(K)) \bullet \left[(u'_{hi}\, \Delta b_{i-1} + u'_{li}\, \Delta b_i)_{i\geq 1} \right] = \Psi'_\infty\, (u'_l + u'_h)\Delta b$$

If Ψ is non-saturable at infinity, we have necessarily $u'_l + u'_h = 0$. Then K converges to k^* .
With the help of proposition7 we deduce:

Theorem2 *If the criterion defining a consensual optimum is non-saturable at infinity, all steady state bequests plans that are interior consensual optima converge necessarily to* k^* *. Let K be such an interior consensual optimum such that* $D_2U(k_{i-1}, k_i) \prec 0$ *for all i≥1. If Ψ is not locally egalitarian at G(K), then K is Pareto-optimal.*■

Hence, as long as non-saturability at infinity, which means sensitivity at infinity, is respected, changing the criterion Ψ can only change the speed of convergence to k^* (number of generations necessary to get close enough to k^*) but not the "destination" k^* .

The second assertion of theorem2 is somewhat amazing : **local egalitarianism does not warrant efficiency** (Pareto-optimality), **while its opposite** (favouritism) **does**.

To try to find more "optimistic" properties for egalitarianism, let's define global egalitarianism:

Definition3 *Ψ is "globally egalitarian" if and only if*

$$\Psi'_i\, (G) = 0$$

for all i≥1 and for all G in c.

The following proposition clarifies definition3:

Proposition9 *Let s be a one-to-one mapping on N* and define* $\widehat{s}$ *as in proposition8. Then, if Ψ is globally egalitarian, we have:*

$$\Psi(\hat{s}\,(G))=\Psi(G)$$

for all s and for all G in c[6].■

[6] Although the reciprocal implication seems true, I have failed yet to prove it.

Remark: If we suppress the condition *G* in *c* in the definition of global egalitarianism and Δ*G* in *c* in the definition of local egalitarianism, these concepts would be tighter but much more difficult to characterize. So, I kept *G* in *c*.

Let *K* be an interior consensual optimum for the non-saturable-at-infinity criterion Ψ such that Ψ is **not locally egalitarian** at *G*(*K*) and $D_2U(k_{i-1},k_i) \prec 0$ for all $i \geq 1$. We then have (see above) $\Psi'_{i+1} = \Psi'_1 \prod_{j=1}^{i} \frac{-u'_{lj}}{u'_{hj+1}}$. Ψ non locally egalitarian implies that $\Psi'_1 \neq 0$. Since $\lim_n \Psi'_n = 0$, we then have $\lim_i \prod_{j=1}^{i} \frac{-u'_{lj}}{u'_{hj+1}} = 0$. This implies that (k_{n-1},k_n) never reaches endlessly (k^*,k^*) which means that there is, at least episodically, a **deviant generation** that gets away from k^*. However, *K* is Pareto-optimal.

If Ψ is **globally egalitarian**, we see that every plan which components are equal to k^* from a given index is a consensual optimum (but not necessarily an interior one). In this case, all consensual optima are not necessarily Pareto-optima, but some of them are. For example, the plan where first generations bequeath their maximum bequest until they reach k^* and then the lasting generations stay at k^*, is a consensual optimum and a Pareto optimum. This plan is the faster way to reach k^*.

Observing that this plan is the **best state for long run interest**, we can assert:

Proposition10 *A globally egalitarian non-saturable-at-infinity consensual criterion enables the fastest consensus-optimal and Pareto-optimal attainment of the best state for long run interest* k^*. *Whereas with a non locally egalitarian, non-saturable-at-infinity consensual criterion, every consensual optimum is Pareto optimal but there is endlessly a deviant generation from* k^* ■

6.4-Optimal growth path

As said in section2, an optimal growth path has to be Pareto-optimal and consensus-optimal. We limit henceforth the concept of optimal growth path to steady state ones.

Denote k_∞ the limit of a steady state bequests plan. $k_\infty = (k^* =)$ means that from a given index n we have $k_n = k^*$, $k_\infty = (k^* -)$ means that $k_n \to k^*$ with $k_n \prec k^*$, $k_\infty = (k^* +)$ means that $k_n \to k^*$ with $k_n \succ k^*$, $k_\infty = (k^* -+)$ means that $k_n \to k^*$ with oscillations round k^*.

With a globally egalitarian criterion (non saturable at infinity), the following table characterizes steady state plans to show up which ones are optimal growth paths.

k_∞	$\prec k^*$	$k^* -$	$k^* =$	$k^* -+$	$k^* +$	$\succ k^*$
Pareto optimality	yes	?	yes	?	?	no
Consensual optimality	no	yes	yes	Yes	yes	no

For the cases $k^{*}+$, $k^{*}-$ and $k^{*}-+$, $K\notin \overset{0}{D}\cap(L_{-}\cup L_{+})$ we cannot apply theorem1.

However, for the case $k^{*}-$, we can apply proposition4 if $\sum_{p=0}^{+\infty}\prod_{n=0}^{p}\left|\frac{u'_{\ln+1}}{u'_{hn+2}}\right|$ converges. Since we cannot clearly see that, we may use Rabee-Duhamel criterion which says that if we can write $\left|\frac{u'_{\ln+1}}{u'_{hn+2}}\right|=1-\frac{a}{n+1}+r_{n+1}$, with $a\succ 1$ and $\sum r_n \prec +\infty$, then $\sum_{p=0}^{+\infty}\prod_{n=0}^{p}\left|\frac{u'_{\ln+1}}{u'_{hn+2}}\right|\prec +\infty$. The interpretation is that the bequests plan must not tend "too rapidly" to k^{*}.

Consequently, **only** $k^{*}=$ **is doubtlessly an optimal growth path**. On top of that, the plan $k^{*}=$ is the fastest to reach k^{*}.

7-Golden rule

The golden rule : capital marginal productivity = rate of capital depreciation, characterizes the best steady state in an economy governed by a Solow model, which is one of the first, the simplest, but more insightful models in growth theory. In this model, the sharing between consumption and saving is not the result of an optimization decision, but a fixed parameter as supposed in keynesian theory. Indeed, Solow's goal was to criticize Harrod-Domar model by accepting all its keynesian assumptions except that of fixed proportions production . Solow's economy converges to a steady state depending on the savings rate. The golden rule steady state is obtained by imposing the golden savings rate equal to the quotient of marginal productivity on average productivity.

Here, under intra-life time neutrality assumption, we find similar results, but with a dynamic of **savings behaviours** governed by the needs of **consensual optimality** and **Pareto optimality**.

Intra-life time neutrality means that a generation does not care when it consumes, as long as it is during its lifetime. Thus, the discount rate ρ in the expression of life utility $\int_0^T u(c(t))e^{-\rho t}dt$ is taken 0.[7]

Proposition11 *Suppose that U(h,l) comes from the resolution of the following program:*

$$U(h,l)=\max_c \int_0^T u(c(t))dt \qquad \text{subject to : } k+c(t)=f(k(t))-a\cdot k(t)\ ;$$

$$k(0)=h\ ;\ k(T)=l \text{ and } c\in[0,f(k)]$$

where u is a concave function, C1*on*]*0,*+∞[*, such that all paragraph3-2 and 4-2 conditions on U are verified. Then any optimal growth path verifies the golden rule, and the asymptotic savings rate is the golden savings rate. Moreover, this is true for any bequests plan tending to* k^{*} ■

[7] This assumption may look as naive but ρ≠0 would mean that consumption in the beginning of life weighs more than in other periods of life, which would not reflect life utility, but utility at a given time.

The savings rate tends to $f'(k^*)\dfrac{k^*}{f(k^*)}$.

We look now to the savings behaviours necessary to reach k^*.

If $k_0 \prec k^*$: Suppose that before generation1, economy was stationary at $k = k_0$. A generation who will enhance bequest will have to enhance its savings rate above $a\dfrac{k}{f(k)}$. So it will necessarily sacrifice utility as we have seen in section4. In figure2, curved arrows represent trajectories of "sacrificed "generations and discontinuities represent jumps from a generation to the next one.

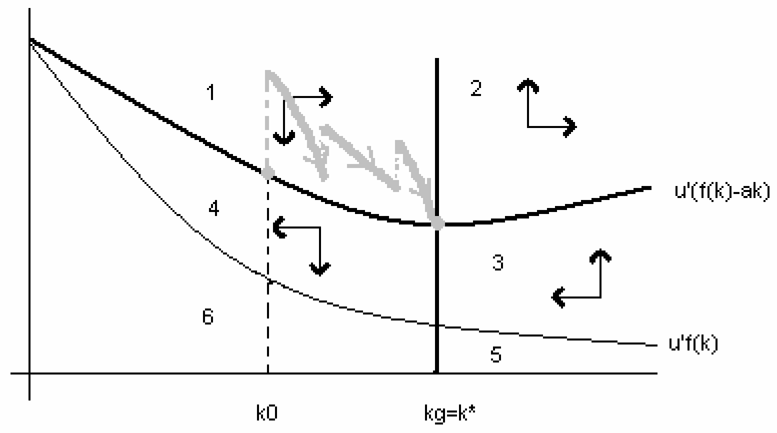

Figure 2: The optimal growth path in a (k,λ) [8]diagram

As we get closer to k^*, the savings rate will decrease back to $a\dfrac{k}{f(k)}$ and the life utility *U*(*h,l*) will increase. Hence, **the efforts accepted by immediate generations will benefit to the infinity of remote generations and there is not another way to reach k^*.**

If $k_0 \succ k^*$: As said in section4, staying in this situation is not Pareto-optimal. **Decreasing savings rate is good for immediate generations as for remote ones**. Economy will depart from region3 and end to k^* and savings rate will decrease to its golden rule level.

If $k_0 = k^*$: **Generations will just have to keep a constant consumption** $c^* = f(k^*) - ak^*$ which warrants the golden savings rate.

8-Welfare analysis

8.1-Selfish and altruistic utility

If anyone behaves only according to what his pure selfishness dictates, the optimal growth path would not be stable. The first generation that is free to deviate would do it and bequeath the minimum capital. This capital $l_{\min}$ is determined by

$$U_l'(k^*, l_{\min}) = 0$$

[8] λ is the "shadow price" of the capital *k*. In mathematical words, it is the adjoint variable of k.

If the generation does not bequeath $l_{\min}$, it must be because of one of the three following reasons:

First, it can be compelled by the planner. But it is hard to imagine that. Planner can fix prices or production, but not bequests.

Second, it can be so altruistic that it maximizes the consensual criterion Ψ instead of its utility. This supposition is not completely utopian because some people determine their behaviour according to collective interests. For example, some Europeans don't buy Japanese cars although they may be competitive, because it would lead to European GDP decrease. We can also combine individual utility *U* an consensual criterion Ψ and use something like $\lambda U+(1-\lambda)\Psi$.

Thirdly, we can suppose that individual utility depends on heirs' utility. This assumption appears as the more appropriate to our problem. We can name it : familial altruism or intergenerational altruism. This concept have been used by Brenheim-Ray(1987) or Lakshmi(2002). The point is then to see if this limited altruism could lead to optimal growth path. In other words, **we try to find out what familial altruism intensity can lead to the optimal growth path**.

We suppose that we can decompose the individual utility V as follows:

$$V=U+A$$

where *U* is the classical selfish utility depending on personal consumption and *A* is the altruistic utility depending on the capital bequeathed to next generation. Thus

$$V(h,l)=U(h,l)+A(l)$$

where *U* is the utility function used in previous sections (with assumptions of paragraph3-2) and *A* is the altruistic utility.

A has to be increasing with *l* since altruistic feelings are satisfied when bequest increases. We also suppose that *A* is C1 and strictly concave[9]on R.

8.2-Honour your heir, but not more than yourself!

Define as **spontaneous equilibrium** a bequests plan where each generation g_n chooses k_n solution to:

$$\max_l U(h,l)+A(l) \quad \text{where } h=k_{n-1}$$

Denote $l_{\min}(h)$ and $l_{\max}(h)$ respectively the lower bound and the upper bound of $\{l/(h,l)\in D_u\}$. If[10]

---

[9] Concavity requirement on A means decreasing marginal altruistic utility.

[10] The condition $U'_l(h,l_{\min}(h))+A'(l_{\min}(h))\geq 0$ means that it is always better to bequest more than $l_{\min}(h)$. Without this condition, adding *A* to *U* would not change anything to spontaneous equilibrium and economy would go in under-accumulation. The condition $U'_l(h,l_{\max}(h))+A'(l_{\max}(h))\leq 0$ means that it is always better to bequest less than $l_{\max}(h)$. Without this condition, *U* would be useless and economy would go in over-accumulation.

$$U'_l(h, l_{\max}(h)) + A'(l_{\max}(h)) \leq 0 \text{ and } U'_l(h, l_{\min}(h)) + A'(l_{\min}(h)) \geq 0$$

then

Proposition12 *If a spontaneous equilibrium is an optimal growth path, then*

$$U'_h(k^*, k^*) = A'(k^*)$$

■

The left hand side of the equation above is the increase of selfish utility resulting from an increase of heritage by one unit. The right hand side is the increase of altruistic utility resulting from an increase of bequest by one unit. The interpretation is that **spontaneous equilibrium cannot be optimal growth path unless generations feel** (asymptotically) **about their heirs as they feel about themselves.**

If $U'_h(k_\infty, k_\infty) \succ A'(k_\infty)$, then $U'_h(k_\infty, k_\infty) + U'_l(k_\infty, k_\infty) \succ 0$ which implies $k_\infty \prec k^*$. So **if feelings toward heirs are deficient, economy will stay in under accumulation.**

Similarly, if $U'_h(k_\infty, k_\infty) \prec A'(k_\infty)$ then $k_\infty \succ k^*$. **If feelings toward heirs are excessive, economy will go in over accumulation.**

8.3-Transitory state

Denote $k_n = \varphi(k_{n-1})$ the relation between k_{n-1} and k_n in a spontaneous equilibrium. The condition $U'_h(k^*, k^*) = A'(k^*)$ is necessary for a spontaneous equilibrium to be an optimal growth path, but it is not sufficient. We have to make sure that the sequence $k_n = \varphi(k_{n-1})$ converges.

Assume that *U* and *A* are C1 on their definition sets.

If there is α in]0,1[such that

$$A''(\varphi(h)) \prec -U''_{ll}(h, \varphi(h)) - \frac{|U''_{lh}(h, \varphi(h))|}{\alpha}$$

for all *h* for which the last expression is defined, then, under the condition $U'_h(k^*, k^*) = A'(k^*)$, consensual optimality is warranted but still not Pareto optimality.

The last condition can be replaced with

$$A''(k^*) \prec -U''_{ll}(k^*, k^*) - |U''_{lh}(k^*, k^*)|$$

if we want only to have convergence for k_0 close enough to k^*.

If $\varphi'(k) = 0$ in a neighbourhood of k^*, which is equivalent to

$$U''_{lh}(k, k) = 0$$

and if k_0 is in this neighbourhood, we will be in the case ($k^* =$) which is an optimal growth path.

9-Conclusion

The need to specify the concept of optimal growth path without infinite horizon sum, has lead me to try to mathematically characterize optimality between generations in an OLG economy.

For Pareto-optimality, we establish that, under some conditions, that for "most" generations we should have:

$$-D_2U(k_{n-1},k_n) \leq D_1U(k_n,k_{n+1})$$

which means that if generation g_n decreases its bequest by one unit, it wins less than what is lost by generation g_{n+1}.

To complete the concept of Pareto-optimality, this paper introduces consensual optimality with egalitarian and non-saturable-at-infinity criterion. These concepts are used for the definition of optimal growth paths, which are shown to converge necessarily to the capital k^* defined by

$$D_2U(k^*,k^*) + D_1U(k^*,k^*) = 0$$

Moreover, with intra-life time neutrality, k^* observes the golden rule.

Then, with the use of familial altruistic utility, we have shown that if marginal altruistic utility of bequest is equal to marginal selfish utility of heritage, spontaneous equilibrium is consistent with optimal growth path.

However, bequests plans which are not in $L_- \cup L_+$ have not been examined here, particularly Pareto optima that cross cyclically the line L. They don't meet regularity requirement, but their study should be interesting. For example, it could help know to what extent changes in bequeathing behaviour affects long period economic cycles.

It should also be interesting to drop intra-life time neutrality assumption and see consequences on golden rule observance.

Appendix

The infinite part of the differential of a Frechet-differentiable function on l_∞

Proposition1 *Let c be the set of real converging sequences, f a function from* l_∞ *to R, Frechet-differentiable at* $x_0 \in l_\infty$*, and* $r_n(h)$ *the sequence of c obtained by setting to 0 the n first terms of a sequence h of c.*

Then, the following limit exists:

$$\lim_{\|h\|\to 0, h\in c, \lim h\neq 0} \frac{\limsup_n f(x_0 + r_n(h)) - f(x_0)}{\lim h}$$

We denote it :

$$\frac{\partial f}{\partial \infty}(x_0)$$

■

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