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RESEARCH REPORT

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STOCHASTIC MODEL OF THIN MARKET WITH DIVISIBLE COMMODITY

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Abstract

We suggest a model of (a thin) market at which the number of participants is random with Poisson distribution. We provide a formula for joint distribution of the market price and the traded volume. We derive an asymptotic distribution of the quantities. We find that, according to our model, with increasing intensity of the participants' number, the fluctuations of the market price vanish while the variance of the traded volume increases.

Keywords thin market, market price, traded volume, asymptotic distribution

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1 Introduction

The standard economic theory teaches us that changes of the market price are caused by movements of the aggregate demand and of the aggregate supply curves (the movements itself are caused by changes of agents' preferences) or by a delay of producers' reaction to the demand (various cobweb models are trying to describe the situation, see [1]).

We point at another possible cause of the price (and of the traded volume) fluctuation - the varying number of market participants.

We suggest a model in which the individual demand function of each buyer and the individual supply function of each seller remains constant but the number of buyers coming to trade and the number of sellers coming to trade are random. We assume Poisson distribution of both quantities.

We suppose our model to be quite realistic (especially on the side of the demand). For instance, a consumer may have constant demand for milk but he/she may not come to the same milkman every day. Another exemplary situation is the following: The consumers have constant (weekly) demand for carrot but they buy it only once a week. Similarly, the farmers may provide constant supply of carrot but they come to sell it only once a week. In this situation, both the number of buyers and the number of sellers may be modelled using random variables so that our model may be suitable.

2 Definitions

Consider a market with perfectly divisible commodity being peopled by two types of agents - the buyers and the sellers. Suppose all buyers to have the same (linear) demand function

$$d(p) = \max(d_1(d_2 - p), 0)$$

where $d_1 > 0$, $d_2 > 0$ are some constants. Further, suppose that all sellers have the same (linear) supply function

$$s(p) = \max(s_1(p - s_2), 0),$$

where $s_1 > 0$, $s_2 > 0$ and $s_2 < d_2$.

Denote by X the number of buyers coming to trade and by Y the number of sellers coming to trade and assume that X and Y are random both having Poisson distribution with parameters δ, σ respectively. We recall that the Poisson distribution of agents' arrival is usually used in mathematical models of situations with a large number of agents, each coming with the same small probability.

Under our assumptions, the (random) aggregate demand function and the (random) aggregate supply function are

$$D(p) = Xd(p), \qquad S(p) = Ys(p)$$

with expected values $ED(p) = \delta d(p), ES(p) = \sigma s(p)$. Assume further, that the market price is determined by the intersection of the aggregate demand function with the aggregate supply function, provided that at least one buyer and at least one seller came to trade. If no seller came or no buyer came then the price remains undefined. Then, the price equals to

$$P = \begin{cases} \frac{d_1 d_2 X + s_1 s_2 Y}{d_1 X + s_1 Y} = d_2 \left(1 - \frac{1 - s_2 d_2^{-1}}{1 + d_1 s_1^{-1} X Y^{-1}}\right) & \\ \qquad = s_2 \left(1 + \frac{d_2 s_2^{-1} - 1}{s_1 d_1^{-1} Y X^{-1} + 1}\right) & \text{if } X > 0 \wedge Y > 0 \\ \textit{undefined} & \text{otherwise} \end{cases} \tag{1}$$

(symbol $\wedge$ denotes logical *and*) while the traded volume is

$$Q = \begin{cases} D(P) = S(P) = \frac{d_2 - s_2}{s_1^{-1} Y^{-1} + d_1^{-1} X^{-1}} & \text{if } X > 0 \wedge Y > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Since $X > 0 \wedge Y > 0 \Rightarrow Q > 0$, we have

$$[P = \textit{undefined}] \Leftrightarrow [Q = 0] \Leftrightarrow [Y = 0 \vee X = 0] \tag{2}$$

(symbol $\vee$ means logical *or*).

3 Distribution of the market price and the traded volume

If $p > s_2$ then

$[P \neq \mathit{undefined}] \wedge [P \leq p]$

$$\begin{aligned} &\Leftrightarrow \quad [X > 0 \wedge Y > 0] \wedge [S(p) \geq D(p)] \\ &\Leftrightarrow \quad [X > 0 \wedge Y > 0] \wedge \left[s_1^{-1} Y^{-1} \frac{d_2 - p}{p - s_2} \leq d_1^{-1} X^{-1} \right]. \end{aligned} \tag{3}$$

Further, if $q > 0$ then

$[P \neq \mathit{undefined}] \wedge [Q \leq q]$

$$\Leftrightarrow [X > 0 \wedge Y > 0] \wedge \left[\frac{d_2 - s_2}{q} \leq s_1^{-1} Y^{-1} + d_1^{-1} X^{-1} \right]. \tag{4}$$

Using the equivalences above and the fact that $P \neq \mathit{undefined} \Rightarrow P \geq s_2$ we may describe the distribution of the random vector[1] (P, Q) as follows:

Lemma 1 *If $p > s_2$ and $q > 0$ then*

$\mathcal{P}\{0 < Q \leq q, P \leq p\}$

$$\begin{aligned} = \quad & \mathcal{P}\{[X > 0, Y > 0] \\ & \wedge [\max\left(\frac{d_2 - s_2}{q} - s_1^{-1} Y^{-1}, s_1^{-1} Y^{-1} \left(\frac{d_2 - s_2}{p - s_2} - 1 \right) \right) \leq d_1^{-1} X^{-1}]\}, \end{aligned}$$

if $p \leq s_2$ then

$$\mathcal{P}\{0 < Q \leq q, P < p\} = 0 \tag{5}$$

and it holds that

$$\begin{aligned} \mathcal{P}\{Q = 0\} &= \mathcal{P}\{P = \mathit{undefined}\} = \mathcal{P}\{[Q = 0] \wedge [P = \mathit{undefined}]\} \\ &= \mathcal{P}\{X = 0 \vee Y = 0\}. \end{aligned}$$

[1]Since P may take not only real values but also the value *undefined*, it is not a real random variable but the random element defined on space $\mathrm{R} \cup \{\mathit{undefined}\}$ with σ-algebra $\sigma(\mathcal{B}(R), \{\mathit{undefined}\})$ where the symbol $\mathcal{B}$ denotes Borel σ-algebra (see [3] for information how to handle random elements on general spaces).

Lemma 2 *If X and Y are independent, then*

$$\begin{aligned}&\mathcal{P}\{0 < Q \le q, P \le p\}\\ &= e^{-\sigma-\delta}\sum_{i=1}^{\infty}\sum_{\left\{j\in\mathrm{N}:1\le j\le\left(d_1\max\left\{\frac{d_2-s_2}{q}-s_1^{-1}i^{-1},s_1^{-1}i^{-1}\left(\frac{d_2-s_2}{p-s_2}-1\right)\right\}\right)^{-1}\right\}}\frac{\delta^i}{i!}\frac{\sigma^j}{j!}\end{aligned} \quad (6)$$

for each $p > s_2$ and $q > 0$ and it holds that

$$\mathcal{P}\{Q = 0\} = e^{-\delta} + e^{-\sigma} - e^{-\delta-\sigma}. \quad (7)$$

Proof. We may get (6) by summing the probabilities of all combinations of X and Y values fulfilling conditions (3) and (4). The relation (7) is straightforward. □

4 Asymptotic distribution of market price and traded volume

Fix d_1, d_2, s_1, s_2 and denote P_n and Q_n the market price and the traded volume reached at the market defined by d_1, d_2, s_1, s_2, $X = X_n \sim \mathrm{Po}(n\kappa)$ and $Y = Y_n \sim \mathrm{Po}(n\lambda)$ where $n \in \mathrm{N}$, $\kappa \in \mathrm{R}^+$, $\lambda \in \mathrm{R}^+$ are some constants (the symbol $\mathrm{Po}(z)$ denotes Poisson distribution with parameter z). Assume that X and Y are independent. The subject of our interest is the limit behavior of vector (P_n, Q_n) as $n \to \infty$.

Lemma 3 *Denote*

$$p^* = \frac{\kappa d_1 d_2 + \lambda s_1 s_2}{\kappa d_1 + \lambda s_1}, \qquad q^* = \frac{d_2 - s_2}{d_1^{-1}\kappa^{-1} + s_1^{-1}\lambda^{-1}}.$$

It holds that

$$\left(\sqrt{n}(P_n - p^*), \frac{1}{\sqrt{n}}(Q_n - nq^*)\right) \overset{n\to\infty}{\longrightarrow} Z$$

(we mean the convergence in distribution) where $Z = (Z^1, Z^2)$ is a random vector defined by

$$\begin{aligned}&\mathcal{P}\{Z^1 < c_1, Z^2 < c_2\}\\ &= \mathcal{P}\left\{\xi_1 < \min\{\kappa^{1/2}Ac_1 + \kappa^{1/2}\lambda^{-1/2}\xi_2, s_1^{-1}\kappa^{1/2}\lambda^{-1}Ac_2 - s_1^{-1}d_1\kappa^{3/2}\lambda^{-3/2}\xi_2\}\right\}\\ &= \int_{-\infty}^{C}\phi\left(\kappa^{1/2}Ac_1 + \kappa^{1/2}\lambda^{-1/2}x\right)d\phi(x)\\ &\quad + \int_{C}^{\infty}\phi\left(s_1^{-1}\kappa^{1/2}\lambda^{-1}Ac_2 - s_1^{-1}d_1\kappa^{3/2}\lambda^{-3/2}x\right)d\phi(x)\end{aligned}$$

$$A = \frac{(\kappa d_1 + \lambda s_1)^2}{\kappa\lambda d_1 s_1 (d_2 - s_2)}, \qquad C = A\frac{s_1^{-1}\lambda^{-1}c_2 - c_1}{\lambda^{-1/2} + s_1^{-1} d_1 \kappa \lambda^{-3/2}}$$

for each $c_1 \in \mathrm{R}$ *and* $c_2 \in \mathrm{R}$*, where* $\xi_1 \sim \mathcal{N}(0,1)$*,* $\xi_2 \sim \mathcal{N}(0,1)$ *are independent random variables and* ϕ *is standard normal distribution function.*

Corollary

$$\sqrt{n}\frac{A}{\sqrt{\kappa^{-1} + \lambda^{-1}}}(P_n - p^*) \overset{n\to\infty}{\longrightarrow} \mathcal{N}(0,1) \tag{8}$$

$$\frac{1}{\sqrt{n}}\frac{A}{\sqrt{\kappa^2\lambda^{-1}d_1^2 + \kappa^{-1}\lambda^2 s_1^2}}(Q_n - nq^*) \overset{n\to\infty}{\longrightarrow} \mathcal{N}(0,1) \tag{9}$$

Proof. See Appendix A. □

5 Interpretation

Not surprisingly, randomness of market participants' number causes fluctuations both of the price and of the traded volume. Another not surprising fact is that if the demand or the supply function shifts to the right (i.e. the parameter d_2 or s_2 increases) then the price also increases (for all possible positive values of X and Y) and vice versa (those facts follow from (1)).

It may be seen from (1) that the distribution of P does not depend directly on d_1 and s_1 but on their ratio s_1/d_1. Hence, the distribution of the price (not of the volume) remains unchanged if the slope of the individual demand function and the slope of the individual supply function are multiplied by the came constant.

As the market tends to be more liquid (i.e. the intensity of buyers arrival and the intensity of sellers arrival increase at the same rate) then it follows from Corollary of Lemma 1 that the market price converges to the intersection of expected demand and supply curves (which is not changing if the intensities grow accordingly). Hence, at liquid markets, the fluctuations of the price caused by random arrival of the agents are small, hence may be neglected.

The situation is different in case of the traded volume: even if the expectation of traded volume is roughly equal to the intersection of expected demand and supply curves, the variability of the traded volume is increasing with increasing intensity. Hance the fluctuations of the traded volume may not be neglected at the liquid market if the number of agents is random.

6 Concluding remarks

We have assumed Poisson distribution of sellers and buyers. However, Lemma 1 holds for each positive discrete distribution of X and for each positive discrete distribution of Y including deterministic X and/or Y.

Moreover, the model may be easily generalized to the situation that the slopes of the individual demand and supply functions are random: If we denote by D_i the (possibly zero) slope of the i-th demand function and by S_j the (possibly zero) slope of the j-th supply function then the aggregate demand function is $D(p) = R(d_2 - p)$, $R = \sum_i D_i$ whilde the aggregate supply one is $S(p) = T(p - s_2)$, $T = \sum_j S_j$ and it cam be easily shown that Lemma 1 holds with R instead of $d_1 X$ and with T instead of $s_1 Y$.

A Appendix - proof of Lemma 3 and its Corollary

Ad. the Lemma. Fix c_1 and c_2 and denote $p_n \stackrel{\triangle}{=} p^* + c_1/\sqrt{n}$ and $q_n \stackrel{\triangle}{=} nq^* + \sqrt{n}c_2$. It holds that

$$\begin{aligned}
\eta_n &\stackrel{\triangle}{=} \mathcal{P}\left\{ \left[\sqrt{n}(P_n - p^*) \in (-\infty, c_1)\right] \wedge \left[\frac{1}{\sqrt{n}}(Q_n - nq^*) \in (-\infty, c_2)\right] \right\} \\
&= \mathcal{P}\left\{[P_n \in (-\infty, p_n)] \wedge [Q_n \in (-\infty, q_n)]\right\} \\
&= \mathcal{P}\{[P_n \neq \mathit{undefined}] \wedge [Y_n s_1(p_n - s_2) > X_n d_1(d_2 - p_n)] \\
&\qquad \wedge([Q_n = 0] \vee [(Q_n \neq 0) \wedge (d_2 - s_2) d_1 X_n s_1 Y_n < q_n (s_1 Y_n + d_1 X_n)])\} \\
&\stackrel{(2)}{=} \mathcal{P}\{[P_n \neq \mathit{undefined}] \wedge [Y_n s_1(p_n - s_2) > X_n d_1(d_2 - p_n)] \\
&\qquad \wedge[(d_2 - s_2) d_1 X_n s_1 Y_n < q_n (s_1 Y_n + d_1 X_n)]\}
\end{aligned}$$

for each $c_1 \in R$ and $c_2 > -\sqrt{n}q^*$. The subject of our interest is $\lim_{n\to\infty} \eta_n$. Since

$$\begin{aligned}
\mathcal{P}\{P_n = \mathit{undefined}\} &= \mathcal{P}\{X_n = 0 \vee Y_n = 0\} \leq \mathcal{P}\{X_n = 0\} + \mathcal{P}\{Y_n = 0\} \\
&= e^{-n\kappa} + e^{-n\lambda} \stackrel{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

we have

$$\begin{aligned}
\lim_{n\to\infty} \eta_n &= \lim_{n\to\infty} \mathcal{P}\{[Y_n s_1(p_n - s_2) > X_n d_1(d_2 - p_n)] \\
&\qquad \wedge[(d_2 - s_2) d_1 X_n s_1 Y_n < q_n (s_1 Y_n + d_1 X_n)]\} \qquad (10)
\end{aligned}$$

(indeed, if $\lim_n \mathcal{P}\{A_n\} = 0$, then it holds that $\lim_n \mathcal{P}\{\neg A_n \wedge B_n\} = \lim_n [\mathcal{P}\{B_n\} - \mathcal{P}\{A_n \wedge B_n\}] = \lim_n \mathcal{P}\{B_n\}$). Further we have

$$
\begin{aligned}
&Y_n s_1(p_n - s_2) > X_n d_1(d_2 - p_n) \\
&\quad\Leftrightarrow \quad X_n d_1(d_2 - p_n) - Y_n s_1(p_n - s_2) < 0 \\
&\quad\Leftrightarrow \quad (X_n - n\kappa) d_1(d_2 - p_n) - (Y_n - n\lambda) s_1(p_n - s_2) \\
&\qquad\qquad < n\lambda s_1(p_n - s_2) - n\kappa d_1(d_2 - p_n) \\
&\quad\Leftrightarrow \quad (X_n - n\kappa) d_1(d_2 - p_n) - (Y_n - n\lambda) s_1(p_n - s_2) \\
&\qquad\qquad < n\left[p_n(\lambda s_1 + \kappa d_1) - (\lambda s_1 s_2 + \kappa d_1 d_2)\right] \\
&\quad\Leftrightarrow \quad (X_n - n\kappa) d_1(d_2 - p_n) - (Y_n - n\lambda) s_1(p_n - s_2) \\
&\qquad\qquad < \sqrt{n} c_1(\lambda s_1 + \kappa d_1) \\
&\quad\Leftrightarrow \quad \frac{X_n - n\kappa}{\sqrt{n}} \frac{d_1(d_2 - p_n)}{\lambda s_1 + \kappa d_1} - \frac{Y_n - n\lambda}{\sqrt{n}} \frac{s_1(p_n - s_2)}{\lambda s_1 + \kappa d_1} < c_1 \\
&\quad\Leftrightarrow \quad U_n \kappa^{1/2} \frac{d_1(d_2 - p_n)}{\lambda s_1 + \kappa d_1} - V_n \lambda^{1/2} \frac{s_1(p_n - s_2)}{\lambda s_1 + \kappa d_1} < c_1 \qquad (11)
\end{aligned}
$$

where

$$
U_n = \frac{X_n - n\kappa}{\sqrt{\kappa n}}, \qquad V_n = \frac{Y_n - n\lambda}{\sqrt{\lambda n}}.
$$

and, since

$$
\kappa \frac{d_1(d_2 - p^*)}{\kappa d_1 + \lambda s_1} = \lambda \frac{s_1(p^* - s_2)}{\kappa d_1 + \lambda s_1} = \frac{\kappa\lambda d_1 s_1(d_2 - s_2)}{(\kappa d_1 + \lambda s_1)^2} = \frac{1}{A} \qquad (12)
$$

we have

$$
Y_n s_1(p_n - s_2) > X_n d_1(d_2 - p_n) \overset{(11),(12)}{\Leftrightarrow} U_n \kappa^{-1/2} - V_n \lambda^{-1/2} < A c_1. \qquad (13)
$$

Further, since

$$
\begin{aligned}
n^{-3/2} d_1 X_n s_1 Y_n &= n^{-3/2}(X_n - \kappa n + \kappa n)(Y_n - \lambda n + \lambda n) \\
&= n^{-1/2}\sqrt{\kappa} U_n + \kappa\sqrt{n})(\sqrt{\lambda} V_n + \lambda\sqrt{n}) \\
&= n^{-1/2}\sqrt{\kappa\lambda} U_n V_n + \kappa\sqrt{\lambda} V_n + \lambda\sqrt{\kappa} U_n + \kappa\lambda\sqrt{n} \\
&= n^{-1/2}\sqrt{\kappa\lambda} U_n V_n + \kappa\sqrt{\lambda} V_n + \lambda\sqrt{\kappa} U_n + \kappa\lambda\sqrt{n} \\
&= O(n^{-1/2}) + \kappa\sqrt{\lambda} V_n + \lambda\sqrt{\kappa} U_n + \kappa\lambda\sqrt{n} \qquad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
&n^{-3/2} q_n \left(d_1 X_n + s_1 Y_n\right) \\
&\quad = n^{-3/2} q_n \left[d_1(X_n - \kappa n + \kappa n) + s_1(Y_n - \lambda n + \lambda n)\right] \\
&\quad = n^{-1} q_n(d_1\sqrt{\kappa} U_n + s_1\sqrt{\lambda} V_n) + n^{-1/2} q_n(d_1\kappa + s_1\lambda) \\
&\quad = (q^* + O(n^{-1/2}))(d_1\sqrt{\kappa} U_n + s_1\sqrt{\lambda} V_n) + n^{-1/2} q_n(d_1\kappa + s_1\lambda) \\
&\quad = (q^* + O(n^{-1/2}))(d_1\sqrt{\kappa} U_n + s_1\sqrt{\lambda} V_n) + n^{1/2} q^*(d_1\kappa + s_1\lambda) + c_2(d_1\kappa + s_1\lambda) \\
&\quad = (q^* + O(n^{-1/2}))(d_1\sqrt{\kappa} U_n + s_1\sqrt{\lambda} V_n) + d_1 s_1(d_2 - s_2)\kappa\lambda\sqrt{n} \\
&\qquad\quad + c_2(d_1\kappa + s_1\lambda) \qquad (15)
\end{aligned}
$$

we may write, for $a = d_1 s_1 (d_2 - s_2)$,

$$
\begin{aligned}
& aX_n s_1 Y_n < q_n \left(s_1 Y_n + d_1 X_n\right) \\
\overset{(14),(15)}{\Leftrightarrow} \quad & a(\lambda\sqrt{\kappa}U_n + \kappa\sqrt{\lambda}V_n) + a\kappa\lambda\sqrt{n} \\
& \quad < q^*(d_1\sqrt{\kappa}U_n + s_1\sqrt{\lambda}V_n) + c_2(d_1\kappa + s_1\lambda) + a\kappa\lambda\sqrt{n} + O(n^{-1/2}) \\
\Leftrightarrow \quad & \sqrt{\kappa}(a\lambda - q^* d_1)U_n + \sqrt{\lambda}(a\kappa - q^* s_1)V_n < c_2(d_1\kappa + s_1\lambda) + O(n^{-1/2}) \\
\Leftrightarrow \quad & \sqrt{\kappa}\frac{a\lambda - q^* d_1}{d_1\kappa + s_1\lambda}U_n + \sqrt{\lambda}\frac{a\kappa - q^* s_1}{d_1\kappa + s_1\lambda}V_n < c_2 + O(n^{-1/2}) \\
\Leftrightarrow \quad & \sqrt{\kappa}a\frac{\lambda - \frac{\kappa\lambda d_1}{d_1\kappa + s_1\lambda}}{d_1\kappa + s_1\lambda}U_n + \sqrt{\lambda}a\frac{\kappa - \frac{\kappa\lambda s_1}{d_1\kappa + s_1\lambda}}{d_1\kappa + s_1\lambda}V_n < c_2 + O(n^{-1/2}) \\
\Leftrightarrow \quad & \frac{\sqrt{\kappa}a s_1\lambda^2}{(d_1\kappa + s_1\lambda)^2}U_n + \frac{\sqrt{\lambda}a d_1\kappa^2}{(d_1\kappa + s_1\lambda)^2}V_n < c_2 + O(n^{-1/2}) \\
\Leftrightarrow \quad & (s_1\kappa^{-1/2}\lambda U_n + d_1\kappa\lambda^{-1/2}V_n) < Ac_2 + O(n^{-1/2}) \qquad (16)
\end{aligned}
$$

From the computations above it follows

$$
\begin{aligned}
\eta_n \overset{(10),(13),(16)}{=} \; & \mathcal{P}\{[U_n\kappa^{-1/2} - V_n\lambda^{-1/2} < Ac_1] \\
& \wedge[s_1\kappa^{-1/2}\lambda U_n + d_1\kappa\lambda^{-1/2}V_n < Ac_2 + O(n^{-1/2})]\}
\end{aligned}
$$

and, since $(U_n, V_n) \overset{n\to\infty}{\longrightarrow} (\xi_1, \xi_2)$ by the Multivariate Central Limit Theorem, we have, using the Continuous Mapping Theorem (see [2] for both the Theorems),

$$
\begin{aligned}
\lim_{n\to\infty} \eta_n &= \mathcal{P}\left\{[\xi_1\kappa^{-1/2} - \xi_2\lambda^{-1/2} < Ac_1] \wedge [s_1\kappa^{-1/2}\lambda\xi_1 + d_1\kappa\lambda^{-1/2}\xi_2 < Ac_2]\right\} \\
&= \mathcal{P}\{\xi_1 < \min\{\kappa^{1/2}Ac_1 + \kappa^{1/2}\lambda^{-1/2}\xi_2, \\
&\qquad s_1^{-1}\kappa^{1/2}\lambda^{-1}Ac_2 - s_1^{-1}d_1\kappa^{3/2}\lambda^{-3/2}\xi_2\}\} \\
&= \int \phi(\min\{\kappa^{1/2}Ac_1 + \kappa^{1/2}\lambda^{-1/2}x \\
&\qquad , s_1^{-1}\kappa^{1/2}\lambda^{-1}Ac_2 - s_1^{-1}d_1\kappa^{3/2}\lambda^{-3/2}x\})d\phi(x) \\
&= \int_{-\infty}^{C} \phi\left(\kappa^{1/2}Ac_1 + \kappa^{1/2}\lambda^{-1/2}x\right)d\phi(x) \\
&\quad + \int_{C}^{\infty} \phi\left(s_1^{-1}\kappa^{1/2}\lambda^{-1}Ac_2 - s_1^{-1}d_1\kappa^{3/2}\lambda^{-3/2}x\right)d\phi(x). \qquad (17)
\end{aligned}
$$

Ad. the Corollary. From the Lemma it follows that

$$
\begin{aligned}
\mathcal{P}\left\{\sqrt{n}(P_n - p^*) < c_1\right\} \overset{n\to\infty}{\longrightarrow} \; & \mathcal{P}\left\{Z^1 < c_1\right\} = \lim_{c_2\to\infty}\mathcal{P}\left\{Z^1 < c_1, Z^2 < c_2\right\} \\
\overset{(17)}{=} \; & \mathcal{P}\left\{\kappa^{-1/2}\xi_1 - \lambda^{-1/2}\xi_2 < Ac_1\right\} \\
= \; & \mathcal{P}\left\{\mathcal{N}\left(0, \kappa^{-1} + \lambda^{-1}\right) < Ac_1\right\} \\
= \; & \mathcal{P}\left\{\sqrt{\kappa^{-1} + \lambda^{-1}}A^{-1}\mathcal{N}(0,1) < c_1\right\}
\end{aligned}
$$

which proves (8). Similarly,

$$\begin{aligned}\mathcal{P}\left\{n^{-1/2}(Q_n - nq^*)\right\} &\stackrel{n\to\infty}{\longrightarrow} \mathcal{P}\left\{[s_1\kappa^{-1/2}\lambda\xi_1 + d_1\kappa\lambda^{-1/2}\xi_2 < Ac_2]\right\} \\ &= \mathcal{P}\left\{\mathcal{N}\left(0, s_1^2\kappa^{-1}\lambda^2 + d_1^2\lambda^{-1}\kappa^2\right) < Ac_2\right\} \\ &= \mathcal{P}\left\{A^{-1}\sqrt{s_1^2\kappa^{-1}\lambda^2 + d_1^2\lambda^{-1}\kappa^2}\mathcal{N}(0,1) < c_2\right\}.\end{aligned}$$

□

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