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# Generic Determinacy of Equilibria with Local Substitution

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Abstract

Consumption of a good at one point in time is a substitute for consumption of the same good an instant earlier or later. Utility functions which conform to this fact must necessarily be non-time separable, as Hindy, Huang, and Kreps show. When agents' utility functions are non-time separable in the required way, the price space consists of semimartingales with an absolutely continuous compensator. In general, this space is not closed under taking pointwise maxima, that is, it is not a lattice. Therefore, neither the Mas-Colell/Richard existence theorem nor the determinacy theorem by Shannon/Zame apply. In a paper with Peter Bank, existence is established for such intertemporal economies; here, I show that generically, the number of equilibria is finite and that equilibrium allocations depend continuously on endowments. The notion of genericity is (finite) prevalence as developed by Anderson/Zame.

Keywords: Hindy-Huang-Kreps preferences, prevalence, local substitution, determinacy

JEL Classification: D91

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# Introduction

Two cornerstones of General Equilibrium Theory are existence and generic finiteness of equilibria. In finite dimensions, Arrow and Debreu [3] use Kakutani's fixed point theorem to establish existence, and Debreu [6] shows that Sard's theorem yields local uniqueness for almost all initial endowment vectors. For infinite–dimensional commodity spaces, existence is ensured by the Mas-Colell/Richard Theorem ([12], [1]), and generic finiteness is established by Shannon and Zame[14]. These theorems cover most relevant economic models, with one notable exception: the important class of intertemporal economies where preferences exhibit local substitution ([8]). Peter Bank and I [4] show that equilibria for local substitution economies exist. Here I establish generic finiteness.

Two assumptions Shannon and Zame make are not satisfied here: in general, the price space is not a lattice, and aggregate endowment is not strictly positive. The main contribution of the present paper is therefore to prove generic determinacy without these assumptions.

Strict positivity of aggregate endowment may seem an innocuous assumption, and the results do of course hold true if one retains it; however, there might be reason to avoid it in the present context where endowments are (random) measures on the time axis. I would like to cover the cases where there is only positive supply at some fixed points in time, or, for that matter, only at one point in time, thus including discrete time models. Of course, a measure with a finite support is not strictly positive (in the sense that the order ideal it generates is weakly dense in the commodity space).

Strict positivity of aggregate endowment is used by [14] in order to extend certain price functionals from the order ideal to the whole commodity space. In the present context, one can use another approach; since the structure of the model provides explicitly a candidate for a price functional, it must only be shown that the candidate has indeed the desired properties (cf. [4], [7]), and the abstract extension problem does not arise. The relevant arguments for that part have been developed in my paper with Peter Bank [4].

The lattice property of the price space is used by Shannon and Zame in order to establish joint continuity of the excess spending map. The critical point here is that the weighted maximum of utility gradients is not necessarily a continuous linear functional on the commodity space. However, it turns out that continuity is needed only on the order interval formed by zero consumption and aggregate endowment, and, as I show, maxima of continuous linear functional are indeed continuous on order intervals. A sketch of the argument is as follows. The order ideal generated by aggregate endowment can be identified with a suitable $L^\infty$–space; on this space, the weighted maximum of utility gradients is a continuos linear functional, and order intervals are weak–* compact; this makes it possible to establish continuity in the original topology on the order interval formed by zero and aggregate endowment.

As is well known from the finite–dimensional case ([6]), the number of equilibria can be infinite, even with smooth preferences. However, the set of initial endowment vectors for which this occurs is of Lebesgue measure zero, and it is said that *generically*, the number of equilibria is finite. In the absence of a Lebesgue measure (i.e. a translation invariant measure which assigns positive measure to all nonempty open sets) for infinite dimensional spaces, another concept of genericity is needed. Here, 'generically' will mean that the set of initial endowment vectors with finitely many equilibria is *finitely prevalent* in the set of nondegenerate initial endowment vectors, a concept introduced into the economics literature by Anderson and Zame [2].

The paper is organized as follows. The next section presents the model and examples of utility functions covered in this paper. Section 2 states the main result and contains its proof. The appendix collects some supplementary proofs.

# 1 Assumptions, Preliminaries, Examples

Consider a stochastic pure exchange economy with a finite number $m$ of agents. Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t,\ 0 \leq t \leq T), \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right continuity and completeness; $\mathcal{F}_0$ is $\mathbb{P}$-a.s. trivial.

A nonnegative, nondecreasing, rightcontinuous and adapted process $C = (C_t)_{t\geq 0}$ is called an optional random measure. If $X = C - C'$ for two optional random measures, then $X$ is called a signed optional random measure, and the commodity space $\mathcal{X}$ is the space of all signed optional random measures. Agents' consumption set $\mathcal{X}_+$ consists of all optional random measures $C$ with $\mathbb{E}C_T < \infty$. An ordering $\leq$ is given on $\mathcal{X}$ via $X \leq Y$ iff $X - Y \in \mathcal{X}_+$. For $C \in \mathcal{X}_+$, the order interval is

$$[0, C] = \{D \in \mathcal{X}_+ : 0 \leq D \leq C\}.$$

The Hindy–Huang–Kreps norm on $\mathcal{X}$ is given by $\|X\|_{HHK} =$

$\mathbb{E}\int_0^T |X_t|dt+\mathbb{E}|X_T|$ . On the consumption set, this norm induces the topology of weak convergence in probability plus $L^1$–convergence of total cumulative consumption. This topology captures the notion of local substitutability of consumption as shown by Hindy, Huang [8] and Hindy, Huang, and Kreps [9].

The $m$ agents are described by their utility functions $U^i : \mathcal{X}_+ \to \mathbb{R}, i = 1,\dots,m$. Throughout, we take utility functions and aggregate endowment $\bar{E} \in \mathcal{X}_+, \bar{E} \neq 0$ as fixed, and we vary the vector of initial endowments $(E^i)_{i=1,\dots,m}$ in the set

$$\mathcal{D} = \left\{ \left(E^i\right)_{i=1,\dots,m} \in \mathcal{X}_+^m : E^i \neq 0, \sum_{i=1}^m E^i = \bar{E} \right\} .$$

Therefore, one can identify an economy with an endowment vector $(E^i) \in \mathcal{D}$ in the sequel. Optional, nonnegative processes $\psi$ with

$$0 < \langle \psi, \bar{E} \rangle \triangleq \mathbb{E} \int_0^T \psi_t d\bar{E}_t < \infty$$

are called price processes. The set of all price processes is denoted $\Psi$. An allocation is a vector $(C^i)_{i=1,\dots,m} \in \mathcal{X}_+^m$. It is feasible if $\sum_{i=1}^m C^i \le \bar{E}$. The set of feasible allocations will be denoted by $\mathcal{Z}$. An (Arrow–Debreu) equilibrium for the economy $(E^i) \in \mathcal{D}$ consists of a feasible allocation $(C^i)_{i=1,\dots,m}$ and a price process $\psi$ such that, for any $i = 1,\dots,m$, the consumption plan $C^i$ maximizes agent $i$'s utility over all $D^i$ satisfying the budget–constraint $\langle \psi, D^i \rangle \le \langle \psi, E^i \rangle$.

A vector of initial endowments $(E^i) \in \mathcal{D}$ is called determinate if the number of equilibria for the economy $(E^i)$ is finite and the equilibrium allocation correspondence

$$(F^i) \in \mathcal{D} \mapsto \left\{ (C^i) \in \mathcal{D} : (C^i) \text{ is equilibrium allocation for } (F^i) \right\}$$

is continuous at $(E^i)$. Here, I shall show that the set of determinate endowments is *finitely prevalent in* $\mathcal{D}$, a concept introduced into the conomics literature by [2]. A Borel subset $\mathcal{D}_0 \subset \mathcal{D}$ is called finitely prevalent in $\mathcal{D}$ if there exists a finite–dimensional subspace $\mathcal{V}$ of $\mathcal{X}$ and an element $F \in \mathcal{X}$ such that the intersection $\mathcal{V} \cap (F + \mathcal{D})$ is not a Lebesgue null set in $\mathcal{V}$, and for all $G \in \mathcal{X}$ the sets $\mathcal{V} \cap (G + (\mathcal{D} \backslash \mathcal{D}_0))$ *are* Lebesgue null sets in $\mathcal{V}$. The subspace $\mathcal{V}$ is called a probe (cf. [10]). Prevalence is defined for completely metrizable sets only, so we note as a preliminary fact proved in the appendix:

**Lemma 1** *The metric space* $(\mathcal{X}_+, \|\cdot\|_{HHK})$ *is complete.*

Another requirement for determinacy is that utility functionals are smooth and strictly concave. As in [14], we assume *quadratic concavity* with respect to an adapted norm in the sense of the following definition.

**Assumption 2** *The utility functions* $U^i$ *have the following properties:*

1. $U^i$ *is strictly increasing and continuous,*
2. *there exist a mapping* $\nabla U^i : \mathcal{X}_+ \to \Psi$*, a norm* $\|\cdot\|_i$ *on* $\mathcal{X}$ *and constants* $B, K > 0$ *such that*

   (a) *on* $[0, \bar{E}]$*, the topology induced by* $\|\cdot\|_i$ *coincides with the topology induced by* $\|\cdot\|_{HHK}$*,*

   (b) *for all* $C \in [0, \bar{E}]$ *and all* $X \in \mathcal{X}$

   $$\left|\langle \nabla U^i(C), X\rangle\right| \leq B \|X\|_i \ ,$$

   (c) *for all* $C, C' \in \mathcal{X}_+$*,* $C'' \in [0, \bar{E}]$

   $$\left|\langle \nabla U^i(C) - \nabla U^i(C'), C''\rangle\right| \leq B \|C - C'\|_i \ ,$$

   (d) *for all* $C, C' \in [0, \bar{E}]$

   $$U^i(C') - U^i(C) \leq \langle \nabla U^i(C), C' - C\rangle - K \|C' - C\|_i^2 \ .$$

**Example 3** Hindy–Huang–Kreps utility functions satisfy Assumption 2. For $\beta, \eta > 0$ and $X \in \mathcal{X}$ set

$$\begin{aligned} z_t^X &= e^{-\beta t} \int_0^t e^{\beta s} dX_s \\ y_t^X &= \eta e^{-\beta t} + z_t^X \ . \end{aligned}$$

Let $u, v : \mathbb{R}_+ \to \mathbb{R}$ be twice continuously differentiable with strictly positive first and strictly negative second derivative. Set for $\delta > 0$ and $C \in \mathcal{X}_+$

$$U(C) = \mathbb{E} \int_0^T e^{-\delta t} u\left(y_t^C\right) dt + \mathbb{E} v\left(y_T^C\right) \ .$$

Define a norm on $\mathcal{X}$ via

$$\|X\|_R \triangleq \mathbb{E} \int_0^T \left|z_t^X\right| dt + \mathbb{E}\left|z_T^X\right| .$$

The subgradient of $U$ at $C \in \mathcal{X}_+$ is

$$\nabla U(C)_t = \mathbb{E}\left[ \int_t^T e^{-\delta s} u'\left(y_s^C\right) e^{-\beta(s-t)} ds \,\middle|\, \mathcal{F}_t \right] + \mathbb{E}\left[ e^{-\beta(T-t)} v'\left(y_T^C\right) \middle|\, \mathcal{F}_t \right] .$$

We claim that $U$, $\nabla U$ and $\|\cdot\|_R$ satisfy Assumption 2 if aggregate endowment $\bar{E}$ is bounded a.s. This is shown in the appendix.

# 2 Generic Determinacy

Here is the road to determinacy. In a first step, efficient allocations are parametrized by a finite–dimensional set of so–called utility weights. This is the usual welfare approach to general equilibrium pioneered by Negishi [13]. The weighted maximum of utility gradients at an efficient allocation is a candidate for an equilibrium price since it supports the associated efficient allocation. Moreover, and most importantly, all equilibria correspond to some utility weights vector. This is shown in Subsection 2.1. The proof of the parallel results in [14] uses their assumption that aggregate endowment is strictly positive. Hence, I present a different proof of these facts here, which relies on my previous work with Peter Bank [4].

Every equilibrium corresponds to a zero of the (finite–dimensional!) excess spending map, and determinacy will follow from regularity of this map. As Shannon and Zame show, it sufiices to have Lipschitz continuity in the utility weights and continuity in endowments. Lipschitz continuity follows from quadratic concavity of utility functions, the concept introduced by Shannon and Zame. For continuity in endowments, however, Shannon and Zame rely on the lattice structure of the price space, what is impossible in the present framework. The important remark is that the maximum of continuous linear functionals (albeit not being necessarily continuous on the commodity space) is continuous on order intervals — and that is all what is needed here (Subsection 2.2).

**Theorem 4** *Under Assumption 2, the set of determinate endowments is finitely prevalent in the set $D$ of all nondegenerate endowments.*

The proof of generic determinacy is given in Subsection 2.3.

## 2.1 Parametrizing Efficient Allocations and Equilibria

Introduce the set

$$\Lambda \triangleq \left\{ \lambda \in \mathbb{R}^m : \forall i \; \lambda^i > 0, \sum_{i=1}^{m} \lambda^i = 1 \right\}$$

and its closure $\bar{\Lambda}$. A feasible allocation $(C^i) \in \mathcal{Z}$ is called $\lambda$–efficient for $\lambda \in \bar{\Lambda}$ if it maximizes the weighted sum of utilities $\sum \lambda^i U^i(D^i)$ over all feasible allocations $(D^i) \in \mathcal{Z}$.

The following results are as in [14]. Unfortunately, we cannot use their proofs, because they assume strictly positive aggregate endowment. Fortunately, however, one can rely on our existence paper [4].

**Lemma 5** *For $\lambda \in \bar{\Lambda}$ there exists a unique $\lambda$–efficient allocation $(C^i_\lambda)$. It satisfies*

$$\langle \psi_\lambda - \lambda^i \nabla U^i \left( C^i_\lambda \right), C^i_\lambda \rangle = 0 \quad (i = 1, \ldots, m) \tag{1}$$

*for*

$$\psi_\lambda \triangleq \max \lambda^i \nabla U^i \left( C^i_\lambda \right) \, . \tag{2}$$

Proof : [4, Lemma 1]. □

**Lemma 6** *1. Let $(C^i) \in \mathcal{X}^m_+$ and $\psi \in \Psi$ form an equilibrium. Then there exist $\lambda \in \Lambda$ and $L > 0$ such that*

$$C^i = C^i_\lambda \tag{3}$$

$$\psi = L\psi_\lambda \quad \mathbb{P} \otimes d\bar{E} - a.e. \tag{4}$$

$$\langle \psi_\lambda, C^i_\lambda - E^i \rangle = 0 \, . \tag{5}$$

*2. If (5) holds true for some $\lambda \in \Lambda$, then $(C^i_\lambda)$ and $\psi_\lambda$ form an equilibrium.*

Proof : For 1. , let $((C^i), \psi)$ be an equilibrium. In particular, $\psi$ supports the allocation $(C^i)$. Lemma 3 in [4] yields nonnegative $k^i \geq 0$ such that

$$\psi = \max k^i \nabla U^i (C^i) \quad \mathbb{P} \otimes d\bar{E} - a.e. \tag{6}$$

$$\text{and } \langle \psi - k^i \nabla U^i (C^i), C^i \rangle = 0 \, . \tag{7}$$

Note that not all $k^i$ can be zero, because else $\psi = 0 \notin \Psi$. Moreover, strict monotonicity of utility functions implies that

$$\sum C^i = \bar{E}\,. \tag{8}$$

Now, by Lemma 1 in [4], (6), (7), and (8) show that $(C^i)$ is $\lambda$–efficient for $\lambda^i \triangleq \frac{k^i}{\sum k^j}$, and by uniqueness, $C^i = C^i_\lambda$. I show next that $\lambda^i > 0$. $\lambda^i = 0$ implies $C^i_\lambda = 0$, and therefore $U^i(C^i) < U^i(E^i)$, a contradiction to $(C^j)$ being an equilibrium. Hence, $\lambda \in \Lambda$ and $C^i = C^i_\lambda$, which establishes (3). From this, and (6), we obtain (4), and (5) is the equilibrium budget constraint.

Now assume that (5) holds true for some $\lambda \in \Lambda$. $(C^i_\lambda)$ is a feasible allocation by definition. We must show that $C^i_\lambda$ maximizes utility over agent $i$'s budget set. So, assume that $\langle \psi_\lambda, D\rangle \leq \langle \psi_\lambda, E^i\rangle$. Concavity of $U^i$, (1), (5), and the budget constraint imply

$$\begin{aligned} U^i(D) - U^i(C^i_\lambda) &\leq \langle \nabla U^i(C^i_\lambda), D - C^i_\lambda\rangle \\ &\leq \frac{1}{\lambda^i}\langle \psi_\lambda, D - C^i_\lambda\rangle \\ &= \frac{1}{\lambda^i}\langle \psi_\lambda, D - E^i\rangle \leq 0\,. \end{aligned}$$

Therefore, $C^i_\lambda$ is agent $i$'s demand given $\psi_\lambda$, and the proof is complete. □

## 2.2 Continuity of the Excess Spending Map

The topological dual $\mathcal{X}^*_{HHK}$ consists of semimartingales $\psi = M + A$, where $M$ is a bounded martingale and $A$ is an absolutely continuous process whose derivative $A'$ is bounded a.e., see [8]. In general, this space is not a lattice, that is, it is not closed with respect to taking pointwise maxima. To see this, suppose the information filtration is generated by a Brownian motion $M$. Set $\tau \triangleq \inf\{t \geq 0 : |M_t| = 1\}$ and $\psi \triangleq M^\tau$, Brownian motion stopped at 1 or $-1$. Then $\psi \in \mathcal{X}^*_{HHK}$, and, of course $0 \in \mathcal{X}^*_{HHK}$. By the Tanaka formula, $\max\{\psi, 0\} = M' + L$, where $M'$ is a bounded martingale, and $L$ is Brownian local time stopped at $\tau$. One of the remarkable features of diffusions is that their local time, while being nondecreasing, is not absolutely continuous ([11]). Therefore, $\max\{\psi, 0\} \notin \mathcal{X}^*_{HHK}$ —the topological dual is not a lattice.

So one of the Shannon/Zame assumptions (A2) is not satisfied here, and we must find a way to proceed without it. A careful reading of Shannon and Zame's proof shows that this assumption is needed only at one, but crucial point, where one has to show that the excess spending map, or $E \mapsto \langle \psi_\lambda, E \rangle$, is continuous in endowments. Since $\mathcal{X}^*_{HHK}$ is not a lattice, we cannot hope that this holds true on the whole commodity space $\mathcal{X}$. Fortunately, we need it only on the order interval $[0, \bar{E}]$. As the next lemma shows, we do have continuity there.

**Lemma 7** *For $\lambda \in \Lambda$, the mapping $E \mapsto \langle \psi_\lambda, E \rangle$ is continuous on $[0, \bar{E}]$.*

Proof : Let $\lambda \in \Lambda$ be given and assume that $\|E_n - E\|_{HHK} \to 0$ in $[0, \bar{E}]$. Denote by $a_n \triangleq \langle \psi_\lambda, E_n \rangle$. By Lemma 5, (1), we have $0 \leq a_n \leq \langle \psi_\lambda, \bar{E} \rangle = \sum_i \lambda^i \langle \nabla U^i(C^i_\lambda), C^i_\lambda \rangle$. Since $\nabla U^i(C^i_\lambda)$ is a positive linear functional, and by Assumption 2, 2 (b), we have $\langle \nabla U^i(C^i_\lambda), C^i_\lambda \rangle \leq \langle \nabla U^i(C^i_\lambda), \bar{E} \rangle \leq B \left\|\bar{E}\right\|_i$. Thus, the sequence $(a_n)$ is bounded, and has, therefore, limit points. Let $a$ be such a limit point, that is, $a = \lim_k \langle \psi_\lambda, E_{n_k} \rangle$ for some subsequence $(n_k)$. We must show that $a = \langle \psi_\lambda, E \rangle$. This is not clear, at first, since we do not know whether $\psi_\lambda$ belongs to the HHK–dual or not. However, by the same reasoning as above, $\langle \psi_\lambda, \bar{E} \rangle \leq B \sum_i \lambda^i \left\|\bar{E}\right\|_i < \infty$. Therefore, $\psi_\lambda$ belongs to $L^1(\mathbb{P} \otimes d\bar{E})$, and is thus a continuous linear functional on the space $L^\infty(\mathbb{P} \otimes d\bar{E})$.

The order interval $[0, \bar{E}]$ is compact in the weak*–topology on $L^\infty(\mathbb{P} \otimes d\bar{E})$. Hence, we may assume without loss of generality that $\lim_k \langle \psi_\lambda, E_{n_k} \rangle = \langle \psi_\lambda, F \rangle$ for some $F \in [0, \bar{E}]$. It only remains to be shown that $F = E$. This follows from the fact that the HHK dual $\mathcal{X}^*_{HHK} \subset L^1(\mathbb{P} \otimes d\bar{E})$ because $\mathcal{X}^*_{HHK}$ consists of bounded semimartingales. For every $\psi \in \mathcal{X}^*_{HHK}$, we have therefore $\langle \psi, E \rangle = \lim \langle \psi, E_{n_k} \rangle = \langle \psi, F \rangle$, which implies $E = F$. □

For later use, we record the following continuity results from Shannon/Zame which do not use their critical assumptions A2 (strict positivity of aggregate endowment) and A5 (price space is a lattice).

**Lemma 8 (Shannon/Zame)** *1. The mapping $\lambda \mapsto C^i_\lambda$ is locally Lipschitz continuous with respect to $\|\cdot\|_i$ and continuous with respect to $\|\cdot\|_{HHK}$.*

*2. the mapping $\lambda \mapsto (\langle \psi_\lambda, C^i_\lambda \rangle)$ is locally Lipschitz,*

3. *For* $E \in [0, \bar{E}]$, *the mapping* $\lambda \mapsto (\langle \psi_\lambda, E \rangle)$ *is locally Lipschitz, uniformly in* $E$.

PROOF : 1. is Lemma 5.1., 2. is Lemma 6.1., (i) and 3. is Lemma 6.1., (ii) in [14]. □

**Corollary 9** *The mapping* $(\lambda, E) \mapsto (\langle \psi_\lambda, E \rangle)$ *is jointly continuous on* $\Lambda \times [0, \bar{E}]$.

PROOF : Let $\lambda_n \to \lambda$ in $\Lambda$ and $\|E_n - E\|_{HHK} \to 0$ in $[0, \bar{E}]$. Set $\psi_n \triangleq \psi_{\lambda_n}$. By Lemma 7, we know $|\langle \psi_\lambda, E_n - E \rangle| \to 0$. Lemma 8 yields a constant $L > 0$, independent of $(E_n)$ and $E$, such that $|\langle \psi_n - \psi, E_n \rangle| \leq L \|\lambda_n - \lambda\| \to 0$. Altogether, we obtain

$$|\langle \psi_n, E_n \rangle - \langle \psi, E \rangle| \leq |\langle \psi_n - \psi, E_n \rangle| + |\langle \psi, E_n - E \rangle| \to 0 \,.$$

□

## 2.3 Proof of the Theorem

PROOF : As a preliminary, we need to show that the set $\mathcal{D}_d \triangleq \{(E^i) \in \mathcal{D} : (E^i) \text{ is determinate}\}$ of determinate endowments is a Borel set. The argument is as in [14], p. 650. Note that joint continuity of the mapping $(\lambda, (E^i)) \mapsto \langle \Psi_\lambda, C^i_\lambda - E^i \rangle$ is needed for that. This follows from Lemma 8 and Corollary 9.

We introduce the $m-1$–dimensional set

$$\mathcal{V} \triangleq \left\{ \left(\alpha^i \bar{E}\right) : \alpha^i \in \mathbb{R}, \sum \alpha^i = 1 \right\} \,.$$

The set $\mathcal{V} \cap \mathcal{D} = \left\{ \left(\alpha^i \bar{E}\right) : \alpha^i > 0, \sum \alpha^i = 1 \right\}$ has positive Lebesgue measure in $\mathcal{V}$. In order to establish $\mathcal{V}$ as a probe, it remains to be shown that for every $F \in \mathcal{X}^m$ the set $\mathcal{V} \cap (F + \mathcal{D} - \mathcal{D}_0)$ has Lebesgue measure zero in $\mathcal{V}$. So fix some $F \in \mathcal{X}^m$ and assume that $G \in \mathcal{V} \cap (F + \mathcal{D} - \mathcal{D}_0)$. This implies that $G^i = \alpha^i \bar{E} = F^i + E^i$ for some positive $\alpha^i$ with $\sum \alpha^i = 1$ and some $E \in \mathcal{D}$.[1]

[1] In particular, $\sum F^i = 0$ follows.

Introduce the open set $\Delta \stackrel{\Delta}{=} \left\{\lambda \in \mathbb{R}^{m-1} : \lambda^i > 0, \sum_{i=1}^{m-1} \lambda^i < 1\right\}$ in $\mathbb{R}^{m-1}$ and the function

$$\begin{aligned}\sigma :& \Delta \to \mathbb{R}^{m-1} \\ \lambda \mapsto & \left(\frac{\langle \psi_\lambda, C^i_\lambda \rangle}{\langle \psi_\lambda, \bar{E} \rangle}\right)_{i=1,\ldots,m-1} .\end{aligned}$$

$\sigma$ is well–defined because $\langle \psi_\lambda, \bar{E}\rangle \geq \lambda^1 \langle \nabla U^1(C^1_\lambda), \bar{E}\rangle > 0$, since $\nabla U^1(C^1_\lambda) \in \Psi$ by Assumption 2. By Lemma 8, $\sigma$ is locally Lipschitz continuous. By Lemma 6, the number of equilibria of the $G$–economy is equal to the number of $\lambda \in \Lambda$ with $\sigma(\lambda) = \alpha$. Hence, it is enough to show that the number of solutions to $\sigma(\lambda) = \alpha$ is finite for almost every $\alpha$. By Sard's theorem for Lipschitz functions, almost every $\alpha \in \mathbb{R}^{m-1}$ is a regular value for $\sigma$. For a regular value $\alpha$, solutions of $\sigma(\lambda) = \alpha$ are locally unique. Due to utility maximization, the solutions belong to the set $\{\lambda \in \Delta : U^i(C^i_\lambda) \geq U^i(\alpha^i \bar{E})\}$. Since $U^i(\alpha^i \bar{E}) > 0$ (strict monotonicity) and $C^i_\lambda$ is continuous in $\lambda$ by Lemma 8, this set is compact. Every closed and locally isolated subset of a compact set is finite, so the number of equilibria for the $G$-economy is finite.

The proof that the equilibrium allocation correspondence is continuous at $G$ is as in Shannon/Zame. For the sake of completeness, I prove here in a slightly different way that the equilibrium allocation correspondence is lower hemicontinuous at $(\alpha^i \bar{E})$ for regular values $\alpha$. Let $\alpha$ be a regular value of $\sigma$, and $\sigma(\lambda^*) = \alpha$. Fix $\epsilon > 0$. We have to find $\delta > 0$ such that the mapping $S(\lambda, (E^i)) \stackrel{\Delta}{=} \langle \psi_\lambda, C^i_\lambda - E^i \rangle$ has a zero in $B_\epsilon \stackrel{\Delta}{=} \{\lambda \in \Lambda : \|\lambda - \lambda^*\| < \epsilon\}$ for every $(E^i) \in \mathcal{D}$ with $\left\|E^i - \alpha^i \bar{E}\right\|_{HHK} < \delta, i = 1, \ldots, m$. Since $\alpha$ is a regular value, we know from the above that $\lambda^*$ is locally unique; choose $\eta \leq \epsilon$ with $B_\eta \cap \{S(\cdot, (\alpha^i \bar{E})) = 0\} = \{\lambda^*\}$. This implies that the degree at zero $deg(S(\cdot, (\alpha^i \bar{E})), B_\eta) \neq 0$. It is enough to show that this translates to $S(\cdot, (E^i))$ when $(E^i)$ is close to $(\alpha^i \bar{E})$. Let $\mu \stackrel{\Delta}{=} \min_{\|\lambda - \lambda^*\| = \eta} S(\lambda, (\alpha^i \bar{E}))$. Set

$$T((E^i)) \stackrel{\Delta}{=} \max_{\lambda \in \bar{B}_\eta} \left\|S(\lambda, (E^i)) - S(\lambda, (\alpha^i \bar{E}))\right\| = \max_{\lambda \in \bar{B}_\eta} \left\|\langle \psi_\lambda, \alpha^i \bar{E} - E^i \rangle\right\| .$$

By the Maximum Theorem ([5]) and Corollary 9, $T$ is continuous, and $T((\alpha^i \bar{E})) = 0$. Hence, there exists $\delta > 0$ such that $|T((E^i))| < \mu$ whenever $\left\|E^i - \alpha^i \bar{E}\right\|_{HHK} < \delta$. For such an initial endowment $(E^i)$, consider the homotopy $H(t, \lambda) \stackrel{\Delta}{=} tS(\lambda, (E^i)) - (1-t)S(\lambda, (\alpha^i \bar{E}))$. $H(t, \cdot)$ has no zeros on the boundary of $B_\eta$, since for $\lambda \in \partial B_\eta$,

$$|H(t, \lambda)| \geq \left\|S(\lambda, (\alpha^i \bar{E}))\right\| - t \left\|S(\lambda, (E^i)) - S(\lambda, (\alpha^i \bar{E}))\right\| > \mu - t\mu \geq 0 .$$

By homotopy invariance of the degree, we conclude that $deg(S(\cdot,(E^i)),B_\eta) = deg(S(\cdot,(\alpha^i\bar{E})),B_\eta) \neq 0$, and we are done. □

# A Proof of Lemma 1

PROOF : Let $(C_n)$ be a Cauchy sequence in $\mathcal{X}_+$. The metric space $(\mathcal{X}_+, \|\cdot\|_{HHK})$ is a subspace of the metric space $(L^1(\Omega\times[0,T],\mathcal{O},\mathbb{P}\otimes(dt+\delta_T)), \|\cdot\|_{HHK})$, where $\mathcal{O}$ denotes the optional $\sigma$–field. Since $L^1$–spaces are complete, there exists an optional process $Z \in L^1(\Omega\times[0,T],\mathcal{O},\mathbb{P}\otimes(dt+\delta_T))$ with $\lim\|C_n - Z\|_{HHK} = 0$. By passing to a subsequence if necessary, we may assume without loss of generality that $C_n \to Z$ $\mathbb{P}\otimes(dt+\delta_T)$– a.e. This shows that $Z$ is nonnegative, nondecreasing, and rightcontinuous a.e. Denote by $\bar{Z}$ the optional, nonnegative, and nondecreasing process which satisfies $\bar{Z} = Z$ $\mathbb{P}\otimes(dt+\delta_T)$– a.e. Then $\lim\left\|C_n - \bar{Z}\right\|_{HHK} = 0$ and and $\bar{Z} \in \mathcal{X}_+$, which shows completeness. □

# B On Hindy–Huang–Kreps Utility Functions

I prove here the claim made in Example 3. The utility function $U$ is strictly increasing because so are $u$ and $v$. Continuity of $U$ with respect to $\|\cdot\|_{HHK}$ is shown in [4] (for $v = 0$, but it is easy to adapt the argument). It is easy to see that $\|\cdot\|_R$ is indeed a norm on $\mathcal{X}$, and that the topology coincides with the $\|\cdot\|_{HHK}$–topology on $\mathcal{X}_+$.

Therefore, I focus on the remaining conditions 2, (b) through (d) in Assumption 2. $y^C$ is uniformly bounded away from zero since $\eta > 0$. Therefore, $u'(y_t^C)$ and $v'(y_T^C)$ are uniformly bounded by some constant $B$. Note that by partial integration $\langle\nabla U(C), X\rangle = \mathbb{E}\int_0^T e^{-\delta t}u'\left(y_t^C\right)z_t^X dt + \mathbb{E}v'\left(y_T^C\right)z_T^X$. Therefore,

$$|\langle\nabla U(C), X\rangle| \leq B\mathbb{E}\int_0^T \left|z_t^X\right|dt + \mathbb{E}\left|z_T^X\right|,$$

which is 2 (b).

Next, assume that aggregate endowment is bounded a.e. This implies, of course, that $z^{\bar{E}}$ is bounded a.e., say by $B > 0$. Lipschitz continuity of $u'$ and

$v'$ yields a constant $L$ such that

$$
\begin{aligned}
|\langle \nabla U(C) - \nabla U(C'), C'' \rangle| &\leq L\mathbb{E} \int_0^T \left| y_t^C - y_t^{C'} \right| z_t^{\bar{E}} dt + L\,\mathbb{E} \left| y_T^C - y_T^{C'} \right| z_T^{\bar{E}} \\
&\leq LB \left\| C - C' \right\|_R .
\end{aligned}
$$

$u$ and $v$ are quadratically concave since they are twice continuously diffentiable and strictly concave (compare [14]). Hence, there exists a constant $K > 0$ such that

$$
\begin{aligned}
u(y) - u(z) &\leq u'(z)(y-z) - K(y-z)^2 \\
v(y) - v(z) &\leq v'(z)(y-z) - K(y-z)^2
\end{aligned}
$$

for all $y, z \in [0, B]$. This yields quadratic concavity, as the following inequalities show for $C, D \in [0, \bar{E}]$:

$$
\begin{aligned}
U(D) - U(C) &= \mathbb{E} \int_0^T e^{-\delta t} \left( u\left(y_t^D\right) - u\left(y_t^C\right) \right) dt + \mathbb{E} \left( v\left(y_T^D\right) - v\left(y_T^C\right) \right) \\
&\leq \mathbb{E} \int_0^T e^{-\delta t} u'\left(y_t^C\right) \left(y_t^D - y_t^C\right) dt + \mathbb{E} v'\left(y_T^C\right) \left(y_T^D - y_T^C\right) \\
&- K\,\mathbb{E} \int_0^T e^{-\delta t} \left(y_t^D - y_t^C\right)^2 dt - K\mathbb{E} \left(y_T^D - y_T^C\right)^2 \\
&\leq \langle \nabla U(C), D - C \rangle - K e^{-\delta T} \left\| D - C \right\|_R^2 .
\end{aligned}
$$

The last line uses the Cauchy–Schwarz inequality.

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