Closed form solutions to a generalization of the Solow growth model.

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Abstract

The Solow growth model assumes that labor force grows exponentially. This is not a realistic assumption because, exponential growth implies that population increases to infinity as time tends to infinity. In this paper we propose replacing the exponential population growth with a simple and more realistic equation - the Von Bertalanffy model. This model utilizes three hypotheses about human population growth: (1) when population size is small, growth is exponential; (2) population is bounded; and (3) the rate of population growth decreases to zero as time tends toward infinity. After making this substitution, the generalized Solow model is then solved in closed form, demonstrating that the intrinsic rate of population growth does not influence the long-run equilibrium level of capital per worker. We also study the revised model's stability, comparing it with that of the classical model.

Keywords: Solow growth model, population growth

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1 Introduction

This paper describes a generalization of the Solow economic growth model that allows population growth rate to diminish over time. The original Solow growth model assumes that the labor force L grows at a constant rate $n > 0$. In continuous time, the population growth rate used in the original Solow model is

$$n = \frac{\dot{L}}{L} = \frac{\frac{\partial L}{\partial t}}{L} \tag{1}$$

for any initial level L_0, at time t the level of the labor force is

$$L(t) = L_0 e^{nt} \tag{2}$$

This assumption is realistic only for small values of the labor force because, with unlimited exponential growth, as time tends toward infinity, so too does the

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labor force. Several studies support the hypothesis that the world's population growth rate is deceasing and tends toward zero.[1] Natural resources are limited, implying that food shortages, unemployment, and pollution eventually limit population growth. This limit is usually called the carrying capacity of the environment (denoted herein as L_∞) and forms a numerical upper bound on the population size.

Therefore, as described by Maynard Smith [9], a more realistic model of the growth of the labor force $L(t)$ must exhibit the following properties:

1. when population is small enough in proportion to environmental carrying capacity L_∞, then L grows at a constant rate $n > 0$.

2. when population is large enough in proportion to environmental carrying capacity L_∞, the economic resources become scarcer, reducing the rate of population growth.

3. population growth rate decreases to 0 over time.

In this paper we assume that the labour force $L(t)$ exhibits all these properties. In particular, we introduce the von Bertalanffy equation (von Bertalanffy (1938)).[2] This model is widely used in population studies and data analysis [3] and is one of the simplest realistic models of population dynamics that incorporate all the properties previously introduced.

The von Bertalanffy function is the solution of the initial value problem:

$$\begin{cases} \dot{L} = r\left(L_\infty - L\right) \\ \quad L(0) = L_0 \end{cases} \tag{3}$$

where L_∞ is a theorical maximum asymptote size of the labor force (carrying capacity), L_0 is the labor force at time t_0 and r is a constant which determines the speed at which the labor force reaches the asymptote. This equation shows that the growth rate is a decreasing linear function of the population size. In addition, the equation verifies four relevant facts:

1. The growth rate is greatest when the population size is smallest;

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[1] See, for example, Day (1996)

[2] See Mingari Scarpello et al. (2003) for a similar generalization, which uses the logistic model of population growth. In this paper the authors focus upon the solutions to the equation of motion in a closed form by using the special functions.

[3] Cloern et al. (1978) apply the von Bertalanffy equation to predict body growth for Macoma balthica. Jurado-Molina et al. (1992) conduct a study in which the von Bertalanffy equation is applied to Mugil curema to calculate its weight growth. Xiao (2000) uses the von Bertalanffy model to calculate the parameters from a set of tagging data concerning times at liberty, lengths at release, and lengths at recapture of a Lates calcarifer. Anislao, Auró, and González (2002) present a work in which the speed of growth of Cyprinus carpio was estimated with data including length and weight of Cyprinus carpio, a work in which they employ the von Bertalanffy equation. Finally, De Graaf and Prein (2005) compare three approaches to the multivariate analysis of Oreochromis niloticus growth, based upon the von Bertalanffy equation.

2. The growth rate decrease to zero as the population size approaches the carrying capacity;

3. If the population size is greater than the carrying capacity, the growth rate is negative; and

4. There is a constant upper limit on population size (L_∞).

This equation has only one steady state L_∞ and the solution is given by:

$$L(t) = L_\infty - (L_\infty - L_0)e^{-rt} \tag{4}$$

Note that, with the von Bertalanffy model, the growth rate becomes:

$$n(t) = \frac{\dot{L}(t)}{L(t)} = \frac{r\,(L_\infty - L)}{L_\infty - [L_\infty - L_0]\,e^{-rt}} = \frac{r\,(L_\infty - L_0)}{(L_0 - L_\infty) + L_\infty e^{rt}} \tag{5}$$

which decreases monotonically to 0 as t tends to infinity.

2 The Model

The original Solow model assumes that:

1. There is an aggregate production function $Y = F(K, L)$, which is assumed to satisfy a series of technical conditions, such as:

    (a) it is increasing in both factors;
    (b) it shows decreasing marginal returns for each factor;
    (c) it displays constant returns to scale; and
    (d) it satisfies the Inada conditions.

Among all the possible production functions satisfying these properties, we shall assume the Cobb-Douglas function, as it is the most often cited in the literature:

$$F\,(K, L) = K^\alpha L^{1-\alpha}; 0 < \alpha < 1 \tag{6}$$

2. The change in capital stock equal the gross investment $I = sF\,(K, L)$ minus the capital depretiaton δK. It is

$$\dot{K} = sF\,(K, L) - \delta K \tag{7}$$

3. There exists a law of motion for the stock of capital per worker. In continuous time, this law is:

$$\dot{k} = sk^\alpha - (\delta + n)\,k \tag{8}$$

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4. The population grows at rate n, which equals:

$$L(t) = L_0 e^{nt} \qquad (9)$$

In this paper the last assumption is set aside. In its place we assume that the labor force follows a von Bertalanffy model:

$$\begin{cases} \dot{L} = r\left(L_\infty - L\right) \\ \quad L(0) = L_0 \end{cases} \qquad (10)$$

In growth theory it is convenient to express all the variables of interest in per capita terms. Thus, we shall use small letters to denote the variables in per worker terms. If $k = \frac{K}{L}$ is the capital per worker we have

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \qquad (11)$$

and then

$$\frac{\dot{k}}{k} = \frac{sF\left(K, L\right) - \delta K}{K} - \frac{r\left(L_\infty - L_0\right)}{\left(L_0 - L_\infty\right) + L_\infty e^{rt}} = sk^{\alpha - 1} - \delta - n(t) \qquad (12)$$

From this, we obtain the equation of motion for the stock of capital per worker for the modified Solow growth model, which uses the von Bertalanffy labor growth mode. This describes how capital per worker varies over time:

$$\dot{k} = sk^\alpha - \left(\delta + n(t)\right)k \qquad (13)$$

where $n(t) = \frac{r(L_\infty - L_0)}{(L_0 - L_\infty) + L_\infty e^{rt}}$. In the next section, we solve the differential equation (13) and analyze the stability of the model obtaining the asymptotic value of the variables.

3 Closed form solutions and stability analysis

It has recently been pointed out that the classical Solow growth model with Cobb-Douglas technology has closed form solutions. See, for example, the recent textbook of Barro and Sala-i-Martin [2] and the references [7] and [11]. In a recent paper, Mingari Scarpello et al. [10] showed that this result can be extended when the population growths following the logistic law. In this section, we will show that equation 13 can also be solved in closed form. Equation (13) is a Bernoulli type equation that can be transformed by the change of variables

$$u = k^{1-\alpha} \qquad (14)$$

into the linear equation

$$\dot{u} + \left(1 - \alpha\right)\left(\delta + n(t)\right)u = \left(1 - \alpha\right)s. \qquad (15)$$

In order to find the solutions to this linear differential equation, one must remember that, given the continuous functions $a(t)$ and $b(t)$, the solution of a linear differential equation:

$$\dot{x} + a(t)x = b(t) \tag{16}$$

with the initial condition $x_0 = x(0)$ is

$$x(t) = e^{A(t)} \left(x_0 + \int_0^t b(\tau)e^{-A(\tau)}d\tau \right) \tag{17}$$

where

$$A(t) = \int_0^t a(\tau)d\tau. \tag{18}$$

Observe that

$$|x_0 - x_1| \, e^{A(t)} \tag{19}$$

gives the difference between two different solutions with initial conditions x_0 and x_1. Then, a solution of the linear differential equation (16) is stable if and only if the function $A(t)$ is bounded from above in $[0, +\infty)$. If, besides, it is

$$\lim_{t \to +\infty} A(t) = -\infty \tag{20}$$

then the solutions are globally asymptotically stable. Observe that the solutions of equation (16) have an horizontal asymptote if there exists the limit:

$$\lim_{t \to +\infty} \frac{b(t)}{a(t)} = x_\infty \tag{21}$$

and, in this case we have:

$$\lim_{t \to +\infty} x(t) = -x_\infty \tag{22}$$

In fact, by the L'Hopital's rule,

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} e^{A(t)} \left(x_0 + \int_0^t b(\tau)e^{-A(\tau)}d\tau \right) = \lim_{t \to +\infty} \frac{b(t)}{-a(t)} = -x_\infty \tag{23}$$

Now we can employ these observations to solve the equation (15). In this case, the continuous functions $a(t)$ and $b(t)$ are:

$$a(t) = (1 - \alpha)(\delta + n(t)) = (1 - \alpha)\left(\delta + \frac{r(L_\infty - L_0)}{(L_0 - L_\infty) + L_\infty e^{rt}} \right) \tag{24}$$

$$b(t) = (1 - \alpha)s$$

and then we have that

$$A(t) = (1 - \alpha)\left[\delta t + \int_0^t \frac{r(L_\infty - L_0)}{(L_0 - L_\infty) + L_\infty e^{r\tau}}d\tau \right]$$
$$= (1 - \alpha)\left[\delta t - rt + \ln\left(\frac{(L_0 - L_\infty) + L_\infty e^{rt}}{L_0} \right) \right], \tag{25}$$

$$e^{A(t)} = e^{(1-\alpha)\left[\delta t - rt + \ln\left(\frac{(L_0 - L_\infty) + L_\infty e^{rt}}{L_0}\right)\right]}$$
$$= e^{(1-\alpha)(\delta - r)t}\left(\frac{(L_0 - L_\infty) + L_\infty e^{rt}}{L_0}\right)^{1-\alpha} \tag{26}$$

and

$$\int_0^t b(\tau)e^{-A(\tau)}d\tau = \int_0^t (1-\alpha)\, se^{(\alpha-1)(\delta-r)\tau}\left(\frac{(L_0 - L_\infty) + L_\infty e^{r\tau}}{L_0}\right)^{\alpha-1} d\tau$$
$$= \frac{(1-\alpha)s}{L_0^{\alpha-1}}\int_0^t e^{(\alpha-1)(\delta-r)\tau}\left((L_0 - L_\infty) + L_\infty e^{r\tau}\right)^{\alpha-1} d\tau. \tag{27}$$

Then, to obtain the closed form solution of equation (15) we have to compute the integral

$$I(t) = \int_0^t e^{(\alpha-1)(\delta-r)\tau}\left((L_0 - L_\infty) + L_\infty e^{r\tau}\right)^{\alpha-1} d\tau. \tag{28}$$

This can be done in terms of the hypergeometric function $_2F_1$ obtaining that.

$$I(t) = \frac{\left(\frac{L_\infty - L_0}{L_\infty}\right)^{\alpha-1}}{(\alpha-1)(\delta-r)}\left[A\left(\frac{L_\infty}{L_0 - L_\infty}\right) - e_2^{(\alpha-1)(\delta-r)t}A\left(\frac{e^{rt}L_\infty}{L_0 - L_\infty}\right)\right] \tag{29}$$

where

$$A(Z) =_2 F_1\left(1-\alpha, 1-\alpha+\frac{\delta}{r}, 2-\alpha+\frac{\delta}{r}, Z\right).$$

In the paper [10] there is a detailed description of the evaluation of integral $I(t)$ and an Appendix with a short outline on the hypergeometric function.

Then we have that the solutions to equation (15) are given by

$$u(t) = e^{(1-\alpha)(\delta-r)t}\left(\frac{(L_0 - L_\infty) + L_\infty e^{rt}}{L_0}\right)^{1-\alpha}\left[k_0^{1-\alpha} + \frac{(1-\alpha)s}{L_0^{\alpha-1}}I(t)\right] \tag{30}$$

where k_0 is the initial value verifying $k_0 = k(0)$ and $I(t)$ is given by equation (29). This is the closed form solution of equation (15). From this and the change of variables $k = u^{\frac{1}{1-\alpha}}$ we can obtain the closed form solution of equation (13), representing the generalized Solow model.

Note that $A(t)$ is bounded from above in $[0, +\infty)$ and tends to $-\infty$ as $t \to +\infty$. Thus, the solutions of equation (15) are globally asymptotically stable. Finally, we have that:

$$\lim_{t\to+\infty}\frac{b(t)}{a(t)} = \lim_{t\to+\infty}\frac{(1-\alpha)s}{(1-\alpha)\left(\delta + \frac{r(L_\infty - L_0)}{(L_0 - L_\infty) + L_\infty e^{rt}}\right)} = \frac{s}{\delta} \tag{31}$$

and then all the solutions of (15) have the horizontal asymptote at level $\frac{s}{\delta}$ as $t \to +\infty$. The change of variables $k = u^{\frac{1}{1-\alpha}}$ transforming solutions of equation (15) into solutions of (13) is continuous. This implies that equation (13) is globally asymptotically stable, and, as time approaches to infinity, capital per

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worker k tends to the long run limit value $\left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$. This equilibrium value $\hat{k} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$ is not a steady state, since it is not a solution of equation (13). Nevertheless, it is the long run value of the per worker level of capital k. Observe that the intrinsic rate of population growth $n(t)$ does not influence the long-run equilibrium per worker level of capita k. For any initial condition, capital per worker converges to the value $\hat{k}$. This is true since the model is asymptotically stable.

Now we will contrast the long run levels of capital per worker $\hat{k}$ (modified Solow model with von Bertalanffy equation) and $\tilde{k}$ (original Solow model),we can see that:

$$\hat{k} = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}} > \left(\frac{s}{\delta + n}\right)^{\frac{1}{1-\alpha}} = \tilde{k} \tag{32}$$

comparing the long run levels of output per worker $\hat{y}$ (modified Solow model with von Bertalanffy equation) and $\tilde{y}$ (original Solow model) we have that:

$$\hat{y} = \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} > \left(\frac{s}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}} = \tilde{y} \tag{33}$$

finally, we show the long run levels of consumption per capital $\hat{c}$ (modified Solow model with von Bertalanffy equation) and $\tilde{c}$ (original Solow model)

$$\hat{c} = (1 - s)\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} > (1 - s)\left(\frac{s}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}} = \tilde{c} \tag{34}$$

That is, if population growth follows the von Bertalanffy law, the long run levels of the Solow model are improved.

4   Concluding Remarks

The original Solow model assumes that population growth is exponential. This is not a realistic assumption because, with limited resources, population growth must be bounded. In this paper we have used the von Bertalanffy equation instead of the exponential equation to model labor growth in the Solow model. The von Bertalanffy equation is the simplest model of population growth that has the following characteristics: (1) population size is bounded; and (2) the rate of population growth decreases to zero as time tends toward infinity. These are most remarkable hypotheses concerning human population growth. Adapting this model to the labor force, we have solved the generalized Solow model in closed form in terms of the hypergeometric function, and we have analyzed the stability of this model. This paper demonstrates that, using the von Bertalanffy equation, the intrinsic rate of population growth does not influence the long-run equilibrium levels of the per capita variables, and that the equilibrium levels of consumption per capita, capital, and output are improved.

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