# Logical Pitfalls of Assuming Bounded Solutions to

# Expectational Difference Equations

by David Eagle, Ph.D. and Elizabeth Murff, Ph.D.

**Abstract**

**The precedent for solving expectational difference equations has been to solve converging equations backwards and diverging equations forward by assuming the solution is bounded. This precedent often leads to incorrect solutions and has less than rigorous foundations. More rigorous procedures would be to determine the terminal condition in a finite model and take the limit of that terminal condition as the horizon goes to infinity. Also, whether one solves forward or backwards depends on the context of the difference equation, not on convergence or divergence. These new procedures reveal Woodford's (2003) model of a cashless economy to be incomplete.**

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# Logical Pitfalls of Assuming Bounded Solutions to Expectational Difference Equations

**By David Eagle, Ph.D. and Elizabeth Murff, Ph.D.**

## I. Introduction

Many mathematical techniques used in economics came from applications in rocket science including Ito calculus, the Wiener process, and much of control theory. Hence, this paper uses a rocket example to assess the reasonableness of the rational expectations precedent of assuming a solution is bounded when solving expectational difference equations.

Suppose a rocket is located in space at $x_0$. The rocket has one thrust to send it off at a constant speed. While the control center has targeted a path for the rocket, it has no mechanism to control the rocket's direction. Define $x_t$ and $x_t^*$ to respectively represent the actual and targeted location vectors of the rocket at time t.

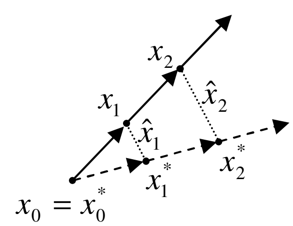

Figure 1. Actual vs. targeted time paths

Define $\hat{x}_t \equiv \left| x_t - x_t^* \right|$, which is the distance between $x_t$ and $x_t^*$. For the moment, assume the solution of $\hat{x}_t$ is bounded as is required by the rational expectations precedent. If the rocket's actual trajectory differs from the targeted trajectory, then $\lim_{t \to \infty} \left| x_t - x_t^* \right| = \infty$ (See Figure 1). Therefore, the only bounded solution of $\hat{x}_t$ is where the rocket is forever on target. It would be unreasonable to infer that this unique bounded solution to $\hat{x}_t$ means the rocket's direction is "determined" even though no mechanism exists to control the rocket's direction. To make that inference would be an abuse of mathematics. We write this paper to put an end to this type of abuse in economics where inferences like this are being made.

Woodford (2003) develops a model of a cashless economy as the basis for his book *Interest and Prices*, a cornerstone of the New Keynesian Economic literature. Just as there is no steering mechanism in the rocket example, Woodford's model has no monetary frictions and hence no mechanism by which monetary policy affects nominal aggregate demand and hence prices. However, by assuming his solution is bounded, Woodford claims prices are determined in his model under various interest-rate policies.

The precedent for solving expectational difference equations, as established by the rational expectations literature, is that one should solve stable roots backwards, unstable roots forward, and declare indeterminacy for unit roots. The precedent also states that if one solves an equation forward, then one should assume the solution of that difference equation is bounded.

This paper traces the history of the precedent for solving expectational difference equations. We find that the justification for making the assumption of bounded solutions was not met through any rigorous mathematical proof, but rather through a few examples, an incorrect claim to rule out "speculative bubbles", and another incorrect claim that we are free to make that assumption.

We find examples where this precedent leads to incorrect solutions. Then we propose a more rigorous approach to solve expectational difference equations. For each expectational difference equation one solves forward, one should determine the appropriate terminal condition based on a finite version of the model, and then take the limit of that terminal condition as the horizon goes to infinity. We also argue that the direction that we solve an expectational difference equation depends not on divergence or convergence, but on the context of the difference equation.

Section II presents Woodford's infinite implicit function theorem, which incorporates the assumption of a bounded solution. Section III presents Woodford's argument that his model is complete. Section IV, the major section of this paper, reviews the literature behind the precedent for solving expectational difference equations, discusses several examples where the precedent leads to incorrect solutions, and proposes new rigorous procedures. Section V applies these revised procedures to Woodford's model revealing that it is incomplete. Section VI clears away the smoke and mirrors to show how Woodford's logic is based on self-destructive policies. Section VII discusses additional problems with Woodford's argument. Section VIII concludes and reflects on the implications of the conclusions.

## II. Woodford's Infinite Implicit Function Theorem

Woodford's basis for calling his model complete rests on how to solve expectational difference equations. While this paper's emphasis is on expectational difference equations, the points we discuss here apply to regular difference equations as well. To simplify things initially, consider the following deterministic difference equation where b>1:

$$P_{t+1} = bP_t \tag{1}$$

Imagine that (1) applies at times t=0,1,2,…,∞. Also, let $\{P_t\}$ denote the sequence of $P_t$ for t=0,1,2,…, ∞. There are two different schools of thought concerning how to solve these "infinite" difference equations. The "normal" school of thought[1] believes that to solve for $\{P_t\}$,

[1] See Enders (1995) for a sense of the "normal" school of thought. However, because the current thinking in economics has been contaminated with the second school of thought, Ender's discussion of difference equations is a blending of these two schools of thought.

we need another condition such as an initial condition that states the value of $P_0$. For example, if we assumed the initial condition that $P_0 = 100$, then we would conclude that $P_{t+1} = 100b^t$ for t=0,1,2,…, ∞.

However, another school of thought believes that with only equation (1), they can "determine" or solve for $\{P_t\}$; that no other condition such as an initial condition is needed. They believe that the solution to the difference equation in (1) is that $P_t$=0 for t=0,1,2,…, ∞.

As part of this second school of thought, Woodford presents an infinite form of the implicit function theorem that he uses to show that this unique solution exists.[2] To see how this theorem applies, assume the following:

$$\hat{f}_t = f_1\hat{p}_{t-1} + f_2\hat{p}_t + f_3\hat{p}_{t+1}\,. \tag{A.8}$$

where $\hat{f}_t$ is an n-dimensional vector. Woodford (2003, p. 632) describes the requirements of his infinite form of the implicit function theorem by saying:

> "In the infinite-dimensional case, the requirement of continuity of the linear operator … requires that the operator be bounded …. That is, the sequence $\{\hat{f}_t\}$ must be bounded in the case of any bounded sequence $\{\hat{p}_t\}$. Assuming the $l_\infty$ (or sup norm) topology on the linear space of sequences, one then requires that $\hat{f}_t$ satisfy some uniform bound for all t in the case that each of the $\hat{p}_t$ satisfy some uniform bound. … In the case considered here, the condition obtains (generically) if and only if the characteristic equation
>
> $\mathrm{Det}[f_3\lambda^2+f_2\lambda+f_1]=0$     (A.9)
>
> has exactly n roots inside the unit circle (i.e., such that $|\lambda|<1$) and n roots outside ($|\lambda|>1$).

Please note the bounded requirements of Woodford's infinite form of the implicit function theorem. In particular, the theorem requires that the solution sequence $\{\hat{p}_t\}$ be

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There is a claim that if you have an initial condition or some other condition, then use it, but if you don't then assume the solution is bounded.

[2] In a footnote, Woodford refers to Lang (1983, Chapter 6) for some implicit function theorems connected with infinite spaces. However, we were unable to find the theorem Woodford presents. We have asked Woodford directly to clarify where he got his infinite implicit function theorem, but he has not responded. It is as though Woodford is presenting a theorem, without proof, that is not developed elsewhere. In this paper, we are not, however, claiming his theorem is wrong; rather we are saying it is misapplied.

bounded. It is generally not good mathematical practice to make an assumption about your solution.

The bounded requirement separates this second school of thought from the "normal" school of thought. Note that the use of the $l_\infty$ topology is not justification of the bounded requirement. Instead, the use of the $l_\infty$ topology is only justified in circumstances where the bounded requirement is appropriate.

To apply Woodford's infinite form of the implicit function theorem to (1), define $\hat{f}_t = bP_t - P_{t+1}$. Here $f_1$=0, $f_2$=b, and $f_3$= -1. Next, determine the roots to $-\lambda^2 + b\lambda = 0$. Since in our case $\hat{f}_t$ is one dimensional, we need one root inside the unit circle and one root outside. Those roots are λ=0 and λ=b. The λ=0 root is inside the unit circle. Since we assumed b>1, then the λ=b is outside the unit circle. Therefore, Woodford's infinite form of the implicit function theorem implies that there exists a unique bounded solution to (1).

Solving (1) forward results in

$$P_t = \frac{P_T}{b^{T-t}} \qquad (2)$$

Since b>1, the assumption that $\{P_t\}$ is bounded implies that $\lim_{T\to\infty} \frac{P_T}{b^{T-t}} = 0$, leading to the conclusion that $P_t$=0 for t=0,1,2,…, ∞. This contradicts the "normal" approach to difference equations. Figure 2 shows what the "normal" school of thought considers to be solutions to (1). While it is true there is only one bounded solution, the "normal" school of thought considers the unbounded solutions to be legitimate solutions as well.[3]

[3] There is some indication that the second school of thought thinks that if one has an initial condition then one can then use the initial condition to determine the solution even if that solution is unbounded, but if one does not have an initial condition then one would be free to assume a bounded solution in order to apply Woodford's infinite implicit function theorem. This approach would not be rigorous since if someone forgot they had the initial condition, then they would be led to a zero solution as being the unique solution whereas if they did remember the initial condition, they would be lead to a nonzero solution (assuming $P_0$ is not zero). Furthermore, no where in

Woodford's use of the $l_\infty$ topology means that he believes that it is appropriate to assume that the solution must be bounded. He provides no justification or reference to any other works to support his bounded-solution assumption.[4] At this time, we can only guess that Dr. Woodford is making the assumption because the rational expectations precedent for solving expectational difference equations is to assume the solution is bounded. However, section IV shows that precedent to be defective.

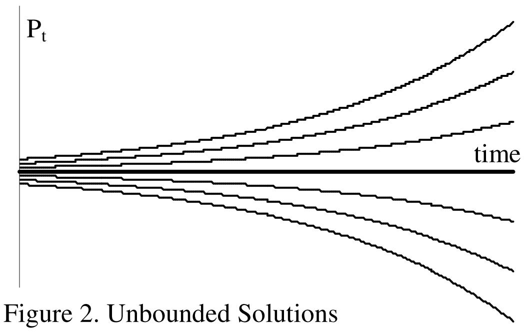

Figure 2. Unbounded Solutions

## III. Woodford's Argument That His Model Is Complete

In Chapter 2 of his book *Interest and Prices*, Woodford (2003) presents his model of a cashless economy. This model forms the basis for the rest of his book as well as literature outside his book (e.g., Aoki, 2003). If Woodford is clear about anything, it is that the basis for price determination in his model is the Fisher equation. On page 73, Woodford states that, "I then have a system of two equations at each date, (1.15) and (1.21), to determine the two endogenous variables $P_t$ and $i_t$ …" Woodford's (1.15) just states that the nominal interest rate is determined by the central bank.[5] Therefore, with that nominal interest rate determined, Woodford's (1.21) is his supposed basis for price determination in his model, but if and only if the central bank follows certain policies according to Woodford.

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the literature has anyone explained why assuming a solution is bounded is appropriate if one has no initial condition, but is not appropriate when one has an initial condition.

[4] In February and again at the beginning of April 2004, we asked Dr. Woodford directly why he thinks it is appropriate to assume a bounded solution. However, as of the time this paper is written (May 7, 2004); Dr. Woodford has not responded.

[5] Eagle (2004a) argues that the central bank in Woodford's model in fact cannot affect the nominal interest rate that will be paid on loans not issued by the central bank.

Woodford's (1.21) states that $1+i_t=\beta^{-1}\left\{E_t\left[\frac{u_c(Y_{t+1};\xi_{t+1})}{u_c(Y_t;\xi_t)}\frac{P_t}{P_{t+1}}\right]\right\}^{-1}$ where $i_t$, $P_t$, and $Y_t$ are respectively the nominal interest, the price level, and aggregate supply (income) at time t. Woodford assumes a representative consumer who has a time preference factor of β and a time-separable utility function, whose derivative with respect to consumption is $u_c$. The stochastic component to utility, $\xi_t$, and aggregate supply are exogenous to Woodford's model. Woodford (2003, p. 71) states that his equation (1.21) "takes the form of a 'Fisher equation for the nominal interest rate, where the intertemporal marginal rate of substitution of the representative household plays the role of the real interest factor."

For the rest of this paper, we simplify the presentation of Woodford's (1.21) by defining $\Gamma_{t+1}\equiv\frac{u_c(Y_{t+1};\xi_{t+1})}{u_c(Y_t;\xi_t)}$, which then is an exogenous stochastic variable. The reciprocal of Woodford's (1.21) can then be written as:

$$\frac{1}{1+i_t}=\beta\cdot E_t\left[\Gamma_{t+1}\frac{P_t}{P_{t+1}}\right] \tag{3}$$

Equation (3) is a nonlinear difference equation. Woodford used a first-order Taylor's approximation to log linearize (3) into the following linear difference equation:

$$\hat{i}_t=\hat{r}_t+E_t[\hat{P}_{t+1}]-\hat{P}_t+E_t[\pi^*_{t+1}] \tag{4}$$

where $E_t[\pi^*_{t+1}]$ is related to the targeted inflation rate.[6] For now, just think of $\hat{i}_t$, $\hat{r}_t$, and $\hat{P}_t$ as the log-linearized values for the nominal interest rate, the real interest rate, and the price level. In particular, $\hat{P}_t\equiv\ln\left(\frac{P_t}{P^*_t}\right)$ where $P^*_t$ is the targeted price level at time t. Note that $\hat{P}_t$ is therefore a distance measure of the actual price level from the targeted price level, just as in the rocket example in the beginning of this paper, $\hat{x}_t \equiv \left| x_t - x_t^* \right|$.

Woodford then assumes an interest-rate targeting policy with feedback from prices:

$$\hat{i}_t = \varphi_P \hat{P}_t + v_t \tag{5}$$

Substituting this into (4), he gets:

$$(1+\varphi_P)\hat{P}_t = \hat{r}_t + E_t[\hat{P}_{t+1}] + E_t[\pi^*_{t+1}] - v_t \tag{6}$$

He then solves this forward to get:

$$\hat{P}_t = \frac{E_t[\hat{P}_T]}{(1+\varphi_P)^{T-t}} + \sum_{j=0}^{T-t-1} \left( \frac{E_t[\hat{r}_{t+j} + \pi^*_{t+j+1} - v_{t+j}]}{(1+\varphi_P)^{j+1}} \right) \tag{7}$$

Next, he assumes his solution is bounded, which implies that $\lim_{t\to\infty} \frac{E_t[\hat{P}_T]}{(1+\varphi_P)^{T-t}} = 0$. He concludes that his unique bounded solution for prices is:

$$\hat{P}_t = \sum_{j=0}^{\infty} \left( \frac{E_t[\hat{r}_{t+j} + \pi^*_{t+j+1} - v_{t+j}]}{(1+\varphi_P)^{j+1}} \right) \tag{8}$$

Because he derived a unique bounded solution, Woodford concludes that prices are therefore determined in his model and his model is complete. This is analogous to the rocket example of concluding that the direction of the rocket is determined because there is a unique bounded solution for the distance $\hat{x}_t$ even though no mechanism exists to steer the rocket.

For finite models, a necessary condition for a model to be complete is that the model must have at least as many equations as unknowns. However, for infinite models counting the number of equations and unknowns often will not work. If you have an infinite number of

[6] We are merely presenting Woodford's log linearization; we are not stating it is correct. We have some suspicions that the term $E_t[\pi^*_{t+1}]$ should not be included, but that would be a digression from the theme of this paper.

equations and an infinite number of unknowns, then the number of equations equals the number of unknowns. However, the infinite number of unknowns also is n times the infinite number of equations for any finite number n. This is true even though there is a one-to-one mapping of the number of unknowns to the number of equations, since any countable infinity is a one-to-one mapping of any other countable infinity. The veil of infinity and the assumption of a bounded solution are hiding the incompleteness of Woodford's model. The next section argues against the rational expectations precedent of assuming a bounded solution. Section V then removes the veil of infinity to demonstrate the incompleteness of Woodford's model.

## IV. Solving Expectational Difference Equations Correctly

Consider the following simple expectational difference equation:

$$E_{t-1}[y_{t+1}] = aE_t[y_t] + c_t x_t$$

where $y_t$ is some endogenous variable and $x_t$ is an exogenous variable. The rational expectations literature has established the following precedent for solving this difference equation:[7]

> Assume that the sequence $\{E_{t-1}[y_s]\}$ for s=t,t+1,… is bounded. Then (i) if |a|>1, we solve forward and assume the solution is bounded, (ii) if |a|<1, we solve backward, (iii) if |a|=1, we cannot solve the expectational difference equation.

This section shows that the foundations for this precedent are less than rigorous and the precedent will often lead to erroneous conclusions.

Much of the foundations for this precedent are in Sargent (1979). On page 177, Sargent claims that we can assume the solution is bounded "as we are free to do if no other side condition has been imposed …" On page 195, Sargent states, "Thus far, our advice to solve stable roots

[7] See Sargent (1979, p. 177)

backward and unstable roots forward has been to a certain extent arbitrary, being partly based on the desire to have solutions that are bounded…" On pages 195-200, Sargent provides an example where the assumption of a bounded solution is justified.

Other rational expectations literature supporting this precedent include Sargent and Wallace (1975) where they assume the solution is bounded to rule out "speculative bubbles." Later in this paper, we show that Sargent and Wallace's conclusion is correct because of their assumption about other exogenous stochastic processes not being too explosive. "Speculative bubbles," however, have no relevance.

Sargent (1986) did use a particular norm $l_\infty$, which does require solutions be bounded. However, to use such a norm, it must be appropriate to assume that the solutions are bounded. One should not try to apply the logic in the opposite direction. While one can and should use the appropriateness of assuming bounded solutions in order to use this norm, one should not use the norm to justify the assumption of bounded solutions.

In summary, our literature review into the foundations for the assumption that the solution be bounded included (i) a couple of examples where the assumption of a bounded solution was appropriate, (ii) a claim that "we are free" to assume the solution is bounded, and (iii) a possible misuse of the $l_\infty$ norm to justify the assumption. We found no mathematical proof to support making this assumption. These foundations are less than rigorous. Many users of this precedent have treated it as accepted fact and do make no references to any literature supporting this precedent.

Having weak foundations for the precedent does not, however, mean that the precedent is wrong. However, we now show that the precedent is wrong by presenting several examples where this precedent leads to incorrect solutions.

Our first example is an example from finance, which consists of two securities consistent with the following difference equation:

$$P_t = \beta \cdot E_t[P_{t+1}] + \beta \cdot C \tag{9}$$

where $P_t$ is the price of the security, β is the discount factor, and C is the annual constant payment on the security. The first security is an ordinary perpetuity, a security that pays C dollars at the end of each year forever.

The second security is a perpetuity combined with a put option to cash the security in at $S_t$, which we call the striking price of this security. At time 0, the striking price of the security is $S_0$. At the end of each year during which the security is held, the holder receives C dollars. Immediately after the C dollar payment, a coin is flipped. If tails, the striking price remains unchanged. However, if the flipped coin lands on heads, the new striking price at time t equals $\frac{2-\beta}{\beta} S_{t-1} - 2C$. In order that we need not worry about the strike price falling below zero, we assume that $S_0 \geq \frac{\beta C}{1-\beta}$, which insures that the strike price is nondecreasing. The one-period-ahead expected striking price equals $E_{t-1}[S_t] = \frac{1}{2}\left(\frac{2-\beta}{\beta} S_{t-1} - 2C\right) + \frac{1}{2} S_{t-1}$ which implies:

$$E_{t-1}[S_t] = \frac{S_{t-1}}{\beta} - C \tag{10}$$

The price of the second security must be at least the striking price because one can always cash the security in at the striking price.

The expectational difference equation (9) applies to both the first security and the second security. Equation (9) says that the price of the security today equals the present value of the price next year plus the present value of the constant annual payment.

If we solve forward (9), we conclude that $P_t = \lim_{n\to\infty} \beta^n E_t[P_{t+n}] + \sum_{j=0}^{\infty} \beta^{j+1} \cdot C$. Since $\beta < 1$, the rational expectations precedent's assumption that the solution is bounded implies that the $\lim_{n\to\infty} \beta^n E_t[P_{t+n}] = 0$. However, doing so results with the following price:

$$P_t = \frac{\beta C}{1-\beta} \tag{11}$$

While equation (11) is the price of the ordinary perpetuity, it is not the price of the second security, since the price of the second security cannot be less than its striking price. Therefore, the rational expectations precedent produces an erroneous price for the second security.

Since the rational expectations precedent for assuming a solution is bounded is not universally justified, we need more rigorous procedures for solving expectational difference equations. We propose always following the approach used by Sargent (1979, pp. 195-2000) where he assumes a finite horizon to determine the terminal condition and then takes the limit of that terminal condition as the horizon goes to infinity.

To see what is appropriate for the terminal conditions of the two securities in this example, assume finite versions of both of these securities to determine the appropriate terminal conditions, and then take the limit as the horizon of the securities goes to infinity. A finite version of the ordinary perpetuity would be an ordinary annuity, a security that pays C dollars at the end of each year until time T. Equation (9) applies for t=0,1,2, …,T-1 only. Therefore, we can only solve (9) forward until time T to get:

$$P_t = \beta^{T-t} E_t[P_T] + \sum_{j=0}^{T-t-1} \beta^{j+1} \cdot C \tag{12}$$

At time T, the price of this first security would be zero, because there will be no further payments of C beyond time T. Hence,

$$P_t = \sum_{j=0}^{T-t-1} \beta^{t+1} \cdot C = C\beta\left(\frac{1-\beta^{T-t}}{1-\beta}\right) \tag{13}$$

The above is the value of an ordinary annuity with T-t payments remaining. Taking the limit of (13) as T goes to infinity does give (11) as the price or present value of an ordinary perpetuity.

Now, look at a finite version of the second security. Again, equation (9) applies for t=0,1,2, …,T-1 only. Solving forward again gives us (12). At time T, the holder of the second security will exercise the option at the strike price at that time. Therefore, $P_T = S_T$ and hence, $E_t[P_T] = E_t[S_T]$. We can derive $E_t[S_T]$ from equation (10), which is another expectational difference equation, the backward solution of which is:

$$E_t[S_T] = \frac{S_t}{\beta^{T-t}} - C\sum_{j=0}^{T-t-1} \frac{1}{\beta^j} \tag{14}$$

Since, $E_t[P_T] = E_t[S_T]$, substitute (14) into (12) to get

$P_t = \beta^{T-t}\left(\frac{S_t}{\beta^{T-t}} - C\sum_{j=0}^{T-t-1} \frac{1}{\beta^j}\right) + \sum_{j=0}^{T-t-1} \beta^{j+1} \cdot C$. This implies that $P_t = S_t$ regardless of the time horizon. Therefore, in the limit as T approaches infinity, $P_t = S_t$.

This example illustrates that applying the rational expectation precedent can lead to erroneous results. We cannot generalize from Sargent's one example that all solutions must be bounded for all examples and for all models. Instead, for any difference equation we try to solve forward, we need to study the difference equation in a model with a finite horizon in order to determine the appropriate terminal condition, and then take the limit of that terminal condition as horizon goes to infinity.

The example of these two securities demonstrates that the other part of the rational expectations precedent about solving expectational difference equations is also incorrect. The direction we solve an expectational difference equation depends not on whether that difference equation converges or diverges, but on the context of that difference equation. In financial models we solve forward as current prices derive their values from expected future prices. In other contexts, we need to solve backwards. For example, we solved equation (10) for the striking price backwards not forward because the striking price depends on current and past coin flips.

Now consider a different infinitely repeated game where the player starts out with $W_0$ amount of money at time 0. In each period, a coin is flipped. If heads, the player's money will change to $a(1+b)W_{t-1}$, and if tails, the player's money will change to $a(1-b)W_{t-1}$ where a is a positive constant and b is between 0 and 1. The player's expected money after the coin flip will be $E_{t-1}[W_t] = aW_{t-1}$. This is an expectational difference equation. Regardless whether a is greater than, less than, or equal to one; we still need to solve the difference equation back to $W_0$. Doing so gives us the solution that $E_0[W_t] = a^t W_0$.

If a>1, then the limit as t goes to infinity of expected money the player would have from this game is infinity. Imposing a restriction that the winnings are bounded would the winnings to always equal zero and that the person started the game with no money, contradicting our assumption that she did start with a positive amount of money. If a=1, then the player's expected money would equal her initial money holdings. If a<1, then the player's expected money holdings will go to zero as the game's horizon goes to infinity. Regardless of the value of a, we still need to solve the expectational difference equation backwards because the expected winnings is determined by current and past coin flips.

On the other hand, with financial models, usually we should solve the expectational difference equation forward rather than backward because the current price is determined by the future expected price. In particular, consider a stock that pays no dividend and will never pay any dividend. Its current price should equal:

$$P_t = \beta E_t[P_{t+1}] \tag{15}$$

where $\beta$ is a discount factor. While we normally think about the discount factor as being less than one, it could be greater than one in some circumstances, such as if this price was an inflation-adjusted price when the consumers' time preference factor is positive. Regardless, of the value of $\beta$; we should still solve (15) forward.

To solve the difference equation forward, assume a model with a finite horizon T. Equation (15) only applies for periods t=0,1,2,…,T-1; since the price level at time T cannot equal a function of $E_t[P_{t+1}]$ since there will be no time T+1. As a result, something else is needed to determine the price of the security at time T. When a firm's life ends, it is liquidated. Therefore, the price of the firm at time T should equal its liquidation value at time T. Assuming that we expect the liquidation value to grow at the rate of g over time, we get that $E_{t-1}[L_t] = L_t(1+g)$. This latter expectational difference equation is one that we should solve backwards to get that $E_t[L_T] = L_t(1+g)^{T-t}$. Therefore, $E_t[P_T] = E_t[L_T] = L_t(1-g)^{T-t}$. In order to solve for the current price of this stock, we need to solve the expectational difference equation (15) forward, which gives $P_t = \beta^{T-t}E_t[P_T]$. Substituting $L_t(1-g)^{T-t}$ for $E_t[P_T]$ gives the following terminal condition:

$$P_t = \beta^{T-t} L_0 (1+g)^{T-t} \tag{16}$$

Next, take the limit of the terminal condition (16) as T goes to infinity. We get three cases for this limit:

**Case 1**. $(1+g) = \beta^{-1}$. In this case, $P_t = L_t$.

**Case 2**. $(1+g) < \beta^{-1}$. In this case, it would be better to liquidate the firm rather than keep the firm as an ongoing concern. If the firm chooses to liquidate, then $P_t$ still equals $L_t$. However, if the firm chooses not to liquidate the firm even though that decision makes the shareholders worse off, then $P_t$ will be less than $L_t$.

**Case 3**. $(1+g) > \beta^{-1}$. In this case, the price will be infinite. However, the assumption that this expected growth rate is constant forever above the discount rate is highly unlikely. Nevertheless, if we do so assume, then the price should be infinite.

Assuming the solution is bounded would be inappropriate in this example. Making this assumption would have imply the price of stock always equals zero, a conclusion inconsistent with current financial theory.

Sargent and Wallace (1975, p. 248) solved a difference equation forward, to give them their equation (12). We reproduce their equation (12) below with some inconsequential formatting changes and with T-t-1 instead of n:

$$(1-J_0)E_{t-1}[p_t] = \sum_{j=0}^{T-t-1}\left(\frac{J_1}{1-J_0}\right)^j \left(E_{t-1}[X_{t+j}] + J_2 E_{t-1}[m_{t+j}]\right) + \left(\frac{J_1}{1-J_0}\right)^{T-t} E_{t-1}[p_T]$$

Sargent and Wallace assume that $\lim_{T\to\infty}\left(\frac{J_1}{1-J_0}\right)^{T-t} E_{t-1}[p_T] = 0$ to supposedly rule out "speculative bubbles."

Instead, apply the principles of this section and assume a finite model to determine the terminal condition. The price level at time T is determined by their equation (17), which some would call a money demand function, but which we call the structural velocity equation (See Eagle, 2004b). However, because the economy ends at time T, there is no interest rate from time T to the nonexistent time T+1. Therefore, let's assume Sargent and Wallace's (17) applies at time T but without the interest rate term. Then $p_T = -c_1 y_T - u_{3T} + m_T$. Taking expectations based on the information set at time t-1 gives $E_{t-1}[p_T] = -c_1 E_{t-1}[y_T] - E_{t-1}[u_{3T}] + E_{t-1}[m_T]$.

Based on Sargent and Wallace's definition of the exogenous variable $X_T$, we conclude that $E_{t-1}[X_T] = -c_1 E_{t-1}[y_T] - E_{t-1}[u_{3T}]$ and therefore, $E_{t-1}[p_T] = E_{t-1}[X_T] + E_{t-1}[M_T]$. Substituting this into Sargent and Wallace's equation (12) gives:

$$(1 - J_0)E_{t-1}[p_t] = \sum_{j=0}^{T-t} \left( \frac{J_1}{1 - J_0} \right)^j \left( E_{t-1}[X_{t+j}] + J_2 E_{t-1}[m_{t+j}] \right)$$

Taking the limit of this as T goes to infinity and making Sargent and Wallace's assumption that the processes of $E_{t-1}[X_{t+j}]$ and $E_{t-1}[m_{t+j}]$ "not be too explosive," we get the following finite value for $E_{t-1}[p_t]$:

$$E_{t-1}[p_t] = \sum_{j=0}^{\infty} \left( \frac{J_1}{1 - J_0} \right)^j \frac{\left( E_{t-1}[X_{t+j}] + J_2 E_{t-1}[m_{t+j}] \right)}{(1 - J_0)}$$

Sargent and Wallace were justified in reaching the above conclusion, not by anything to do with "speculative bubbles," but by their assumption of "not too explosive" processes for $E_{t-1}[X_{t+j}]$ and $E_{t-1}[m_{t+j}]$,

In this section, we found that the rational expectations precedent for solving expectational difference equations is neither correct nor rigorous. In its place, this section provides the following two principles:

1. Whether we solve an expectational difference equation forward or backward depends not on whether the difference equation diverges or converges but rather on the context of the difference equation.
2. When we do need to solve an expectational difference equation forward, then we need to assume a version of the model with a finite horizon to determine the appropriate terminal condition, and then take the limit of that terminal condition as we let the horizon approach infinity.

These principles are not meant to be all encompassing. Future research is likely to reveal additional principles and possible qualifications to the ones listed here.

## V. Applying Revised Procedures to Woodford's Model

The best information we have indicates that Woodford assumed his solution is bounded because of the flawed precedent of doing so when solving expectational difference equations. Based on our analysis in the last section, we reject this precedent and replace it with determining the terminal condition in a finite model and then taking its limit as the horizon goes to infinity. This section applies this requirement to Woodford's model.

We reproduce (7) below:

$$\hat{P}_t = \frac{E_t[\hat{P}_T]}{(1+\varphi_P)^{T-t}} + \sum_{j=0}^{T-t-1}\left(\frac{E_t[\hat{r}_{t+j} + \pi^*_{t+j+1} - v_{t+j}]}{(1+\varphi_P)^{j+1}}\right) \tag{7}$$

This is the result of Woodford's solving his expectational difference equation (6) forward. Instead of assuming that the sequence $\{P_t\}$ is bounded, assume a version of Woodford's model with a finite horizon T, and then determine the terminal condition for $\hat{P}_T$.

If we assume today is time 0, then Woodford's Fisher equation (3) and hence log-linearized Fisher equation (4) only apply to times t=0,1,…T-1. Since there is no time T+1, there can be no loans from time T to T+1, and hence there is no interest rate $i_T$. Hence, Woodford's equation (4) cannot be used to determine $\hat{P}_T$. All that exists in Woodford's model of a cashless economy to supposedly determine prices is Woodford's Fisher equation or a log linearization of it. Thus, nothing exists to determine $\hat{P}_T$ in any finite version of Woodford's model. Without the flawed rational expectations literature's precedent for assuming that the solution of an expectational difference equation is bounded, Woodford's model is revealed to be incomplete.

Determining whether finite models are complete or not is relatively straight forward. A necessary condition for model completeness is that there must be at least as many equations as unknowns. If we count the number of equations and unknowns and find that the number of unknowns exceeds the number of equations, the incompleteness of the model is clear. However, counting tells us very for models where both the number of equations and unknowns are countably infinite. While one could state the number of equations and unknowns are in some sense equal, one can also state that the number of unknowns equals n times the number of equations for any finite and positive integer n.

The logical remedy is to look at a finite version of the model, count the number of equations and unknowns and then take the limit as the horizon goes to infinity. To count the number of unknowns and equations, we need to take into account the states of nature. Assume there are n states of nature per node in a decision tree as would be the case with an n-side die thrown at each point in time to determine the various outcomes.

First, count the unknown prices. There would be one at time 0, n at time 1, $n^2$ at time 2, …, $n^t$ at time t, etc. Therefore, with a finite horizon T, the number of unknowns would be $1+n+n^2+...+n^t+...+n^T$, which equals $\sum_{t=0}^{T} n^t$.

Other than transversality conditions, the sole basis for determining prices in Woodford's model is Woodford's Fisher equation. There would be one Fisher equation at time 0, n at time 1, $n^2$ at time 2, …, $n^t$ at time t, etc. However, there would be no equations at time T with no time T+1 to settle loans, no interest rate and hence no Fisher equation exists at time T. Therefore, the number of equations equals $\sum_{t=0}^{T-1} n^t$.

The ratio of unknowns to the number of equations will be $\frac{\sum_{t=0}^{T} n^t}{\sum_{t=0}^{T-1} n^t} = n + \frac{1}{\sum_{t=0}^{T-1} n^t}$. In other words, no matter how many periods that consumers live, there will be more than n times as many unknowns as there are equations. In the limit, as the number of time periods go to infinity, the ratio of unknowns to equations will be n. Thus, Woodford's Fisher equation (3) falls drastically short of being able to determine prices for any finite time horizon, and is unable to determine prices when the horizon goes to infinity. Once we see through the veil of infinity, we see Woodford's model is incomplete.

## VI. Seeing Through the Smoke and Mirrors of Woodford's Logic

In the previous section, we showed that Woodford's model does not determine prices because his model is incomplete. In this section we show that if one still believes that his model does determine prices, then the reason why Woodford's policy rule (5) with $\varphi_P > 0$ leads to a unique <u>bounded</u> solution is that this policy rule is exploding or self destructive. As did Woodford, define $P_t^*$ to be the central bank's targeted price level. Also, define $\Pi_{t+1}^* \equiv \frac{P_{t+1}^*}{P_t^*}$, which equals one plus the targeted inflation rate from time t-1 to time t. Now define $\tilde{P}_t \equiv \frac{P_t}{P_t^*}$, which is the ratio of the price level to its target. We can then rewrite (3) as:

$$\frac{1}{1+i_t} = \beta \cdot E_t\left[\Gamma_{t+1} \frac{P_t^* \tilde{P}_t}{P_{t+1}^* \tilde{P}_{t+1}}\right] = \beta \cdot \frac{P_t^*}{P_{t+1}^*} E_t\left[\Gamma_{t+1} \frac{\tilde{P}_t}{\tilde{P}_{t+1}}\right] = \beta \cdot \frac{1}{\Pi_{t+1}^*} E_t\left[\Gamma_{t+1} \frac{\tilde{P}_t}{\tilde{P}_{t+1}}\right]$$

(We assume the public knows the targeted prices levels.). Next define $\bar{i}_t \equiv \frac{\Pi^*_{t+1}}{\beta} - 1$. Note that this is defined more generally than Woodford's (2003, p. 78) definition to allow different targeted inflation rates for different periods. We then get:

$$\frac{1}{1+i_t} = \frac{1}{1+\bar{i}_t} E_t\left[\Gamma_{t+1}\frac{\tilde{P}_t}{\tilde{P}_{t+1}}\right] \tag{17}$$

For now, let's assume perfect foresight to the above, so that we can rewrite it as:

$$\tilde{P}_{t+1} = \frac{1+i_t}{1+\bar{i}_t}\Gamma_{t+1}\tilde{P}_t \tag{18}$$

Taking the natural logarithm of both sides gives $\ln(\tilde{P}_{t+1}) = \ln\left(\frac{1+i_t}{1+\bar{i}_t}\right) + \ln(\Gamma_{t+1}) + \ln(\tilde{P}_t)$, which we can rewrite as:

$$\hat{P}_{t+1} = \hat{i}_t + \gamma_{t+1} + \hat{P}_t \tag{19}$$

where $\hat{P}_t \equiv \ln(\tilde{P}_t)$, $\hat{i}_t \equiv \ln\left(\frac{1+i_t}{1+\bar{i}_t}\right)$,[8] and $\gamma_{t+1} \equiv \ln(\Gamma_{t+1})$. Once again, $\hat{P}_t$ is a measure of the distance of the actual price level from the targeted price level and is analogous to the $\hat{x}_t \equiv \left|x_t - x_t^*\right|$ distance measure in the rocket example at the beginning of this paper.

Now consider the solutions to (19) by case.[9] Case 1 is where $\hat{i}_t + \gamma_{t+1}$ is a positive constant for all t. Then $\hat{P}_t$ increases over time with the constant slope of $\hat{i}_t + \gamma_{t+1}$ and only

---

[8] Woodford defines $\hat{i}_t \equiv \ln\left(\frac{1+i_t}{1+\bar{i}}\right)$ where $\bar{i} = \frac{\bar{\pi}}{\beta} - 1$ where $\bar{\pi}$ is an assumed constant inflation rate plus one. He later assumes that $\bar{\pi}$ =0. Our formulation of $\hat{i}_t$ is really a generalization of Woodford's formulation.

[9] The solutions to (11) should correspond to Woodford's Proposition 2.5, however, Woodford's language of that proposition is confusing. It appears to us that the "tight-enough bounds … on the interest-rate target process" in essence restricts those interest rates to Woodford's $\bar{i}$ , not just close but exactly.

unbounded solutions exist. Case 2 is where $\hat{i}_t + \gamma_{t+1}$ is a negative constant for all t. Then $\hat{P}_t$ decreases over time with the constant slope of $\hat{i}_t + \gamma_{t+1}$ and again only unbounded solutions exist. Case 3 is where $\hat{i}_t + \gamma_{t+1}$=0 for all t. Then $\hat{P}_{t+1} = \hat{P}_t$. In this case, there will be an uncountably infinite number of solutions as can be seen in figure 3.[10]

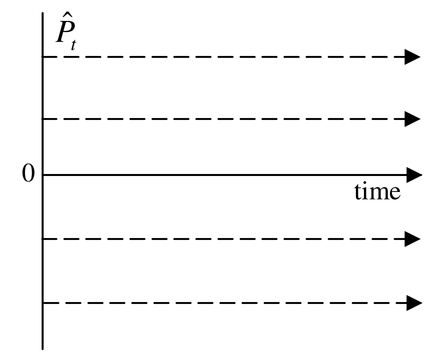

Figure 3: Solutions in case 3 when $\hat{i}_t + \varphi_{t+1} = 0$ for all t.

While he acknowledges that prices are not uniquely determined for exogenous interest rates, Woodford (2003, p. 87) argues that a policy rule that sets this variable with a feedback from the system can uniquely determine prices at least in a bounded sense. Consider the policy rule that the central bank sets the interest rate variable $\hat{i}_t$ so that $\hat{i}_t = \varphi(\hat{P}_t)$ where $\varphi(0) = 0$. To simplify, let's also assume that $\gamma_t = 0$ for all t. Then equation (4) becomes:

$$\hat{P}_{t+1} = \hat{P}_t + \varphi(\hat{P}_t) \tag{20}$$

Consider such a policy where $\varphi'(\hat{P}_t) < 0$ as shown in Figure 4. Under this policy, if the price level is below its target (i.e. $\hat{P}_t < 0$), then $\varphi(\hat{P}_t) > 0$ and (20) implies that $\hat{P}_{t+1} > \hat{P}_t$ so that the price level will increase towards its targeted level. On the other hand, if the price level is above its target (i.e., $\hat{P}_t > 0$), then $\varphi(\hat{P}_t) < 0$ and (20) implies that

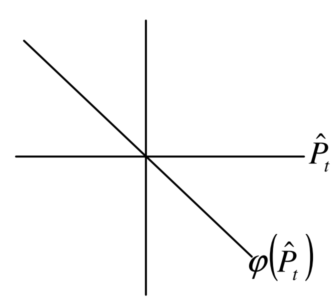

Figure 4: Graph of $\varphi(\hat{P}_t)$ when $\varphi(0) = 0$ and $\varphi'(\hat{P}_t)$<0.

[10] There are other cases in addition to these three depending on how $i_t + \varphi_t$ change over time.

$\hat{P}_{t+1} < \hat{P}_t$ so that the price level will decrease towards its targeted level. This policy rule would be a self-correcting policy rule as can be seen in Figure 5.

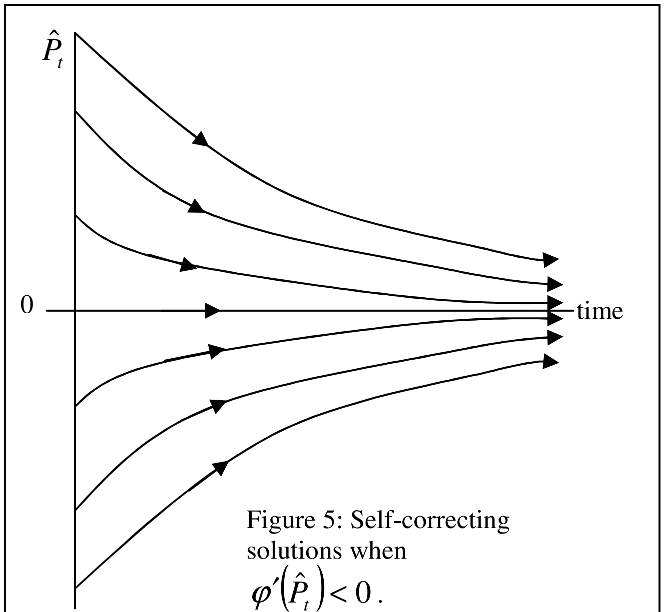

Figure 5: Self-correcting solutions when $\varphi'(\hat{P}_t) < 0$.

Normally, a self-correcting policy is considered to be a good policy. However, because a self-correcting policy has many bounded solutions, Woodford could not claim this policy would determine prices even according to his infinite implicit function theorem.

Now consider the policy Woodford actually did recommend where $\varphi'(\hat{P}_t) > 0$ as shown in Figure 6. Under this policy, if the price level is below its target (i.e. $\hat{P}_t < 0$), then $\varphi(\hat{P}_t) < 0$ and (20) implies that $\hat{P}_{t+1} < \hat{P}_t$ so that the price level will decrease away from its targeted level. On the other hand, if the price level is above its target (i.e., $\hat{P}_t > 0$), then $\varphi(\hat{P}_t) > 0$ and (20) implies that $\hat{P}_{t+1} > \hat{P}_t$ so that the price level will increase away from its targeted level. Rather than self-correcting, this policy rule is self-destructive as seen in Figure 7. However, because all non-zero solutions are self-destructive in this case, there is a unique bounded solution where $\hat{P}_t = 0$ for all t.

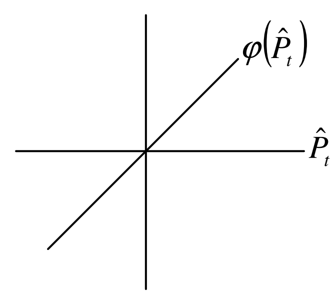

Figure 6: Graph of $\varphi(\hat{P}_t)$ when $\varphi(0) = 0$ and $\varphi'(\hat{P}_t)$>0.

By having the central bank follow a self-destructive policy, Woodford made all the other solutions “not count” so that he could claim his model “determines prices.” This line of logic, if it is viewed as being sound, would encourage economists to recommend self-destructive policies in order to claim their models to be complete and all the unknowns in their model are determined.

Remember the rocket example at the beginning of the paper. If the rocket had a course correction mechanism to get it on target, then there would be an infinite number of bounded solutions in that example similar to Figure 5. However, if it had a mechanism to push it even more off course whenever it was off course, then that would help guarantee a unique bounded solution in that example.

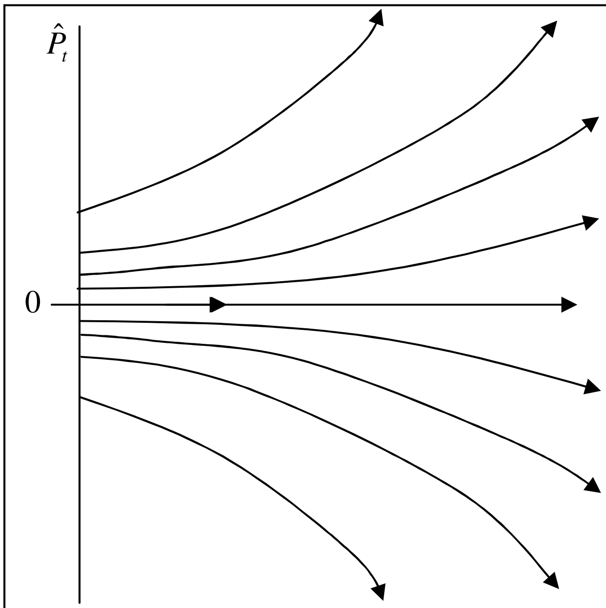


Figure 7. Self-destructive solutions when $\varphi'\left(\hat{P}_t\right) > 0$

For current real world economies with monetary frictions, Woodford's policy is useful and is not self destructive. The previous discussion about his policy being self destructive is only relevant to Woodford's model of a cashless economy. In that model, his policy is self destructive if his policy does affect prices. However, for the reasons we presented in section V, we do not think Woodford's policy is able to affect prices in his model.

## VII. Other Problems with Woodford's Argument

This section discusses the following additional problems with Woodford's argument:

(i) Discontinuities exist between the finite implicit function theorem and Woodford's infinite implicit function theorem.

(ii) Inferring causality from Woodford's unique bounded solution would lead one to conclude that central banks have a "magical power" of thought.

(iii) Woodford's logic would lead one to conclude that economists should direct policymakers to ignore feedback that may indicate their policy is off track

(iv) Inferring causality from the exact bounded solution to Woodford's model would lead one to conclude that the current price level is determined by the realized values of future events.

Woodford presents his infinite implicit function theorem in his Appendix A.3 as The Infinite Implicit Function Theorem rather than a theorem. While we do not deny the validity of this theorem, we do consider Woodford's use of it to be unsound.

Discontinuities exist between the finite implicit function theorem and Woodford's infinite implicit function theorem as can be exemplified with the difference equation (1). Suppose that equation (1) is all the information we have. Now consider time going from 1,2,…,T where T represents the finite time horizon of our system. With T periods, there are T unknowns, the $P_t$'s for t=0,1,2,…,T. Equation (1) does not apply to time t=T because time t=T+1 does not exist. Therefore, there are only T-1 equations, one less equation than the T unknowns. By the finite version of the implicit function theorem, our model is incomplete. In the limit as T goes to infinity, the difference between the number of unknowns and the equations is one. This means that no matter how large but finite T gets, we will always be short one equation.

The reason we cannot get a unique solution in the finite case is because all the examples in Figure 2 would be considered as legitimate solutions when T is finite. However, when T becomes infinite and we apply Woodford's infinite version of the implicit function theorem, all the $\{P_t\}$ sequences which were allowed under the finite implicit function theorem become unacceptable because they are unbounded except for the one where $P_t$=0 for all t. This is a discontinuity between the finite implicit function theorem and Woodford's infinite version, as in any neighborhood of infinity, we can find a finite T such that an uncountably infinite number of acceptable solutions exist, yet when T equals infinity only one exists.

A second discontinuity exists when we do have an initial condition $P_0 > 0$. From this initial condition and (1) we can compute $\{P_t\}$ for any finite time horizon T. However, if T becomes infinity, then Woodford's infinite implicit function theorem says that no acceptable solution exists because the resulting "solution" is unbounded (unless $P_0$ happens to be zero). This is a discontinuity, because in any neighborhood of infinity, we can find a finite T to which an acceptable solution does exist by the finite version of the implicit function theory. However, since no bounded solution consistent with $P_0 > 0$ exists, Woodford's infinite version concludes no acceptable solution exists.

Woodford uses his infinite implicit function theorem to infer that prices are "determined" in his model. That there is only one bounded solution does not mean any mechanism exists to cause this unique bounded solution to occur rather than some unbounded solution. This is analogous to the rocket example at the beginning of this paper, where there was nothing to cause the direction of the rocket to be the targeted direction.

Consider the special case where $E_t\left[\hat{r}_{t+j} + \pi^*_{t+j+1} - v_{t+j}\right] = 0$ for all t and for all j=0,1,2,… . Then, (8) implies that $\hat{P}_t = 0$ for all t. Since $\hat{P}_t \equiv \ln\left(\frac{P_t}{P_t^*}\right)$, this means that the price level always equals its targeted level. If we infer causality from (8), then if the central bank thinks that its targeted prices should forever be 100, then the solution $\hat{P}_t = 0$ for all t says that in fact the price will forever be 100; if the central bank thought the targeted prices should forever be 200 instead of 100, then $\hat{P}_t = 0$ for all t says that in fact the price will forever be 200 instead of 100. In both cases, if $\hat{P}_t = 0$ and $v_t$=0 for all t, $\hat{i}_t$=0 for all t by Woodford's policy rule in (5). In other words, the interest rate is the same regardless whether the price is 100 or 200. If the central bank could change prices just by thinking about different targets, it would be like the control center in the rocket example, changing the rocket's direction by thinking about a different targeted direction.

For another example, define the following alternative policy rule in place of (5):

$$\text{Policy rule \#2: } \hat{i}_t = \begin{cases} \varphi_P(\hat{P}_t) \text{ if } \hat{\mathrm{P}}_0 = 0 \\ \varphi_P(\hat{P}_t - \hat{P}_0) \text{ otherwise} \end{cases}$$

In policy rule #2, the central bank checks to see if $\hat{P}_0 = 0$ as (8) says it should in this example. If $\hat{P}_0 = 0$, then the central bank will continue with Woodford's policy rule (5). Only if $\hat{P}_0 \neq 0$, will the central bank follow $\hat{i}_t = \phi_P(\hat{P}_t - \hat{P}_0)$. This second rule differs from Woodford's rule by including a correction in case $\hat{P}_0 \neq 0$, which is something that should never happen according to (8).. In essence the second rule corrects the price targets for when $\hat{P}_0 \neq 0$.

There are an uncountably infinite number of bounded solutions with this second policy rule. One solution is $\hat{P}_t = 0$ for all t, which was the unique bounded solution for policy rule (5). In addition, $\hat{P}_t = \hat{P}_0$ for all t would also be bounded solutions for policy rule #2 for any $\hat{P}_0 \in \mathrm{R}$. Note that the central bank only thinks differently when following policy #2 instead of policy rule (5). When $\hat{P}_t = \hat{P}_0$ for all t, the log-linearized interest rates set by policy rule #2 are zero for all t, just as they were under policy rule (5).

In order to get a unique bounded solution, Woodford would have to direct the central bank not to use policies like policy rule #2 that have feedback. Giving too much credence to

Woodford's infinite implicit function theorem would not only result in economists designing self-destructive policies rather than self-corrective ones, but it would also encourage economists to direct policy makers to put their heads in the sand rather than looking at feedback after implementing a policy to make sure it is on track.

While Woodford relied on a Taylor's approximation in order to obtain price determination, we have derived the exact unique bounded solution to Woodford's model. Before we show that solution, consider a special case where nominal aggregate demand is known one period in advance. This special case could be justified by nominal income targeting by the authority that affects nominal aggregate demand.[11] Let $N_t$ represent the nominal aggregate demand at time t. Therefore, $N_{t+1}$, which equals $P_{t+1}Y_{t+1}$, is known with certainty at time t. However, because $Y_{t+1}$ is not known, both $P_{t+1}$ and $Y_{t+1}$ are stochastic. Assume also that consumers' utility is the natural logarithm. In other words, $u(Y_t;\xi_t)=\ln(Y_t)$. Then $u_c(Y_t;\xi_t)=1/Y_t$.

By our definition of $\Gamma_{t+1}$, we conclude that $\Gamma_{t+1} \equiv \frac{u_c(Y_{t+1};\xi_{t+1})}{u_c(Y_t;\xi_t)} = \frac{1/Y_{t+1}}{1/Y_t} = \frac{Y_t}{Y_{t+1}}$. Therefore, (3) can be written as $\frac{1}{1+i_t} = \beta \cdot E_t\left[\frac{Y_t}{Y_{t+1}}\frac{P_t}{P_{t+1}}\right]$. At time t, we know $P_t$ and $Y_t$. Also, by our nominal income targeting assumption, the public knows the product of $P_{t+1}Y_{t+1}$. Therefore, $\frac{1}{P_tY_t} = \beta \cdot (1+i_t) \cdot \frac{1}{P_{t+1}Y_{t+1}}$. Where $\tilde{P}_t \equiv \frac{P_t}{P_t^*}$, $\Pi_{t+1}^* \equiv \frac{P_{t+1}^*}{P_t^*}$, and $\bar{i}_t \equiv \frac{\Pi_{t+1}^*}{\beta} - 1$, we again get:

$$\tilde{P}_{t+1} = \frac{1+i_t}{1+\bar{i}_t}\Gamma_{t+1}\tilde{P}_t \qquad (18)$$

[11] See Eagle and Domian (2003, and 2004) for a model of a cashless economy with temporary money. The temporary money authority then is the authority that determines nominal aggregate demand.

Earlier in this paper, we derived (18) when we assumed perfect foresight. Here, however, both $\tilde{P}_{t+1}$ and $\Gamma_{t+1} = \frac{Y_t}{Y_{t+1}}$ are stochastic. Taking natural logarithms of both sides of (18) and defining $\hat{i}_t \equiv \ln\left(\frac{1+i_t}{1+\bar{i}_t}\right)$, and $\gamma_{t+1} \equiv \ln(\Gamma_{t+1}) = \ln\left(\frac{Y_t}{Y_{t+1}}\right)$, we get:

$$\hat{P}_{t+1} = \hat{i}_t + \gamma_{t+1} + \hat{P}_t \qquad (19)$$

Once again, please realize that $\hat{P}_{t+1}$ and $\gamma_{t+1}$ are stochastic and hence unknown at time t. Solving (19) forward gives $\hat{P}_t = \frac{\hat{P}_{t+k}}{(1+\phi_P)^k} - \sum_{j=0}^{k-1} \frac{\gamma_{t+j+1} + v_{t+j}}{(1+\varphi_P)^{j+1}}$. Taking the limit as k goes to infinity, we get $\hat{P}_t = \lim_{k\to\infty} \frac{\hat{P}_{t+k}}{(1+\varphi_P)^k} - \lim_{k\to\infty} \sum_{j=0}^{k-1} \frac{\gamma_{t+j+1} + v_{t+j}}{(1+\varphi_P)^{j+1}}$ Now if we assume the solution is bounded, then we get that:

$$\hat{P}_t = -\sum_{j=0}^{\infty} \frac{\gamma_{t+j+1} + v_{t+j}}{(1+\varphi_P)^{j+1}} \qquad (21)$$

However, (21) expresses $\hat{P}_t$ as a function of future values of $\gamma_t$ and v=t. To interpret Woodford's infinite implicit function theorem in a causal sense would be a mistake. Equation (21) shows a condition that must hold in order for the sequence of $\hat{P}_t$ to be bounded; we should not interpret it to represent causation.

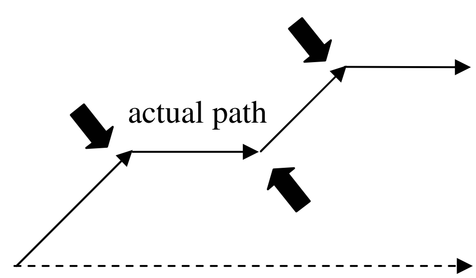


Figure 8. Target and actual rocket paths with future forces

In the original rocket example at the beginning of this paper, we assumed that there were no future forces on the rocket. Instead, assume the rocket is subject to the three future forces in Figure 8 regardless of the direction it is initially headed. Then in order for the distance between the actual and targeted rocket locations to be bounded, the initial rocket's direction would have to be such that after the rocket encounters the forces, the path from then on would be parallel to the targeted path. As can be seen in Figure 8, rotating the actual path at all clockwise or counterclockwise would result with the distance between the actual and targeted rocket locations going to infinity as time goes to infinity. Hence, in order for the distance between the rocket's actual and desired locations to be bounded, the rocket's trajectory at time t must be off from the desired trajectory exactly by the amount required to offset all the future disturbances. Clearly, this does not mean those future disturbances caused the initial direction of the rocket. Similarly, we should make no causal inference from (21) being a function of future realized values.

That we should not interpret the unique bounded solution in a causal sense is even clearer with the precise solution to Woodford's model. Define $r_t \equiv \dfrac{1}{\tilde{P}_{t+1} E_t\left[\dfrac{\Gamma_{t+1}}{\tilde{P}_{t+1}}\right]}$. Note that $\tilde{P}_{t+1}$ is a random variable, its value may change for different states of nature. On the other hand, $E_t\left[\dfrac{\Gamma_{t+1}}{\tilde{P}_{t+1}}\right]$ is a constant at time t. Since $\tilde{P}_{t+1}$ is a random variable at time t, not a constant, we should not make the mistake of trying to bring the random variable $\tilde{P}_{t+1}$ inside the expectations operator. Also, note that $r_t$ is a random variable at time t. Some may argue that this should be labeled as $r_{t+1}$. However, in financial economics, $r_t$ is closely related to the real return from time t to time t+1, which using Woodford's symbolization should be $r_t$ not $r_{t+1}$.

From this definition of $r_t$, we easily conclude that $E_t\left[\dfrac{\Gamma_{t+1}}{\tilde{P}_{t+1}}\right] \equiv \dfrac{1}{r_t \tilde{P}_{t+1}}$. Below we reproduce (17), where we define $\bar{i}_t \equiv \dfrac{1+\pi^*_{t+1}}{\beta} - 1$:

$$\frac{1}{1+i_t} = \frac{1}{1+\bar{i}_t} E_t\left[\Gamma_{t+1}\frac{\tilde{P}_t}{\tilde{P}_{t+1}}\right] \tag{17}$$

Since at time t, $\tilde{P}_t$ is known, we can rewrite this as $\frac{1}{\tilde{P}_t} = \frac{1+i_t}{1+\bar{i}_t} E_t\left[\frac{\Gamma_{t+1}}{\tilde{P}_{t+1}}\right]$. Since $E_t\left[\frac{\Gamma_{t+1}}{\tilde{P}_{t+1}}\right] \equiv \frac{1}{r_t\tilde{P}_{t+1}}$, we conclude that $\frac{1}{\tilde{P}_t} = \frac{1+i_t}{1+\bar{i}_t}\left[\frac{1}{r_t\tilde{P}_{t+1}}\right]$. Taking natural logarithms of both sides gives $-\ln(\tilde{P}_t) = \ln\left(\frac{1+i_t}{1+\bar{i}_t}\right) - \ln(r_t) - \ln(\tilde{P}_{t+1})$, which can be rewritten as:

$$-\hat{P}_t = \hat{i}_t - \hat{r}_t - \hat{P}_{t+1} \tag{22}$$

where $\hat{r}_t \equiv \ln(r_t)$. Please note that both $\hat{r}_t$ and $\hat{P}_{t+1}$ are random variables, unknown at time t.

Rearranging (22) with elementary algebra and substituting Woodford's policy (5) that $\hat{i}_t = \varphi_P \hat{P}_t + v_t$, we get:

$$\hat{P}_{t+1} = (1+\varphi_P)\hat{P}_t + v_t - \hat{r}_t\ . \tag{23}$$

Solving this forward gives $\hat{P}_t = \frac{\hat{P}_{t+k}}{(1+\varphi_P)^k} + \sum_{j=0}^{k-1}\frac{\hat{r}_{t+j} - v_{t+j}}{(1+\varphi_P)^{j+1}}$. Taking the limit as k goes to infinity on both sides gives $\hat{P}_t = \lim_{k\to\infty}\frac{\hat{P}_{t+k}}{(1+\varphi_P)^k} + \lim_{k\to\infty}\sum_{j=0}^{k-1}\frac{\hat{r}_{t+j} - v_{t+j}}{(1+\varphi_P)^{j+1}}$. Since $\varphi_P > 0$, if we apply the bounded-assumption requirement of Woodford's infinite implicit function theorem to $\{\hat{P}_t\}$, it must be the case that $\lim_{k\to\infty}\frac{\hat{P}_{t+k}}{(1+\varphi_P)^k}$ =0. Therefore, we conclude that

$$\hat{P}_t = \sum_{j=0}^{\infty}\frac{\hat{r}_{t+j} - v_{t+j}}{(1+\varphi_P)^{j+1}} \tag{24}$$

This says that the current price level is a function of future values of $r_t$ and $v_t$, their realized values not their expected values.[12] Once again, we should not interpret (24) in a causal sense, but rather as a condition that must hold in order for the sequence of $\hat{P}_t$ to be bounded. However, if we don't interpret (24) as a causal relationship, we cannot use it to argue that prices are determined in Woodford's model.

Two examples that validate (24) are as follows:

**Example 1**: Define $G(t) \equiv u_c(Y_t;\xi_t)$. Then $\Gamma_{t+1} = \frac{G(t+1)}{G(t)}$. Let the random process that determines G(t) be such that $G(t) = \begin{cases} 0.9 \; \textit{with probability} \; 0.5 \\ 1.1 \quad \text{otherwise} \end{cases}$. Also, let $\varphi_P = 1.01$, and $\beta$=0.95.

In this example, it turns out that if $\hat{P}_t < -0.05$ then $\Gamma_{t+1} = \frac{1.1}{0.9}$, if $\hat{P}_t > 0.05$ then $\Gamma_{t+1} = \frac{0.9}{1.1}$, and otherwise $\Gamma_{t+1} = 1.0$. In other words,

$$\Gamma_{t+1} = \begin{cases} \frac{1.1}{0.9} & \text{if } \hat{P}_t < -0.05 \\ 1 & \text{if } -0.05 < \hat{P}_t < 0.05 \\ \frac{0.9}{1.1} & \text{if } \hat{P}_t > 0.05 \end{cases}$$

Basically, this is an example of one-period-ahead perfect foresight. If $\hat{P}_t$ is consistent with (24) and we know that ahead of time, we can deduce exactly the value of $\Gamma_{t+1}$ from the realization of $\hat{P}_t$, which enables us to perfectly predict $\hat{P}_{t+1}$. This example is built on the assumption that we know ahead of time that $\hat{P}_t$ is consistent with (24).

[12] While $r_{t+j}$ is an expected value, it is a conditional expected value given the information at time t+j. Therefore, (24) is saying that $\hat{P}_t$ is a function of expected values that are based on future information sets.

**Example 2**: This example assumes that (24) is just an equation of consistency, not one of causality. Under this assumption, Woodford's model is not complete. We complete it with temporary money as in Eagle and Domian (2003, 2004), assume logarithmic utility functions, and assume that the authority in charge of temporary money follows nominal income targeting. We have discussed this example at the beginning of this section. We now show that this example is consistent with the general case. In this example, $\Gamma_{t+1} = \frac{Y_t}{Y_{t+1}}$. Therefore,

$$E_t\left[\frac{\Gamma_{t+1}}{\tilde{P}_{t+1}}\right] = E_t\left[\frac{Y_t}{\tilde{P}_{t+1}Y_{t+1}}\right] = Y_t E_t\left[\frac{1}{\tilde{P}_{t+1}Y_{t+1}}\right] = Y_t E_t\left[\frac{P^*_{t+1}}{P_{t+1}Y_{t+1}}\right] = Y_t P^*_{t+1} E_t\left[\frac{1}{P_{t+1}Y_{t+1}}\right] = Y_t P^*_{t+1}\frac{1}{P_{t+1}Y_{t+1}}.$$

since $\tilde{P}_{t+1} = \frac{P_{t+1}}{P^*_{t+1}}$ and the nominal-income-targeting assumption. Therefore,

$$r_t \equiv \frac{1}{\tilde{P}_{t+1}E_t\left[\frac{\Gamma_{t+1}}{\tilde{P}_{t+1}}\right]} = \frac{1}{\tilde{P}_{t+1}Y_t P^*_{t+1}\frac{1}{P_{t+1}Y_{t+1}}} = \frac{1}{P_{t+1}\frac{Y_t}{P_{t+1}Y_{t+1}}} = \frac{Y_{t+1}}{Y_t}. \text{ Hence,}$$

$$\hat{r}_t \equiv \ln(r_t) = \ln\left(\frac{Y_{t+1}}{Y_t}\right) = -\varphi_{t+1} \equiv \ln\left(\frac{Y_t}{Y_{t+1}}\right).$$ Equations (24) and (21) are the same.

## VIII. Conclusion

The Fisher equation combined with various policies is the sole basis for price determination in Woodford's model of a cashless economy. His argument that his model is complete and determines prices relies on the precedent of assuming a solution is bounded when solving expectational difference equations. We traced the foundations of this precedent to a couple of examples and incorrect claims about "speculative bubbles," and being free to do so.

We demonstrated that the assumption of a bounded solution leads to incorrect solutions in several examples. We proposed replacing this precedent with the more rigorous approach of assuming a finite model to determine the appropriate terminal condition and then taking the limit of that terminal condition as the horizon goes to infinity. When we applied this to Woodford's model, we found that any finite version of the model is incomplete. In addition to replacing the assumption that the solution is bounded, we also argue that the direction in which we solve an expectational difference equation depends not on the divergence or convergence of the equation but by the context of the equation.

While it is true that there is a unique bounded solution to Woodford's model, we cannot infer causality from that unique bounded solution. Trying to do so will lead to all types of nonsensical results including (i) economists recommending self-destructive policies and policies without feedback, (ii) central banks changing price levels merely by thinking about different price targets, and (iii) today's price level being determined by future realized values.

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