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Emission Targets and Equilibrium Choice of Technique

by

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Date September 2002

Abstract We study the technological pre-conditions for a cost-minimizing choice of technique in the presence of government emission targets on by-products of production. Whether a by-product is a desirable commodity or an undesirable pollutant is determined endogeneously as part of the price-quantity equilibrium solution. Non-trivial counter-examples highlight the potential risk of over-ambitious pollution targets. We show that pollution targets can be supported by the appropriate taxes providing that technology allows for a certain type of labour-intensive pollution abatement activities. Our proof is *constructive*: the tax equilibria we posit can be computed by the Lemke Complementary Pivoting Algorithm.

Keywords Multisectoral Growth Theory, Choice of Technique, Pollution Taxes, Permit Markets, Lemke Algorithm

J.E.L. Class C61, D57, O33,Q38

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Emission Targets and Equilibrium Choice of Technique

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§ 1 Introduction

In an innovative series of papers, Lager (1999, 2001) has applied rent-theoretical techniques to the issue of pollution abatement in a multi-sectoral economy. Lager's results are based on a somewhat restrictive existence result from rent theory, and as such in need of extension. The present paper provides one such extension in the form of a novel existence result, and offers some structural insights into the technological pre-conditions for implementing a pollution targets via permit markets of pollution quotas or dually via a system of pollution taxes.

We consider a situation where the regulator issues a system of annual pollution quotas to firms, in the form of tradeable pollution permits. The market price of such permits corresponds to the rent payments needed to obtain scarce land as an input. If the government can estimate these permit prices, it can replace the actual permit market by a corresponding system of pollution taxes.

Is there a system of commodity prices and pollution taxes that makes the technological choices of cost-minimizing producers consistent with the pollution target? Non-trivial examples of environmentally constrained economies where *no* consistent system of equilibrium prices exists have long been known and have puzzled the multi-sectoral theorist; see d'Agata (1983, 1984) for instructive early discussions. One of the most advanced existence results in this area is by Salvadori (1986),originally given in a rent context. Lager (2001) has translated Salvadori's result into a pollution context. Salvadori proves that a given final demand vector can be supported by a price-quantity equilibrium if the money interest rate is lower than the maximum balanced growth rate that is feasible under the environmental constraint — an instructive but weak result, since the potential for balanced growth is very limited if environmental constraints are tight.

We extend the earlier results Salvadori and Lager and show that a given final demand vector can be supported by a competitive price-tax-quantity equilibrium if the money interest rate does not exceed the unconstrained maximum growth rate and if, in addition, there is a sufficiently strong *substitution potential* between labour and environmental inputs at the given demand vector. Balanced growth still serves as a key reference point, but only as a means of checking the productivity of the current technology, not as an actual feasibility requirement on the actual growth path of the economy.

The natural framework for our analysis is Linear Complementarity Theory — we formulate our equilibrium conditions as a Linear Complementarity Problem and show that this problem admits a solution. Our proof is *constructive;* we show that the equilibria we discuss can be computed by the Lemke Complementary Pivoting Algorithm.

The paper is organised as follows. *Section 2* sets up framework and notation. *Section 3* states and interprets our equilibrium conditions. *Section 4* gives numerical examples. *Section 5* provides the background from linear complementarity theory and establishes the link to Dantzig and Manne's Complementarity Construction Theorem. *Section 6* discusses earlier existence results by Bidard, Salvadori and Lager. *Section 7* states our technological substitution assumption. *Section 8* states our existence theorem: a price-quantity solution to the equilibrium conditions from Section 3 does exist if technology satisfies the substitution assumption from Section 7, and this solution can effectively be computed by the Lemke algorithm. *Section 9–10* then furnish the proof of the theorem. *Section 12* gives the conclusion. The *Appendix* presents an implementation of the rent algorithm in the Python programming language.

§ 2 Framework and Notation

2.1 Motivation We wish to identify the equilibrium choice of technique and choice of production levels of cost-minimizing producers in a self-contained period.

Production generates by-prtoducts that need to be disposed in the environment. The environment has a limited "carrying capacity" for the absorption of such by-products.

To protect the environment, the government imposes emission standards on producers. These standards may be implemented either directly, as a system of tradable emission permits, or dually, as a system of emission taxes that reflect the shadow price of the carrying capacity.

We refer to the imposed emission targets as "carrying capacities". Every output is a potential emission into the environment, and there is one carrying capacity for each potential emission. We refer to the rental price of a carrying capacity as a "emission charge". Such charges may come about as market prices of tradable permits or as emission taxes.

Producers choose their activities $\boldsymbol{x}$ from the common "book of blueprints" $(\boldsymbol{A}, \boldsymbol{\ell}, \boldsymbol{B})$ that lists all known production processes. Each production process has constant returns and is of type

$$(\,\boldsymbol{a} \quad \boldsymbol{\ell}\,) \rightarrow \boldsymbol{b},$$

where $\boldsymbol{a}$ are commodity inputs, $\boldsymbol{\ell}$ are labour inputs and $\boldsymbol{b}$ are commodity outputs (including emissions). Inputs precede outputs by one period.

Within any individual process, we allow for arbitrary patterns of joint production (in $\boldsymbol{b}$) Any process may generate outputs that need to be absorbed by the existing carrying capacity. There is no *a priori* distinction between desirable commodities and undesirable pollutants; whether a particular type of commodity is an economic "good" or an economic "bad" is determined endogeneously as part of the overall price-quantity equilibrium solution.

A commodity turns out to be desirable if it is produced without the need to make emissions into the environment and will have a positive price but a zero emission charge. By contrast, if a commodity turns out to be undesirable, it will lead to a binding constraint in carrying capacity and will carry a positive emission charge but a zero price.

Processes that use desirable commodities as inputs will have to *pay* for these commodities as a cost element; processes that use undesirable commodities act as abatement processes and receive revenue for the reduction in emission charges that they bring about.

Technology allows for a wide range of substitution possibilites, and "choice of technique" is endogenous to the equilibrium solution. The number N of processes in the book-of-blueprints is much larger than the number $M + Q$ of commodities and emissions, and only a small subset of the available processes will be active in equilibrium; cost-minimizing producers will choose processes that are profitable and reject processes that make losses. In competitive equilibrium, there are no pure profits and all active processes will *break even* under commodity prices $\boldsymbol{p}$, emission charges $\boldsymbol{y}$, wage rate w and interest rate r:

$$\boldsymbol{pb} = (1 + r)\boldsymbol{pa} + \boldsymbol{y}[\boldsymbol{b} - \boldsymbol{a}] + w\ell.$$

The supply of labour is perfectly elastic at nominal wage rate w. The interest rate is treated as given. In addition to consumption goods, final demand $\boldsymbol{d}$ may also include additional capital goods for an unbalanced expansion of producton that goes beyond the replacement of worn-out stocks. For given carrying capacity $\boldsymbol{s}$ and given final demand $\boldsymbol{d}$, we are seeking a price-quantity equilibrium in which commodity prices $\boldsymbol{p}$, dosposal charges $\boldsymbol{y}$ and activity levels $\boldsymbol{x}$ are simultaneously determined by the Rule of Free Goods and the Rule of Profitability.

Emission charges $\boldsymbol{y}$ act as a penalty on the use of carrying capacity and ensure that processes that save on carrying capacity but have high unit costs can compete with cheaper but more pollute-intensive processes. As aggregate production expands and the pollution-constraint becomes tighter, higher emission charges force cost-minimizing producers to adopt production processes that are sufficiently emission-saving to make increased output levels consistent with the fixed carrying capacity supplies.

2.2 Notation and Assumptions There are M commodities and hence potential emissions 1 type of labour (in unlimited supply), N processes, and Q types of carrying capacity ($Q = M$). Typical labels: Process n, commodity m, carrying capacity q.

In technology matrices $\boldsymbol{A}$, $\boldsymbol{B}$, columns refer to processes, rows refer to commodities and resources. We write a_i^j for the entry in row i and column j of matrix $\boldsymbol{A}$. The jth column and ith row of matrix $\boldsymbol{A}$ are denoted by $\boldsymbol{A}^{[j]}$ and $\boldsymbol{A}_{[i]}$, respectively. Similarly, we write p^j or $\boldsymbol{p}^{[j]}$ for the jth entry of row vector $\boldsymbol{p}$ and x_i or $\boldsymbol{x}_{[i]}$ for the ith entry of column vector $\boldsymbol{x}$. Transposition of $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{\mathrm{T}}$.

$\boldsymbol{A}$	$(M \times N)$	Commodity inputs. Nonnegative. Every column $\boldsymbol{A}^{[n]}$ of $\boldsymbol{A}$ is nonzero (no process without capital inputs).
$\boldsymbol{B}$	$(M \times N)$	Commodity outputs. Nonnegative. Every row $\boldsymbol{B}_{[m]}$ of $\boldsymbol{B}$ is nonzero (all commodities are potential outputs).
$\boldsymbol{\ell}$	$(1 \times N)$	Labour inputs. Strictly positive (no process without labour inputs).
$\boldsymbol{x}$	$(N \times 1)$	Activity levels of processes during the period. Nonnegative.
$\boldsymbol{d}$	$(M \times 1)$	Final demand for commodities, at the end of the period. Nonnegative and nonzero.
$\boldsymbol{s}$	$(Q \times 1)$	Carrying capacities during the period. Strictly positive. $(Q = M)$

$\boldsymbol{p}$	$(1 \times M)$	Commodity prices during the period (stationary). Nonnegative.
$\boldsymbol{y}$	$(1 \times M)$	Emission charges at end of period. Nonnegative.
w	scalar	Nominal wage rate at end of period, positive.
r	scalar	Nominal interest rate, strictly positive.

§ 3 Equilibrium Conditions

Our setup is similar to Salvadori (1986), Bidard (1991) and Lager (2001). For given demand $\boldsymbol{d}$, carrying capacity $\boldsymbol{s}$ and interest rate r, we wish to find nonnegative prices $\boldsymbol{p}$, emission charges $\boldsymbol{y}$, wages w and activity levels $\boldsymbol{x}$ that satisfy Conditions 3.1–3.5 below.

3.1 Condition [Physical Feasibility] *Under activity levels* $\mathbf{x}$*, no commodity or carrying capacity must be in excess demand.*

$$\boldsymbol{Bx} \geq \boldsymbol{Ax} + \boldsymbol{d}, \tag{1}$$

$$[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x} \leq \boldsymbol{s} + \boldsymbol{d}. \tag{2}$$

Commodities m or carrying capacities q for which the weak inequality in 3.1 is an *equality* are not in excess supply and are called *scarce.*

3.2 Condition [No Excess Profits] *Under commodity prices* $\boldsymbol{p}$ *and emission charges* $\boldsymbol{y}$*, no process must make profits in excess of the interest rate.*

$$\boldsymbol{pB} \leq (1 + r)\boldsymbol{A} + \boldsymbol{y}[\boldsymbol{B} - \boldsymbol{A}] + w\boldsymbol{\ell}. \tag{1}$$

Processes n for which the weak inequality in 3.2 is an *equality* are not loss making and are called *profitable.*

3.3 Condition [Rule of Free Goods] *Every commodity m or carrying capacity q that is in excess supply under activities* $\mathbf{x}$ *must be free under prices* $\boldsymbol{p}$ *and emission charges* $\boldsymbol{y}$.

$$\boldsymbol{B}_{[m]}\boldsymbol{x} > \boldsymbol{A}_{[m]}\boldsymbol{x} + \boldsymbol{d}_{[m]} \quad \Rightarrow \quad \boldsymbol{p}^{[m]} = 0. \tag{1}$$

$$\boldsymbol{s}_{[q]} + \boldsymbol{d}_{[q]} > [\boldsymbol{B} - \boldsymbol{A}]_{[q]}\boldsymbol{x} \quad \Rightarrow \quad \boldsymbol{y}^{[q]} = 0. \tag{2}$$

By 3.3, commodities m and carrying capacities q are either *priced* (if scarce) or *free* (if in excess supply).

3.4 Condition [Rule of Profitability] *Every process n that makes losses under commodity prices* $\boldsymbol{p}$ *and emission charges* $\boldsymbol{y}$ *must be idle under actvities* $\boldsymbol{x}$*.*

$$\boldsymbol{pB}^{[n]} < \boldsymbol{pA}^{[n]}(1+r) + \boldsymbol{y}[\boldsymbol{B}-\boldsymbol{A}]^{[n]} + w\boldsymbol{\ell}^{[n]} \quad \Rightarrow \quad \boldsymbol{x}_{[n]} = 0. \tag{1}$$

By 3.4, processes n are are either *active* (if profitable) or *idle* (if loss making).

3.5 Normalisation *All prices and emission charges are in terms of labour:* $w = 1$.

3.6 Definition [Cost-Minimizing Equilibrium] A triple $(\boldsymbol{p}, \boldsymbol{y}, \boldsymbol{x})$ satisfying 3.1–3.5 is called a *Cost-Minimizing Equilibrium* (for the given final demand $\boldsymbol{d}$, carrying capacity $\boldsymbol{s}$ and interest rate r).

By standard complementarity arguments (exploiting the nonnegativity of all choice variables), conditions 3.3–3.4 can be replaced by the following two *balance conditions*:

3.7 Balance Condition [Output equals Demand] *The value of output* $\boldsymbol{pBx}$ *equals the sum of capital replacement* $\boldsymbol{pAx}$*, and final demand* $\boldsymbol{pd}$*. Similarly, the value of the available carrying capacity* $\boldsymbol{ys}$ *equals the value of the carrying capacity that is used by emissions,* $\boldsymbol{y}[\boldsymbol{B}\text{-}\boldsymbol{A}]\boldsymbol{x}$*, net of final demand for emissions,* $\boldsymbol{yd}$*.*

$$\boldsymbol{pBx} = \boldsymbol{pAx} + \boldsymbol{pd}, \tag{1}$$

$$\boldsymbol{ys} + \boldsymbol{yd} = \boldsymbol{y}[\boldsymbol{B}-\boldsymbol{A}]\boldsymbol{x}. \tag{2}$$

3.8 Balance Condition [Output equals Income] *The value of output* $\boldsymbol{pBx}$ *equals the sum of capital replacement* $\boldsymbol{pAx}$*, profits* $r\boldsymbol{pAx}$*, emission charges* $\boldsymbol{y}[\boldsymbol{B}\text{-}\boldsymbol{A}]$ *and wage bill* $w\boldsymbol{\ell x}$*.*

$$\boldsymbol{pBx} = (1+r)\boldsymbol{pAx} + \boldsymbol{y}[\boldsymbol{B}-\boldsymbol{A}]\boldsymbol{x} + w\boldsymbol{\ell x}. \tag{1}$$

3.9 Observation [Symmetric Duality if r=0] In the special case where the interest rate r is set to zero in 3.2 and 3.4, system 3.1–3.4 reduces to a dual pair of linear programmes, and we can solve *separately* for prices and quantities. For quantities, we merely need to solve

$$\min_x \boldsymbol{\ell x} \quad \text{s.t. } 3.1, \tag{P}$$

(disregarding prices $\boldsymbol{p}$ and charges $\boldsymbol{y}$), and for prices, we merely solve

$$\max_{p,y} \boldsymbol{pd} - \boldsymbol{ys} \quad \text{s.t. } 3.2 \text{ (with } r = 0), \tag{D}$$

(disregarding quantities $\boldsymbol{x}$). Solutions $\boldsymbol{x}^*$ from (P) and $(\boldsymbol{p}^*, \boldsymbol{y}^*)$ from (D) will then also solve system 3.1–3.4. By the Duality Theorem of Linear Programming, programs (P)/(D) will admit optimal solutions if the quantity constraint 3.1 and the price constraint 3.2 are both feasible. Since 3.2 is trivially feasible by setting all prices and rents to zero, in the special case of zero interest rates the mere feasibility of the quantity constraint 3.1 is a sufficient condition for the existence of a price-quantity equilibrium solution to system 3.1–3.4.

3.10 Observation [Asymmetric Duality if r>0] Under *positive* interest rates ($r > 0$), the price constraint (3.2) is no longer symmetrically dual to the quantity constraint (3.1) and the symmetry between prices and quantities is *broken;* prices and quantities then can no longer be found *separately* but need to be determined *jointly.* As a consequence of the asymmetry between (3.1) and (3.2), under $r > 0$ the Duality Theorem of Linear Programming no longer applies and mere feasibility of quantity constraint (3.1) is no longer a sufficient condition for a price-quantity equilibrium solution to 3.1–3.4. Instead, we need to find a *Fixed Point* where prices and quantities support each other in a Nash-like fashion: Prices assign value only to those goods that are scarce under the given activity levels, and processes are active only if they are profitable under the given prices. For a classic discussion of such asymmetric duality relationships in the context of the von Neumann model, see Los (1976).

§ 4 Numerical Examples

To simplify our examples, we distinguish between ordinary commodities that do not require carrying capacity, on the one hand, and pure polluting emissions that are never useful in production or consumption.

Matrix $\boldsymbol{C}$ denotes the emission matrix of these pure pollutants.

4.1 Example [Existence: 1 Commodity, 1 Pollutant, r=0] [Based on Bidard (1987).] Consider a technology with one commodity and one type of pollutant. There are *two* processes; $\boldsymbol{A} = (2, 1)$, $\boldsymbol{C} = (2, 3)$, $\boldsymbol{B} = (6, 6)$, $\boldsymbol{\ell} = (2, 2)$. Carrying capacity is $\boldsymbol{s} = 12$. Final demand is $\boldsymbol{d} = 22$. The interest rate is zero, $r = 0$. Condition 3.1 admits feasible solutions: say, $x_1 = 6$, $x_2 = 0$ or $x_1 = 3$, $x_2 = 2$. Hence by Observation 3.9 an equilibrium to 3.1–3.4 must exist. The solution to program (D) is $p^* = 1$, $y^* = 1$, and the solution to program (P) is $x_1^* = 3$, $x_2^* = 2$, hence $p = 4$, $y = 1$, $x_1 = 3$, $x_2 = 12$, is a Cost-Minimizing Equilibrium.

4.2 Example [Non-existence: 1 Commodity, 1 Pollutant, r>0] As Example 4.1, but with an interest rate of $r = 1$ or 100%. Running process 1 on its own satisfies condition 3.1; if $x_1 = 5\,{}^1\!/_2$, $x_2 = 0$, production $[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x}$ exactly meets final demand $\boldsymbol{d}$ and carrying capacity $\boldsymbol{s}$ are not fully utilised by pollutants $\boldsymbol{Cx}$. Prices $p = 1$ and emission charges $y = 0$ would be consistent with condition 3.3 and would make process 1 break even. But under these prices and rents, process 2 would make an extra-profit of 2, thus violating condition 3.2. Running process 2 on its own violates condition 3.1, since process 2 uses too much carrying capacity for the given final demand. Thus, neither process 1 nor process 2 can be run on their own. The only remaining configuaration is to run process 2 *jointly* with process 1. To make both processes break even at the same time, prices and charges would need to be $p = -1$, $y = -2$, which would violate the nonnegativity condition. Thus, no equilibrium exists for this economy. To allow for an equilibrium solution, either interest rate r has to fall (leading to lower commodity prices that don't give extra-profits to process 2 if process 1 breaks even), or demand $\boldsymbol{d}$ has to fall (making it possible to run process 2 on its own). At the given interest rate and the given final demand, technology does not admit a consistent price-quantity solution.

4.3 Example [Ensuring existence by removing a process] As Example 4.2, but *eliminate* process 2 from technology: $\boldsymbol{A} = (2)$, $\boldsymbol{C} = (2)$, $\boldsymbol{B} = (6)$, $\boldsymbol{\ell} = (2)$. As before, $\boldsymbol{s} = 12$, $\boldsymbol{d} = 22$, $r = 1$. In this reduced technology, $x_1 = 5\,{}^1\!/_2$, $p = 1$ and $y = 0$ are a Cost-Minimizing Equilibrium.

4.4 Example [Ensuring existence by adding a process] As Example 4.2, but this time *add* an additional third process: $\boldsymbol{A} = (2, 1, 1)$, $\boldsymbol{C} = (2, 3, 2\,{}^1\!/_2)$, $\boldsymbol{B} = (6, 6, 6)$, $\boldsymbol{\ell} = (2, 2, 4)$. As before, $\boldsymbol{s} = 12$, $\boldsymbol{d} = 22$, $r = 1$. In this extended technology, $p = 3\,{}^1\!/_2$, $y = 4$, $x_1 = 0$, $x_2 = 2$, $x_3 = 2\,{}^2\!/_5$ is a Cost-Minimizing Equilibrium.

We see that in these simple examples, non-existence of a Cost-Minimizing Equilibrium is due to an interest rate that makes the available production processes *inconsistent* with each other (Example 4.2). Such inconsistency can be remedied, either by *removing* one of the clashing processes (Example 4.3) and achieving an equilibrium in which carrying capacity is underutilised, or by *adding* a further process that allows for the full utilisation of carrying capacity. The next two examples show similar features for slightly more complex cases.

4.5 Example [3 Commodities, 1 Pollutant] [d'Agata (1983)] Consider a technology with three types of commodities and one type of pollutant. There are five processes;

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 4 & 1 & 1 \\ 1 & 6 & 1 & 4 & 3 \end{pmatrix}, \ \boldsymbol{B} = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 10 & 10 \end{pmatrix}, \ \boldsymbol{d} = \begin{pmatrix} 900 \\ 600 \\ 190 \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} 0 & 0 & 10 & 10 & 10 \end{pmatrix}, \ \boldsymbol{\ell} = \begin{pmatrix} 10 & 10 & 10 & 10 & 22 \end{pmatrix}, \ \boldsymbol{s} = \begin{pmatrix} 1000 \end{pmatrix}.$$

The interest rate is $1/2$, $r = 50\%$. Inspection shows that the subsystems comprising only processes $\{1,2,3\}$, $\{1,2,3,4\}$ or $\{1,2,3,5\}$ each admit an equilibrium solution, but the entire system of processes $\{1,2,3,4,5\}$ does not. For example, for subsystem $\{1,2,3,4\}$, the equilibrium solution is: $x_1 = 100$, $x_2 = 97\,1/2$, $x_3 = 91\,2/3$, $x_5 = 8\,1/3$, $p_1 = 2\,1/2$, $p_2 = 10$, $p_3 = 10$, $y = 1\,1/8$. But under these prices and emission charges, process 5 would yield an extra-profit of $27/8$. Similarly, under subsystem $\{1,2,3,5\}$, process 4 would yield an extra profit. Thus as a whole, under emission constraint $\boldsymbol{s}$ the system $(\boldsymbol{A}, \boldsymbol{C}, \boldsymbol{\ell}, \boldsymbol{B})$ does *not* admit an equilibrium for the given demand $\boldsymbol{d}$ and interest rate r, but if certain processes are eliminated from the book-of-blueprints, the resulting subsystem *does* admit an equilibrium.

4.6 Example [2 commodities, 2 pollutants] [d'Agata (1984).] Consider a technology with two types of commodities and two types of pollutant. There are four processes;

$$\boldsymbol{A} = \begin{pmatrix} 3 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \end{pmatrix}, \ \boldsymbol{B} = \begin{pmatrix} 10 & 10 & 0 & 0 \\ 0 & 0 & 10 & 10 \end{pmatrix}, \ \boldsymbol{d} = \begin{pmatrix} 150 \\ 350 \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} 10 & 0 & 20 & 0 \\ 0 & 50 & 0 & 10 \end{pmatrix}, \ \boldsymbol{\ell} = \begin{pmatrix} 10 & 15 & 1 & 5 \end{pmatrix}, \ \boldsymbol{s} = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}.$$

The interest rate is $1\,5/8$, $r = 162.5\%$. The situation is similar to Example 4.5. Inspection shows that the subsystem comprising only processes $\{1,2,4\}$, admits an equilibrium solution, but the entire system of processes $\{1,2,3,4\}$ does not. For subsystem $\{1,2,4\}$, the equilibrium solution is: $x_1 = 20\,10/317$, $x_2 = 11\,13/317$, $x_4 = 44\,252/317$, $p_1 = 15\,1/5$, $p_2 = 11\,1/5$, $y_1 = 0$, $y_2 = 3/10$. But under these prices and rents, process 3 would yield an extra-profit of 17 Similarly for other feasible subsystems. Thus, as in 3.14, the system $(\boldsymbol{A}, \boldsymbol{C}, \boldsymbol{\ell}, \boldsymbol{B})$ does *not* admit an equilibrium for the given $\boldsymbol{s}$, $\boldsymbol{d}$ and r, but if certain processes are eliminated from the book-of-blueprints, the resulting subsystem *does* admit an equilibrium.

Counter-examples 4.2, 4.5, 4.6 highlight the non-trivial nature of the existence issue for equilibrium problem 3.1–3.4. In these counter-examples, non-existence is the result of a fundamental *inconsistency* between the different processes that are available to cost-minimizing producers. Such inconsistencies could never arise if the price system was symmetrically dual to the quantity system (Observation 3.9 and Example 4.1). Because the money interest rate introduces a fundamental *asymmetry* into price-quantity duality (Observation 3.10), even an economy that contains a viable sub-system capable of an equilibrium path may well be unable to support such a path. Intuitively, the interest rate charged on all capital investment "distorts" the physical productivity relationships between processes and sometimes makes processes profitable in price space that should not become active in quantity space.

What are the deeper sources of such inconsistencies between processes? Under what circumstances will the intervention of the interest rate *not* allow for "the wrong" processes to become profitable? This question has been waiting for a comprehensive answer ever since counter-examples like 4.5 and 4.6 were discovered some two decades ago. We shall assess the scope of various existence theorems on land rents (and hence emission charges) in the light of their ability to explain these classic counter-examples.

§ 5 Skew-Dual LCPs

From a programming viewpoint, system 3.1–3.5 forms a *Linear Complementarity Problem*; see Lemke (1968) for an incisive classic article and Cottle/Pang/Stone (1992) for a deep and comprehensive survey. Our existence result will be based on Dantzig and Manne's (1974) far-reaching Complementarity Construction Theorem, which we re-state here in the context of our model.

5.1 Complementarity Problem [LCP] Consider two given constraint matrices $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ of the same order (say, $m \times m$), with associated constraint vectors $\boldsymbol{q}_1$ $(m \times 1)$ and $\boldsymbol{q}_2$ $(n \times 1)$ and slack vectors $\boldsymbol{w}_1$ $(m \times 1)$ and $\boldsymbol{w}_2$ $(n \times 1)$. Find nonnegative choice variables $\boldsymbol{z}_1$ $(n \times 1)$ and $\boldsymbol{z}_2$ $(m \times 1)$ that satisfy S1–S3:

$$\begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{O} & \boldsymbol{M}_1 \\ -\boldsymbol{M}_2^{\mathrm{T}} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{bmatrix}, \tag{S1}$$

$$\boldsymbol{w} \geq \boldsymbol{O}, \tag{S2}$$

$$\boldsymbol{w}^{\mathrm{T}}\boldsymbol{z} = 0. \tag{S3}$$

A more general version of the Dantzig/Manne result allows for quadratic valuations, by allowing for the two zero submatrices on the main diagonal of $\boldsymbol{M}$ to become *nonzero;* see Jones (1982).

5.2 Lemke Algorithm Consider constraint (S1) from problem 5.1, and extend it to (L1) by a covering vector $\boldsymbol{c}$ of order $(m+n) \times 1$:

$$\boldsymbol{w} = \boldsymbol{q} + \boldsymbol{M}\boldsymbol{z} + \boldsymbol{c}\theta, \tag{L1}$$

where entry $\boldsymbol{c}_{[i]}$ of covering vector $\boldsymbol{c}$ is strictly positive. Scalar θ is required to be nonnegative. The set of nonnegative triples $(\boldsymbol{z}, \boldsymbol{w}, \theta)$ saisfying (L1) forms an unbounded convex polyhedron (the *feasible set* of Problem 5.1). Any point in the feasible set that satisfies complementarity condition (S3) is called an *almost-complementary* solution (a.c.) to Problem 5.1. If furthermore $\theta = 0$, then $(\boldsymbol{z}, \boldsymbol{w}, \theta)$ is a *complementary* solution to Problem 5.1. The basic a.c. solutions are vertices of the feasible set, and every such vertex has either one or two neighbouring a.c. vertices. The Lemke algorithm starts by setting $\theta = 1$; this identifies a convenient initial a.c. vertex:

$$\boldsymbol{z} = 0, \quad \boldsymbol{w} = \boldsymbol{q} + \boldsymbol{c} \geq \boldsymbol{O}.$$

From this initial vertex, the algorithm then proceeds by pivoting along a uniquely identified path of neighbouring a.c. vertices (by changing the covering variable θ), until it reaches a terminal a.c. vertex with *no* second neighbouring a.c. vertex. At this terminal a.c. vertex, the covering variable θ is either zero or positive. If $\theta = 0$, the vertex represents a *complementary* solution and the problem is solved. By contrast, if $\theta > 0$, the algorithm has not succeeded in finding a complementary solution (termination in a ray). Linear Complementarity Theory is concerned with finding conditions under which one can be sure that the termination in a ray will not happen.

5.3 Definition [Skew Duality] Suppose that in S1, constraint matrices $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ satisfy the condition

$$[\boldsymbol{M}_1 - \boldsymbol{M}_2] \geq \boldsymbol{O}; \tag{S4}$$

we then say that Problem S1–S3 is a *skew-dual* Linear Complentarity Problem.

5.4 Observation [3.1–3.4 is a Skew Dual LCP] Inspection of 3.1–3.4 shows that it satisfies S1–S4 from 4.1–4.2. Set

$$\boldsymbol{q}_1 = \begin{bmatrix} -\boldsymbol{d} \\ \boldsymbol{s} + \boldsymbol{d} \end{bmatrix}, \quad \boldsymbol{q}_2 = \boldsymbol{\ell}^{\mathrm{T}}, \quad \boldsymbol{z}_1 = \begin{bmatrix} \boldsymbol{p}^{\mathrm{T}} \\ \boldsymbol{y}^{\mathrm{T}} \end{bmatrix}, \ \boldsymbol{z}_2 = \boldsymbol{x},$$

$$\boldsymbol{M}_1 = \begin{bmatrix} [\boldsymbol{B} - \boldsymbol{A}] \\ -[\boldsymbol{B} - \boldsymbol{A}] \end{bmatrix}, \quad \boldsymbol{M}_2 = \begin{bmatrix} [\boldsymbol{B} - (1+r)\boldsymbol{A}] \\ -[\boldsymbol{B} - \boldsymbol{A}] \end{bmatrix}.$$

We then have:

$$[\boldsymbol{M}_1 - \boldsymbol{M}_2] = \begin{bmatrix} r\boldsymbol{A} \\ \boldsymbol{O} \end{bmatrix}.$$

Since $\boldsymbol{A}$ is nonnegative and r is positive, this confirms that $[\boldsymbol{M}_1 - \boldsymbol{M}_2] \geq \boldsymbol{O}$, as required by condition S4.

Intuitively, positive values of θ represent proportional reductions in final demand requirements $\boldsymbol{d}$ and emission targets $\boldsymbol{s}$. The sequence of almost-complementary solutions produced by the Lemke Algorithm can be interpreted as the equilibrium solutions to a sequence of economies with identical technology but with variously reduced final demand and emission requirements. In the terminall complementary solution, this demand reduction has vanished and final demand has returned to its true level.

Performing the Lemke pivoting process on this sequence of demand-reduced almost-complementary economies has some resemblance with the "market algorithm" discussed in Bidard's (1990) algorithmic approach to the choice-of-technique problem.

5.5 Cross-Dual Constraints Consider the skew-dual LCP S1–S4 from 5.1–5.2. We define a dual set of constraints, obtained by exchanging the roles of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$: Find nonegative $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ that satisfy

$$\begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{O} & \boldsymbol{M}_2 \\ -\boldsymbol{M}_1^{\mathrm{T}} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{bmatrix}, \tag{D1}$$

$$\boldsymbol{w} \geq \boldsymbol{O}. \tag{D2}$$

5.6 Theorem [Dantzig/Manne 1974] *Consider a skew-dual LCP. If S1–S2 from 5.1 and D1–D2 from 5.4 each admit a nonnegative solution, then there exist a complementary solution satisfying S1–S3. Moreover, this solution can effectively be computed by the Lemke Complementary Pivoting Algorithm.* ▪

In the cost-minimizing equilibrium 3.1–3.4, the cross-dual constraints D1–D2 obtain by treating the quantity constraint 3.1 as a price constraint and treating the price constraint 3.2 as a quantity constraint. It is instructive to write out the details of these cross-dual constraints:

5.7 Cross-Dual Quantity Condition *Even if all input requirements $\boldsymbol{A}$ are increased by a uniform expansion factor $(1+r)$, it must be possible to find a feasible nonnegative activity vector $\boldsymbol{x}$ that satisfies final demand without violating the land constraint:*

$$\boldsymbol{Bx} \geq (1 + r)\boldsymbol{Ax} + \boldsymbol{d}, \tag{1}$$

$$\boldsymbol{s} + \boldsymbol{d} \geq [\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x}. \tag{2}$$

5.8 Cross-Dual Price Condition *Even if the interest rate is zero, it must be possible to find a nonnegative price vectors $\boldsymbol{p}$ and $\boldsymbol{y}$ that give no excess profits to any process:*

$$\boldsymbol{pB} \leq \boldsymbol{pA}+\boldsymbol{y}[\boldsymbol{B} - \boldsymbol{A}] + w\boldsymbol{\ell}. \tag{1}$$

Thus, with regards to problem 3.1–3.4, the Dantzig/Manne Theorem 5.6 can be re-stated:

5.9 Theorem [Dantzig/Manne Restated] *If 3.1 and 5.6 each admit a feasible nonnegative quantity solution $\boldsymbol{x}$, and if 3.2 and 5.7 each admit a feasible nonnegative price solution $(\boldsymbol{p}, \boldsymbol{y})$, then system 3.1–3.4 admits a complementary price-quantity equilibrium solution $(\boldsymbol{p}, \boldsymbol{y}, \boldsymbol{x})$; moreover, this equilibrium solution can effectively be computed by the Lemke Complementary Pivoting Algorithm.* ▪

Theorem 4.8 gives a *sufficient,* not a necessary condition for equilibrium. Failure to meet the requirements of the Theorem does *not* imply nonexistence of equilibrium.

Recall from Observation 3.10 that in problem 3.1–3.4, the symmetric duality relationship between prices and quantities is *broken,* resulting in the need to solve for prices and quantities simultaneously. The skew-duality condition S4 from 5.3 requires that the asymmetry between prices and quantities is such that the price constraint becomes *weaker* (input costs are augmented by the interest rate). Conditions 5.7–5.8 then require that the asymmetry between prices and quantities is "not too strong" — the quantity matrix $[\boldsymbol{B} - \boldsymbol{A}]$ can still serve as a price constraint (5.7), and the price matrix $[\boldsymbol{B} - (1 + r)\boldsymbol{A}]$ can still serve as a quantity constraint (5.8). Theorem 5.9 asserts that a joint price-quantity solution can be found, provided that the asymmetry between prices and quantities is kept in check by these cross-dual conditions.

5.10 Observation [Trivial Solution to 3.2/4.7] *Because labour inputs $\boldsymbol{\ell}$ are positive, 3.2 and 5.8 always admit a feasible solution: $\boldsymbol{p} = \boldsymbol{O}$, $\boldsymbol{y} = \boldsymbol{O}$.*

5.11 Observation [5.7 implies 3.1] *If 5.7 admits a feasible solution, then so does 3.1.*

In the light of these observations, we only have to worry about condition 5.7. Because 5.7 requires the possibility of a uniform expansion of all processes $\boldsymbol{x}$ (at expansion rate r), it potentially clashes with the emission constraint. Only for very low values of interest rate r or low values of final demand will this condition be feasible.

5.12 Example [4.3 revisited] In Example 4.3, with $r = 1$, process 1 has a net product of $6 - 1 = 5$ ssociated with 6 units of gross output. The process requires $2/6 = {}^1/_3$ units of land per unit of gross output. There are 12 units of lands available. Thus, the process could produce up to $12/\,{}^1/_3 = 36$ units of gross output. 36 units of gross output become $36 \times {}^5/_6{=}30$ units of net output, and equilibrium would be possible for final demand of up to 30 units. Howver, the cross-dual feasibility constraint requires that balanced growth at rate r be feasible. Under $r = 1$, process 1 a balanced-growth r-net product of $6 - (1 + r)1 = 4$, compared with its net product of $6 - 1 = 5$. Thus, under balanced growth at rate r, the process could produce only a net product of $36 \times {}^4/_6 = 24$ units. The Dantzig/Manne Theorem correctly predicts equilibria for demand below 24 units, but is unable to predicts to predict the equilibria between 24 and 30 units.

§ 6 Previous Existence Results

Salvadori (1986) was the first to apply the Dantzig/Manne theorem to the land rent problem; see also Kurz/Salvadori (1995). Lager (2002) has applied the same result to permit markets. Salvadori's approach was simply to *posit* that 4.6 is feasible.

6.1 Theorem [Salvadori 1986] *Assume that Condition 5.7 is feasible. Then there exists an equilibrium solution to system 3.1–3.4.*

Proof By Observations 5.10–5.11. ∎

Salvadori's direct application of the Dantzig/Manne result to problem 3.1–3.4 is effective but very limited in scope. As illustrated in Example 5.12, for reasonsably large interest rates r, the cross-dual quantity constraint 4.6 will only be feasible if final demand $\boldsymbol{d}$ is small. Thus, for 3.1–3.4 to satisfy the conditions of Theorem 6.1, either the interest rate has to be much smaller than the productivity of $[\boldsymbol{B} - \boldsymbol{A}]$ suggests, or the level of final demand has to be much smaller than is physically feasible. This restrictiveness of condition 5.7 is clearly undesirable. As Salvadori pointed out in his paper, there are many relevant cases where an equilibrium is known to exist and yet 5.7 is *not* feasible.

§ 7 Substitution Potential

In Section 4, we encountered cases where a positive interest rate created cost-inconsistencies between processes that made an economic price-quantity equilibrium impossible. Under what circumstances would the intervention of the interest rate remain compatible with equilibrium? Salvadori's Theorem 6.1 gives a partial answer, by identifiying a strong physical feasibility requirement that serves as a sufficient condition for existence (and whose violation therefore serves as a necessary condition for non-existence). However, being based on a physical *feasibility* requirement, Salvadori's result fails to address one of the key features of the counter-examples from Section 4: the fact that *expanding* the book-of-blueprints (and hence expanding the range of feasible production plans) may well *destroy* the possibility of an equilibrium solution, rather than confirming it.

In order to explore the deeper sources of equilibrium, we need to study the precise *inter-relationships* between the various individual processes, rather than merely checking for global feasibility. This is what we do in the present section. We impose a substitution requirement on the range of possible *relationships* between alternative feasible production plans, and we later show that this requirement ensures equilibrium. Cases of non-existence of equilibrium, like Examples 4.4–4.5, must then necessarily *violate* this substitution requirement; if "the wrong" processes have become profitable under the intervention of the interest rate, then it must be the case that there is a lack of "Input Flexibility" between the various alternative production plans.

More precisely, our condition requires that whenever a commodity or land-type is scarce under current activities, it must be possible to provide the same net outputs with slightly lower levels of non-labour inputs (but presumably with higher levels of labour inputs).

7.1 Assumption [Input Flexibility] *Consider an activity vector* $\boldsymbol{x}$ *under which there would be at least one scarce commodity in 3.1(1) and at least one carrying capacity that is in short supply or excess demand in 3.1.(2):*

$$\exists m : [\boldsymbol{B} - \boldsymbol{A}]_{[m]}\boldsymbol{x} = \boldsymbol{d}_{[m]}, \quad \exists q : \boldsymbol{s}_{[q]} + \boldsymbol{d}_{[q]} \leq [\boldsymbol{B} - \boldsymbol{A}]_{[q]}\boldsymbol{x}.$$

Then we can find an activity vector $\boldsymbol{x}'$ *that satisfies:*

$$[\boldsymbol{B} - \boldsymbol{A}]_{[m]}\boldsymbol{x}' \geq \boldsymbol{d}_{[m]} \qquad \forall m \text{ for which } [\boldsymbol{B} - \boldsymbol{A}]_{[m]}\boldsymbol{x} = \boldsymbol{d}_{[m]} \tag{1}$$

$$[\boldsymbol{B} - \boldsymbol{A}]_{[q]}\boldsymbol{x} > [\boldsymbol{B} - \boldsymbol{A}]_{[q]}\boldsymbol{x}', \qquad \forall q \text{ for which } [\boldsymbol{B} - \boldsymbol{A}]_{[q]}\boldsymbol{x} \geq \boldsymbol{s}_{[q]} + \boldsymbol{d}_{[q]} \tag{2}$$

$$\boldsymbol{A}_{[m]}\boldsymbol{x} \geq \boldsymbol{A}_{[m]}\boldsymbol{x}' \qquad \forall m \text{ for which } \boldsymbol{A}_{[m]}\boldsymbol{x} > 0. \tag{3}$$

That is, under activities $\boldsymbol{x}'$ *final demand is met for all scarce commodities, there are strictly smaller emissions for all scarce carrying capacities, and the use of scarce capital stocks is not larger than under activities* $\boldsymbol{x}$*.*

Intuitively, vector $\boldsymbol{x}'$ saves on emissions and capital inputs, while possibly using larger amounts of labour.

The reader will recognise a distant family relationship between the statement of our flexibility condition 7.1 and the statement of the familiar non-tightness condition from Malinvaud-type multisectoral theory, as in Kurz (1969). Nontightness requires that the use of nonproducible inputs can be reduced by increasing the use of producible inputs; flexibility condition 6.1 requires that the use of scarce inputs (capital goods and carrying capacity) can be reduced by increasing the use of non-scarce inputs (labour).

7.2 Assumption [Balanced Growth] *In the absence of emission constraint, balanced growth at rate r covering* $\boldsymbol{d}$ *would be feasible. That is, there exists a nonnegative activity vector* $\boldsymbol{x}$ *satisfying:*

$$[\boldsymbol{B} - (1 + r)\boldsymbol{A}]\boldsymbol{x} \geq \boldsymbol{d}.$$

7.3 Observation [Convexity] *The set of final demand vectors* $\boldsymbol{d}$ *for which conditions 7.1 and 7.2 admit a solution is nonempty and convex and contains the origin.*

7.4 Example [4.2 revisited] Inspection of Example 4.2 shows that condition 6.1(2) will be violated if $\boldsymbol{x}$ is given by $x_1 = 3$, $x_2 = 2$. In this case, land-use could not be reduced without increasing capital use.

7.5 Example [4.4 revisited] Inspection of Example 4.4 shows that the introduction of the third process removes the lack of input flexibility from Example 4.2. By adding a process that uses less capital but more labour than process 2, we allow for a reduction in land use (compared with $\boldsymbol{x}$ $x_1 = 3$, $x_2 = 2$), without increasing the use of capital goods. (say, $\mathbf{x}$pp $x_1 = 3$, $x_2 = 0, x_3 = 2$).

7.6 Example [4.3 revisited] Alternatively, in Example 4.3, removing process 2 also removes the lack of input flexibility, simply by removing the scarcity of land (under the given final demand).

§ 8 Existence Theorem

We are now able to state our main result. If there is input flexibility between land and labour, the restrictions of Salvadori's Theorem 6.1 can be removed:

8.1 Theorem [Equilibrium for Given Demand] *Let final demand $\boldsymbol{d}$, carrying capacity $\mathbf{s}$ and interest rate r be given. Suppose Assumptions 7.1–7.2 on the set of feasible activity vectors $\mathbf{x}$ are satisfied for the given $\boldsymbol{d}$, $\mathbf{s}$ and r. Then there exists a price-quantity equilibrium solution $(\mathbf{x}, \boldsymbol{p}, \boldsymbol{y})$ to 3.1–3.5; this solution can be computed by applying the Lemke Algorithm to a related "Extended System" (described in Section 9).*

8.2 Corollary [Existence of Equilibrium] *Let interest rate r be given. An economy in which balanced growth at rate r would be feasible if carrying capacity was abundant will admit a cost-minimizing equilibrium for some final demand vector $\boldsymbol{d}$.*

Proof Theorem 8.1 is proved in Section 10 below. Corollary 8.2 follows from Theorem 8.1 and Observation 7.3. ■

§ 9 The Extended System

To prove our Theorem from Section 8, we need to work in an extended version of the original Linear Complementarity Problem 3.1-3.4, the so-called Extended System. The present section sets up the Extended System; Section 10 then does the proof of Theorem 8.1.

We wish to extend the scope of Dantzig and Manne's Theorem from Section 5 to a wider range of economies than those covered by Salvadori's Theorem 6.1. Recall that the Cross-Dual Feasibility Conditions 5.7–5.8 are merely a *sufficient,* not a necessary condition for a solution to 3.1–

3.4. From an algorithmic point of view, Conditons 5.7–5.8 ensure that the pivoting sequence of the Lemke algorithm can proceed through the entire sequence of almost-complementary solutions without terminating in a ray. We shall show that an unblocked progression of the Lemke pivoting sequence can be ensured by embedding the original problem 3.1–3.4 in a wider problem, called the *Extended System*. By construction, the Extended System always satisfies the extended versions of conditions 5.7–5.8, and thus it permits an equilibrium solution. Our existence theorem then shows that under the substitution conditions from Section 7, the equilibrium solution to the Extended System will also be an equilibrium solution to the original system 3.1–3.4.

The Extended System relaxes the emission constraint by adding a new "pure abatement process" to technology $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{\ell})$. The new process has *no* commodity inputs and produces *no* commodity outputs; its only function is to absorb emissions in proportion to carrying capacity $\boldsymbol{s}$. This artificial process has very high labour inputs. We stress that this new process is entirely artificial. If conditions 7.1–7.2 are met, then any equilibrium solution to the Extended System will discard the artificial process as too expensive and will employ only the processes available in the original system 3.1–3.4.

Extended System *In the statement of the Dantzig-Manne theorem, extend system 3.1–3.4 by an additional fictitious process, labelled 0. This process requires no commodity inputs and produces no commodity outputs, but at the end of the period it absorbs inputs in proportion to carrying capacity* $\boldsymbol{s}$*. (In other words, it creates carring capacity.) Process zero has finite but arbitrarily large labour requirements, L. We denote the activity level of Process 0 by* γ*. Thus, for system S1–S3 from 4.1, we define:*

$$\boldsymbol{q}_1 = \begin{bmatrix} -\boldsymbol{d} \\ \boldsymbol{s}+\boldsymbol{d} \end{bmatrix}, \quad \boldsymbol{q}_2 = \begin{bmatrix} \boldsymbol{\ell}^{\mathrm{T}} \\ L \end{bmatrix}, \quad \boldsymbol{z}_1 = \begin{bmatrix} \boldsymbol{p}^{\mathrm{T}} \\ \boldsymbol{y}^{\mathrm{T}} \end{bmatrix}, \; \boldsymbol{z}_2 = \begin{bmatrix} \boldsymbol{x} \\ \gamma \end{bmatrix},$$

$$\boldsymbol{M}_1 = \begin{bmatrix} [\boldsymbol{B}-\boldsymbol{A}] & \boldsymbol{O} \\ -[\boldsymbol{B}-\boldsymbol{A}] & \boldsymbol{s} \end{bmatrix}, \quad \boldsymbol{M}_2 = \begin{bmatrix} [\boldsymbol{B}-(1+r)\boldsymbol{A}] & \boldsymbol{O} \\ -[\boldsymbol{B}-\boldsymbol{A}] & \boldsymbol{s} \end{bmatrix}.$$

Linear Complementarity Theory traditionally uses "augmented systems" that are quite similar to our Extended System; see Section 3.7 in Cottle/Pang/Stone (1992) for a range of relevant results. The traditional aim of such extensions is to find a convenient *initial vertex* for a pivoting sequence. Our aim is different; we do not need to *start* the pivoting sequence (which is already ensured by setting θ to unity in the Lemke algorithm), but we might need to prevent termination in a ray at the intermediate stages. These different aims are reflected in different extensions. In traditional "augmented systems", the artificial process covers the *negative* part of constraint vector $\boldsymbol{q}_1$ (corresponding to final demand $\boldsymbol{d}$ in our system), whereas in our Extended System, the artificial process covers the *positive* part of $\boldsymbol{q}_1$ (corresponding to land stocks $\boldsymbol{s}$).

In more explicit form, the Extended System requires that we augment 3.1–3.4 to:

9.1 Condition [Quantities]

$$\boldsymbol{Bx} \geq \boldsymbol{Ax} + \boldsymbol{d}. \tag{1}$$

$$\boldsymbol{s} + \boldsymbol{s}\gamma + \boldsymbol{d} \geq [\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x}. \tag{2}$$

9.2 Condition [Prices]

$$\boldsymbol{pB} \leq \boldsymbol{pA}(1 + r) + \boldsymbol{y}[\boldsymbol{B} - \boldsymbol{A}] + w\boldsymbol{\ell}. \tag{1}$$

$$\boldsymbol{ys} \leq L. \tag{2}$$

9.3 Condition [Complementarity on Quantities]

$$\boldsymbol{pBx} = \boldsymbol{pAx} + \boldsymbol{pd}. \tag{1}$$

$$\boldsymbol{ys} + \boldsymbol{ys}\gamma + \boldsymbol{yd} = \boldsymbol{y}[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x}. \tag{2}$$

9.4 Condition [Complementarity on Prices]

$$\boldsymbol{pBx} = (1 + r)\boldsymbol{pAx} + \boldsymbol{y}[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x} + w\boldsymbol{\ell x}. \tag{1}$$

$$\boldsymbol{ys}\gamma = L\gamma. \tag{2}$$

9.5 Lemma [Solution to Extended System] *Under Assumption 7.2, the Extended System admits a nonnegative equilibrium solution* $(\boldsymbol{p}, \boldsymbol{y}, \boldsymbol{x}, \gamma)$.

Proof We apply Theorem 5.9 to the Extended System 8.1–8.4. By extension of Observations 5.10–5.11, the cross-dual price condition can trivially be satisfied by setting all prices and emission charges to zero. The extended cross-dual quantity condition (corresponding to Condition 5.7) is as follows:

Cross-Dual Quantity Condition

$$\boldsymbol{Bx} \geq (1 + r)\boldsymbol{Ax} + \boldsymbol{d}, \tag{1}$$

$$\boldsymbol{s} + \boldsymbol{s}\gamma + \boldsymbol{d} \geq [\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x}. \tag{2}$$

By setting γ sufficiently large, the emission constraint from Condition (2) effectively becomes nonbinding, and Assumption 7.2 then ensures that Condition (1) is feasible. The Lemma then follows from Theorem 5.9 and extension of Observations 5.10–5.11. ▪

9.6 Lemma [Extended System and Cost-Minimizing Equilibrium] *The Extended System 9.1-9.4 reduces to a Cost-Minimizing System 3.1–3.4 if*

$$\boldsymbol{ys} < L.$$

Proof By 9.4(2), $\boldsymbol{ys} < L\gamma$ implies that $\gamma = 0$ (Process 0 is idle). All other equilibrium relationships from 9.1–9.4 then reduce to 3.1–3.4. ▪

§ 10 Proof of Theorem 8.1

Consider the feasibility constraints

$$[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x} \geq \boldsymbol{O}, \tag{1}$$

$$[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x} \leq \boldsymbol{s} + \boldsymbol{d}. \tag{2}$$

This system may be bounded or unbounded. We consider each of these two cases in turn.

First, suppose that constraints (1)–(2) are *unbounded* (for nonnegative activities $\boldsymbol{x}$). In that case, the original (non-extended) problem 3.1–3.5 satisfies the cross-dual quantity condition 5.7 by virtue of Assumption 7.1 and we are done without referring to the Extended System.

Second, suppose that constraints (1)–(2) are *bounded* (for nonnegative activities $\boldsymbol{x}$). In that case, we apply the Lemke algorithm to the Extended System.

Let $(\boldsymbol{p}, \boldsymbol{y}, \boldsymbol{x}, \gamma)$ be a solution to 9.1–9.4. By Lemma 9.5 such a solution does exist. By Lemma 7.6, we need to show that $\boldsymbol{ys} < L$. The proof is by contradiction. If no carrying capacity is scarce under $\boldsymbol{x}$ (that is, $[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x} < \boldsymbol{s}+\boldsymbol{s}\gamma+\boldsymbol{d}$ in 9.1(2)), then complementarity condition 9.3(2) requires $\boldsymbol{y} = \boldsymbol{O}$ and we are done. Suppose therefore that some carrying capacity is in fact scarce and $\boldsymbol{y}$ is nonzero, and suppose $\boldsymbol{ys} = L$, contrary to our claim. By 9.1(2), scarce carrying capacity implies that equilibrium activities $\boldsymbol{x}$ must be nonzero. For $\boldsymbol{x}$ to be nonzero, the positivity of $\boldsymbol{\ell}$ and profitability 9.4(1) then implies that at least one produced commodity must be scarce. By complementarity 9.3(1), this commodity must have a positive price. Consider a vector $\boldsymbol{x}'$ that satisfies Assumption 7.1; by construction, compared with $\boldsymbol{x}$ this activity vector will weakly under-utilize all priced commodities and will strictly under-utilize all carrying capacities that carry a positive rent. Premultiply 7.1(1) by $\boldsymbol{p}$ and 7.1(2) by $\boldsymbol{y}$:

$$\boldsymbol{pBx}' \geq \boldsymbol{pAx}' + \boldsymbol{pd} \quad \text{and} \tag{A1}$$

$$\boldsymbol{y}[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x} > \boldsymbol{y}[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x}'. \tag{A2}$$

In A2, the inequality of 7.1(2) is preserved because by complementarity 9.3(2) emission charge $y^{[i]}$ is positive only if $[\boldsymbol{B}-\boldsymbol{A}]_{[i]}x$ is positive. Similarly, postmultiply the inequalities in 9.2(1) by $\boldsymbol{x}'$:

$$\boldsymbol{pBx}' \leq (1 + r)\boldsymbol{pAx}' + \boldsymbol{y}[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x}' + w\boldsymbol{\ell x}'. \tag{A3}$$

By A1 and A3:

$$\boldsymbol{pd} \leq r\boldsymbol{pAx}' + \boldsymbol{y}[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x}' + w\boldsymbol{\ell x}'. \tag{A4}$$

By 9.3–9.4:

$$\boldsymbol{pd} = r\boldsymbol{pAx} + \boldsymbol{y}[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x} + w\boldsymbol{\ell x}. \tag{A5}$$

Hence, combining A4 with A5,

$$r\boldsymbol{pA}[\boldsymbol{x}' - \boldsymbol{x}] + w\boldsymbol{\ell}[\boldsymbol{x}' - \boldsymbol{x}] \geq \boldsymbol{y}\Big[[\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x} - [\boldsymbol{B} - \boldsymbol{A}]\boldsymbol{x}'\Big]. \tag{A6}$$

By A2, A6, 7.1(2) and 9.3(2), $\boldsymbol{ys} = L$ then implies

$$r\boldsymbol{pA}[\boldsymbol{x}' - \boldsymbol{x}] + w\boldsymbol{\ell}[\boldsymbol{x}' - \boldsymbol{x}] > \alpha(L + L\gamma) \tag{A7}$$

for some positive α. Hence, since $\gamma \geq 0$,

$$r\boldsymbol{pA}[\boldsymbol{x}' - \boldsymbol{x}] + w\boldsymbol{\ell}[\boldsymbol{x}' - \boldsymbol{x}] > \alpha L. \tag{A8}$$

On the lefthand side of A8, the labour term $w\boldsymbol{\ell}[\boldsymbol{x}' - \boldsymbol{x}]$ is bounded above by the positivity of $\boldsymbol{\ell}$ and by the boundedness assumption on the constraint system (1)–(2). The capital term $r\boldsymbol{pA}[\boldsymbol{x}' - \boldsymbol{x}]$ is negative by virtue of Assumption 7.1(3). Thus, the lefthand side of A8 is bounded above. By contrast, the righhand side of A8 can be made arbitrarily large by setting L arbitrarily large, since α is independent of L. Thus, for sufficiently large values of L, the inequality in A8 will be violated, yielding the desired contradiction. Hence, we must have $\boldsymbol{ys} < L$, and the fictitious Process 0 can not be active in equilibrium, $\gamma = 0$, as desired. This ends the proof. ▪

§ 11 Conclusion

The framework discussed in this paper offers a complementary pivoting framework to finding a cost-minimizing choice of technique for an environmentally constrained multisectoral economy, thereby extending earlier results by Salvadori (1986) and Bidard (1991) and expanding the scope of algorithmic approaches to the choice-of-technique problem (Bidard 1990).

Existence Theorem 8.1 shows that in an emission-constrained economy, a cost-minimizing equilibrium does exists for a wide range of final demand vectors, provided that the available technology allows for a sufficiently strong substitution potential between labour and other inputs. The ability of producers to switch to less pollution-intensive production methods puts an effective check on emission charges, and the presence of this price check stabilises the search for an equilibrium solution in the pivoting sequence of the Lemke algorithm. The "Extended System" from Section 9 exploits this natural price check.

By using a constructive existence proof based on a well-understood algorithmic technique (the Lemke algorithm), Theorem 8.1 not only identifies the actual equilibrium conditions (Conditions 7.1–7.2), it also offers a deeper analytical understanding of the the sources of equilibrium, as revealed by the ultimate replacement of the artificial abatement process 0 by a real production process in the pivoting sequence of the Extended System.

References

Bidard, C (1987): "Il n'y plus de rente", in C Bidard / G Abraham-Frois (eds), *La Rente*, Paris: Economica.

Bidard, C (1990): "An algorithmic theory of the choice of techniques", *Econometrica,* 58.

Bidard, C (1991): *Prix, Reproduction, Rarete.* Paris: Dunod.

Bidard, C (1996): "All-engaging systems" *Economic Systems Research*, 8.

Bidard, C / G Erreygers (1998) "Sraffa and Leontief on Joint Production" *Review of Political Economy*, 10

Bidard, C / Erreygers, G (1998): "The Adjustment Property", *Economic Systems Research*, 10.

Cottle, RW / Pang, J-S / Stone, RE (1992): *The Linear Complementarity Problem.* (Computer Science and Scientific Computing Series.) Boston etc.: Academic Press.

d'Agata, A (1983): "The Existence and Uniqueness of Cost-Minimizing Systems in Intensive Rent Theory." *Metroeconomica,* 35.

d'Agata (1984): "Moltiplicita di merci agricole e rendita differenziale estensiva", *Ricerche Economiche,* 38.

Barker, T (1999): "Achieving a 10% cut in Europe's CO2 emissions using additional excise duties", *Economic Systems Research*, 11.

Barker, T / Ekins, P (2001): "Carbon taxes and carbon emission trading, *Journal of Economic Surveys*, 15.

Coggins, JS / Swinton, JR (1996): "The price of pollution: A dual approach to valuing SO2 allowances." *Journal of Environmental Economics and Management*, 30.

Dales, JH (1968): "Land, water, and ownership", *Canadian Journal of Economics*, 1.

Dantzig, GB / Manne, AS (1974): "A complementarity algorithm for an optimal capital path with invariant proportions", *Journal of Economic Theory,* 9.

Duchin, F (1988): "Analysing structural change in the economy", in *Input-Output Analysis: Current Developments,* Ciaschini, M.(ed.): London: Wiley.

Duchin, F (1988): "Analyzing technological change: an engineering data base for input-output models of the economy", *Engineering with Computers*, 4. Duchin, F (1990): "Framework for the evaluation of scenarios for the conversion of biological materials and wastes to useful products: An input-output approach." *Structural Change and Economic Dynamics*, 1.

Fossell, O / Polenske, KR (1998): "Input-output and the environment//, *Economics Systems Research*, 10.

Jones, PC (1982): "Computing an optimal invariant capital stock", *SIAM Journal of Algebraic and Discrete Methods*, 3.

Joskow, PL / Schmalensee / EM Baley (1999): "The market for sulfur dioxide emissions", *American Economic Review*, 89.

Kurz, HD / Salvadori, N (1995): *Theory of Production. A long-period analysis.* Cambridge: Cambridge University Press.

Kurz, M (1969): "Tightness and substitution in the theory of capital", *Journal of Economic Theory*, 4.

Lager, C (1998): "Prices of 'goods' and 'bads': An application of the theory of differential rent" *Economic Systems Research*, 10

Lager, C (2001): "Joint Production with Restricted Free Disposal", *Metroeconomica.*

LEMKE, CE (1968): "On complementary pivot theory," in GB Dantzig / AF Veinott (eds), *Mathematics of the Decision Sciences.* Providence,Rhode Island: AMS.

LENZEN, M (2002): "Environmentally important paths, linkages and key sectors in the Australian economy", *Structural Change and Economic Dynamics*, in press.

LEONTIEF, W (1985): "TECHNOLOGICAL CHANGE, PRICES, WAGES, AND RATES OF RETURN ON CAPITAL IN THE U.S. ECONOMY", IN LEONTIEF, W.: *Input-Output Economics*, 2ND ED., OXFORD ETC.: OXFORD UNIVERSITY PRESS 1986.

LEONTIEF, W (1970): "ENVIRONMENTAL REPERCUSSIONS AND THE ECONOMIC STRUCTURE: AN INPUT-OUTPUT APPROACH", IN LEONTIEF, W.: *Input-Output Economics*, 2ND ED., OXFORD ETC.: OXFORD UNIVERSITY PRESS 1986. LEONTIEF, W (1973): "NATIONAL INCOME, ECONOMIC STRUCTURE, AND ENVIRONMENTAL EXTERNALITIES," IN LEONTIEF, W.: *Input-Output Economics*, 2ND ED., OXFORD ETC.: OXFORD UNIVERSITY PRESS 1986.

LEONTIEF, W / FORD, D (1972): "AIR POLLUTION AND THE ECONOMIC STRUCTURE: EMPIRICAL RESULTS OF INPUT-OUTPUT COMPUTATIONS", IN LEONTIEF, W.: *Input-Output Economics*, 2ND ED., OXFORD ETC.: OXFORD UNIVERSITY PRESS 1986.

LOS (1976): "Extended von Neumann models and game theory", in J Los / MW Los (eds), *Computing Equilibria: How and Why?*, Amsterdam: North-Holland.

LUTZ, C (2000): "NOx emissions and the use of advance pollution abatement techniques in West Germany", *Economics Systems Research*, 12

SALVADORI, N (1986): "Land and Choice of Techniques within the Sraffa framework", *Australian Economic Papers,* 25.

SCHMALENSEE , R / PL LOSKOW ETAL, (1998): "An interim evaluation of sufur dioxide emissions trading", *Journal of Economic Perspectives*, 12

SWINTON, JR (1998) "AT WHAT COST DO WE REDUCE POLLUTION? SHADOW PRICES OF SO2 EMISSIONS." *Energy Journal*, 19.

STEENGE, Æ (1978): "Environmental repercussions and the economic structure: further comments", *Review of Economics and Statistics*, 60.

VON STENGEL, B (2000): "Computing equilibria for two-person games", *Handbook of Game Theory*, Vol 3, eds RJ Aumann, S Hart North Holland: Amsterdam.

SYMONS, L / PROOPS J / GAY, P (1994): "Carbon taxes, consumer demand and carbon dioxide emissions: a simulation analysis for the UK", *Fiscal Studies*, 15.

ZHANG, ZX (1998): "Economic modelling approaches to cost estimates for the control of carbon-dioxide emissions", *Energy Economics*, 20.

Appendix:

LCP Implementation in Python

The attached printouts show an effective implementation of the LCP algorithm for computing emission charges, written in the Python language. The programme refers to "land" and land rents; "land" stands for carrying capacity, and "rent" stands for the corresponding emission charge. Matrix $\boldsymbol{C}$ is the emission matrix. Appendix A shows the data file, Appendix B shows the underlying programme file, and Appendix C shows the screen output of the Python session. All files can be downloaded from: www.keele.ac.uk/depts/ec/web/People/martind.html

A1 Input File

```
#####################################################
##  file lcpdata.py by Martin Diedrich, 28 Oct 2001 ##
#####################################################
##  Use jointly with program module lcpcomp.py
##  and raional-number module yarny.py
#####################################################
##  Requires Python, for use with an interactive shell
##  (preferably: use in the IDLE shell)
#####################################################
##  Author:  Martin Diedrich, Keele Economics
##  email:   m.e.diedrich@econ.keele.ac.uk
####################################################
#  INPUT OF REQUIRED FILES: lcpcomp.py and yarny.py
#from lcpcomp import *
from lcpcomp import *
from yarny import Rat
#####################################################


####################################################
#  BASIC PARAMETERS
# Interest Factor (1+r)
R=Rat(2,1,2)
# Number of Commodities
COMM=2
# Number of Land types
LAND=2
# Number of Processes
PROC=4
# Number of extra processes for profitability check
EXTRA=2

####################################################
#  TECHNOLOGY  Inputs A, Outputs B, Land C
# Commodity Inputs (nonnegative)
# Think of a1, a2 as column vectors.
a1=[3, 1]
a2=[1, 3]
a3=[1, 2]
a4=[2, 1]
# Commodity Outputs (nonnegative)
b1=[10, 0]
b2=[10, 0]
# NB: disable process 3, else b3=[0,10]
b3=[ 0, 0]
b4=[ 0, 10]
# Land Inputs (nonnegative)
c1=[10, 0]
c2=[0 ,50]
c3=[20, 0]
c4=[ 0,10]
# Input-Output Matrices
# Adjoin a,b,c columns into A,B,C matrices
A=[a1,a2,a3,a4]
B=[b1,b2,b3,b4]
C=[c1,c2,c3,c4]

####################################################
## Constraint vectors of LCP. Vectors l,d,s will be
##   stacked into constraint vector q=[-d,s,l].
# Labour Costs (strictly positive)
l=[Rat(10),Rat(15),Rat(1),Rat(5)]
# Final Demand (nonnegative, nonzero)
d=[Rat(150),Rat(350)]
# Land Stocks (nonnegative)
s=[Rat(1000),Rat(1000)]


#####################################################
# CHECK EXTRA PROCESS for profitability under (p,y)
#####################################################
# Here: e1,e2 identical to processes 3&4 from example
# Inputs
ea1=[1, 2]
ea2=[2, 1]
# Outputs
eb1=[0, 10]
eb2=[0, 10]
# Land Inputs
ec1=[20, 0]
ec2=[0, 10]
# Extra Matrices
EA=[ea1,ea2]
EB=[eb1,eb2]
EC=[ec1,ec2]
# Labour Inputs
EL=[Rat(1),Rat(5)]

#####################################################
# Show intermediate steps? =1 if YES, =0 if NO
showsteps=0
#####################################################

#####################################################
#  CONSTRUCTING AND PROCESSING LCP
#  Don't touch this!
#####################################################
setupM(R,A,B,C,d,s,l,COMM,LAND,PROC,showsteps,\
        EA,EB,EC,EL,EXTRA)
#####################################################
# Done

```

A2 Program File

```
######################################################
##  file lcpcomp.py by Martin Diedrich, 28 Oct 2001  #
##  Python module for Lemke Algorithm in Rent Model  #
######################################################
##  Use jointly with input file lcpdata.py
######################################################
##  Preliminary Version:
##  Version 0.2 (28 Oct 2001)
##  Author:  Martin Diedrich, Keele Economics
##  email:   m.e.diedrich@econ.keele.ac.uk
######################################################
##  Needs to be initiated from the interactive Python
##  shell by data file lcpdata.py.
##  USAGE:
##  Place files lcpcomp.py (this file), lcpdata.py
##  andyarny.py in the Python path.  Then type
##  "import lcpdata" in the Python shell and watch.
##  Modify lcpdata.py to change data input.
######################################################
##  Solves the rent problem as an LCP, using the Lemke
##  Algorithm.  Implements some ideas from J.A.Tomlin,
##  Mathematical Programming Study, Vol 7 (1978).
#####################################################
##  GLOBAL VARIABLES (for link lcpdata->lcpcomp):
##  Function setupM() declares the following
##  variables as global:
##    nLand, nZO, nZT, nZZ, nZC, dD, sS, lL
##    aA, bB, cC, rR, MO, MR, M, q, sss
##    nEZT, nEZZ, lEL, aEA, bEB, cEC
######################################################
######################################################


######################################################
##  Preparatory Functions: Matrix Operations
# Vector plus Vector
def vecadd(x,y):
    z=x[:]
    i=0
    while i<len(x):
        z[i]=x[i]+y[i]
        i=i+1
    return z

# Scalar multiple of vector
def scalvect(a,x):
    z=x[:]
    i=0
    while i<len(x):
        z[i]=a*x[i]
        i=i+1
    return z

# Scalar multiple of matrix
def scalmat(a,m):
    n=m[:]
    j=0
    while j<len(n):
        n[j]=m[j][:]
        i=0
        while i<len(n[j]):
             n[j][i]=a*m[j][i]
             i=i+1
        j=j+1
    return n

# Matrix transposition
def transpo(m):
    n=[]
    r=len(m[0])
    c=len(m)
    i=0
    j=0
    while j<r:
        nn=[]
        i=0
        while i<c:
            nn.append(m[i][j])
            i=i+1
        n.append(nn)
        j=j+1
    return n

# Matrix plus Matrix
def matadd(M,N):
    Z=M[:]
    j=0
    while j<len(M):
        Z[j]=M[j][:]
        i=0
        while i<len(M[0]):
            Z[j][i]=M[j][i]+N[j][i]
            i=i+1
        j=j+1
    return Z

######################################################
#  CONSTRUCTING THE LCP MATRICES
# Check Dimensions
def testMatrix(M,m,n):
    if not len(M)==n:
        print '\n INPUT ERROR: check columns in: ',M
    if not len(M[0])==m:
        print '\n INPUT ERROR: check rows in: ',M

def testVector(X,ll):
    if not len(X)==ll:
        print '\n INPUT ERROR: check: ',X

# Form matrix (B,0)
def ExtendMat(B,C):
    dimC=len(C[0])
    Z=[]
    j=0
    while j<len(B):
        ZZ=B[j][:]
        i=0
        while i<dimC:
             ZZ.append(0)
             i=i+1
        Z.append(ZZ)
        j=j+1
    return Z

# Form matrix (A,C)
def JoinMat(A,C):
    Z=[]
    j=0
    while j<len(A):
        ZZ=A[j][:]+C[j][:]
        Z.append(ZZ)
        j=j+1
    return Z

# Net Product [(B,0)-(A,C)]
# A with interest factor, Disc
def MakeNetmat(B,A,C,Disc):
    BB=ExtendMat(B,C)
    YA=scalmat(-Disc,A)
    YC=scalmat(-1,C)
    AA=JoinMat(YA,YC)
    MM=matadd(BB,AA)
    return MM

# Righthand Side "q"
def makeRHS(d,s,l):
    dd=scalvect(-1,d)
    q=dd+s+l
    return q

# Reduced final demand if covering vector active:
def adjD(a,q):
    ap=1-a
    D=scalvect(ap,dD)
    return D

#####################################################
##   DEFINE matrix "M" from submatrices "m" and "n"
##   m,n onto antidiagonal, rest set to zero
def makeM(m,n):
    M=[]
    a=len(m)
    b=len(n)
    r=a+b
    j=0
    while j<b:
        MM=[]
        i=0
```

```
        while i<b:
            MM.append(0)
            i=i+1
        while i<r:
            MM.append(n[j][i-b])
            i=i+1
        M.append(MM)
        j=j+1
    while j<r:
        MM=[]
        i=0
        while i<b:
            MM.append(m[j-b][i])
            i=i+1
        while i<r:
            MM.append(0)
            i=i+1
        M.append(MM)
        j=j+1
    return M

#####################################################
# INPUT FROM FILE lcpdata.py
def setupM(R,A,B,C,d,s,l,p,y,x,showsteps,\
           EA,EB,EC,EL,EXTRA):
    global nLand, nZO, nZT, nZZ, nZC, dD, sS
    global lL, aA, bB, cC, rR, MO, MR, M, q, sss
    global nEZT, nEZZ, lEL, aEA, bEB, cEC
    sss=showsteps
    [rR,aA,bB,cC,dD,sS,lL]=[R,A,B,C,d,s,l]
    [nZC,nLand,nZT,nZO,nZZ]=[p,y,x,p+y,p+y+x]
    [aEA,bEB,cEC,lEL,nEZT]=[EA,EB,EC,EL,EXTRA]
    nEZZ=p+y+EXTRA
    testMatrix(A,nZC,nZT)
    testMatrix(B,nZC,nZT)
    testMatrix(C,nLand,nZT)
    testVector(l,x)
    testVector(d,p)
    testVector(s,y)
    testMatrix(EA,nZC,nEZT)
    testMatrix(EB,nZC,nEZT)
    testMatrix(EC,nLand,nEZT)
    testVector(EL,EXTRA)
    MO=MakeNetmat(B,A,C,1)
    MR=MakeNetmat(B,A,C,R)
    MMR=scalmat(-1,MR)
    MMR=transpo(MMR)
    M=makeM(MO,MMR)
    q=makeRHS(d,s,l)
    dolcp(M,q)


#####################################################
##  create Lemke Tableau

# I of size m
def slackmat(M):
    S=M[:]
    j=0
    while j<len(M):
        S[j]=M[j][:]
        i=0
        while i<len(M):
            if i==j:
                S[j][i]=1
            else:
                S[j][i]=0
            i=i+1
        j=j+1
    return S

# Artificial covering vector f=-q if q<0
def artivec(q):
    f=q[:]
    j=0
    while j<len(q):
        if q[j]<0:
            f[j]=q[j]
        else:
            f[j]=0  # set zero if not needed
        j=j+1
    return f

## L = [f, I, - M, q]
def lemkemat(M,q):
    S=slackmat(M)
    f=artivec(q)
    C=scalmat(-1,M)
    L=[f]              # covering
    j=0
    while j<len(S):
        L.append(S[j]) # slacks
        j=j+1
    j=0
    while j<len(C):
        L.append(C[j]) # processes
        j=j+1
    L.append(q)        # RHS vector
    return L

#####################################################
## Initial Step

## Find most negative entry "k" in RHS q
def initight(q):
    mm=0
    j=0
    while j<len(q):
        mm=min(mm,q[j])
        if mm==q[j]:
            k=j
        j=j+1
    return k

## Pivot in f-column, about row k
def iniclear(L,q):
    k=initight(q)    # k = tight row
    LL=L[:]
    pivel=L[0][k]    # k-element of f-column
    j=0
    while j<len(L):       # create pivot row
        LL[j]=L[j][:]
        LL[j][k]=1.0*L[j][k]/pivel
        j=j+1
    LLL=LL[:]
    pivcol=LL[0]
    j=0
    while j<len(L):       # subtract x-pivot row
        LLL[j]=LL[j][:]
        i=0
        while i<len(q):
            if not i==k:
                LLL[j][i]=LL[j][i]-pivcol[i]*LL[j][k]
            i=i+1
        j=j+1
    com=[k+1,0]  # [tight constraint, new var]
    return [LLL,com]    #    (1,2,3,...)

#####################################################
##  INDEX OF BASIC VARIABLES
##  Implemented as pair of  "dictionaries":
##  "ConInd" maps constraints (rows) into basic vars
##  "VarInd" maps basic variables into constraints

## Prepare Index Constraint -> Var;
## Initially only slacks active: ConInd[i]=i+1 (all i)
def prepConInd(M):
    ConInd={}
    i=1
    while i<len(M)+1:
        ConInd[i]=i   #Constraint 1 covered by slack 1
        i=i+1
    ConInd['p']=0
    return ConInd

## Prepare Index for Var -> Constraint:
## Initially only slacks active: VarInd[i+1]=i all i
def prepVarInd(M):
    VarInd={}
    VarInd[0]='p'
    i=1
    while i<len(M)+1:
        VarInd[i]=i  #Constraint 1 covered by slack 0
        i=i+1
    return VarInd

#####################################################
##  MAIN PIVOT STEP
## Update the Index: Re-assign leaving constraint
##      and identify new entering variable
def upIndy(ConInd,VarInd,com,h,Alist,rhs):
    inVar='X'
    outVar='Y'
    NewConInd=ConInd.copy()
    NewVarInd=VarInd.copy()
    oldPiVar=ConInd['p']
    outVar=ConInd[com[0]]      # old tight constraint
    if outVar==0:
        del NewConInd['p']
```

```
    elif outVar<h+1:
        inVar=outVar+h         # slack out?
        NewConInd['p']=inVar
        NewVarInd[inVar]='p'
    else:
        inVar=outVar-h         # process out?
        NewConInd['p']=inVar
        NewVarInd[inVar]='p'
    NewConInd[com[0]]=com[1]   # new var 0,1,2,...
    del NewVarInd[oldPiVar]
    del NewVarInd[outVar]
    NewVarInd[com[1]]=com[0]
    if not VarInd[0]=='p':     # store pivot details
        art=rhs[VarInd[0]-1]   # in "Alist"
        if art>1:
            art='leave 0'
    else:
        art='enter 1'
    Alist.append([outVar,inVar,art])
    return [NewConInd,NewVarInd,Alist]

## Index of Pivot Column
def PivColIn(ConInd):
    kc=ConInd['p']
    return kc

## Index of Pivot Row
def PivRowIn(L,kc):
    qq=L[len(L)-1][:]
    cc=L[kc][:]
    mm=100000000000000000L
    i=0
    kr=-1                            #-1 if stop
    while i<len(qq):
        if cc[i]>0.00000000001:      # entry>0?
            rr=1.0*qq[i]/cc[i]  # MinRatio test
            mm=min(mm,rr)
            if mm==rr:
                kr=i
        i=i+1
    return kr

## Main Pivoting Operation
def pivotstep(L,ConInd,r,h):
    kc=PivColIn(ConInd)
    kr=PivRowIn(L,kc)
    if kr==-1:      #stop!
        OFFF=1
        LLL=L[:]
        com=[0,0]   # finished
    else:
        OFFF=0
        pivel=L[kc][kr]        # Pivot Element
        LL=L[:]
        j=0
        while j<r:             # Create Pivot Row
            LL[j]=L[j][:]
            LL[j][kr]=1.0*L[j][kr]/pivel
            j=j+1
        LLL=LL[:]
        pivcol=LL[kc]
        j=0
        while j<r:          # Create 0 in Piv Column
            LLL[j]=LL[j][:]
            i=0
            while i<h:
                if not i==kr:
                    LLL[j][i]=LL[j][i]-pivcol[i]*LL[j][kr]
                i=i+1
            j=j+1
        com=[kr+1,kc]   #[Leave Constraint, Enter Var]
    return [LLL,com,OFFF]


######################################################
#######          Display Output            ###########
######################################################

def PrintMat(a):
    j=0
    while j<len(a):
        print 'Process',j+1
        print '    ',a[j]
        j=j+1

def PrintSystem(L,g,h):
    print '\n Main LCP System'
    print '\n Commodity Inputs:'
    PrintMat(aA)
    print '\n Commodity Outputs:'
    PrintMat(bB)
    print '\n Land Inputs:'
    PrintMat(cC)
    print '\n Labour Costs:'
    print lL
    print '\n Interest Factor (1+r):'
    print rR
    print '\n Final Demand:'
    print dD
    print '\n Land Stocks:'
    print sS

def PrintInitialStep(L,ConInd,VarInd,step):
    print '\n Initial Index: Step', step
    print ConInd
    print VarInd

def PrintStep(ConInd,VarInd,step):
    h=nZO+nZT
    print '\n ########################################'
    print ' #############    STEP    ################'
    print '\n PIVOTING STEP: ', step
    print '("p" is entering variable)'
    print '\n Constraint -> Basic Variable'
    print 'P= 1 ..',nZC, '; Y= ', nZC+1,' ..',nZO,\
           '; X= ',nZO+1,' ..',nZO+nZT
    print ConInd
    print '\n Basic Variable -> Constraint'
    print 'W= 1 ..',nZO+nZT,'P= ', h+1,' ..', h+nZC,\
          '; Y= ', h+nZC+1,' ..', h+nZO,\
          '; X= ',h+nZO+1,' ..', h+nZO+nZT
    print VarInd

def zType(a):
    h=nZZ
    if a==0:
        zn=['Covering ', a]
    elif a<nZC+1:
        zn=[' p-slack ',a]
    elif a<nZO+1:
        zn=[' y-slack ',a-nZC]
    elif a<nZZ+1:
        zn=[' x-slack ',a-nZO]
    elif a<h+nZC+1:
        zn=[' price p ',a-h]
    elif a<h+nZO+1:
        zn=[' rent y  ',a-h-nZC]
    elif a<h+nZZ+1:
        zn=[' act x   ',a-h-nZO]
    else:
        zn=['start/stop', '']
    return zn

def printSwitch(a):
    print zType(a[0])[0],zType(a[0])[1],'out ->',\
          zType(a[1])[0],zType(a[1])[1],'in',\
          '  |  theta =',a[2]

def printPivot(aMESSG,Alist):
    print '\n', aMESSG,\
          'PIVOT RECORD: (var out -> var in)'
    i=0
    while i<len(Alist):
        printSwitch(Alist[i])
        i=i+1

def printBasis(L,VarInd,h,g,Alist,aMESSG):
    slacks=[]
    prices=[]  # only if >0
    rents=[]   # only if >0
    aprices=[] # all
    arents=[]  # all
    activs=[]
    i=1
    while i<h+1:
        if i in VarInd.keys():
            if not VarInd[i]=='p':
                slacks.append([i,L[g+1][VarInd[i]-1]])
        i=i+1
    while i<h+nZC+1:
        if i in VarInd.keys():
            if not VarInd[i]=='p':
                prices.append([i-h,L[g+1]\
                              [VarInd[i]-1]])
                aprices.append(L[g+1]\
                                [VarInd[i]-1])
        else:
            aprices.append(0)
        i=i+1
```

```
    while i<h+nZO+1:
        if i in VarInd.keys():
            if not VarInd[i]=='p':
                rents.append([i-h-nZC,L[g+1]\
                             [VarInd[i]-1]])
                arents.append(L[g+1]\
                               [VarInd[i]-1])
        else:
            arents.append(0)
        i=i+1
    while i<h+nZZ+1:
        if i in VarInd.keys():
            if not VarInd[i]=='p':
                activs.append([i-h-nZO,L[g+1]\
                                        [VarInd[i]-1]])
        i=i+1
    print '\n ',aMESSG,'SOLUTION: '
    print '\n Up to', nZC,\
          'commodity prices [Commodity,Price]:'
    print prices
    print '\n Up to',nLand,\
          'land rents [Land Type, Rent]:'
    print rents
    print '\n Up to',nZO,'from',nZT,\
          'activities [Process, Activity Level]:'
    print activs
    printPivot(aMESSG,Alist)
    return  [aprices, arents]

def showStepsol(L,VarInd,h,g,Alist,q):
    print '\n INTERMEDIATE STEP SOLUTION'
    print '\n Scale of Covering Vector:'
    if not VarInd[0]=='p':
        art=L[g+1][VarInd[0]-1]
    if not art>1:
        print art
        DD=adjD(art,q)
        print '\n Reduced Demand (by covering vector):'
        print dD,'reduced to', DD
    else:
        print ' covering vector off'
    printBasis(L,VarInd,h,g,Alist,'INTERMEDIATE')

def showsol(L,VarInd,h,g,OFFF,Alist):
    print '\n #######################################'
    print ' #######################################'
    print '\n Final Step:  LCP SOLUTION'
    if OFFF==1:
        print '\n NO SOLUTION - Lemke blocked!'
        printPivot('PIVOTING ABORTED ---', Alist)
        print 'Termination in ray: theta->inf'
    else:
        pricerent=printBasis(L,VarInd,h,g,Alist,\
                            'COMPLETE')
    return pricerent


######################################################
##  MAIN RUNNING COMMAND:  Do Lemke LCP on (M,q)
######################################################

def dolcp(M,q):
    MM=M[:]
    Alist=[['Y',0,'inf']]
    h=len(q)
    g=2*h
    step=0
    L=lemkemat(M,q)
    PrintSystem(L,g,h)
    r=len(L)
    a=iniclear(L,q)
    ConInd=prepConInd(M)
    VarInd=prepVarInd(M)
    if sss==1:
        PrintInitialStep(a[0],ConInd,VarInd,step)
    Indy=upIndy(ConInd,VarInd,a[1],h,Alist,a[0][g+1])
    [ConInd,VarInd,Alist]=Indy
    step=step+1
    OFFF=0
    while OFFF==0 and 0 in ConInd.values():
        if sss==1:
            PrintStep(ConInd,VarInd,step)
        a=pivotstep(a[0],ConInd,r,h)
        if sss==1:
           showStepsol(a[0],VarInd,h,g,Alist,q)
        OFFF=a[2]
        if OFFF==0:
            Indy=upIndy(ConInd,VarInd,a[1],h,\
                        Alist,a[0][g+1])
            [ConInd,VarInd,Alist]=Indy
        step=step+1
    PSOL=showsol(a[0],VarInd,h,g,OFFF,Alist)
    if nEZT>0:
        ProfBasis(PSOL,h,g,aEA,bEB,cEC,lEL,nEZT)


######################################################
##  EXTRA COMMAND:  Do profit checks on extra procs
######################################################

def ProfBasis(psol,h,g,xa,xb,xc,xl,xxx):
    print '\n #######################################'
    print ' #######################################'
    print '\n CHECK EXTRA PROCESSES FOR PROFITS'
    pprices=psol[0]
    prents=psol[1]
    print 'prices ', pprices
    print 'rents ', prents
    n=0
    while n<xxx:
        print '\n Extra Process',n+1
        print 'Commodity Inputs' ,xa[n]
        print 'Land Inputs' ,xc[n]
        print 'Labour Inputs' ,xl[n]
        print 'Commodity Outputs' ,xb[n]
        capval=0
        landval=0
        outval=0
        m=0
        while m<nZC:
            ent=xa[n][m]
            pri=pprices[m]
            capval=capval+ent*pri
            m=m+1
        valr=(rR-1)*capval
        m=0
        while m<nLand:
            ent=xc[n][m]
            pri=prents[m]
            landval=landval+ent*pri
            m=m+1
        labval=xl[n]
        m=0
        while m<nZC:
            ent=xb[n][m]
            pri=pprices[m]
            outval=outval+ent*pri
            m=m+1
        totcost=capval+valr+labval+landval
        profit=outval-capval-valr-labval-landval
        print '\n Costs and Revenues of Extra Process',\
              n+1
        print 'Capital Value:', capval,' Interest:',valr
        print 'Land Costs:',landval,\
              ' Labour Costs:', labval
        print 'Total Costs:', totcost,\
              ' Total Revenue:', outval
        print 'Profits:', profit
        n=n+1
# done
```

A3 Screen Output

```
>>> import lcpdat

 Main LCP System

 Commodity Inputs:
Process 1
    [3, 1]
Process 2
    [1, 3]
Process 3
    [1, 2]
Process 4
    [2, 1]

 Commodity Outputs:
Process 1
    [10, 0]
Process 2
    [10, 0]
Process 3
    [0, 0]
Process 4
    [0, 10]

 Land Inputs:
Process 1
    [10, 0]
Process 2
    [0, 50]
Process 3
    [20, 0]
Process 4
    [0, 10]

 Labour Costs:
[10, 15, 1, 5]

 Interest Factor (1+r):
2+1/2

 Final Demand:
[150, 350]

 Land Stocks:
[1000, 1000]

 #######################################
 ########################################

 Final Step:  LCP SOLUTION

  COMPLETE SOLUTION:


 Up to 2 commodity prices [Commodity,Price]:
[[1, 15+1/5], [2, 11+1/5]]

 Up to 2 land rents [Land Type, Rent]:
[[2, 3/10]]

 Up to 4 from 4 activities [Process, Activity Level]:
[[1, 20+10/317], [2, 11+13/317], [4, 44+252/317]]

COMPLETE PIVOT RECORD: (var out -> var in)
start/stop  out -> Covering  0 in   |  theta = inf
 p-slack  2 out ->  price p  2 in   |  theta = enter 1
 x-slack  4 out ->  act x    4 in   |  theta = 1
 p-slack  1 out ->  price p  1 in   |  theta = 1
 x-slack  2 out ->  act x    2 in   |  theta = 1
 y-slack  2 out ->  rent y   2 in   |  theta = 127/277
 x-slack  1 out ->  act x    1 in   |  theta = 127/277
Covering  0 out -> start/stop  in   |  theta = leave 0

 #######################################
 ########################################

 CHECK EXTRA PROCESSES FOR PROFITS
prices  [15+1/5, 11+1/5]
rents  [0, 3/10]

 Extra Process 1
Commodity Inputs [1, 2]
Land Inputs [20, 0]
Labour Inputs 1
Commodity Outputs [0, 10]

 Costs and Revenues of Extra Process 1
Capital Value: 37+3/5  Interest: 56+2/5
Land Costs: 0  Labour Costs: 1
Total Costs: 95  Total Revenue: 112
Profits: 17

 Extra Process 2
Commodity Inputs [2, 1]
Land Inputs [0, 10]
Labour Inputs 5
Commodity Outputs [0, 10]

 Costs and Revenues of Extra Process 2
Capital Value: 41+3/5  Interest: 62+2/5
Land Costs: 3  Labour Costs: 5
Total Costs: 112  Total Revenue: 112
Profits: 0
>>>
```

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