An Algorithm for Solving Arbitrary Linear Rational Expectations Model

Paweł Kowal*

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Abstract

We consider solutions to general linear dynamic systems, possibly singular and non square with general stability conditions. Besides constructing a general algorytm for finding solutions we provide necessary and sufficient conditions for existence of a solution.

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1 Introduction

In this paper we present a method of solving linear rational expectations models. The method is based on computing deflating subspace associated with an appropriate matrix pair. Such a procedure is common for most of the methods of solving linear rational expectations models, i.e. Uhlig, (1995), Sims, (2001), Hansen, McGrattan and Sargent (1994), Blanchard and Kahn (1980). Existing methods can however be applied only to small subset of linear systems. One of the exception is the method proposed by Sims (2000) which can be applied to any regular system. These methods are based on generalized Schur decomposition or QZ decomposition, which are known to be numerically stable for any regular matrix pair, but numerically unstable for singular matrix pairs. In this way these methods cannot be extended directly to singular problems.

*Department of Economics, Warsaw School of Economics, email: pkowal3@sgh.waw.pl

In this paper we propose generalization of the method proposed by Sims which allow us considering nonsingular systems, in particular rectangular linear systems. This method is based on the GUPTRI decomposition proposed by Demmeland and Kågström, (1993), a generalization of generalized Schur decomposition for any matrix pair. Besides a standard solution, proposed method allows also for considering systems with many equilibria. Such a systems allow for sunspot equilibria in which non-fundamental stochastic disturbances influence model dynamics. The proposed method can deliver set of all sunspot equilibria.

Presented method is a generalization of the Sims algorithm also in another important dimension, i.e. we analize stability conditions (boundary conditions at infinity) more carefully. In case of general stability conditions there may exists solution to the model which cannot be constructed by appropriate selection of eigenvalues in the generalized Schur decomposition or the GUPTRI decomposition. This possibility is not considered in Sims, (2001).

The rest of the paper is organized as follows. Section 2 states the problem. Section 3 presents definitions and basic results from computational linear algebra. Section 5 presents properties of eigenvectors of a matrix pair. In section 6 we consider a matrix equation associated with the problem. Sections 4 and 7 present the method for solving the problem. Section 7 provides also necessary and sufficient conditions for existence of solution to the problem. Section 8 concludes.

2 The Problem

Let us consider the following linear system

$$0 = A_1 x_t + A_2 y_t + B_1 x_{t+1} + B_2 y_{t+1} + E_t \Big\{ C_1 x_{t+1} + C_2 y_{t+1} \Big\} + \epsilon_{t+1} \tag{1}$$

where $x \in \mathbb{R}^n$ is a vector of state variables, $y \in \mathbb{R}^m$ is a vector of control variables and $\{\epsilon_{t+1} \in \mathbb{R}^s\}$ is a vector of i.i.d. random variables, such that $E_t\{\epsilon_{t+1}\} = 0$. Operator E_t is a conditional expectation under information set $I_t = \{x_s, y_s, w_s; s \leq t\}$, which consists of all state and control variables up to period t as well as additional variables, w_t, discussed later.

Definition 2.1. *Solution to the problem (1) is defined as a set of, possibly time dependent, maps* $\{Y_t, P_t\}_{t=0}^{\infty}$

$$Y_t : \mathbb{R}^n \ni x \mapsto y = Y_t(x) \in \mathbb{R}^m$$

and transition matrices

$$P_{t+1} : \mathbb{R}^n \ni x_t \mapsto x_{t+1} = P_{t+1}(x_t) \in \mathbb{R}^n$$

such that

$$\begin{aligned} x_{t+1} &= P_{t+1}(x_t) \\ 0 &= A_1 x(t) + A_2 Y_t(x_t) + B_1 x_{t+1} + B_2 Y_{t+1}(x_{t+1}) \\ &\quad + E_t\{C_1 x_{t+1} + C_2 Y_{t+1}(x_{t+1})\} + \epsilon_{t+1} \end{aligned}$$

for each $x_t \in \mathbb{R}^n$, $\epsilon_{t+1} \in \mathbb{R}^s$, *and for each* t.

Definition 2.2. *The solution* $\{Y_t, P_t\}_{t=0}^{\infty}$ *is called linear if*

$$\begin{aligned} Y_t(x_t) &= Y_1 x_t + Y_2 w_t \\ P_t(x_t) &= P_1 x_t + P_2 w_t + P_3 \epsilon_{t+1} + P_4 \upsilon_{t+1} \end{aligned} \tag{2}$$

where $w_t \in \mathbb{R}^p$ *and*

$$w_{t+1} = S_1 x_t + S_2 w_t + \upsilon_{t+1}$$

and υ_{t+1} *is an i.i.d. random variable, possibly dependent on* ϵ_{t+1}.

Assumption 2.3. *We are looking for linear solutions to the system (1) such that for given matrices* H *and* Ξ *the following growth restriction holds*

$$\lim_{t\to\infty} E_0\Big\{\Xi^t H \begin{bmatrix} x_t \\ y_t \end{bmatrix}\Big\} = 0 \tag{3}$$

for any x_0 *and* w_0, *where,* $\Xi = \operatorname{diag}(\xi_1, \dots, \xi_k) \in \mathbb{R}^{k\times k}$ *is a diagonal matrix,* $H \in \mathbf{R}^{k,n+m}$.

The problem (1) can be represented in the following form

$$\begin{aligned} 0 &= \mathcal{A}\begin{bmatrix} x_t \\ y_t \end{bmatrix} + \mathcal{B}\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} + \omega_{t+1} \\ \omega_{t+1} &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} E_t\{x_{t+1}\} - x_{t+1} \\ E_t\{y_{t+1}\} - y_{t+1} \end{bmatrix} + \epsilon_{t+1} \end{aligned}$$

where

$$\mathcal{A} = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} B_1 + C_1 & B_2 + C_2 \end{bmatrix} \tag{4}$$

If $\{Y_t, P_t\}_{t=0}^{\infty}$ is a linear solution in the form (2) such that (3) holds then $x_{t+1} - E_t\{x_{t+1}\} = P_3\epsilon_{t+1} + P_4\upsilon_{t+1}$, $y_{t+1} - E_t\{y_{t+1}\} = Y_1(P_3\epsilon_{t+1} + P_4\upsilon_{t+1}) + Y_2\upsilon_{t+1}$ and

$$\begin{aligned} 0 &= \left(\mathcal{A}\begin{bmatrix} I & 0 \\ Y_1 & Y_2 \end{bmatrix} - \mathcal{B}\begin{bmatrix} I & 0 \\ Y_1 & Y_2 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ S_1 & S_2 \end{bmatrix}\right)\begin{bmatrix} x_t \\ w_t \end{bmatrix} + \tilde{\omega}_{t+1} \\ \tilde{\omega}_{t+1} &= (I - (C_1 + C_2 Y_1)P_3)\epsilon_{t+1} - ((C_1 + C_2 Y_1)P_4 + C_2 Y_2)\upsilon_{t+1} \end{aligned} \tag{5}$$

Conditions (5) must be fulfilled for each x_t, w_t, ϵ_{t+1}, υ_{t+1}. In this way we have the following theorem:

Definition 2.4. *Let* $U = \mathrm{col}(U_x, U_y)$ *is a partition of the matrix* U*, such that the matrix* U_x *consists of the first* n *rows of the matrix* U*, where* n *is a dimension of the vector of state variables,* x_t*.*

Theorem 2.5. *If there exists a linear solution* $\{Y_t, P_t\}_{t=0}^{\infty}$*, such that (3) holds then there exist matrices* U*,* Q *satisfying*

$$\mathcal{A}U = \mathcal{B}UQ$$

matrices U_x*,* $[C_1,\ C_2]U$*, have full row rank and the matrix* Q *satisfies*

$$\lim_{t\to\infty} \Xi^t HUQ^t = 0 \tag{6}$$

Proof. Let

$$U = \begin{bmatrix} I & 0 \\ Y_1 & Y_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} P_1 & P_2 \\ S_1 & S_2 \end{bmatrix}$$

Then, from (5), $\mathcal{A}U = \mathcal{B}UQ$, U_x has full row rank and Q satisfies (6). Observe that

$$I = (C_1 + C_2Y_1)P_3 = \begin{bmatrix} C_1 & C_2 \end{bmatrix} U \begin{bmatrix} I \\ 0 \end{bmatrix} P_3$$

because the identity matrix I has full rank, thus also the matrix $[C_1,\ C_2]U$ has full row rank. □

3 Preliminaries

In this section we present definitions and some basic results from computational linear algebra.

Definition 3.1. *Let us denote by* $\mathrm{null}\, A$*, where* A *is any matrix, an orthonormal basis of the null space of* A*.*

Definition 3.2. *Let* $A \in R^{m\times n}$ *is any matrix. Let us denote*

$$\dim_1 A = m \qquad \dim_2 A = n$$

Theorem 3.3. ***The Moore-Penrose pseudoinverse****. For each matrix* $A \in \mathbb{R}^{m\times n}$*, there exists an unique matrix* $X \in \mathbb{R}^{n\times m}$ *such that*

$$AXA = A \qquad XAX = X \qquad (AX)' = AX \qquad (XA)' = XA$$

this matrix X *is denoted as* $A^\dagger$*.*

Proof. See [5]. $\square$

Proposition 3.4. *If a matrix $A \in \mathbb{R}^{m\times n}$ has ful row rank, then $AA^\dagger = I$.*

Proof. We have $AA^\dagger A = A$, thus $A'(AA^\dagger - I)' = 0$. Since $\ker A = 0$, thus $AA^\dagger = I$. $\square$

Theorem 3.5. ***The singular value decomposition (svd).*** *For each matrix $A \in \mathbb{R}^{m\times n}$, there exist matrices U, V, D, such that*

$$A = UDV'$$

where $D \in \mathbb{R}^{m\times n}$ is a diagonal matrix with only nonnegative diagonal elements sorted in decreasing order, and U, V are orthogonal matrices.

Proof. See [5]. $\square$

Definition 3.6. *A matrix pair $(\mathcal{A}, \mathcal{B})$ is called regular if $\mathcal{A}$ and $\mathcal{B}$ are square, and $\det(\alpha\mathcal{A} - \beta\mathcal{B}) \neq 0$ for some $(\alpha, \beta) \in \mathbb{C}^2$. Otherwise, the matrix par $(\mathcal{A}, \mathcal{B})$ is called singular. A pair $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{(0, 0)\}$ is said to be an eigenvalue of $(\mathcal{A}, \mathcal{B})$ if $\det(\alpha\mathcal{A} - \beta\mathcal{B}) = 0$. If $\alpha \neq 0$, then, the pair (α, β) represents a finite eigenvalue $\lambda = \beta/\alpha$ of the par $(\mathcal{A}, \mathcal{B})$. The pair $(0, \beta)$ represents an infinite eigenvalue of $(\mathcal{A}, \mathcal{B})$.*

Definition 3.7. *Let $\mathcal{A} \in \mathbb{R}^{m\times n}$, $\mathcal{B} \in \mathbb{R}^{m\times n}$. For any $\lambda \in \mathbb{C}$ a vector $x_0 = 0$ is called an eigenvector of order $k = 0$ of the matrix pair $(\mathcal{A}, \mathcal{B})$ associated with the eigenvalue λ.*

A vector x_k is called an eigenvector of order k, $k \geq 1$, of the matrix pair $(\mathcal{A}, \mathcal{B})$ associated with the eigenvalue λ if there exists an eigenvector of order $k-1$ of the matrix pair $(\mathcal{A}, \mathcal{B})$ associated with the eigenvalue λ, x_{k-1}, such that $\mathcal{A}x_k = \lambda\mathcal{B}x_k + \mathcal{B}x_{k-1}$.

Proposition 3.8. *Consider a matrix pair $(\mathcal{A}, \mathcal{B})$ and invertible matrices P, Q. Let $A = P\mathcal{A}Q$, $B = P\mathcal{B}Q$. If x_k is an eigenvector of order k of the matrix pair $(\mathcal{A}, \mathcal{B})$ associated with an eigenvalue λ, then $Q^{-1}x_k$ is an eigenvector of order k of the matrix pair (A, B) associated with an eigenvalue λ.*

Proof. Let $m = 1$ and let x_m is an eigenvector of order m of $(\mathcal{A}, \mathcal{B})$ associated with the eigenvalue λ. Then $P^{-1}AQ^{-1}x_m = \lambda P^{-1}BQ^{-1}x_m$. Thus $Q^{-1}x_m$ is an eigenvector of order $k = 1$ of (A, B) associated with the eigenvalue λ. Let for $m = 1, \dots, k-1$ if x_m is an eigenvector of $(\mathcal{A}, \mathcal{B})$ of order m associated with the eigenvalue λ, then $Q^{-1}x_m$ is an eigenvector of order m of (A, B) associated with the eigenvalue λ. Then there exists an eigenvector of order y_{k-1} such that $P^{-1}AQ^{-1}x_k = \lambda P^{-1}BQ^{-1}x_k + P^{-1}BQ^{-1}y_{k-1}$. Thus $AQ^{-1}x_k =$

$\lambda BQ^{-1}x_k + BQ^{-1}y_{k-1}$. By the assumption $Q^{-1}y_{k-1}$ is an eigenvector of order $k-1$ of (A, B), thus also $Q^{-1}x_k$ is an eigenvector of order k of (A, B) associated with λ. $\square$

Theorem 3.9. ***The generalized Schur decomposition****. For each matrices $\mathcal{A}, \mathcal{B} \in R^{n\times n}$ there exist orthogonal matrices U, V, and real matrices R_A, R_B, such that R_B is upper-triangular, R_A is quasi-upper triangular and*

$$\mathcal{A}U = VR_A \qquad\qquad \mathcal{B}U = VR_B$$

Additionally, eigenvalues of R_A, R_B can be sorted in any order.

Proof. See [5]. $\square$

Theorem 3.10. ***The Kronecker decomposition****. For each matrix pair $(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{m\times n}$, there exists a canonical representation*

$$\begin{aligned} P\mathcal{A}Q &= \operatorname{diag}\{G_\epsilon, I, J_f, H_\eta^T\} \\ P\mathcal{B}Q &= \operatorname{diag}\{H_\epsilon, J_\infty, I, G_\eta^T\} \end{aligned} \tag{7}$$

where:

1. $G_\epsilon, H_\epsilon \in \mathbb{R}^{\epsilon_1\times\epsilon}$, $J_\infty \in \mathbb{R}^{k_j\times k_j}$, $J_f \in \mathbb{C}^{f\times f}$, $H_\eta^T, G_\eta^T \in \mathbb{R}^{\eta_1\times\eta}$.
2. *matrices P, Q are invertible.*
3. $G_\epsilon = \operatorname{diag}\{G_{\epsilon 1}, \dots, G_{\epsilon p}\}$, $H_\epsilon = \operatorname{diag}\{H_{\epsilon 1}, \dots, H_{\epsilon p}\}$, $G_\eta^T = \operatorname{diag}\{G_{\eta 1}^T, \dots, G_{\eta q}^T\}$, $H_\eta^T = \operatorname{diag}\{H_{\eta 1}^T, \dots, H_{\eta q}^T\}$, *where G_i and H_i are $i \times (i+1)$, $i \geq 0$, matrices*

$$G_i = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \qquad H_i = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

4. $J_\infty = \operatorname{diag}\{J_{v1}(0), \dots, J_{vs}(0)\}$ *and $J_i(0)$ is the Jordan block of order i corresponding to the null eigenvalue. Notice, that the matrix J_∞ is nilpotent.*
5. *J_f is a matrix in the Jordan canonical form.*

The matrix par is regular iff $P\mathcal{A}Q = \operatorname{diag}\{I, J_f\}$, $P\mathcal{B}Q = \operatorname{diag}\{J_\infty, I\}$.

Proof. See [4]. $\square$

Theorem 3.11. ***The GUPTRI decomposition***. *For any matrix pair $(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{m\times n}$, there exist orthogonal matrices $P \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{n\times n}$, such that*

$$P'\mathcal{A}Q = \begin{bmatrix} A_r & * & * \\ 0 & A_{reg} & * \\ 0 & 0 & A_l \end{bmatrix} \qquad P'\mathcal{B}Q = \begin{bmatrix} B_r & * & * \\ 0 & B_{reg} & * \\ 0 & 0 & B_l \end{bmatrix}$$

where the asterisk denotes arbitrary conforming submatrices. The matrix pair (A_{reg}, B_{req}) is regular and has the same regular structure (i.e. contains all finite and infinite eigenvalues of $(\mathcal{A}, \mathcal{B})$) as $(\mathcal{A}, \mathcal{B})$. The rectangular blocks (A_r, B_r) and (A_l, B_l) contain the singular structure, right and left minimal indices, of the pair $(\mathcal{A}, \mathcal{B})$ and are block quasi-upper triangular.

Proof. See [2], [3]. □

4 The matrix equation $\mathcal{A}U = \mathcal{B}U\Sigma$

Proposition 4.1. *The only eigenvector of the matrix pair (I, J_∞) is $x = 0$. The only eigenvector of the matrix pair (H_η^T, G_η^T) is $x = 0$.*

Proof. By simple calculations. □

Proposition 4.2. *Let $L_k \in \mathbb{R}^{\epsilon\times 1}$ is a matrix, that contains zero on all positions except the k-th position, and one on the k-th position. Then L_k is an eigenvector of order k the matrix pair (G_ϵ, H_ϵ) associated with eigenvalue $\lambda = 0$.*

Proof. By simple calculations. □

Theorem 4.3. *Consider a matrix pair $(\mathcal{A}, \mathcal{B})$. There exists a matrix Ψ, such that for all matrices U, Σ, satisfying $\mathcal{A}U = \mathcal{B}U\Sigma$ and $\lim_{t->\infty} \Xi^t HU\Sigma^t = 0$, there exist a matrix Λ, such that $U = \Psi\Lambda$.*

Proof. Consider the Kronecker decomposition of the matrix pair $(\mathcal{A}, \mathcal{B})$, $P\mathcal{A}Q = A$, $P\mathcal{B}Q = B$, where A, B are in canonical form (7). Let matrices U and Σ satisfy assumptions. Consider the Jordan decomposition of the matrix Σ, $\Sigma = V\tilde{\Sigma}V^{-1}$, where $\tilde{\Sigma}$ is in the Jordan canonical form. By assumption

$$\lim_{t->\infty} \Xi^t HUV\tilde{\Sigma}^t = 0$$

Let $\tilde{\Sigma} = \mathrm{diag}(\tilde{\Sigma}_1, \tilde{\Sigma}_2, \ldots, \tilde{\Sigma}_q)$, where $\tilde{\Sigma}_i$ is a Jordan block and let $V = [V_1, V_2, \ldots, V_q]$ is corresponding partition of the matrix V. Then for $i = 1, 2, \ldots, q$ we have $\lim_{t->\infty} \Xi^t HUV_i \tilde{\Sigma}_i^t = 0$. Let λ_i is an eigenvalue of $\tilde{\Sigma}_i$. We have

$$\tilde{\Sigma}_i^t = \lambda_i^t \begin{bmatrix} 1 & \binom{t}{1}\lambda_i^{-1} & \cdots & \cdots & \binom{t}{m_i-1}\lambda_i^{-m_i+1} \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \binom{t}{1}\lambda_i^{-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \equiv \lambda_i^t \bar{\Sigma}_{it}$$

where m_i is size of the Jordan block $\tilde{\Sigma}_i$. If for $j \in \{1, 2, \ldots, k\}$ we have $\lim_{t->\infty} \xi_j^t \lambda_i^t = 0$, then also $\lim_{t->\infty} \lambda_i^t \xi_j^t H_j UV_i \bar{\Sigma}_{it} = 0$, because $\bar{\Sigma}_i^t$ is a polynomial with respect to t. If $\lim_{t->\infty} \xi_j^t \lambda_i^t \neq 0$ then $\lim_{t->\infty} H_j UV_i \bar{\Sigma}_{it} = 0$. Partitioning $H_j UV_i$ on columns and multiplying $H_j UV_i$ and $\bar{\Sigma}_{it}$ we can see that $H_j UV_i = 0$. Let V_i^p is the p-th column of V_i. Observe that UV_i^p is an eigenvector of $(\mathcal{A}, \mathcal{B})$ of order p associated with the eigenvalue λ_i.

Thus, UV_i consists of eigenvectors associated with eigenvalue λ_i, such that for each $j \in \{1, 2, \ldots, k\}$, $\lim_{t->\infty} \xi_j^t \lambda_i^t = 0$ or $H_j UV_i = 0$.

We have

$$\begin{bmatrix} G_\epsilon & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & J_f & 0 \\ 0 & 0 & 0 & H_\eta^T \end{bmatrix} \begin{bmatrix} \tilde{U}_i^1 \\ \tilde{U}_i^2 \\ \tilde{U}_i^3 \\ \tilde{U}_i^4 \end{bmatrix} = \begin{bmatrix} H_\epsilon & 0 & 0 & 0 \\ 0 & J_\infty & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & G_\eta^T \end{bmatrix} \begin{bmatrix} \tilde{U}_i^1 \\ \tilde{U}_i^2 \\ \tilde{U}_i^3 \\ \tilde{U}_i^4 \end{bmatrix} \tilde{\Sigma}_i$$

where $\tilde{U}_i = Q^{-1} UV_i$. Thus

$$G_\epsilon \tilde{U}_i^1 = H_\epsilon \tilde{U}_i^1 \tilde{\Sigma}_i \qquad \tilde{U}_i^1 = J_\infty \tilde{U}_i^2 \tilde{\Sigma}_i$$
$$J_f \tilde{U}_i^3 = \tilde{U}_i^3 \tilde{\Sigma}_i \qquad H_\eta^T \tilde{U}_i^4 = G_\eta^T \tilde{U}_i^4 \tilde{\Sigma}_i$$

From (4.1) we have $\tilde{U}_i^2 = 0$, $\tilde{U}_i^4 = 0$.

Let μ_i, $i \in \{1, 2, \ldots, r\}$ are distinct eigenvalues of J_f. Let $i \in \{1, 2, \ldots, r\}$. Let $J_i = \{j \in \{1, 2, \ldots, q\} : |\xi_j \mu_i| \geq 1\}$ and H^{J_i} is a matrix that consists of rows of the matrix H with index belonging to the set J_i. Let columns of a matrix $\tilde{\Phi}_i^1$ span the space of eigenvectors of order 1 of $(\mathcal{A}, \mathcal{B})$ associated with the eigenvalue μ_i belonging to $\ker H^{J_i}$. For $k > 1$ let columns of a matrix $\tilde{\Phi}_i^k$ span the space of eigenvectors, x, of order k of $(\mathcal{A}, \mathcal{B})$ associated with the eigenvalue μ_i belonging to $\ker H^{J_i}$, such that there exist a matrix Π_i satisfying $\mathcal{A}x = \lambda_i \mathcal{B}x + \mathcal{B}\tilde{\Phi}_i^{k-1}\Pi_i$. Let $\tilde{\Phi}_i = [\tilde{\Phi}_i^1, \ldots, \tilde{\Phi}_i^m]$, where m is such a number that each eigenvector of $(\mathcal{A}, \mathcal{B})$ associated with the eigenvalue λ_i has order lower or equal to m. Let $\tilde{\Phi}_0 = Q\,\mathrm{col}(I_\epsilon, 0, 0, 0)$, where $I_\epsilon \in \mathbb{R}^{\epsilon \times \epsilon}$ is an identity matrix. Finally let $\Phi = [\tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\Phi}_2, \ldots, \tilde{\Phi}_k]$.

Let $i \in \{1, 2, \ldots, k\}$. Then for each $j \in \{1, 2, \ldots, k\}$, $\lim_{t->\infty} \xi_j^t \lambda_i^t = 0$ or $H_j UV_i = 0$. All columns of UV_i are eigenvectors of $(\mathcal{A}, \mathcal{B})$ associated with λ_i and $UV_i \in \ker H^{J_i}$. If λ_i is not an eigenvalue of J_f, then $\tilde{U}_i^3 = 0$, and there exists a matrix Λ_i, such that $UV_i = \tilde{\Phi}_0 \Lambda_i$.

Let $\lambda_i = \mu_l$ for some $l \in \{1, 2, \ldots, r\}$. Consider the first column of V_i. The vector UV_i^i is an eigenvector of $(\mathcal{A}, \mathcal{B})$ of order 1. From the definition of $\tilde{\Phi}_l^1$ we obtain that there exists a matrix $\tilde{\Lambda}_i^1$ such that $UV_i^1 = \Phi_l^1 \tilde{\Lambda}_i^1$. Assume that for $k - 1 > 1$ there exists a matrix $\tilde{\Lambda}_i^{k-1}$ such that $UV_i^{k-1} = \Phi_l^{k-1} \tilde{\Lambda}_i^{k-1}$. Then we have $\mathcal{A}UV_i^k = \mu_l \mathcal{B}UV_i^k + \mathcal{B}UV_i^{k-1} = \mu_l \mathcal{B}UV_i^k + \mathcal{B}\Phi_l^{k-1}\tilde{\Lambda}_i^{k-1}$. Then, from the definition of $\tilde{\Phi}_i^k$ we obtain that there exists a matrix $\tilde{\Lambda}_i^k$ such that $UV_i^k = \Phi_l^k \tilde{\Lambda}_i^k$. In this way we have proved that there exist a matrix $\tilde{\Lambda}_i$ such that $UV_i = \Phi\tilde{\Lambda}_i$. Thus $U = \Phi\tilde{\Lambda}$, where $\tilde{\Lambda} = [\tilde{\Lambda}_1, \tilde{\Lambda}_2, \ldots, \tilde{\Lambda}_k]V^{-1}$. □

5 Maximal solution

5.1 Singular pencil

If a pencil $(\mathcal{A}, \mathcal{B})$ is singular then generalized Schur decomposition is not reliable. Small perturbation of matrices $\mathcal{A}$, $\mathcal{B}$ may drastically change eigenvalues of $(\mathcal{A}, \mathcal{B})$. In this case we use GUPTRI decomposition.

Let consider GUPTRI decomposition of a matrix pair $(\mathcal{A}, \mathcal{B})$

$$\mathcal{A}U = VR_A \qquad\qquad \mathcal{B}U = VR_B$$

where R_A and R_B are in GUPTRI canonical form. Let λ_i^A, λ_i^B are i-th eigenvalues of regular blocks A_{reg} and B_{reg} respectively. Let $\lambda_i = \lambda_i^A / \lambda_i^B$ and let λ is a set of all distinct finite eigenvalues λ_i. Let q is a size of the set λ.

Consider the i-th eigenvalue belonging to the set λ, μ_i. Let us sort eigenvalues of A_{reg} and B_{reg} in such a way, that all eigenvalues μ_i appears in left upper block of A_{reg} and B_{reg}. Then

$$\begin{bmatrix} V_1 & V_2^i & V_3^i \end{bmatrix} \begin{bmatrix} A_r & A_{12}^i & * \\ 0 & \tilde{A}_{reg}^i & * \\ 0 & 0 & * \end{bmatrix} = \mathcal{A} \begin{bmatrix} U_1 & U_2^i & U_3^i \end{bmatrix}$$

$$\begin{bmatrix} V_1 & V_2^i & V_3^i \end{bmatrix} \begin{bmatrix} B_r & B_{12}^i & * \\ 0 & \tilde{B}_{reg}^i & * \\ 0 & 0 & * \end{bmatrix} = \mathcal{B} \begin{bmatrix} U_1 & U_2^i & U_3^i \end{bmatrix}$$

Proposition 5.1.

$$A_r A_r^\dagger = I \qquad\qquad B_r B_r^\dagger = I$$

Proof. There exist unitary matrices P, Q such that $A_r = PGQ$. Since $\ker G' = 0$, thus also $\ker A_r' = 0$, and A_r has full row rank. Now, we can use the proposition (3.4). Similarly for B_r. $\square$

From the GUPTRI decomposition we have

$$V_1 A_r = \mathcal{A} U_1 \qquad\qquad V_1 B_r = \mathcal{B} U_1 \tag{8}$$

and

$$V_1 A_{12}^i + V_2^i \tilde{A}_{reg}^i = \mathcal{A} U_2^i \qquad\qquad V_1 B_{12}^i + V_2 \tilde{B}_{reg}^i = \mathcal{B} U_2^i \tag{9}$$

5.1.1 Singular part

We are looking for an invertible matrix W and a matrix J with only zero eigenvalues, such that $A_r W = B_r W J$. Let $\tilde{U}_1 = \operatorname{null} A_r$. Let $\tilde{U}_i = A_r^\dagger B_r \tilde{U}_{i-1}$ for $i = 2, \ldots, m$. Then $A_r \tilde{U}_i = B_r \tilde{U}_{i-1}$. For each i let $\tilde{U}_i = [\tilde{U}_i^1, \ldots, \tilde{U}_i^\upsilon]$, where $\upsilon = \dim_2 \tilde{U}_1$. Let $V_j^m = [\tilde{U}_1^j, \ldots, \tilde{U}_m^j]$. for for $j = 1, \ldots, \upsilon$.

Let $k_1^m \leq m$ is the biggest number, such that all the first k_1^m columns or V_1^m are linearly independent and let W_1^m consists of the first k_1^m columns of V_1^m. For $j = 2, \ldots, \upsilon$, let $k_j^m \leq m$ is the biggest number such that all first k_j^m columns of V_j^m and all colums of W_{j-1}^m are linearly independent and let W_j^m consists of the first k_j^m columns of V_j^m. Let $W^m = [W_1^m, \ldots, W_\upsilon^m]$ and $J^m = \operatorname{diag}\{J_1^m, \ldots, J_\upsilon^m\}$, where J_i^m is a Jordan block of zero eigenvalue and $\dim_1 J_i^m = k_i^m$ for $i = 1, \ldots, \upsilon$. Then for any m, $A_r W^m = B_r W^m J^m$, all eigenvalues of J^m are zero, and all columns of W^m are linearly independent.

Proposition 5.2. *For any eigenvector, x, of (A_r, B_r) of order m associated with zero eigenvalue there exists a matrix Λ, such that $x = W^m \Lambda$.*

Proof. Let x_1 is an eigenvector of (A_r, B_r) of order 1 associated with the zero eigenvalue, then $A_r x_1 = 0$, and there exists a matrix X_1, such that $x_1 = W^1 X_1$. Let x_{k-1} is an eigenvector of (A_r, B_r) of order $k-1$, $k > 1$, associated with the zero eigenvalue, and let $x = W^{k-1} X_{k-1}$, for some matrix X_{k-1}. Let x_k is an eigenvector of order k. There exists an eigenvector, y_{t-1}, of order $k-1$ such that $A_r x_k = B_r y_{k-1}$. Let $\tilde{x}_k = A_r^\dagger B_r y_{k-1}$, then $A_r(x_k - \tilde{x}_k) = 0$, hence $x_k = \tilde{x}_k + \tilde{U}_1 Y_k$ for some matrix Y_k. There exists a matrix Y_{k-1} such that $y_{k-1} = W^{k-1} Y_{k-1}$. Thus, $x_k = \tilde{U}_1 Y_k + A_r^\dagger B_r W^{k-1} Y_{k-1}$. Consider i-th column of W^{k-1}, w_i. By construction $w_i = \tilde{U}_q^p$ for some p, q, $q \leq k-1$. Then, using the definition of $\tilde{U}_i$, $A_r^\dagger B_r w_i = \tilde{U}_{q+1}^p$. But then $\tilde{U}_{q+1}^p$ is one of the columns of W^k, or $\tilde{U}_{q+1}^p = W^k \Lambda_i$ for some matrix Λ_i. Thus, in both cases, there exists a matrix Λ_i, such that $\tilde{U}^p_{q+1} = W^k\Lambda_i$. Hence, $A_r^\dagger B_r w_i = W^k\Lambda_i$ and $A_r^\dagger B_r W^{k-1} = W^k\Lambda$, where $\Lambda = [\Lambda_1,\ldots,\Lambda_q]$, $q = \dim_1 W^{-1}$. Additionally, there exists a matrix I_1^k, such that $\tilde{U}_1 = W^k I_1^k$. In this way we have, $x_k = W^k(I_1^k Y_k + \Lambda Y_{k-1})$. This ends the proof. □

Proposition 5.3. *Let $A_r = PGQ$, $B_r = PHQ$ is the Kronecker decomposition of the pair (A_r, B_r). Let $G = \operatorname{diag}\{G_1,\ldots,G_k\}$, $\dim_2 G_i = m_i$, for $i = 1,\ldots,k$, and $m = \max\{m_1,\ldots,m_k\}$. Then W_m is a square, invertible matrix and $\operatorname{rank} W_m = n$, where $n = \dim_2 A_r$.*

Proof. Let x is an eigenvector of order $k \le m$ of (A_r, B_r) associated with zero eigenvalue. Then the exists a matrix Λ, such that $x = W_m\Lambda$. By propositions (4.2) and (3.8) each column of the matrix Q^{-1} is an eigenvector of (A_r, B_r) of order $k \le m$ associated with the zero eigenvalue. Lex q_i, $i = 1,\ldots,n$ is the i-th column of Q^{-1}. There exists Λ_i such that $q_i = W_m\Lambda_i$. Thus, $Q^{-1} = W_m\Lambda$, where $\Lambda = [\Lambda_1,\ldots,\Lambda_n]$.

By construction, $\operatorname{rank} W_m = \dim_2 W_m$. Since $\operatorname{rank} Q^{-1} = n$, thus $\operatorname{rank} W_m \ge n$. On the other hand $\dim_1 W_m = n$, thus, $\operatorname{rank} W_m \le n$. Hence, $n = \operatorname{rank} W_m = \dim_1 W_m = \dim_2 W_m$, and W_m is a square, invertible matrix. □

Let $W = W_m$, where m is given by the proposition (5.3). From (8) we have

$$\mathcal{A}U_1 = \mathcal{B}U_1 WJW^{-1}$$

By construction, the matrix WJW^{-1} contains only zero eigenvalues.

5.1.2 Regular part

Because $\tilde{B}^i_{reg}$ is invertible, from (9) we have

$$V_2^i = \mathcal{B}U_2^i(\tilde{B}^i_{reg})^{-1} - V_1 B_{12}(\tilde{B}^i_{reg})^{-1}$$

hence

$$\mathcal{A}U_2^i = \mathcal{B}(U_1\Xi_1^i + U_2^i\Xi_2^i)$$

where $\Xi_1^i = B_r^\dagger(A^i_{12} - B^i_{12}\Xi_2^i)$, $\Xi_2^i = (\tilde{B}^i_{reg})^{-1}\tilde{A}^i_{reg}$, and

$$\begin{aligned}\mathcal{A}\left[\begin{array}{cc} U_1 & U_2^i\end{array}\right] &= \mathcal{B}\left[\begin{array}{cc} U_1 & U_2^i\end{array}\right]\left[\begin{array}{cc} WJW^{-1} & \Xi_1^i \\ 0 & \Xi_2^i\end{array}\right] \\ &\equiv \mathcal{B}\left[\begin{array}{cc} U_1 & U_2^i\end{array}\right]\Xi^i\end{aligned}$$

We must modify yet matrices U_1 and U_2^i to fulfill stability conditions. Let $J_i = \{j \in \{1,2,\ldots,q\} : |\xi_j\mu_i| \ge 1\}$. Let H^{J_i} consists of rows of the matrix H with indices belonging to J_i. Let $\tilde{U}^i = [U_1, U_2^i]$. We are looking for a matrix Π^i, such that $H^{J_i}\tilde{U}^i\Pi^i = 0$ and there exist a matrix Σ^i, such that $\mathcal{A}\tilde{U}^i\Pi^i = \mathcal{B}\tilde{U}^i\Pi^i\Sigma^i$. Condition $H^{J_i}\tilde{U}^i\Pi^i = 0$ implies that there exists a matrix Λ_1^i, such that $\Pi^i = N^i\Lambda^i$, where $N^i = \mathrm{null}(H^{J_i}\tilde{U}^i)$.

Observe that in case of general stability conditions there may exist solutions to (1) which cannot be obtained by appropriate selection of eigenvalues of the matrix pair $(\mathcal{A}, \mathcal{B})$. Let us consider the following problem

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \Sigma$$

with the stability condition

$$\lim_{t->\infty} 2^t[1, -1] \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \Sigma^t = 0$$

Then $\mathrm{col}(U_1, U_2) = \mathrm{col}(1, 1)$ and $\Sigma = 1$ solves this problem. From the Schur decomposition we can select on of the eigenvalue equal 1. Then selected eigenvectors are $\mathrm{col}(1, 0)$ and $\mathrm{col}(0, 1)$, which do not satisfy stability condition.

Let us consider first eigenvectors of order 1 associated with the eigenvalue μ_i. We are looking for a matrix Λ_1^i such that $\mathcal{A}\tilde{U}^iN^i\Lambda_1^i = \mu_i\mathcal{B}\tilde{U}^iN^i\Lambda_1^i$. On the other hand $\mathcal{A}\tilde{U}^iN^i\Lambda_1^i = \mathcal{B}\tilde{U}^i\Xi^iN^i\Lambda_1^i$, hence $\mathcal{B}\tilde{U}^i(\mu_iI - \Xi^i)N^i\Lambda_1^i = 0$. Hence, $\Lambda_1^i = \mathrm{null}\,\mathcal{B}\tilde{U}^i(\mu_iI - \Xi^i)N^i$.

Let us assume that for $k > 2$ we have a matrix Λ_{k-1}^i, such that $\tilde{U}^i\Lambda_{k-1}^i$ is an eigenvector of order $k-1$ associated with an eigenvalue μ_i, and $\mathcal{A}\tilde{U}^iN^i\Lambda_{k-1}^i = \mu_i\mathcal{B}\tilde{U}^iN^i\Lambda_{k-1}^i + \mathcal{B}\tilde{U}^iN^i\Lambda_{k-2}^i\Psi_{k-1}^i$ for an appropriate matrix Ψ_{k-1}^i. We are looking for matrices Λ_k^i, Ψ_k^i, such that $\mathcal{A}\tilde{U}^iN^i\Lambda_k^i = \mu_i\mathcal{B}\tilde{U}^iN^i\Lambda_k^i + \mathcal{B}\tilde{U}^iN^i\Lambda_{k-1}^i\Psi_k^i$. We have $\mathcal{A}\tilde{U}^iN^i\Lambda_k^i = \mathcal{B}\tilde{U}^i\Xi^iN^i\Lambda_k^i$. Hence $0 = \mathcal{B}\tilde{U}^i[(\mu_iI - \Xi^i)N^i, N^i\Lambda_{k-1}^i]\,\mathrm{col}(\Lambda_k^i, \Psi_k^i)$ and $\mathrm{col}(\Lambda_k^i, \Psi_k^i) = \mathrm{null}\,\mathcal{B}\tilde{U}^i[(\mu_iI - \Xi^i)N^i, N^i\Lambda_{k-1}^i]$.

Let m is the smallest number satisfying $\dim_2 \Lambda_m^i = 0$. Let us take $U^i = [\tilde{U}^iN^i\Lambda_1^i, \tilde{U}^iN^i\Lambda_2^i, \ldots, \tilde{U}^iN^i\Lambda_m^i]$ and

$$\Sigma^i = \begin{bmatrix} \mu_iI & \Psi_1^i & 0 & \cdots & 0 \\ 0 & \mu_iI & \Psi_2^i & \ddots & \vdots \\ \vdots & \ddots & \mu_iI & \ddots & 0 \\ \vdots & & \ddots & \ddots & \Psi_{m-1}^i \\ 0 & \cdots & \cdots & 0 & \mu_iI \end{bmatrix}$$

Then $\mathcal{A}U^i = \mathcal{B}U^i\Sigma^i$. Let $U = [U_1, U^1, U^2, \ldots, U^q]$ and $\Sigma = \mathrm{diag}(WJW^{-1}, \Sigma^1, \Sigma^2, \ldots, \Sigma^q)$.

Proposition 5.4. *Matrices U and Σ satisfy $\mathcal{A}U = \mathcal{B}U\Sigma$, $\lim_{t->\infty} \Xi^t HU\Sigma^t = 0$ and* $\operatorname{rank} U = \operatorname{rank}\Psi$.

Proof. From the construction we have $\mathcal{A}U = \mathcal{B}U\Sigma$.

For $i = 1, 2, \dots, q$, $H^{J_i}U^i = 0$. Let $j \notin J_i$. The only eigenvalue of Σ^i is μ_i, thus $\lim_{t->\infty} \xi_j^t(\Sigma^i)^t = 0$. Additionally, for each j $\lim_{t->\infty} \xi_j^t W(J)^t W^{-1} = 0$.

From (4.3) we have $\operatorname{rank} U \le \operatorname{rank}\Psi$.

There exists a matrix $\tilde{\Psi}_0$, such that $\tilde{\Phi}_0 = U_1\tilde{\Psi}_0$.

Let v is any eigenvector of $(\mathcal{A}, \mathcal{B})$ associated with an eigenvalue μ_i for $i = 1, 2, \dots, q$ such that $H^{J_i}v = 0$. There exists a matrix $\tilde{\Pi}^i$ such that $v = \tilde{U^i}\tilde{\Pi}^i$. Condition $H^{J_i}v = 0$ implies that there exist a matrix $\tilde{\Lambda}^i$ such that $v = \tilde{U^i}N^i\tilde{\Lambda}^i$, where $N_1^i = \operatorname{null}(H^{J_i}\tilde{U^i})$.

Let v is an eigenvector of order 1. Then $\mathcal{A}v = \mu_i\mathcal{B}v$, hence $\mathcal{A}\tilde{U^i}N^i\tilde{\Lambda}^i = \mu_i\mathcal{B}\tilde{U^i}N^i\tilde{\Lambda}^i$. On the other hand $\mathcal{A}\tilde{U^i}N^i\tilde{\Lambda}^i = \mathcal{B}\tilde{U^i}\Xi^iN^i\tilde{\Lambda}^i$. Thus, $0 = \mathcal{B}\tilde{U^i}(\mu_iN^i\tilde{\Lambda}^i - \Xi^iN^i\tilde{\Lambda}^i) = \mathcal{B}\tilde{U^i}(\mu_iI - \Xi^i)N^i\tilde{\Lambda}^i$. Hence, $\tilde{\Lambda}^i \in \ker\mathcal{B}\tilde{U^i}(\mu_iI - \Xi^i)N^i$, and $\tilde{\Lambda}^i = \Lambda_1^i\Gamma^i$ for some matrix Γ^i. In this way, $v = \tilde{U^i}N^i\Lambda_1^i\Gamma^i$ and there exists a matrix $\tilde{\Psi}_i^1$ such that $\tilde{\Phi}_i^1 = \tilde{U^i}N^i\Lambda_1^i\tilde{\Psi}_i^1$.

Let for $k-1 > 1$ there exists a matrix $\tilde{\Psi}_i^{k-1}$ such that $\tilde{\Phi}_i^{k-1} = \tilde{U^i}N^i\Lambda_{k-1}^i\tilde{\Psi}_i^{k-1}$. Let v is an eigenvector of order k, such that there exists an eigenvector of order $k-1$, v_{k-1}, associated with the eigenvalue μ_i, belonging to the space spanned by $\tilde{\Phi}_i^{k-1}$. Then there exists a matrix Π_i^{k-1}, such that, $\mathcal{A}\tilde{U^i}N^i\tilde{\Lambda}^i = \mu_i\mathcal{B}\tilde{U^i}N^i\tilde{\Lambda}^i + \mathcal{B}\tilde{\Phi}_i^{k-1}\Pi_i^{k-1}$. Thus, $0 = \mathcal{B}\tilde{U^i}[(\mu_iI - \Xi^i)N^i, N^i\Lambda_{k-1}^i]\operatorname{col}(\tilde{\Lambda}^i, \tilde{\Psi}_i^{k-1}\Pi_i^{k-1})$. Hence, we have $\operatorname{col}(\tilde{\Lambda}^i, \tilde{\Psi}_i^{k-1}\Pi_i^{k-1}) \in \ker\mathcal{B}\tilde{U^i}[(\mu_iI - \Xi^i)N^i, N^i\Lambda_{k-1}^i]$. In this way there exists a matrix Γ^i, such that $\operatorname{col}(\tilde{\Lambda}^i, \tilde{\Psi}_i^{k-1}\Pi_i^{k-1}) = \operatorname{col}(\Lambda_k^i, \Psi_k^i)\Gamma^i$. Thus, $\tilde{\Lambda}^i = \Lambda_k^i\Gamma^i$, $v = \tilde{U^i}N^i\Lambda_k^i\Gamma^i$, and there exists a matrix $\tilde{\Psi}_i^k$ such that $\tilde{\Phi}_i^k = \tilde{U^i}N^i\Lambda_k^i\tilde{\Psi}_i^k$. Using definition of U we have, that there exists a matrix Ψ such that $\tilde{\Phi} = U\Psi$, and thus $\operatorname{rank} U \ge \operatorname{rank}\Psi$. □

5.2 Regular pencil

In this section we assume that a matrix pair $(\mathcal{A}, \mathcal{B})$ is regular. Let us consider generalized Schur decomposition of the matrix pair $(\mathcal{A}, \mathcal{B})$

$$V'\mathcal{A}U = T_A \qquad\qquad V'\mathcal{B}U = T_B$$

where matrices U and V are orthogonal, the matrix T_A is quasi-upper triangular, and the matrix T_B is upper triangular. Such a decomposition always exists. Let λ_i^A, λ_i^B are i-th eigenvalues of T_A and T_B respectively. Let $\lambda_i = \lambda_i^A/\lambda_i^B$ and let λ is a set of all distinct finite eigenvalues λ_i. Let q is a size of the set λ.

Consider the i-th eigenvalue belonging to the set λ, μ_i. Let us sort eigenvalues of T_A and T_B is such a way that all eigenvalues μ_i appears in left upper block of T_A and T_B. Then

$$\begin{bmatrix} V_1^i & V_2^i \end{bmatrix} \begin{bmatrix} R_A^i & T_{12}^{Ai} \\ 0 & T_{22}^{Ai} \end{bmatrix} = \mathcal{A} \begin{bmatrix} U_1^i & U_2^i \end{bmatrix}$$
$$\begin{bmatrix} V_1^i & V_2^i \end{bmatrix} \begin{bmatrix} R_B^i & T_{12}^{Bi} \\ 0 & T_{22}^{Bi} \end{bmatrix} = \mathcal{B} \begin{bmatrix} U_1^i & U_2^i \end{bmatrix}$$

where R_A^i is quasi-upper triangular, R_B^i is upper-triangular, both matrices have the same size, and all eigenvalues of $(R_B^i)^-1 R_A^i$ are equal μ_i. This implies

$$\mathcal{A}U_1^i = V_1^i R_A^i \qquad\qquad \mathcal{B}U_1^i = V_1^i R_B^i \tag{10}$$

By assumption, the matrix R_B is invertible. Thus,

$$\mathcal{A}U_1^i = \mathcal{B}U_1^i (R_B^i)^{-1} R_A^i$$

Let $J_i = \{j \in \{1, 2, \ldots, q\} : |\xi_j \mu_i| \geq 1\}$. Let H^{J_i} consists of rows of the matrix H with indices belonging to J_i. We are looking for a matrix Π^i that $H^{J_i} U_1^i \Pi^i = 0$ and there exists a matrix Σ^i such that $\mathcal{A}U_1^i \Pi^i = \mathcal{B}U_1^i \Pi^i \Sigma^i$. We can construct matrices Π^i and Σ^i in the same way as in case of singular pencil. Regularity of the pencil $(\mathcal{A}, \mathcal{B})$ does not simplify the problem much.

If $\xi_j = \xi$ for each j and $\ker H = 0$, then we can obtain matrices U and Σ much easier. Consider ordering of eigenvalues in the Schur decomposition, such that all eigenvalues μ_i satisfying $|\mu_i \xi| < 1$ appears in the left upper block of T_A and T_B. Then we can take $\Sigma = (R_B^i)^{-1} R_A^i$ and $U = U_1^i$.

6 Construction of the solution

Assume that matrices U, Q solve $\mathcal{A}U = \mathcal{B}UQ$, the matrix Q satisfies (6), U_x has full row rank, and $[C_1,\ C_2]U$ has full row rank. Consider the svd decomposition of U_x, $U_x = MSN'$. Because U_x has full row rank, thus $S = [\tilde{S},\ 0]$, where $\tilde{S}$ is an invertible matrix. Let

$$\Lambda = N \begin{bmatrix} \tilde{S}^{-1} M' & 0 \\ 0 & I \end{bmatrix}$$

Then Λ is invertible and the matrix $U\Lambda$ takes the form

$$U\Lambda = \begin{bmatrix} I & 0 \\ \bar{U}_{21} & \bar{U}_{22} \end{bmatrix}$$

Moreover $\mathcal{A}U\Lambda = \mathcal{B}U\Lambda\tilde{Q}$, where $\tilde{Q} = \Lambda^{-1}Q\Lambda$ and $\tilde{Q}$ satisfies (6). Let $Y_1 = \bar{U}_{21}$ and $Y_2 = \bar{U}_{22}$. Let

$$\tilde{Q} = \left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array} \right]$$

and let $P_1 = Q_{11}$, $P_2 = Q_{12}$, $S_1 = Q_{21}$, and $S_2 = Q_{22}$. In this way, assuming $\omega_{t+1} = 0$, matrices Y_1, Y_2, P_1, P_2, S_1, S_2 satisfy (5).

Now let us concentrate on the term ω_{t+1} in (5)

$$0 = \Big((C_1 + C_2Y_1)P_3 - I\Big)\epsilon_{t+1} + \Big((C_1 + C_2Y_1)P_4 + C_2Y_2\Big)\upsilon_{t+1}$$

This equation must be fulfilled for all ϵ_{t+1} and υ_{t+1}. Thus

$$I = (C_1 + C_2Y_1)P_3, \qquad 0 = C_2Y_2 + (C_1 + C_2Y_1)P_4 \tag{11}$$

Because $C_1 + C_2Y_1 = [C_1,\ C_2]U\Lambda$, the matrix $[C_1,\ C_2]U$ has full row rank, and Λ is an invertible matrix, hence $C_1 + C_2Y_1$ has full row rank. Then we can take

$$P_3 = (C_1 + C_2Y_1)^\dagger, \qquad P_4 = -(C_1 + C_2Y_1)^\dagger C_2Y_2$$

However if $C_1 + C_2Y_1$ is not square then there exist many solutions to (11). In this way we have

Theorem 6.1. *If there exist matrices U, Q, such that $\mathcal{A}U = \mathcal{B}UQ$, condition (6) is fulfilled, U_x has full row rank, and $[\mathcal{C}_1,\ C_2]U$ has full row rank, then there exists a linear solution to (1) satisfying (3).*

Theorem 6.2. *For any, possibly rectangular, matrix pair $(\mathcal{A}, \mathcal{B})$ consider matrices Q and U constructed in the previous section. There exists a solution to (1) if and only if matrices U_x and $[C_1,\ C_2]U$ have full row rank.*

Proof. From (5.4) we have $\mathcal{A}U = \mathcal{B}UQ$, and condition (6) is fulfilled. Let matrices U_x and $[C_1,\ C_2]U$ have full row rank. Then, by theorem (6.1) there exists a solution to (1) satisfying (3).

Let $\{Y_t, P_t\}_{t=0}^{\infty}$ is a linear solution to (1) satisfying (3). Then, by theorem (2.5) there exist matrices V, S, such that $\mathcal{A}V = \mathcal{B}VS$, condition (6) is satisfied, V_x and $[C_1,\ C_2]V$ has full rank. By the theorem (4.3) there exist matrices Λ, Ξ, such that $V = \Psi\Lambda$, $U = \Psi\Xi$, where Ψ is the matrix from the theorem (4.3). Since $\operatorname{rank} U = \operatorname{rank} \Psi$, (proposition (5.4)), thus Ξ has full row rank. Hence $\Xi\Xi^\dagger = I$, and $\Psi = U\Xi^\dagger$. In this way $V = U\Xi^\dagger\Lambda$. Since $V_x = U_x\Xi^\dagger\Lambda$ and V_x has full row rank, thus also U_x has full row rank. Next $[C_1,\ C_2]V = [C_1,\ C_2]U\Xi^\dagger\Lambda$. Since $[C_1,\ C_2]V$ has full row rank, thus also $[C_1,\ C_2]U$ has full row rank. $\square$

From the proof of this theorem we have also

Proposition 6.3. *Let $\{Y_t, P_t\}_{t=0}^{\infty}$ is any linear solution to (1), such that (3) holds. Then there exist a matrix Λ such that*

$$\begin{bmatrix} Y_1 & Y_2 \end{bmatrix} = U_y \Lambda$$

Proposition (6.3) shows, that any solution to (1) can be constructed from the maximal solution by selecting appropriate eigenvectors from the matrix U.

7 Conclusions

We have developed an algorithm to compute a linear solution to general linear rational expectation problem with general stability conditions. Since the algorithm is based on numerically stable generalized Schur decomposition in case of regular systems and GUPTRI decomposition in case of singular system, also the algorithm is numerically stable. Besides standard solution, the algorithm delivers also all set of sunspot solution. We have also obtained both sufficient and necessary condition for existence of the solution to the problem.

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