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# Volatility, Heterogeneous Agents and Chaos

ORLANDO GOMES

Unidade de Investigação em Desenvolvimento Empresarial [UNIDE/ISCTE]

and

Escola Superior de Comunicação Social

[Campus de Benfica do Instituto Politécnico de Lisboa; 1549-014 Lisboa, Portugal].

ogomes@escs.ipl.pt

**Abstract**

Agent heterogeneity has been used in recent economic literature to justify nonlinear dynamics for the time paths of aggregate economic variables. In this paper, the mechanism through which heterogeneous agents leads to chaotic motion is explained. Adding to a system with initial behavior heterogeneity an adaptive learning rule based on discrete choice theory, one is able to encounter a reasonable explanation for nonlinear motion. The adaptive learning / bounded rationality rule is not the only ingredient necessary for the absence of a long run steady state; heterogeneity must also imply that the several behavior possibilities alternate as the best behavioral choice. Only in such circumstances heterogeneity persists and an unpredictable outcome is likely to arise.

The paper develops two models. The first is a generic approach that exemplifies how heterogeneity concerning the volatility of two stochastic processes may lead to chaotic motion; the second is a utility maximization setup, where the source of heterogeneity is investment decisions.

Keywords: Heterogeneous agents, Bounded rationality, Chaos, Volatility.

JEL classification: C61, D91, E10

## I. INTRODUCTION

Economic agents exhibit distinct behavior patterns and do not share the same expectations about future events. This is a simple remark about reality, but it may have important implications in what concerns the evolution of aggregate economic variables over time. This paper focus on the implications of behavior / expectations heterogeneity. General conditions under which heterogeneity implies significant changes in the evolution patterns of economic variables will be identified and a simple example of behavior heterogeneity describing a setup where consumers may choose between two investment opportunities will be developed.

The most significant issue involving agent heterogeneity relates to the fact that under certain conditions the interplay between the different types of agents could result in equilibria with unusual properties. For instance, Azariadis and Kaas (2002) present a standard intertemporal consumption utility maximization model, similar to the one to be presented in section IV, in which several types of consumers interact; the consumers distinguish from one another in what concerns time preference, that is, they have distinct discount rates relating future consumption. Under such a scenario, a constant long run steady state will not hold.

Another field where heterogeneity and bounded rationality lead to long run results other than stability is expectations in macroeconomics. Following the work of Sargent (1993) about bounded rationality in macroeconomic expectations, several authors have presented models where the introduction of heterogeneity leads to high dimensional systems where a chaotic pattern of evolution for economic variables might arise [see, e.g., Barucci (1999), Negroni (2003)]. This literature is associated with learning mechanisms that may have different origins: Evans and Honkapohja (2001) develop the concept of recursive / econometric learning; Arifovic (1994) initiates an important literature about genetic algorithm learning, that has been developed by Bullard and Duffy (1998, 1999) and Casari (2003), among others; another approach to macroeconomic learning is the one by Kurz (1994, 1997), Kurz and Motolese (2001) and Kurz, Jin and Motolese (2003) who introduce the term rational belief equilibrium.

Also in asset pricing theory the concept of heterogeneity arises as a central piece in the explanation of economic behavior. Because present prices in financial markets are dependent upon expectations about future prices, the co-existence of different expectation rules can lead to price movements that are erratic and impossible to predict, since in different time periods some type of expectation may dominate the market, while in other moments other expectation rules will determine price evolution. In the following sections, asset pricing will not be a concern of our analysis, but the way in which agent heterogeneity is approached in this literature will be a fundamental reference to our arguments. In particular, the path breaking work by Brock and Hommes (1997, 1998), where heterogeneous expectations are linked to an adaptive belief system of bounded rationality serves as an important guiding line. These two authors are the founders of the 'rational routes to randomness' concept, a concept that explains the way in which individuals that are rational but have different beliefs about future events may imply nonlinear evolution of economic aggregates (namely, in what relates their main concern, asset prices).

In the following sections we will emphasize that the absence of a homogeneous behavior among agents with the same economic goals is an important route leading to chaos; nevertheless, it is not the only one – models in various fields of economic analysis produce strange dynamics without taking heterogeneity as a nuclear starting point. This is the case of Benhabib, Schmitt-Grohé and Uribe (2001*a*, 2001*b*, 2001*c*) relating monetary policy and the Taylor (1993) rules concerning interest rates and of Tuinstra and Wagener (2003) who focus the analysis of chaotic motion in an overlapping generations model under which households predict future inflation rates.

The remainder of this paper is organized as follows. Section II claims that behavior heterogeneity is not the only ingredient necessary for an adaptive system leading to nonlinear dynamics; a second feature is equally important – the absence of a fully rational scenario. The concept of bounded rationality is developed. Section III illustrates with a simple example how agent heterogeneity can lead to unpredictable economic behavior over time. Section IV elaborates a more sophisticated example with important economic meaning. An intertemporal utility maximization setup is considered and agents will differ in the portfolio investments they undertake; agents will have distinct consumption opportunities, given that they will obtain different returns from their wealth. The main result about this model is that the existence of distinct investment opportunities implies, under the model's features, an aggregate long run consumption path that is not smooth and predictable. Finally, section V systematizes the most relevant conclusions.

## II. Bounded Rationality

Heterogeneity simply means that individuals or some kind of groups will not all behave in the same way. According to conventional economic theory, such heterogeneous behavior does not tend to persist. After all, if two groups of agents act distinctly concerning some economic phenomenon, one of such groups will have better results and all individuals will change to the better performance group. Rationality means in this way the absence of persistence of all actions besides the one that gives the best result.

Thus, heterogeneity is unlikely to hold under a fully rational setup, because agents certainly do not hold to a behavior or belief that performs poorly. The key point in favour of the idea of heterogeneity persistence is that individuals are not seen as completely rational, instead they follow some kind of bounded rationality rule, that introduces some sluggishness in the way each individual changes his behavior.

The concept of bounded rationality is linked to the discrete choice theory literature [see McFadden (1973), Manski and McFadden (1981) and Anderson, de Palma and Thisse (1993)]. Discrete choice relies on a mechanism through which agents change behavior over time, without eliminating heterogeneity, that is, the discrete choice model quantifies the shares of individuals attached to each behavioral group in each time moment and the evolution of such shares implies an everlasting change in the values of the shares.

We define $n_{ht}$ as the share of individuals that follow some kind of behavioral rule $h$ in a moment of time $t$. In the presence of $H$ possible rules, we will have $H$ shares. We are interested in the way each percentage $n_{ht}$ evolves over time. Discrete choice theory describes a fitness function or performance measure $U_{ht}$ relating to each of the possibilities of behavior. Individuals will change from one alternative behavior to another according to the value of $U_{ht}$. The better the results given by the chosen strategy of action / behavioral rule, the faster agents will change from anyone group to the best performance group. Discrete choice points to the following expression as representing the percentage of individuals attached to behavior group $h$ in a moment of time $t$:

$$n_{ht} = \frac{e^{\beta . U_{ht}}}{\sum_{h=1}^{H} e^{\beta . U_{ht}}} \qquad (1)$$

In expression (1), it is clear that the better the performance of the strategy $h$ relatively to all the other strategies (measured by the $U_{ht}$ functions), the higher will be the value of the share. Simultaneously, this share depends on a parameter $\beta \geq 0$. The parameter $\beta$ is the intensity of choice and it is a measure of the sensitivity of the agents to the differences in results of the various rules of behavior. A high $\beta$ means that individuals change behavior rapidly as other $U$ functions display better results than the $U$ function attached to the behavior followed in a given moment. A value of parameter $\beta$ close to zero implies that individuals have more resistance to change and stick with the same strategy of action, even if this does not perform as good as other strategies for several consecutive periods of time. In this way, parameter $\beta$ is a measure of the time needed for individuals to realize that it is not worth to keep with a behavior that tends to produce worse results than other behaviors. In other words, $\beta$ is a measure of bounded rationality. Individuals will be more rational, in the sense they respond faster to better incentives, as the higher is the value of $\beta$ (note that if $\beta$=0, the agents will have an extreme behavior of accepting no change and thus $n_{ht}=1/H$, $\forall$ $h$=1, …, $H$). Accordingly, an increase in the choice parameter value represents an increase in rationality.

Our main assumption will be that individuals have a certain degree of rationality ($\beta$ is a positive value but relatively far from infinity; an infinite $\beta$ means full rationality). Combining this interpretation of bounded rationality with the existence of several alternatives concerning behavior or beliefs, we will be able to encounter a setup under which one finds an unpredictable time path for the variables underlying such behavior choices. The interesting point is that two or more perfectly understandable time evolution mechanisms, when combined with bounded rationality may result on a time path that is erratic, impossible to predict and has traces of chaos.[1]

Looking at expression (1) it is straightforward to understand that $n_{ht}$ converges to 0 or 1 if the performance of strategy $h$ is worse or better, respectively, than any other

[1] We understand chaos as the situation under which ‘a pair of initial values located arbitrarily close together may lead to completely different time series though they are generated by the same dynamical system.’ [Lorenz (1997), page 119].

possible strategy. Thus, heterogeneity will not hold in the long run under the scenario of a systematic difference in performance of the possible alternative actions.

In this way, we have found two conditions that are essential for agent heterogeneity to persist and thus to have eventually a meaningful impact in aggregate economic behavior:

(1) it is necessary to consider a bounded rationality approach, based on discrete choice theory. Under such a setup agents will change behavior but not instantly. They will look to accumulated results of their strategic choice and they will change it when such accumulated results manifestly point to the other strategy as leading to better results;

(2) given two or more strategies or behavioral rules, the outcome of one cannot be systematically more favourable than the other(s). Functions $U_{ht}$ of accumulated results should intercept systematically over time to avoid the predominance of one of the strategies, case in which heterogeneity will end up disappearing.

## III. A BASIC SETUP

In this section we present a basic example of persistence of agents heterogeneity. We take an undetermined number of agents that may choose between two strategies. The first gives, in each time moment, an unknown result with an expected value of $\mu$ and a variance $\sigma_i^2$. The second result is also a stochastic process with a mean of $\mu$ and a variance $\sigma_j^2 \neq \sigma_i^2$. That is,

$$\delta_{it} \sim iid(\mu, \sigma_i^2) \tag{2}$$

$$\delta_{jt} \sim iid(\mu, \sigma_j^2) \tag{3}$$

The time paths of (2) and (3) are easy to describe. We do not know in each moment of time which value we will have, but over a time interval with some observations we verify that both time paths are constituted by points around $\mu$ that may be further away from this value if the variance parameter is a higher value. These time series, concerning the two choices each agent faces, respect the second of the conditions presented in the last section for heterogeneity to be meaningful. Certainly best results will alternate: in some moments of time (2) will exhibit a higher value than (3), but in other moments the opposite is true.

We assume that the goal of the individuals in the economy is to choose the time series in (2) or the time series in (3) that allows to obtain higher values. They will not change the choice they make between one of the two possibilities instantly, but they will evaluate results according to a fitness function. Let the fitness function in this case be the sum of all the past results and the present result, where past results are discounted at a rate $\rho_\delta$ (individuals value more recent results than far away in the past results),

$$U_{it} = \sum_{t=0}^{T} \delta_{it} \cdot \frac{1}{(1+\rho_\delta)^{T-t}} \tag{4}$$

$$U_{jt} = \sum_{t=0}^{T} \delta_{jt} \cdot \frac{1}{(1+\rho_\delta)^{T-t}} \tag{5}$$

with $T$ the present moment.

The discrete choice model will in this case state that the following expression gives the number of individuals that stick with results (2) in each time moment (recall that this will change faster or slower according to the value of the intensity of choice parameter, $\beta$),

$$n_{it} = \frac{e^{\beta . U_{it}}}{e^{\beta . U_{it}} + e^{\beta . U_{jt}}} \tag{6}$$

Our concern is with the average value of (2) and (3). If $n_{it}$ was a constant value, such time series would present a behavior similar to (2) and (3), with a volatility that would be somewhere between $\sigma_i^2$ and $\sigma_j^2$. But $n_{it}$ changes at every time moment, and thus $\zeta_t = n_{it} . \delta_{it} + (1 - n_{it}) . \delta_{jt}$ will exhibit an erratic, completely unpredictable (or chaotic) behavior. Our conclusion is that in the presence of two results, giving the possibility of each agent to choose the strategy that best performs under a discrete choice framework, the overall result (the weighted average of the two results) ends up by being a time series with clusters of different volatility and thus completely unpredictable.

This is a general setup, and (2) and (3) may be anything. Section IV concretizes these series as being the returns from financial assets. In appendix [I] the results of the setup of this section are presented graphically for concrete values of parameters; these are $\mu$=0, $\beta$=2 and $\rho_\delta$=0,01, with $\sigma_i^2<\sigma_j^2$ (in the case, let $\sigma_i$=0,1 and $\sigma_j$=1). The graphics display three of the infinite time paths that can be presented for $\delta_{it}$, $\delta_{jt}$, $n_{it}$ and $\zeta_t$. As one can observe, the behavior of $\delta_{it}$ and $\delta_{jt}$ follow a same pattern in each case, but in reality they give place to time paths $n_{it}$ and $\zeta_t$ that are enormously different for each time we run the example. For $\zeta_t$, periods of high volatility co-exist with periods of low volatility, which reflects the predominance of one of the two series [(2) and (3)], nevertheless it is unpredictable which of the series will dominate in each moment and how the change of predominance is realized.

## IV. UTILITY MAXIMIZATION AND HETEROGENEOUS PORTFOLIOS

Consider now a standard intertemporal optimization model regarding consumption utility. Instead of a representative agent assume two types of agents in what concerns investment decisions. A first type of agents includes the ones that invest their wealth in a risk free asset. The second group invests in a risky asset with an expected income rate equal to the income of the risk free asset. Considering a utility function that exhibits decreasing marginal utility and a discount factor $\rho$>0, agents are distributed between the two types and the following problems are assumed:

- Agents of type *i*:

$$Max\sum_{t=0}^{\infty}U(c_{it}).\frac{1}{(1+\rho)^t} \text{ subject to } a_{it+1}=(1+r)-a_{it}-c_{it},\ a_{i0} \text{ given;}$$

- Agents of type *j*:

$$Max\sum_{t=0}^{\infty}U(c_{jt}).\frac{1}{(1+\rho)^t} \text{ subject to } a_{jt+1}=(1+r+\delta_t)-a_{jt}-c_{jt},\ a_{j0} \text{ given.}$$

In these problems, $c_{ht}$, $h=i, j$, defines the consumption level of each type of agent, $a_{ht}$, $h=i, j$, is the wealth of an individual in group $h$, and $r$ is the rate of return of the wealth not consumed in each period (expected rate of return in the case of individuals in the $j$ group). We ignore any fixed return (not dependent on the wealth endowment). We consider that the rate of return on wealth of individuals of type *j* has a stochastic component $\delta_t \sim iid(0,\sigma^2)$.

Solving the optimal control problem for both agent types, we reach a constant consumption growth rate for agents of type *i*, and a consumption growth rate with a same constant expected value for agents of type *j* but where a volatility component is present,

$$\frac{\Delta c_{it}}{c_{it}} = \frac{1}{\theta}.(r-\rho) \tag{7}$$

$$\frac{\Delta c_{jt}}{c_{jt}} = \frac{1}{\theta}.(r+\delta-\rho) \tag{8}$$

with $\theta>1$ a concavity parameter of the utility function.

The growth rate of the consumption aggregate is a weighted average of the growth rates (7) and (8). The shares of individuals choosing one of the two investment strategies are determined by a rule like (6), with $U_{it}$ and $U_{jt}$ the fitness functions, defined in terms of utility results,

$$U_{it} = \sum_{t=0}^{T} U(c_{it}).\frac{1}{(1+\tau)^{T-t}} \tag{9}$$

$$U_{jt} = \sum_{t=0}^{T} U(c_{jt}).\frac{1}{(1+\tau)^{T-t}} \tag{10}$$

In (9) and (10), $\tau$ represents a memory parameter (past utility is important for the evaluation of the best investment strategy, but the farther away in the past is the utility result, less valued it will be today). The growth rate of aggregate consumption is then

$$\frac{\Delta c_t}{c_t} = \left(\frac{\Delta c_{it}}{c_{it}}\right)^{n_{it}} . \left(\frac{\Delta c_{jt}}{c_{jt}}\right)^{1-n_{it}} \tag{11}$$

For (11) we can expect the same kind of lack of constancy (or alternate high and low volatility) as for the $\zeta_t$ series in the previous section. The conclusion is that in an economy where agents may choose between applying their savings in risky assets or, alternatively, risk free assets, and there is bounded rationality, the aggregate consumption growth rate will display an erratic / chaotic behavior that is impossible to predict in the initial moment.

To illustrate the previous logic we consider a numerical example. Assuming the following parameter values, $\{r, \rho, \theta, \tau, \beta, \sigma\}$={0,05; 0,01; 2; 0,01; 20; 0,01}, appendix [II] presents three of the infinite results that are possible for (11).

## V. Final Remarks

Agents look at reality from different perspectives. For example, they are risk averse in their investment decisions in different degrees. The economic science had always the conscience about this fact, nevertheless it was never seen as a fundamental source of disturbance over the time path of important economic variables. The mainstream economic thought is based on a notion of rationality that does not leave space for anything more than the choice for the best result attainable at any moment of time. The main rule is that agents, based on available information, have the ability at any moment to choose the behavior that produces the best expected result.

Bounded rationality or discrete choice behavior intends to add an element of inertia to decisions, which is in reality present in many of the economic decisions individuals make. Now, it is under consideration not only the best instantaneous result, but also the way the agents psychologically weight how their behavior (and the other individuals behavior) as performed in the past.

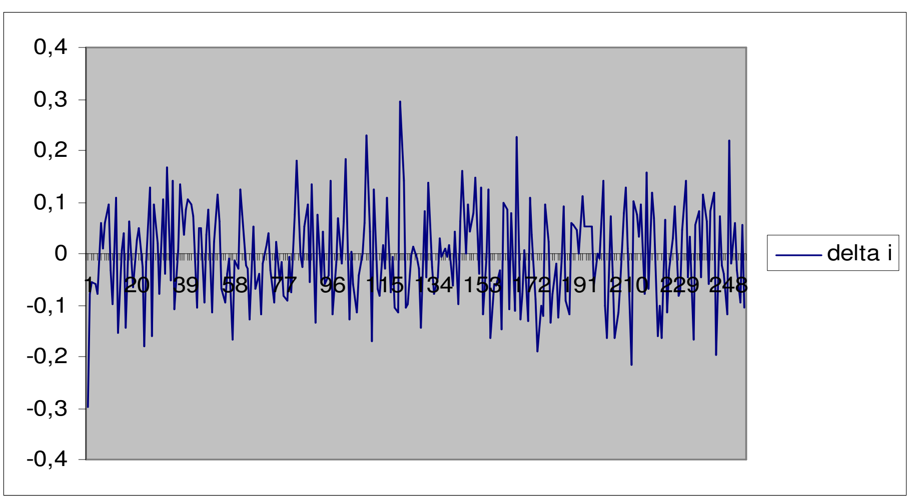
The important point of our analysis is that the heterogeneous agents / bounded rationality setup can lead to unpredictable time paths for economic variables. This was illustrated with a general example, where fitness functions regarding present and past results were assumed and also with an example concerning utility optimization and investment decisions. In each one of the cases, it became clear that it is necessary more than a bounded rationality system for the time series of variables to display strange dynamics. It is also indispensable that the two (or more) time series relating to the two (or more) behavioral rules intersect with some frequency, that is, one of the rules should not be better than the other in all moments of time because in this case heterogeneity will not be maintained and therefore one of the time series will end up by corresponding to aggregate behavior.

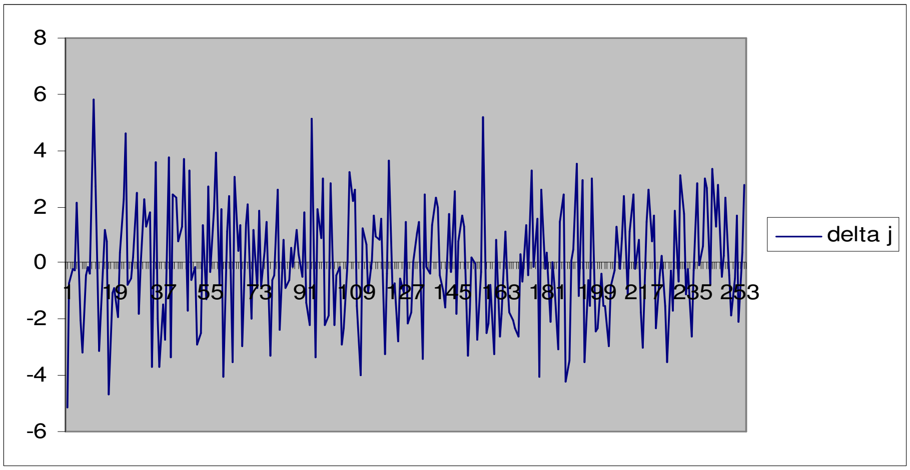
The economic example assumed in section IV fills the previous requisite: a fixed income investment gives sometimes a high income than an investment with risk and the same expected value, but the opposite occurs with precisely the same probability. In this way, consumption growth rates that depend on the rate of return of investment will give place to utility results that also alternate in terms of performance – the utility of agents of type *i* (that invest in riskless assets) is sometimes higher and sometimes lower than the utility of agents of type *j* (that invest in risky assets). The utility results are reflected on the shares of individuals selecting one of the two investment strategies, which in turn has impact over the long run growth rate of consumption of the economy, that has to be an average of individual consumption growth paths.

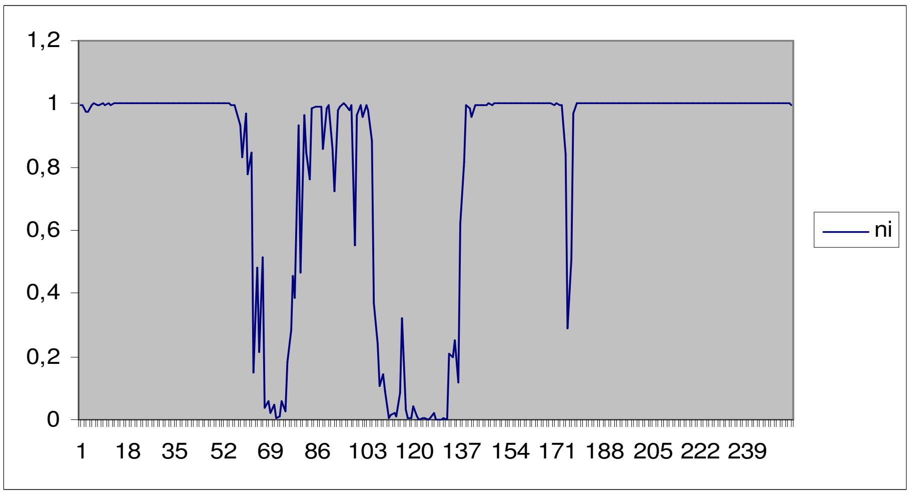
The framework of section III is a general framework and the setup of section IV is meant to be an application. Many other cases where the conditions referred in the previous paragraphs are fulfilled can be considered. For instance, a model where firms can choose between two R&D strategies with uncertain results may be able to explain erratic profit paths or a setup where a world with many countries choosing between two or more trade policies can be a way to present growth paths that are impossible to predict.

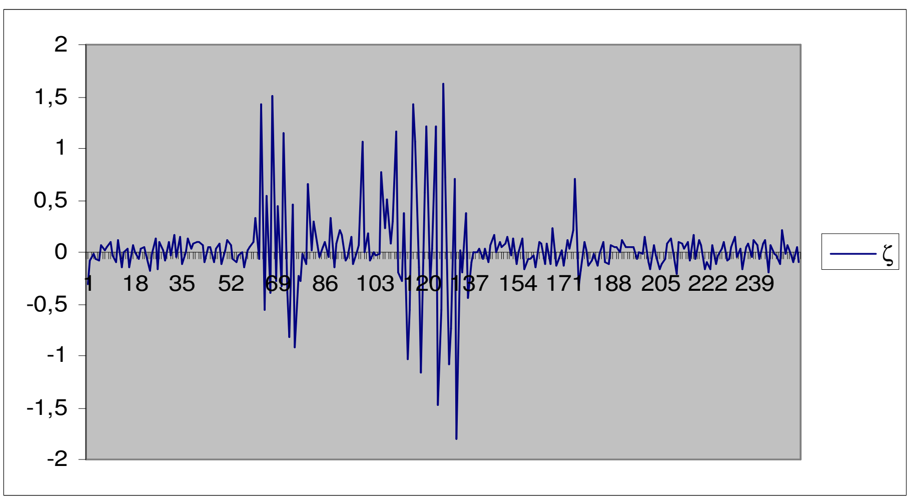
## APPENDIX [I] – THE BASIC SETUP RESULTS

EXAMPLE 1



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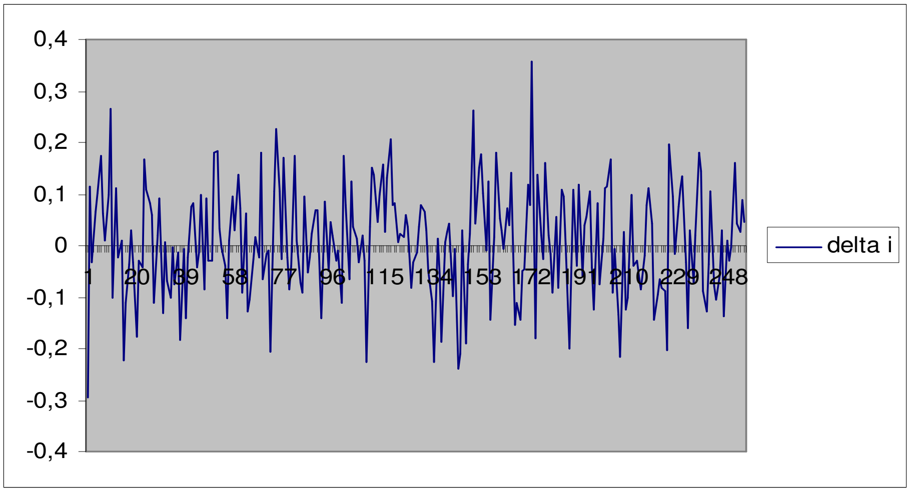
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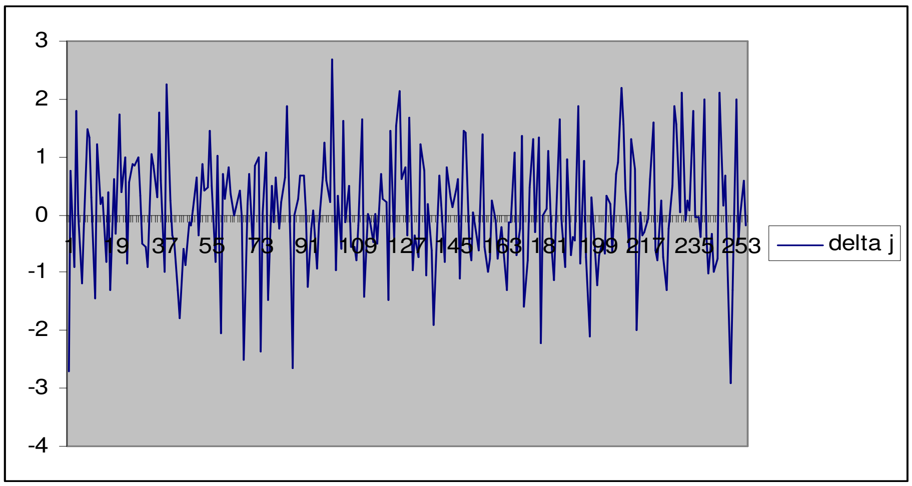
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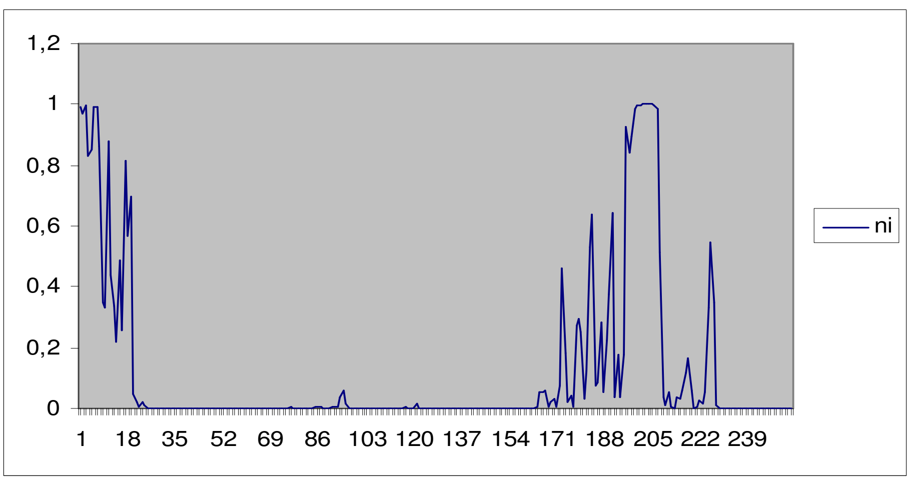
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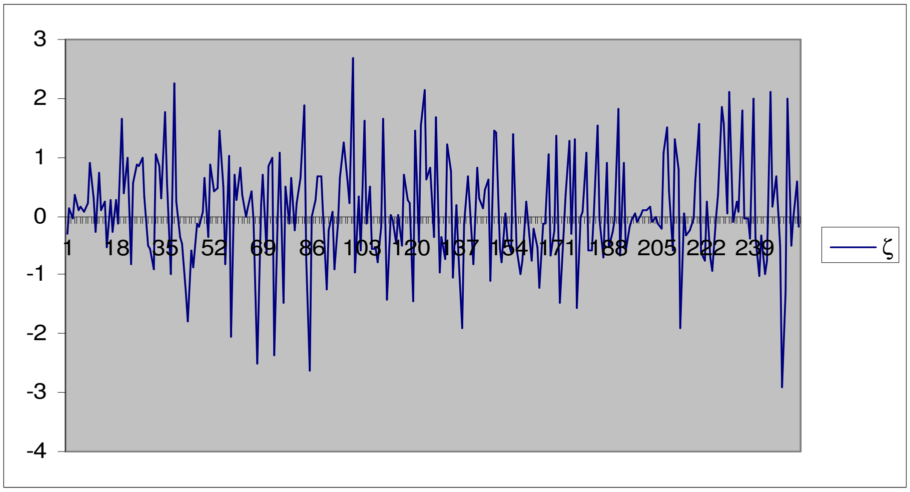
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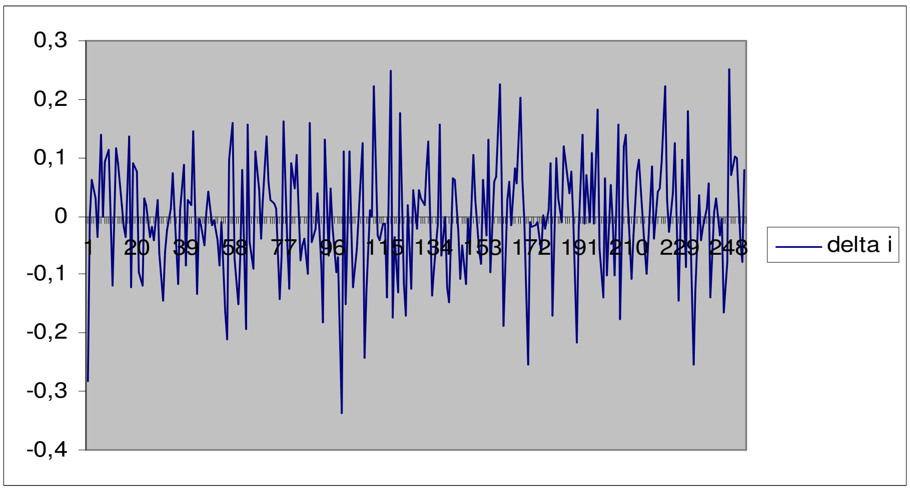
EXAMPLE 3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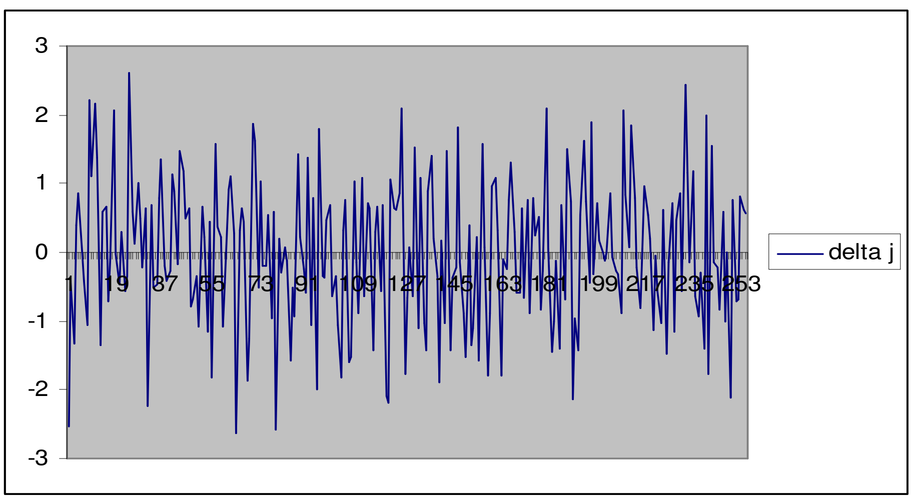
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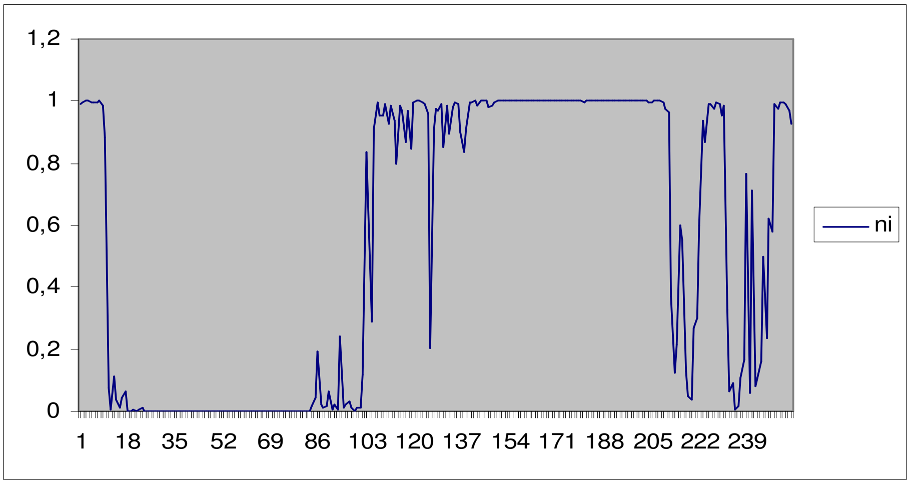
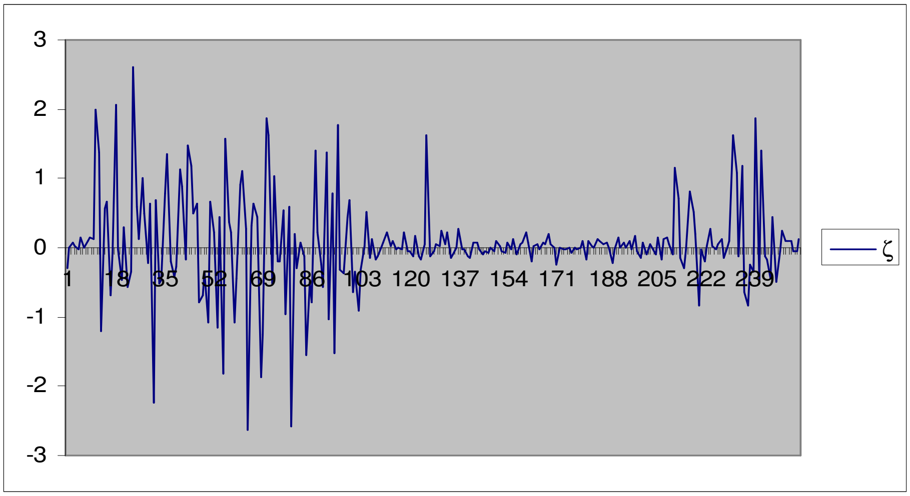
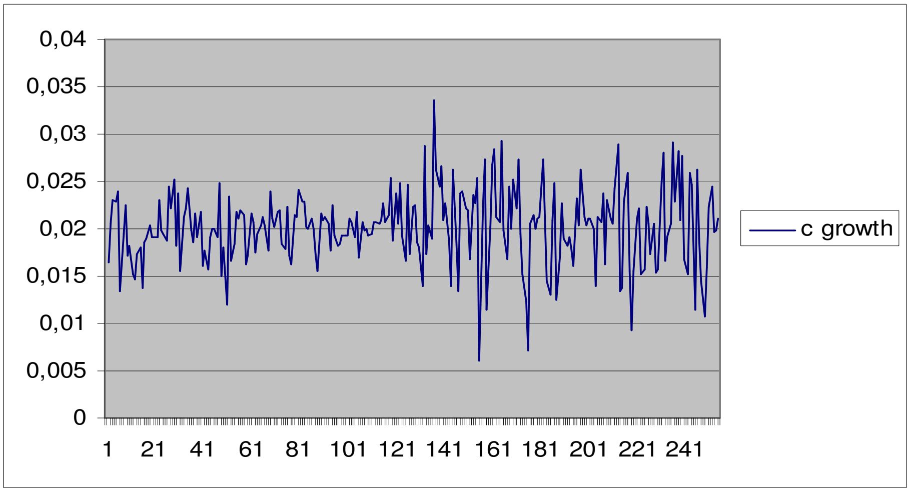
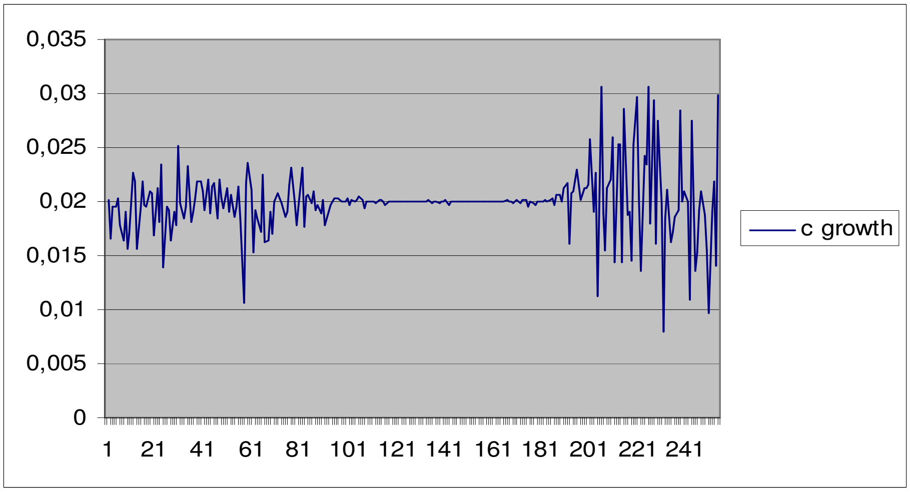
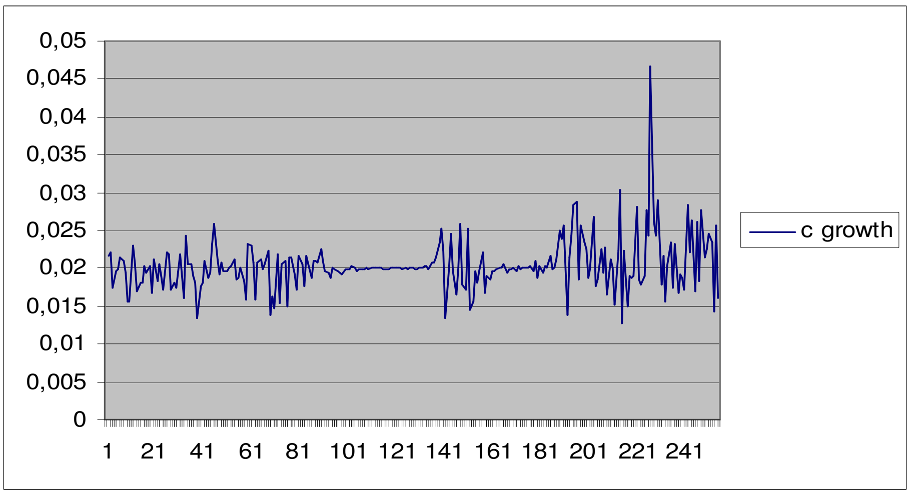



APPENDIX [II] – CONSUMPTION GROWTH RATE IN AN HETEROGENEOUS AGENTS SETUP



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