Non-linear strategies in a linear quadratic differential game[1]

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Abstract

We study non-linear Markov perfect equilibria in a two agent linear quadratic differential game. In contrast to the literature owing to Tsutsui and Mino (1990), we do not associate endogenous subsets of the state space with candidate solutions. Instead, we address the problem of unbounded-below value functions over infinite horizons by use of the 'catching up optimality' criterion. We present sufficiency conditions for existence based on results in Dockner, Jørgenson, Long, and Sorger (2000). Applying these to our model yields the familiar linear solution as well as a condition under which a continuum of non-linear solutions exist. As this condition is relaxed when agents are more patient, and allows more efficient steady states, it resembles a Folk Theorem for differential games. The model presented here is one of atmospheric pollution; the results apply to differential games more generally.

Key words: differential game, non-linear strategies, catching up optimal, Folk Theorem

JEL classification numbers: C61, C73, H41, Q00

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1 Introduction

This paper analyses non-linear strategies in a linear quadratic differential game played by two identical agents whose controls are bounded below by zero. The techniques developed are then applied to two modified versions of the game. This study is motivated by a greenhouse gas emissions problem.

A differential game is a game played in continuous time in which agents' choices cause a state variable to evolve according to a differential equation. The standard solution concept, the Markov Perfect Equilibrium (MPE), allows application of optimal control techniques to the game. Thus, differential games extend static games, repeated games and optimal control problems.

The workhorse of this literature has been the linear quadratic game (LQG). Its name derives from the two equations defining a differential game, the state variable's equation of motion, and agents' instantaneous utility functions. In LQG the former is a linear function of agents' controls and the state variable; the latter are quadratic in the same.[1] Not only do LQG therefore seem to capture some of the spirit of many economic problems[2], but they are known to yield particularly tractable solutions: the singular solution to the differential equation generated by the Hamilton-Jacobi-Bellman's equation is linear and unique within the class of linear MPE under general conditions (Başar and Olsder, 1999, Remark 6.16).

In the absence of further constraints, however, there is no reason to find these singular solutions more appealing than other ones. This observation has generated an interest in non-linear MPE in LQG dating back to Tsutsui and Mino (1990). In their paper, duopolies choose output levels of a homogeneous good, causing a sticky price variable to evolve over time. In addition to the well known linear MPE, they found a continuum of non-linear ones. Economically, these were of particular interest when the steady state of a non-linear MPE was that of the first best - a version of the Folk Theorem for differential games.

Their finding has become the standard reference in papers on non-linear strategies in LQG and has been applied to a variety of settings, including environmental economics (q.v. Dockner and Long, 1993; Wirl and Dockner,

[1] In an affine quadratic game, the equation of motion also includes a constant term. Both specifications will be referred to here as 'linear quadratic' games. This is less precise, but more concise.

[2] Fudenberg and Tirole (1991, 13.3.3) also note the hope that LQG represent "good Taylor approximation[s] to more general games".

1995; Mäler, Xepapadeas, and de Zeeuw, 2003), industrial organisation (q.v. Karp, 1996; Vencatachellum, 1998b) and the economics of the family (q.v. Feichtinger and Wirl, 1993).

At the same time, the paper generated disquiet with its endogenisation of the domain over which strategies were defined, and their performance assessed. The differential equation derived from the HJB equation produced an infinite number of solutions. For each solution, the associated play lay above the control bound over a sub-domain of the state space. Tsutsui and Mino (1990) then evaluated play, and deviations, over these sub-domains.

This endogenisation has serious implications: for a particular strategy to support a Nash equilibrium an agent must regard that strategy as yielding a superior payoff to any other admissible strategy. Such comparison of payoffs can only occur if agents are allowed to consider all possible strategies of play, including those strategies which would cause the state variable to leave the endogenous sub-domain, while remaining in the original state space. By preventing their consideration, this approach ruled out the sort of calculation that underlies the Nash concept. Possibly reflecting these concerns, a leading text on differential games continues to regard the problem of non-linear solutions as an open one (Başar and Olsder, 1999, Remark 6.16).

This paper seeks to remedy this problem. It does so in two steps. First, it presents a sufficiency result for the existence of MPE based on results in Dockner et al. (2000). The result is applicable well beyond LQG. More specifically, it applies to infinite horizon differential games in which the value function may be unbounded below and may not be continuously differentiable.

Each of these latter two features introduces technical complications for sufficiency conditions. Generalised gradients are required at the non-differentiable points in the domain of a candidate value function. The larger problem, and that responsible for concerns about the existing literature, is the unbounded below value function. Over the infinite horizon, this allows integrals associated with control paths to be non-convergent. This, in turn, may require comparison of infinite payoffs. A standard solution to this has been to impose parameter constraints, or Uzawa conditions, to ensure finite valuation. The role of the endogenous sub-domains in Tsutsui and Mino (1990) and its successors is similar: a bounded domain bounds instantaneous utility; with impatience, this ensures finite valuation.

Rather than imposing parameter bounds, we follow a literature which tries to handle infinite values directly. In contrast to our environment, this has been largely motivated by a preference avoid discounting future payoffs or costs rather than by unbounded instantaneous utility. The earliest example may be found in Ramsey (1928), which rejected discounting as an "ethically indefensible ... weakness of the imagination". The 'Ramsey device' (Wan, Jr, 1971) assumed satiation at finite levels of consumption. When actual consumption approached this level, the undiscounted series defined by the extent to which instantaneous utility fell short of 'bliss' was convergent.

To address situations without satiation, von Weizsäcker (1965) and Atsumi (1965) introduced what has become known as the 'overtaking' criterion for comparing programmes with infinite value: a feasible programme is optimal under this criterion if its payoff stream (weakly) exceeds that of any other for all finite horizons beyond some $\bar{T}$.[3]

As optimal programmes may not exist under this criterion, weaker criteria have also been introduced. Best known among these is the 'catching up' criterion of Gale (1967). This stands in the same relation to the overtaking criterion as an ε-equilibrium does to a strict equilibrium.[4]

As such criteria have not been widely applied to games, no consensus exists on their applicability. The sufficiency conditions presented by Dockner et al. (2000) use catching up optimality as their baseline criterion. As the sufficiency result presented here assembles a number of theirs, we also adopt this criterion.

The paper's second step is to apply the sufficiency conditions to the solutions of the HJB equation generated by a standard LQG. The specific model analysed here is closer to that in Dockner and Long (1993) in both form and motivation than it is to Tsutsui and Mino (1990). This has no analytical consequence: the techniques and results presented here are applicable not only to other LQG, but to more general differential games as well. Outside of the LQG framework, singular solutions need not be linear.

Unsurprisingly, we again find the standard linear MPE. We also derive a necessary and sufficient condition for the continuum of non-linear candidates first reported by Tsutsui and Mino (1990) to be MPE. Rowat (2002b) (an earlier version of this paper) and Rubio and Casino (2002) have both identified identified this condition. Rowat (2002b) did not satisfactorily recognise and

[3] Although introduced in the same journal issue, later writers have often unambiguously ascribed the criterion to one author or the other. I am grateful to Jim Mirrlees for suggesting that the criteria were independently defined.

[4] Seierstad and Sydsæter (1977) and Dockner et al. (2000) review these criteria, and a weaker one yet, 'sporadically catching up'. Stern (1984) adds a further five criteria.

address the problem of non-convergent integrals; the possibility of sustained deviation was not recognised in Rubio and Casino (2002).

The condition loosens as agents become more patient; once patience exceeds a threshold, the continuum of non-linear MPE grows continuously. As this threshold depends on other model parameters, even perfect patience may not be sufficient to attain it. When the steady state of the singular solution exceeds that of the first best, the non-linear MPE reach more efficient steady states than the linear. If agents are sufficiently patient, it is even possible to reach the first best steady state. This has the flavour of a Folk Theorem for differential games.

Although multiple equilibria may arise, they are consistent with the unique optima found in the optimal growth literature with unbounded returns (q.v Le Van and Morhaim, 2002, in discrete time): given any fixed play by the second agent, a single optimal control is derived for the first. Multiple equilibria thus result not from multiple best responses to given play, but to multiple (symmetric) fixed points.[5]

The linear quadratic model is presented in Section 2. Section 3 presents and solves its associated Hamilton-Jacobi-Bellman equation. This produces a family of candidate MPE, which are assessed in Section 4. Section 5 concludes. Appendix A presents the sufficiency conditions for equilibrium based on Dockner et al. (2000). These are weaker than required for our present application but should facilitate analysis of more general differential games. The definitions throughout also follow Dockner et al. (2000).

2 The linear quadratic model

Consider a symmetric, stationary differential game. There are two identical agents, $i \in \{1, 2\}$; refer to the agent other than i by $-i$. At each instant in time, t, each selects a control, $u^i(t)$, from its feasible set. With the play of the other, ϕ^{-i}, this influences the evolution of a state variable, x. Each seeks to maximise the present value, discounted at rate $r \in \Re_{++}$, of its utility stream.

[5]The 'technology' in the differential game cannot be classified *ex ante* as being of constant, increasing or decreasing returns, as in the optimal growth, as the transition process depends on both agents' play.

The game thus outlined, $\Gamma(x_0, 0)$, may be formalised as:

$$\max J^i_{\phi^{-i}}\left(u^i(\cdot)\right) = \int_0^\infty e^{-rt} F\left(x(t), u^i(t), \phi^{-i}(x(t))\right) dt \tag{1}$$

$$\text{s.t. } \dot{x} = f\left(x(t), u^i(t), \phi^{-i}(x(t))\right); \tag{2}$$

$$x(0) = x_0 \in X; \tag{3}$$

$$u^i(t) \in U\left(x(t), \phi^{-i}(x(t))\right); \tag{4}$$

The game is symmetric as: agents' instantaneous payoff functions and feasible sets take the same form; their ability to influence the state's evolution is identical.

It is stationary as the instantaneous payoffs, feasible sets and the equation of motion are not explicitly dependent on time. The second argument of $\Gamma(x_0, 0)$ refers to the time at which play begins. As stationary environments may admit non-stationary solutions, we retain the index to recognise this possibility.

The linear quadratic game, LQG, considered here restricts the above as follows:

$$F\left(x(t), u^i(t)\right) \equiv -\left(u^i - \xi\right)^2 - \nu(x - \zeta)^2; \tag{5}$$

$$f\left(x(t), u^i(t), \phi^{-i}(t)\right) \equiv u^i(t) + \phi^{-i}(t) - \delta x(t); \tag{6}$$

$$X \equiv \Re_{++};$$

$$U\left(x(t), \phi^{-i}(x(t))\right) \equiv \Re_+;$$

where δ, ν, ξ and ζ are positive real constants.[6] The parameter restrictions imposed ensure that instantaneous utility is concave in both control and state.

An attractive property of LQG is that, when the state and actions spaces are unbounded, there exist equilibria in which the strategies are linear functions of the state variable alone, yielding value functions that are quadratic in the state. These are typically derived from a system of Riccati equations (Dockner et al. (2000, 7.1.3), Başar and Olsder (1999, Proposition 6.8)). When the strategy spaces are restricted to affine functions of the state variable, these solutions are unique (Başar and Olsder (1999, Remark 6.16), Lockwood (1996)).

[6]The LQG presented here is a special case: more general quadratic functions may include quadratic terms in u^{-i} and cross terms in u^i, ϕ^{-i} and x. Consideration of this simpler case merely facilitates expositional clarity.

Under the commons problem interpretation, u^i is may be thought of as nation i's greenhouse gas emissions, produced incidentally to national production (in a fixed ratio), x the atmospheric stock of greenhouse gasses and δ the decay or assimilation rate. Thus, agents have a production glut point ($u^i = \xi$) and a climate glut point ($x = \zeta$). The former may be consistent with an aggregated neo-classical labour supply trade off between work and leisure or an optimal capacity utilisation ratio. The latter allows agents to have some sense of optimal climate including, but not necessarily, the lunar climate, $\zeta = 0$.

2.1 Some reference payoffs

Two reference payoffs are presented to provide comparisons for payoffs arising from play of the game.

The payoff to being at the glut point, $(x, u^i) = (\zeta, \xi)$, forever is zero. While this is not attainable as a steady state except when $\delta\zeta = 2\xi$, it does impose a finite upper bound on payoffs. Thus, any solution to this problem must have a payoff that is bounded above by zero.

The steady state of the first best when agents are identical and equally weighted by the social planner may also be calculated. The first order necessary conditions of the current value Hamiltonian are

$$\begin{aligned} m(t) &= -F_u\left(x, u^i\right); \\ \dot{m}(t) - (\delta + r)\, m(t) &= 4\nu\,(x - \zeta); \end{aligned}$$

where m is the current value Lagrangian multiplier and F_u is the partial derivative of $F()$ with respect to its second argument. These imply a system of differential equations.[7] Rather than solving the full trajectories note that, in the steady state, $u^i(t) = \frac{\delta}{2} x(t)$ and $m(t) = -4\nu \frac{x-\zeta}{\delta+r}$. Combined with the first order conditions these yield the steady state

$$\left(\bar{x}, \bar{u}^i\right) = \left(2\frac{(\delta + r)\,\xi + 2\nu\zeta}{\delta(\delta + r) + 4\nu}, \delta\frac{(\delta + r)\,\xi + 2\nu\zeta}{\delta(\delta + r) + 4\nu}\right). \tag{7}$$

[7] The dynamic programming approach does not give any clearer an expression for the dynamics. Its differential equation,

$$w'(x) = \frac{(\delta + r)\, w(x) + 4\nu\,(x - \zeta)}{2\xi - \delta x + w}$$

(where $w(x)$ is the derivative of the candidate value function, $W(x)$, and subscripts index agents) has an unwieldy implicit solution.

When

$$\frac{\zeta}{\xi} \leq \frac{2}{\delta} \tag{A1}$$

the first best stock level exceeds the climate glut level; first best output falls below the product glut level. As this is the first best, though, it is optimal by definition and cannot be considered a 'tragedy' result.

As condition A1 recurs throughout the paper we adopt it as an assumption in what follows. For now, we motivate the assumption on strictly expositional grounds, but it will be seen to determine whether the singular solution under or overprovides relative to the first best in steady state.

3 The Hamiltonian-Jacobi-Bellman equation

As the solution to equation of motion 6 depends on agents' play, restrictions on play are necessary to ensure a unique solution, $x(t)$. We impose regularity conditions to ensure this not out of concern that multiple solutions may preclude a maximum solution (q.v. Burton and Whyburn, 1952) but in order to allow agents to associate payoffs to their strategies.

Therefore:

Definition 1. *A control path $u^i : [0, T\rangle \mapsto \Re$ is* feasible *for $\Gamma(x_0, 0)$ if the IV problem defined by equations 2 and 3 has a unique, absolutely continuous solution $x(\cdot)$ such that the constraints $x(t) \in X$ and $u^i(t) \in U(x(t), \phi^{-i})$ hold for all t and the integral in equation 1 is well defined.*

Feasibility may also be referred to as admissibility (Dockner and Sorger, 1996). As in a generalised game (Debreu, 1952), the feasible set for agent i therefore depends on the actions taken by other agents. We shall see that feasible controls are consistent with multiple equilibria; each, however, induces a unique $x(t)$.

As the solution to the initial value problem defined in equation 6 is

$$x(t) = e^{-\delta t}\left\{x_0 + \int_0^t e^{\delta s}\left[u^1(s) + u^2(s)\right] ds\right\}; \tag{8}$$

feasibility merely requires that the $u^i(t)$ be integrable. See Başar and Olsder (1999, pp. 226-7) or Dockner et al. (2000, p. 40) for further discussion.

As the game is stationary, we focus on equilibria supported by stationary strategies. There may also be equilibria supported by non-stationary strategies.[8]

Definition 2. *A* stationary Markov strategy *is a mapping,* $\phi^i : X \mapsto U^i$*, so that the time path of the control* $u^i(t) = \phi^i(x(t))$*.*

Thus, while the payoff-relevant state space may be very large, Markov strategies are functions of the current state alone.

Then:

Definition 3. *A pair of functions* $\phi^i : X \mapsto \Re, i \in \{1,2\}$ *is a* stationary Markov Nash equilibrium *if, for each* $i \in \{1,2\}$*, an optimal control of problem 1 with constraints 2 to 4 exists and is given by the stationary Markov strategy* $u^i(t) = \phi^i(x(t))$*.*

The following restricts the more general definition to stationary games:

Definition 4. *Let* (ϕ^1, ϕ^2) *be a Markov Nash equilibrium of* $\Gamma(x_0, 0)$*. The equilibrium is a* Markov perfect equilibrium *(MPE) if, for each* $(x,t) \in X \times [0,T\rangle$*, the subgame* $\Gamma(x,t)$ *admits a Markov Nash equilibrium* (ψ^1, ψ^2) *such that* $\psi^i(y,s) = \phi^i(y,s)$ *for all* $i \in \{1,2\}$ *and all* $(y,s) \in X \times [t,T\rangle$*.*

When $\Gamma(x,t)$ is stationary, $\Gamma(x,0) = \Gamma(x,t)$. Thus, all stationary Markov Nash equilibria are MPE (Dockner et al., 2000, p.105).

Definition 5. *Let the* value *of game* $\Gamma(x_0, 0)$ *to agent* i *be*

$$V^i(x) = \max_{u^i \geq 0} J^i_{\phi^{-i}}\left(u^i(\cdot)\right).$$

The sufficiency conditions in Theorem 2 require that V^i be locally Lipschitz. By Rademacher's Theorem, Lipschitz continuous functions are almost everywhere differentiable (Clarke, 1983, p. 63). In spite of this weak differentiability assumption, we shall see that the equilibrium value functions derived are members of $\mathcal{C}^\infty$.

Tsutsui and Mino (1990) and Dockner and Long (1993) require the stronger assumption that $V^i(\cdot) \in \mathcal{C}^2$; this will be seen, in our environment, to follow automatically at most points for which our V^i are differentiable. Dockner and Sorger (1996) do not make continuity assumptions; instead, they derive

[8]See Dockner et al. (2000, Exercise 4.5) for an example.

MPE strategies which are discontinuous but which generate a continuous V^i. Başar and Olsder's example 5.2 (Başar and Olsder, 1999, ch 8) demonstrates that value function continuity may fail even in single agent optimisation problems; the optimal control in their example follows a bang-bang pattern. When the value function is finite, Gota and Montruccio (1999) present sufficient conditions for the value function to be $\mathcal{C}^1$ with Lipschitz continuous derivative in spite of the optimal control only being interior for a short time interval.

When agent i's value function is differentiable it solves the HJB equation:

$$rV^i(x) = \max_{u^i \geq 0} \left[-\left(u^i - \xi\right)^2 - \nu (x - \zeta)^2 + V_x^i(x)\left(u^i + \phi^{-i} - \delta x\right)\right]; \qquad (9)$$

given fixed play ϕ^{-i} by agent $-i$. By V_x^i we mean the derivative of $V^i(x)$; later it will refer to a partial derivative. As the equation of motion makes it impossible that $x(t) = 0$ if $x > 0$ no constraints are imposed on the state space in equation 9.

The non-negativity requirement on u^i provides a first order necessary condition for the optimal control:

$$u^{i*} \equiv \max\left\{0, \xi + \frac{V_x^i(x)}{2}\right\}. \qquad (10)$$

As equation 9 is concave in u^i, u^{i*} is unique and a maximiser. Solutions to the HJB equation 9 are not, however, as they introduce a constant of integration.

Refer to situations in which $u^{i*} = 0$ as *corner solutions* and those in which $u^{i*} > 0$ as *interior solutions*. Call the inequality determining the greater term on the RHS of equation 10 the *auxiliary condition*.

3.1 The differential equation

Substitute the conditions of equation 10 into the HJB equation 9. As the differential equation generated produces a family of solutions, denote the family of *candidate value functions* so generated by $\mathcal{W}$; an individual member of that family is referred to as W. Therefore $V^1 \in \mathcal{W}$. Substitute $u^{2*} = u^{1*}$ into the HJB equation to obtain

$$rW(x) = \left\{ \begin{array}{l} -\nu (x - \zeta)^2 + W'(x)(2\xi - \delta x) + \frac{3W'(x)^2}{4}, W'(x) \geq -2\xi \\ -\xi^2 - \nu (x - \zeta)^2 - \delta x W'(x), W'(x) \leq -2\xi \end{array} \right\} \qquad (11)$$

Symmetric play has now been imposed. The remainder of the analysis may be broken into two steps. The first, and standard, solves the two terms of equation 11; this occupies the next subsections. The more difficult and innovative step involves refining $\mathcal{W}$ to identify constants of integration consistent with the requirements of optimal play's value function.

3.1.1 Corner solutions

The solution to equation 11 when $W'(x) \leq -2\xi$ is

$$W(x) = -\frac{\xi^2 + \nu\zeta^2}{r} - \frac{\nu}{2\delta + r}x^2 + \frac{2\nu\zeta}{\delta + r}x + cx^{-\frac{r}{\delta}};$$

where c is a constant of integration. The condition on $W'(x)$ only allows this to hold for values of x satisfying

$$\frac{2\delta}{r}\left[\xi + \frac{\nu\zeta}{\delta + r}\right]x^{\frac{\delta+r}{\delta}} - \frac{2\delta\nu}{(2\delta + r)\,r}x^{\frac{2\delta+r}{\delta}} \leq c. \tag{12}$$

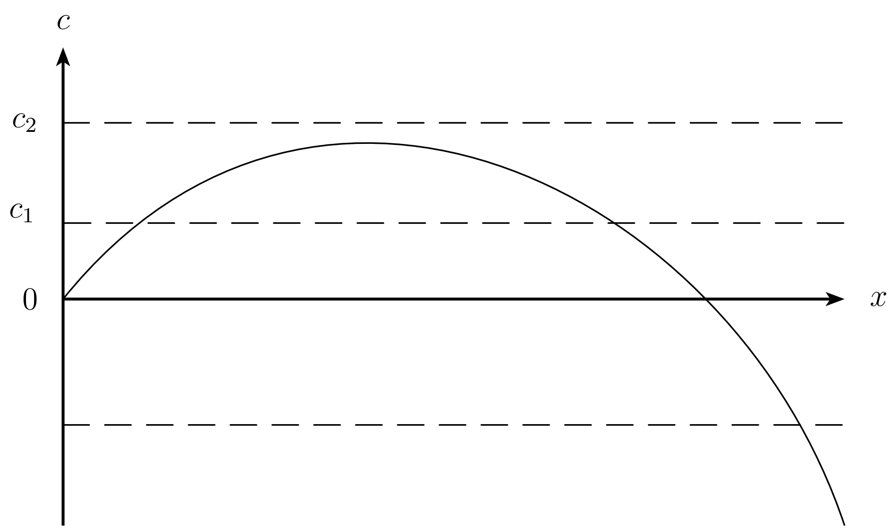

Figure 1: Transitions between the corner and interior solutions

As the exponent on equation 12's first term is smaller than that on the second, it dominates for small values of x. For larger x, though, the second term overpowers it. Figure 1, a stylised plot of equation 12, illustrates the implications of this for solutions. For large values of c (e.g. c_2 in the figure) the condition for the corner solution is always satisfied and $x(x) = 0$ is a solution to the HJB equation. For smaller values, e.g. c_1, it is satisfied for small x, is then violated, and finally is again satisfied for large values of x. For all $c \leq 0$ the condition is violated at small x but eventually comes to hold. It is expected that, when the corner solutions violate condition 12, the strategy will continue in the interior.

3.1.2 Interior solutions

The quadratic interior solution, equation 11 when $W'(x) \geq -2\xi$, is solved by differentiating it again.[9] The next lemma demonstrates when this is legitimate.

Lemma 1. *When $W(x)$ is defined by equation 11 and $W'(x) \geq -2\xi, W(x) \in \mathcal{C}^\infty$ if*

$$\frac{3}{2}W'(x) - \delta x + 2\xi \neq 0. \tag{13}$$

The proof first demonstrates that $W(x) \in \mathcal{C}^2$ when condition 13 holds; it then extends this result to $W(x) \in \mathcal{C}^\infty$.

Proof. Define a function, g, such that $W = g(x, W')$ and note that $g \in \mathcal{C}^1$. At points (x_0, W'_0) where $g_2 \neq 0$ there exists, by the inverse function theorem, an $h(x_0, W_0) \in \mathcal{C}^1$ such that $W' = h(x, W)$ in the neighbourhood of those points. As h and its arguments are members of $\mathcal{C}^1$ then so is W'; hence $W \in \mathcal{C}^2$ in these neighbourhoods.

Derive an expression for $g_2 \equiv \frac{dW(x)}{dW'(x)}$ by differentiating equation 11 when $W'(x) \geq -2\xi$ with respect to $W'(x)$. This yields

$$\frac{dW(x)}{dW'(x)} = \frac{1}{r}\left[\frac{3}{2}W'(x) - \delta x + 2\xi\right];$$

so that $g_2 \neq 0 \Leftrightarrow$ inequality 13.

The result follows by noting that $g \in \mathcal{C}^\infty$. □

Call the locus of points failing to satisfy inequality 13 the *non-invertible locus*. The quadratic term in the interior component of equation 11 causes this to pass through the feasible state-action space. On the other hand,

[9] This approach is also taken by Tsutsui and Mino (1990). Dockner and Sorger (1996) present a case in which direct integration is possible.

when condition 13 is not violated, the relevant portion of equation 11 may be differentiated. For notational convenience define $w(x) \equiv W'(x)$. Therefore:

$$w'(x) = \frac{(\delta + r)\, w(x) + 2\nu (x - \zeta)}{\frac{3}{2} w(x) - \delta x + 2\xi} \text{ when } x \geq 0. \tag{14}$$

Note that the denominator cannot equal zero as that would require $g_2 = 0$ which, by Lemma 1, would have prevented the differentiation performed to reach equation 14.

To solve equation 14 transform the equation into one that is homogeneous of degree zero in its variables by defining $\Omega \equiv w - a$ and $\Psi \equiv x - b$ to remove its constant terms. This requires that

$$a \equiv 2\nu \frac{\delta\zeta - 2\xi}{\delta(\delta + r) + 3\nu}; \text{ and} \tag{15}$$

$$b \equiv \frac{2\xi(\delta + r) + 3\nu\zeta}{\delta(\delta + r) + 3\nu} > 0. \tag{16}$$

These definitions reduce the differential equation to

$$\frac{d\Omega}{d\Psi} = G\left(\frac{\Omega}{\Psi}\right) = \frac{(\delta + r)\,\Omega + 2\nu\Psi}{\frac{3}{2}\Omega - \delta\Psi} = \frac{(\delta + r)\frac{\Omega}{\Psi} + 2\nu}{\frac{3}{2}\frac{\Omega}{\Psi} - \delta}. \tag{17}$$

To exploit the homogeneity of equation 17 define $S \equiv \frac{\Omega}{\Psi}$. Therefore

$$\left[S^2 - \frac{2}{3}(2\delta + r)\, S - \frac{4}{3}\nu\right] d\Psi = \left(\frac{2}{3}\delta - S\right) \Psi dS;$$

which has two constant solutions,

$$S = \{s_a, s_b\} \equiv \frac{1}{3}\left[2\delta + r \pm \sqrt{(2\delta + r)^2 + 12\nu}\right]; \tag{18}$$

with $s_a > 0 > s_b$. These are akin to the algebraic Riccati equations used to derive linear strategies.

Transforming these back into the original variables produces

$$u_a \equiv \xi + \frac{1}{2}\left[a + s_a (x - b)\right]; \tag{19}$$

$$u_b \equiv \xi + \frac{1}{2}\left[a + s_b (x - b)\right]. \tag{20}$$

Defining x_b as the stock level at which the line $u_b(x)$ intersects the steady state locus, $\dot{x}(t) = 0$, yields

$$x_b = \frac{2\xi + a - s_b b}{\delta - s_b}.$$

Therefore, by some tedious arithmetic:

Lemma 2. *Condition A1* $\Leftrightarrow a \leq 0$ *and* $x_b \geq b \geq \bar{x} \geq \zeta$.

When $S \notin \{s_a, s_b\}$, solve

$$\frac{d\Psi}{\Psi} = \frac{\left(\frac{2}{3}\delta - S\right) dS}{(S - s_a)(S - s_b)} = \frac{\gamma_a dS}{S - s_a} + \frac{\gamma_b dS}{S - s_b}; \tag{21}$$

when γ_a and γ_b are determined by the method of partial fractions to be

$$\gamma_a \equiv \frac{r}{3(s_b - s_a)} - \frac{1}{2} < 0; \text{ and } \gamma_b \equiv \frac{-r}{3(s_b - s_a)} - \frac{1}{2} < 0;$$

so that $\gamma_a + \gamma_b = -1$. Integrating equation 21 when $S \notin \{s_a, s_b\}$ then yields

$$\ln|\Psi| = \hat{K} + \gamma_a \ln|S - s_a| + \gamma_b \ln|S - s_b|; \tag{22}$$

where $\hat{K}$ is a real constant of integration. Exponentiation produces

$$|\Psi| = \frac{1}{K}|S - s_a|^{\gamma_a}|S - s_b|^{\gamma_b}; \tag{23}$$

where $K \equiv e^{-\hat{K}} \geq 0$. In terms of x and $W'(x)$ this becomes

$$K = |W'(x) - a - s_a(x - b)|^{\gamma_a}|W'(x) - a - s_b(x - b)|^{\gamma_b}.$$

Equation 10 may be used to rewrite this in terms of $u^1(x)$ instead of $W'(x)$. Doing so does not change the form of the equation. The u_a and u_b solutions identified in equations 19 and 20 correspond to $K = 0$; thus, each of these solutions sets one of the right hand side terms to zero.

To sum up, the solution to differential equation 11 is

$$K = |W'(x) - a - s_a(x - b)|^{\gamma_a}|W'(x) - a - s_b(x - b)|^{\gamma_b} \text{ when } W'(x) \geq -2\xi; \tag{24}$$

$$W(x) = -\frac{\xi^2 + \nu\zeta^2}{r} - \frac{\nu}{2\delta + r}x^2 + \frac{2\nu\zeta}{\delta + r}x + cx^{-\frac{r}{\delta}} \text{ when } W'(x) \leq -2\xi. \tag{25}$$

Equation 24 is still undetermined: integration of $W'(x)$ will produce a second constant of integration. We shall see that K, the first constant of integration, indexes solutions while the second constant adjusts payoffs along given solution paths.

4 Candidate MPE

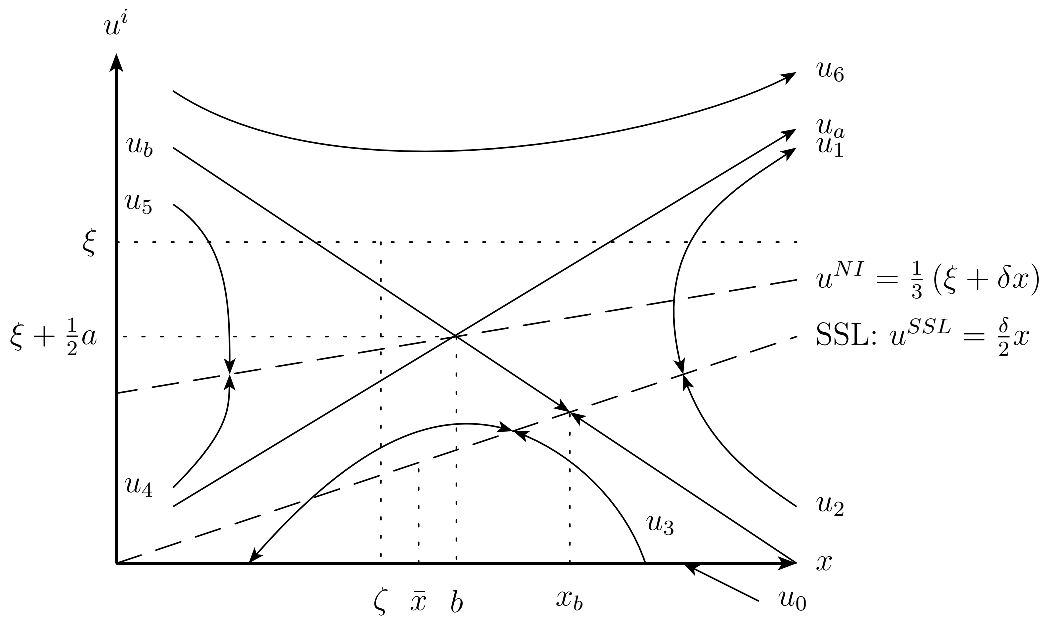

Figure 2: Phase diagram when A1 holds

As K is arbitrary, equation 24 describes a family of infinitely many solutions, $\mathcal{U}$, with members u^i. The upward and downward sloping solutions corresponding to $K = 0$ have already been identified as u_a and u_b, respectively. These intersect at $(x, u^i) = \left(b, \xi + \frac{1}{2}a\right) > \mathbf{0}$, inside the feasible (x, u^i) space. Further, when A1 holds, their intersection is above the climate glut point ($b > \zeta$) and below the product glut point ($\xi + \frac{1}{2}a < \xi$). As there is a non-unique solution to the differential equation at this intersection, call that point a *singularity* and the strategies passing through it *singular solutions.*

Denote the $u^i(x) = 0$ corner strategy of equation 25 by u_0. The remaining six types, denoted $u_1, \ldots, u_6$, are not unique; Figure 2 displays representatives of these families. It also displays the steady state locus (SSL), defined by $\frac{dx}{dt} = 0$, and the non-invertibility locus (NI), along which $\frac{du^i}{dx} = \pm\infty$. In the present case, these are

$$u^{SSL} = \frac{\delta}{2}x \text{ and } u^{NI} = \frac{\delta}{3}x + \frac{\xi}{3};$$

respectively.

Figure 2 can be seen to be equivalent to Figure 1 in Tsutsui and Mino (1990).[10] The diagrams are oriented differently as Tsutsui and Mino's diagram presents the transformed control variable along the vertical axis while that here presents the control variable itself. Thus, their upper bound, $y = \frac{p-c}{s}$, is the present $u^i = 0$.

By the HJB equation's first order condition, equation 10, $W'(x) < 0$ when $u^1 < \xi$, implying that increases in the initial stock, x, always reduce the value of the game when agent 1 plays at less than the product glut level. This may seem particularly surprising when $x < \zeta$ and above the SSL as x increases in time towards the climate glut point. This benefit is apparently balanced by a loss in the product term and, in some cases, a moving more quickly in time beyond the climate glut point.

Candidate strategies must be able to map from any element of the state space, X. The u_6 family of strategies, u_0 and u_a (when it does not intersect the horizontal axis) already do so. The interior solutions u_b, u_3 and u_a (when it does intersect the horizontal axis) are extended by u_0 when they trigger the auxiliary condition, $W'(x) = -2\xi$; denote these extensions by a caret so that $\hat{u}_p \equiv \max\{0, u_p\}$, where p indexes solution families.

As these are all integrable, they all produce unique solutions to equation of motion 6 and therefore represent feasible strategies given given similar play by $-i$. Furthermore, although the candidate strategies will generally be kinked at the corner extension, this is consistent with the requirement that V^i be locally Lipschitz.

4.1 Refining the candidate strategy set

A solution to differential equation 11, $W(x)$, is still two steps removed from describing payoffs under MPE play. First, it must be demonstrated that $W(x) = V^1(x)$, that the candidate value function is a value function. We refine the candidate set against this requirement with two tests: do they

[10] cf. also Figure 1 in Dockner and Long (1993) and in Vencatachellum (1998a).

define functions; are those functions bounded above? Second, we test the remaining candidates against the sufficiency conditions of Theorem 2 in Appendix A.

Lemma 3. *Members of the* u_1, u_2, u_4 *and* u_5 *families of solutions to differential equation 11 cannot form candidate MPE strategies.*

Proof. When members of the u_1, u_2, u_4 and u_5 solution families intersect the non-invertibility locus they cease to be functions. If they are to remain under consideration, some extension to them must be made so that they remain functions over X. They cannot be extended by u_0 as, when they cease to be functions in X, they do not satisfy the auxiliary condition on $W'(x)$. No other extensions are possible. □

It is tempting to consider jumps from one of these solutions to, say, u_0. However, no strategy constructed with jumps like this solves differential equation 11. Similarly, a candidate MPE strategy cannot switch from u_a to u_b (or vice versa) at their intersection as the ensuing path is not one of the trivial solutions presented in equation 18.

Lemma 4. $W(x) \neq V^1(x)$ *along* $\hat{u}_a$ *and the* u_6 *family of strategies.*

Proof. Along $\hat{u}_a$: $x \to \infty \Rightarrow W'(x) \to \infty \Rightarrow W(x) \to \infty$, an impossible integral of the bounded above instantaneous utility function 5. As $u_6(x) > \hat{u}_a(x)$, the u_6 family produces the same contradiction. □

The argument that $\hat{u}_a$ and the $\hat{u}_6$ family do not provide candidate MPE strategies may be illustrated by demonstrating a profitable deviation from their play: as $s_a > \delta$, there is an x such that $\dot{x} > 0$ and $u^i(x) > \xi$ for all greater values of x along these strategies. An agent can then improve its payoff by capping play at $u^i = \xi$; doing so sets the utility loss term in production to zero and slows the climate loss term's growth (as compared to playing $u^i > \xi$).

Discarding $\hat{u}_a$ and the u_6 family leaves only u_0, $\hat{u}_b$ and the $\hat{u}_3$ family of strategies to consider as possible MPE strategies.

Lemma 5. $W(x) \neq V^1(x)$ *in any candidate that satisfies* $u^i(0) = 0$ *and possesses constant of integration* $c \neq 0$ *in that cornered component.*

Proof. By equation 25

$$\lim_{x\to 0} W(x) = -\frac{\xi + \nu\zeta^2}{r} + c \lim_{x\to 0} \frac{1}{x^{\frac{r}{\delta}}}.$$

When $c > 0$, this unbounded limit again contradicts the bounded above instantaneous utility function. As noted in equation 12, which provided the condition for the solution to equation differential 11 to remain in the corner, $c < 0$ and $u^i(0) = 0$ are contradictory. □

The candidate with $c = 0$ is not eliminated by Lemma 5; it leaves $u^i(0) = 0$ immediately.

Again, express this rejection of $u^i(0) = 0$ in terms of profitable deviations by considering play at $x < \zeta$, the glut climate. The cornered strategy requires that agent 1 accept a climate loss as x continues to fall; defection to some small $u^1 > 0$ reduces the climate loss and provides a production gain.[11]

Therefore:

Lemma 6. *When*

$$a - s_a b \leq -2\xi; \tag{26}$$

the only remaining candidate strategy is $\hat{u}_b$.

Proof. As the $\hat{u}_3$ strategies are bounded above by $\hat{u}_a$, Lemma 5 rules out all $\hat{u}_3$ when $u_a(0) \leq 0$; this is equivalent to condition 26. □

This condition parallels that discussed in Rubio and Casino (2002, Section 4). Expanded, its complement is

$$\frac{\zeta}{\xi} < \frac{2}{3}\frac{3\nu + (\delta + r)\left[\delta - r - \sqrt{(2\delta + r)^2 + 12\nu}\right]}{\nu\left[r + \sqrt{(2\delta + r)^2 + 12\nu}\right]}. \tag{27}$$

Thus, the complementary condition holds whenever the climate glut, ζ, is sufficiently small relative to the product glut, ξ.

[11] Similar reasoning would also apply to an $\hat{u}_3$ member for which $u^i(0) > 0$ but which then declined to $u^i(x) = 0$ at some $0 < x < \zeta$. Equation 12 reveals that this is an impossibility: the $\hat{u}_3$ path that passes through $(0, 0)$, and therefore attains $u^i(x) = 0$ at the lowest x, is identified by $c = 0$ along its corner component. This constant sets $u^i(x) = 0$ at $x \in \left\{0, \frac{2\delta+r}{\nu}\left(\xi + \frac{\nu\zeta}{\delta+r}\right)\right\}$. As this second value exceeds ζ for non-negative parameters, the impossibility is established.

Three particular thresholds bear mention. First, condition 26 holds for all ν sufficiently small. In the extreme, when $\nu = 0$, the remaining singular candidate reduces to $\hat{u}_b(x) = \xi$: without a stock effect, there is no interaction between the agents; they optimise with respect to production.[12]

Second, as the derivative of the right hand side is negative with respect to r, the complementary condition holds when agents are sufficiently patient.[13] By Lemma 2, there are $\hat{u}_3$ candidates whose steady states lie closer to $\bar{x}$ than does that of $\hat{u}_b$ (q.v. Figure 2) when condition A1 holds.

This observation motivated the search for 'Folk Theorem' results whereby the efficient solution could be obtained by sufficiently patient agents (Tsutsui and Mino (1990, p.154), Dockner and Long (1993)). When agents are sufficiently impatient, this set of more efficient candidates is eliminated.

Finally, comparison reveals complementary condition 27 to be stricter than assumption A1. Figure 2 illustrates: when the assumption is violated, $a > 0$ (equation 15) so that u_a and u_b intersect at a higher stock and emissions level than the glut levels. As they intersect below the SSL, and as u_a is steeper than the SSL, it must be that $u_a(0) < 0$.

Having discarded various families of solutions from further consideration, we now establish the main result, proving that certain candidates do support MPE.

Theorem 1. *$\hat{u}_b$ and any $\{\hat{u}_3 | \hat{u}_3(0) \geq 0\}$ are MPE strategies.*

The sections of the proof correspond to those of the sufficiency conditions in Theorem 2 (Appendix A):

Proof. 1. as $\hat{u}_b$ and the candidate $\hat{u}_3$ are integrable, they satisfy equation 8. A pair of functions based on these is therefore feasible for $\Gamma(x_0, 0)$.

2. (a) Now consider non-stationary value functions and HJB equations. When $V^i(x(t), t; T)$ is differentiable, it must solve:

$$rV^i(x,t;T) = \left\{ \begin{array}{r} -\nu(x-\zeta)^2 + V^i_x(x,t;T)(2\xi - \delta x) + \frac{3V^i_x(x,t;T)^2}{4} \\ +V^i_t(x,t;T), \\ \text{when } V^i_x(x,t;T) \geq -2\xi; \text{ and} \\ -\xi^2 - \nu(x-\zeta)^2 - \delta x V^i_x(x,t;T) + V^i_t(x,t;T), \\ \text{when } V^i_x(x,t;T) \leq -2\xi \end{array} \right\}$$

[12] This result replicates that of Dockner and Long (1993).

[13] When $\delta^2 > \nu$, perfect patience ensures that the condition holds.

as

$$u^{i*} = \max\left\{0, \xi + \frac{V_x^i(x,t;T)}{2}\right\}.$$

When $u^{i*} = 0$, this is solved by

$$V^i(x(t),t;T) = -\frac{\xi^2 + \nu\zeta^2}{r} - \frac{\nu}{2\delta + r}x^2 + \frac{2\nu\zeta}{\delta + r}x + g\left(t + \frac{\ln x}{\delta}\right)z^{-\frac{r}{\delta}}; \tag{28}$$

where $g(\cdot)$ is an arbitrary function of integration with sufficient differentiability properties. As $u^{i*} = 0$ along this corner solution implies that $x(t) = x(0)e^{-\delta t}$, it follows that $g(\cdot)$ is constant.

When $u^{i*} = \xi + \frac{V_x^i(x,t;T)}{2}$ the differential equation may be solved by separation of variables for

$$V^i(x(t),t;T) = h_1(x) + h_2(t); \tag{29}$$

where $h_1(x)$ and $h_2(t)$ are functions of integration. They satisfy

$$rh_1(x) = \left(k + \nu\zeta^2\right) - \nu(x - \zeta)^2 + h_1'(x)(2\xi - \delta x) + \frac{3h_1'(x)^2}{4}; \text{ and}$$
$$rh_2(t) = \left(rh_2(0) + k + \nu\zeta^2\right)e^{rt} - \left(k + \nu\zeta^2\right);$$

where k is a constant of integration.

Thus

$$rV^i(x(t),t;T) = -\nu(x - \zeta)^2 + h_1'(x)(2\xi - \delta x) + \frac{3h_1'(x)^2}{4} + \left(rh_2(0) + k + \nu\zeta^2\right)e^{rt}. \tag{30}$$

Setting

$$k = -rh_2(0) - \nu\zeta^2;$$

simplifies equation 30 to

$$rV^i(x(t),t;T) = -\nu(x - \zeta)^2 + h_1'(x)(2\xi - \delta x) + \frac{3h_1'(x)^2}{4}; \tag{31}$$

for all t. When $W(x) = h_1(x) = V^i(x,t;T)$, this is identical to the interior component of equation 11. Thus, its solution is that in equation 24.

As there is no constant of integration in equation 28, there are no free parameters with which to satisfy terminal condition 33. Therefore select a T large enough to ensure that transition to $u^{i*} > 0$ occurs by time T. As, along $u^{i*} = 0$, $x(t) = x(0)\, e^{-\delta t}$, a T can be found to do so.

Thus, terminal condition 33 must be satisfied by equation 31. As solutions to this generate second constants of integration, κ, it is possible to set these to satisfy the condition.

(b) as the $\hat{u}_3$ and $\hat{u}_b$ candidate strategies are derived from HJB equations, they are elements of $\Phi(x, t; T)$. The $\hat{u}_3$ candidate strategies never cross the non-invertible locus; $\hat{u}_b$ does once, but remains differentiable through it. Thus, in both cases, the Lebesque measure of times at which equations 34 and 35 are not satisfied is zero.

3. To establish the required limits, first consider the corner solutions. For these, it suffices to demonstrate that, as $T \to \infty$, the RHS of equation 28 converges to that of equation 25. This, in turn, requires that $\lim_{T\to\infty} g(\cdot) = c$ which, as $g(\cdot)$ is independent of T and constant, can be ensured.

   Now consider interior solutions. As already noted, the RHS of equation 31 converges to that of equation 11 when $h_1(x) = V^i(x, t; T)$.

   The limits are also finite: the candidate controls and their induced state variables only take on finite values; discounting ensures finite valuation. The limit value functions are not just locally Lipschitz but $\mathcal{C}^1$ so that, by Lemma 1, they are $\mathcal{C}^\infty$. Finally, the candidates were generated by solving the limit HJB equation 9.

4. as instantaneous objective function 5 only takes on negative values the transversality condition is automatically satisfied.

□

5 Discussion

The problem of endogenised state spaces has dogged the literature on non-singular solutions to differential games since Tsutsui and Mino (1990). Associating each candidate strategy with its own state space has prevented proper consideration of deviations from that candidate, isolating the subsequent literature from the larger body of game theoretical research.

This paper remedies that by applying existing sufficiency results to a linear quadratic differential game. Coupled with other conditions, this allows us to both eliminate solutions to the HJB from consideration as MPE and also to confirm MPE from among them.

Thus, we confirm the existence of the singular solution. As agents' controls are bounded, it no longer generates a quadratic value function. Further, we present a necessary and sufficient condition for the continuum of non-linear solutions first identified by Tsutsui and Mino (1990) to be MPE.

This condition has been identified in previous works, but not properly substantiated. Rowat (2002b) considered modified games in which utility bounds were associated with each initial condition; Rubio and Casino (2002) did not address the requirement that a strategy be defined over the whole of the state space.

In addition to these technical results, some economic insights are obtained. When agents are insensitive to the state ($\nu = 0$), the game reduces to a static optimisation problem. This eliminates the non-linear candidates; the remaining singular MPE sets production loss to zero.

More interestingly, as sufficient patience allows the non-singular MPE, a Folk Theorem for differential games seems to be at work. Here, however, the equilibrium set grows continuously as r decreases. Thus, it may not be possible to attain the first best even once the threshold in condition 27 is exceeded. Note also that assumption A1 is necessary for complementary condition 27 to hold: increases in efficiency through non-singular MPE are only possible when the steady state of the singular MPE exceeds that of the first best.

Our condition is easily derived for other LQGs. In the case of Tsutsui and Mino (1990), the condition, in their notation, is:

$$\beta - \alpha z_a \geq -\frac{c}{s}.$$

Thus, very sticky prices, $s \to 0$, remove the non-linear candidates from the MPE set. As the LHS of the inequality increases in r, patience again allows the possibility of Folk Theorem efficiency.

We conclude by mentioning some avenues for future research. First, the techniques presented here do not depend on the linear quadratic structure of the game. The primary role of that structure is to reduce the singular MPE to a linear MPE. This greatly simplifies derivation of the singular solution, but does not otherwise bear on the existence of non-singular solutions. Thus, we hope that the results presented here ease analysis of a larger class of differential games.

Second, additional or more complicated control bounds could be introduced without new techniques. Tsutsui (1996) considers capacity constraints on firms' production decisions. Analysis of non-constant upper bounds (e.g. a joint savings account game in which maximum aggregate withdrawal is the account balance) may be an interesting generalisation.

Finally, most existing analyses have been symmetric: agents are assumed to be identical; the search for MPE has been confined to ones in which they play identically. Technically, this reflects the possibility that a system of ODEs, one for each player, may not yield an analytical solution.

Rowat (2002a) uses numerical techniques to analyse the game presented here, first allowing identical agents to play differently, then allowing non-identical agents.

A Sufficiency conditions

The Appendix' main Theorem presents a sufficiency result for the existence of Markov Nash equilibria. It also applies to MPE in the in the stationary game $\Gamma(x, 0)$: the time index of any subgame can be re-set to zero. As such, this Theorem allows the conclusions of Theorem 1. We present the Theorem in greater generality than necessary for Theorem 1 to increase its applicability to other environments.

The theorem combines Theorems 4.1, 3.4 and 3.5 of Dockner et al. (2000); the latter two address value functions that are unbounded below and not continuously differentiable, respectively.

Before proceeding, we present two further definitions.

Definition 6. *Define $J^i_{\phi^{-i},T}$ by replacing the upper limit of integration in objective functional 1 with T. A feasible strategy $u^i(\cdot)$ is* catching up optimal *if, for every other feasible strategy $\tilde{u}^i(\cdot)$:*

$$\liminf_{T\to\infty}\left[J^i_{\phi^{-i},T}\left(u^i(\cdot)\right) - J^i_{\phi^{-i},T}\left(\tilde{u}^i(\cdot)\right)\right] \geq 0.$$

As noted by Gale (1967), which introduced the criterion, this is equivalent to: for every $\varepsilon > 0$ there exists a $\bar{T}$ such that, for all $T > \bar{T}$,

$$\left[J^i_{\phi^{-i},T}\left(u^i\left(\cdot\right)\right) - J^i_{\phi^{-i},T}\left(\tilde{u}^i\left(\cdot\right)\right)\right] \geq -\varepsilon.$$

Thus, equilibria in catching up optimal strategies are ε-equilibria for all sufficiently large T. The more stringent criterion of overtaking optimality sets $\varepsilon = 0$.

Stern (1984) presents seven other definitions of optimality in the infinite horizon framework.

To address the possibility of the non-differentiability of V, we also define the generalised gradient, or Clarkian:

Definition 7 (Clarke, 2.5.1)**.** *Let $V : \Re^2 \mapsto \Re$ be Lipschitz continuous in an open neighbourhood of x. The* generalised gradient *of V at x is the set*

$$\partial V\left(x\right) = co\left\{\lim_{i\to\infty} \nabla V\left(x_i\right) | x_i \to x, x_i \notin Z_V\right\};$$

where Z_V is the set of non-differentiable points of V.

Theorem 2. *Consider $\Gamma\left(x_0, 0\right)$, as defined above. Let $\left(\phi^1, \phi^2\right)$ be a given pair of functions $\phi^i : X \mapsto \Re$ and assume that:*

1. *the pair $\left(\phi^1, \phi^2\right)$ is feasible for $\Gamma\left(x_0, 0\right)$.*
2. *for all sufficiently large $T > 0$, and $i \in \{1, 2\}$:*
   (a) *there exist locally Lipschitz continuous functions, $V^i\left(\cdot, \cdot; T\right) : X \times [0, T] \mapsto \Re$ which solve the HJB equations*

$$\begin{aligned} rV^i\left(x, t; T\right) = \max\big\{&F\left(x, u^i, \phi^{-i}\right) + \alpha^i f\left(x, u^i, \phi^{-i}\right) + \beta^i \\ &\mid u^i \in U\left(x, \phi^{-i}\right), \left(\alpha^i, \beta^i\right) \in \partial V^i\left(x, t; T\right)\big\} \\ &\forall \left(x, t\right) \in X \times [0, T]; \end{aligned} \tag{32}$$

   and the terminal condition

$$V^i\left(x, T; T\right) = 0. \tag{33}$$

   Denote by $\Phi^i\left(x, t; T\right)$ the set of all $\left(u^i, \alpha^i, \beta^i\right) \in U\left(x, \phi^{-i}\right) \times \partial V^i\left(x, t; T\right)$ which maximise the RHS of HJB equations 32.

(b) *for the feasible control* u_T^i *against* ϕ^{-i}, *which induces state trajectory* x_T, *there exist* $(\alpha^i(t), \beta^i(t)) \in \Re^2$ *for all* t *such that*

$$\left(u_T^i(t), \alpha^i(t), \beta^i(t)\right) \in \Phi^i(x_T, t; T); \text{ and} \tag{34}$$

$$\frac{d}{dt}V^i(x_T(t), t; T) = \alpha^i(t)\dot{x}_T(t) + \beta^i(t); \tag{35}$$

for almost all $t \in [0, T]$.

3. *for all* $x \in X$ *and* $i \in \{1, 2\}$, *the limits*

$$V^i(x) \equiv \lim_{T\to\infty} V^i(x, t; T) \tag{36}$$

exist, are finite, and are locally Lipschitz continuous functions V^i : $X \mapsto \Re$ *which solve the HJB equations*

$$rV^i(x) = \max\left\{F\left(x, u^i, \phi^{-i}\right) + \alpha^i f\left(x, u^i, \phi^{-i}\right) \mid u^i \in U\left(x, \phi^{-i}\right), \alpha^i \in \partial V^i(x)\right\} \forall x \in X. \tag{37}$$

Denote by $\Phi^i(x)$ *the set of all* $(u^i, \alpha^i) \in U(x, \phi^{-i}) \times \partial V^i(x)$ *which maximise the RHS of HJB equations 37.*

4. $\limsup_{T\to\infty} e^{-rT}V^i(x(T)) \leq 0 \forall i \in \{1, 2\}$.

If $\phi^i(x) \in \Phi^i(x)$ *for each* $i \in \{1, 2\}$ *and almost all* $t \in [0, \infty)$, *then* (ϕ^1, ϕ^2) *is a MPE in the sense of catching up optimality.*

Informally, assumption 1 ensures that each agent faces a well defined problem: given play ϕ^{-i} by the other, the objective functional of agent i is well-defined - although it may take on infinite value. Assumption 2a establishes sufficiency conditions for MPE in the finite horizon problem. Assumption 2b addresses the possibility of the failure of $V^i(\cdot; T)$ to be differentiable; by Rademacher's Theorem, locally Lipschitz continuous functions are differentiable for almost all $t \in [0, T]$ (Clarke, 1983, p. 63). Assumption 4 is a transversality condition. It may be replaced with other conditions to obtain other criteria of optimality mentioned in the introduction. Finally, assumption 3 allows application of the transversality conditions to the other sufficiency conditions. If the optimal controls derived have a Markov representation, then a MPE exists.

Before proving the theorem, we present a lemma guaranteeing that the assumptions in 2a and 2b define an optimal control, $u_T^i(\cdot)$ over $t \in [0, T]$.

Lemma 7. *Under the conditions of Theorem 2,* $(u_T^i, \alpha^i, \beta^i) \in \Phi^i (x_T, t; T)$ *satisfies*

$$J^i_{\phi^{-i},T} \left(u_T^i (\cdot)\right) \geq J^i_{\phi^{-i},T} \left(\tilde{u}^i (\cdot)\right);$$

given ϕ^{-i} *for all feasible* $\tilde{u}^i$*.*

The proof of the Lemma combines those for Theorems 3.1 and 3.5 in Dockner et al. (2000):

Proof. Take ϕ^{-i} as fixed. By assumption 1, any feasible control, $\tilde{u}^i (t)$, induces an absolutely continuous state trajectory, $\tilde{x} (t)$.

By assumption 2a, the ensuing $V^i (\tilde{x} (t), t; T)$ is locally Lipschitz. Thus, its generalised gradients $\partial V^i (\cdot, t; T)$ exist (Clarke, 1983, 2.5.1).

By the chain rule (Clarke, 1983, 2.3.9), the discounted value function satisfies the differential inclusion

$$\begin{aligned} \frac{d}{dt} \left[e^{-rt} V^i (\tilde{x} (t), t; T)\right] \in \Big\{ & e^{-rt} \left[-rV^i (\tilde{x} (t), t; T) + \alpha^i \dot{\tilde{x}} (t) + \beta^i\right] \\ & \mid \left(\alpha^i, \beta^i\right) \in \partial V (\tilde{x} (t), t; T) \Big\}. \end{aligned}$$

Adding terms independent of α^i and β^i to both sides does not alter the inclusion:

$$\begin{aligned} \frac{d}{dt} \left[e^{-rt} V^i (\tilde{x} (t), t; T)\right] & + e^{-rt} F \left(\tilde{x} (t), \tilde{u}^i, \phi^{-i}\right) + re^{-rt} V (\tilde{x} (t), t) \\ & \in \Big\{ e^{-rt} \left[F \left(\tilde{x} (t), \tilde{u}^i, \phi^{-i}\right) + \alpha^i f \left(\tilde{x}, \tilde{u}^i, \phi^{-i}\right) + \beta^i\right] \\ & \quad \mid \left(\alpha^i, \beta^i\right) \in \partial V (\tilde{x} (t), t; T) \Big\}. \end{aligned}$$

As the final term on the left hand side of the inclusion maximises its right hand side, the remaining left hand side terms must be non-positive:

$$e^{-rt} F \left(\tilde{x} (t), \tilde{u}^i, \phi^{-i}\right) \leq -\frac{d}{dt} \left[e^{-rt} V^i (\tilde{x} (t), t; T)\right]. \tag{38}$$

Now consider the candidate control, $u_T^i (t)$, which induces state trajectory $x_T (t)$. Manipulation of equation 35 yields

$$\begin{aligned} & \frac{d}{dt} \left[e^{-rt} V^i (x_T (t), t; T)\right] + e^{-rt} F \left(x_T (t), u_T^i, \phi^{-i}\right) + re^{-rt} V^i (x_T (t), t; T) \\ & = e^{-rt} \left[F \left(x_T (t), u_T^i, \phi^{-i}\right) + \alpha^i (t) f \left(x_T, u_T^i, \phi^{-i}\right) + \beta^i (t)\right]. \end{aligned}$$

As the right hand side is, by inclusion 34, maximal, the first two left hand side terms must sum to zero:

$$e^{-rt}F\left(x_T\left(t\right),u_T^i,\phi^{-i}\right)=-\frac{d}{dt}\left[e^{-rt}V^i\left(x_T\left(t\right),t;T\right)\right] \tag{39}$$

for almost all $t\in[0,T]$, in the sense of Lebesgue.

As the definition of objective functional 1 uses the left hand side terms of equations 38 and 39, the candidate may be compared to other feasible controls:

$$\begin{aligned}J_{\phi^{-i},T}^i\left(u_T^i\left(\cdot\right)\right)&=\int_0^T e^{-rt}\left[F\left(x_T\left(t\right),u_T^i,\phi^{-i}\right)\right]dt\\&=-e^{-rT}V^i\left(x_T\left(T\right),T;T\right)+V^i\left(x_0,0;T\right)\\&=V^i\left(x_0,0;T\right);\end{aligned} \tag{40}$$

and

$$\begin{aligned}J_{\phi^{-i},T}^i\left(\tilde{u}^i\left(\cdot\right)\right)&=\int_0^T e^{-rt}\left[F\left(\tilde{x}\left(t\right),\tilde{u}^i,\phi^{-i}\right)\right]dt\\&\leq-e^{-rT}V^i\left(\tilde{x}\left(T\right),T;T\right)+V^i\left(x_0,0;T\right)\\&=V^i\left(x_0,0;T\right).\end{aligned}$$

In both cases, the final equalities owe to terminal condition 33.

Thus, the difference between the values is

$$J_{\phi^{-i},T}^i\left(u_T^i\left(\cdot\right)\right)-J_{\phi^{-i},T}^i\left(\tilde{u}^i\left(\cdot\right)\right)\geq V^i\left(x_0,0;T\right)-V^i\left(x_0,0;T\right)=0.$$

The result is established. □

Now return to the proof of the Theorem; it is based on that of Theorem 3.4 in Dockner et al. (2000):

Proof of Theorem 2. As $V^i\left(x\left(t\right)\right)$ solves HJB equation 37, we may write the T-truncation of objective functional 1 as

$$\begin{aligned}&J_{\phi^{-i},T}^i\left(u^i\left(\cdot\right)\right)\\&=\int_0^T e^{-rt}\left\{rV^i\left(x\right)-\alpha^i f\left(x,u^i,\phi^{-i}\right)|u^i\in U\left(x,\phi^{-i}\right),\alpha^i\in\partial V^i\left(x\right)\right\}dt\\&=\int_0^T\frac{d}{dt}\left[-e^{-rt}V^i\left(x\left(t\right)\right)\right]dt\\&=V^i\left(x_0\right)-e^{-rT}V^i\left(x\left(T\right)\right).\end{aligned}$$

By equation 40 we may therefore write

$$\begin{aligned}
&\liminf_{T\to\infty}\left[J^i_{\phi^{-i},T}\left(u^i\left(\cdot\right)\right)-J^i_{\phi^{-i},T}\left(u^i_T\left(\cdot\right)\right)\right]\\
&=\liminf_{T\to\infty}\left[V^i\left(x_0\right)-e^{-rT}V^i\left(x\left(T\right)\right)-V^i\left(x_0,0;T\right)\right]\\
&=\liminf_{T\to\infty}\left[-e^{-rT}V^i\left(x\left(T\right)\right)\right]\geq 0;
\end{aligned}$$

by the definition of the limiting value functions, 36, and assumption 4.

As, by Lemma 7,

$$J^i_{\phi^{-i},T}\left(u^i_T\left(\cdot\right)\right)\geq J^i_{\phi^{-i},T}\left(\tilde{u}^i\left(\cdot\right)\right);$$

it follows that

$$\liminf_{T\to\infty}\left[J^i_{\phi^{-i},T}\left(u^i\left(\cdot\right)\right)-J^i_{\phi^{-i},T}\left(\tilde{u}^i\left(\cdot\right)\right)\right]\geq 0.$$

As $\phi^i\left(x\left(t\right)\right)\in\Phi^i\left(x\right)$ for almost all $t\in[0,\infty)$, the set of times for which it fails to be catching up optimal has Lebesgue measure zero. The result therefore follows. □

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