THE ARBITRAGE PRICING THEOREM WITH INCOMPLETE PREFERENCES

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ABSTRACT. This paper proves existence of equilibrium and the arbitrage pricing theorem for an asset exchange economy, where the individual's preferences may be incomplete or intransitive. This extends existing results to a more general set of individual preferences. We also prove the arbitrage pricing theorem for a theory of choice under uncertainty by Bewley [5]. These preferences model Knightian uncertainty by allowing for the possibility that preferences are incomplete.

1. Introduction

The original motivation for this paper is some recent work in the literature of finance theory. A fundamental problem of the analysis of private economies is resolving the prices of assets when their returns are uncertain. Recent studies on this topic take the Arrow-Debreu Model as its starting point. In the present paper, we first study existence and optimality of competitive equilibrium for a wider class of economies originally suggested by Arrow [2], where markets were possibly incomplete. We shall note that our existence theorem is proved as a by-product. We then suggest an analysis on one of the leading theories of asset pricing, namely, arbitrage pricing theorem (APT), which is due to Ross [19]. The main result of our paper extends the APT to a larger class of preferences. In particular, we allow for possible incompleteness of preferences.

1.1. **Arbitrage Pricing.** Ross' APT is based upon three assumptions. The first is that the market does not permit arbitrage opportunities. The second is that the market has a factor structure. The last assumption is that there is a large number of assets. Therefore, the basic result of APT is that under these assumptions, idiosyncratic risk can be fully diversified away and hence asset prices can be expressed in terms of the prices of a small number of factors.

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After Ross' seminal paper there have been a number of studies characterizing the notion of the equilibrium arbitrage pricing theorem (EAPT). With either a finite or infinite number of assets, Connor [6] presented a general theory of equilibrium asset pricing unifying several asset pricing models. Using weak assumptions on induced preferences over assets, Milne [15] proved an EAPT for a finite number of assets. This was extended by Kelsey and Milne [10] to most of the leading non-expected utility theories such as *Gateaux differentiable preferences* (see Machina [12]), *rank dependent expected utility* (see Quiggin [17]), and *Choquet expected utility* (see Schmeidler [20]). They also extended the APT to a larger class of expected utility preferences, namely, the APT was generalized to allow for state dependent utility which arises naturally in multi-period problems and allowed different investors to have different subjective probabilities.

The present paper extends the APT to a still larger class of preferences. Namely, we allow for preferences to be incomplete or intransitive. The assumption that preferences are completely ordered has been questioned by many authors (e.g., Aumann [3] and Suppes [22]). This is reasonable since when individuals have only partial information or when there are many decision makers whose preferences may disagree.

1.2. **Bewley Preferences.** Knightian uncertainty refers to situations in which it is difficult or impossible to assign subjective probabilities. Knight [11] made a distinction between risk, where the probabilities are known, and uncertainty, where probabilities are not known. Knight's distinction does seem to have some intuitive appeal since, in many circumstances, individuals make decisions involving uncertainty rather than risk. Experimental evidence suggests that individuals are repelled by vagueness of probabilities (see Ellsberg [8]).

Some models of Knightian uncertainty relax the independence axiom. However, even with Knightian uncertainty, this axiom may be motivated by observing that consumption in mutually exclusive events cannot be complementary. If an individual is faced with a complex and unfamiliar kind of uncertainty she may well find some decisions difficult or impossible. Thus, there is a case for modelling Knightian uncertainty by relaxing the completeness axiom rather than the independence axiom.

A model of Knightian uncertainty has been proposed by Bewley [5], which retains the independence axiom but relaxes completeness. In his model the decision maker considers a number of probability distributions to be possible. An option a is only preferred to b if a yields higher expected utility with respect to all possible probability distributions. If a yields higher expected utility for some distributions and lower expected utility for others, then a and b are not comparable. Bewley preferences satisfy all the Anscombe and Aumann [1] axioms, apart from completeness. These imply that there exists a closed convex set Π of subjective probabilities such that

$$a \succ b \quad \Leftrightarrow \quad E_\pi \left[u \left(a \right) \right] > E_\pi \left[u \left(b \right) \right]$$

for all $\pi \in \Pi$. Bewley preferences model uncertainty by representing beliefs as sets of probabilities and postulating a cautious decision rule. In his work, the uncertainty is captured by the incompleteness of preferences when probabilities are not well defined and by the structural assumptions which distinguish the Knightian uncertainty from Savage's expected utility theory. Thus, the size of Π is a measure of the Knightian uncertainty about events in a set Ω of finite states of the world. It can be thought of as measuring the decision maker's ignorance (see for instance Rigotti [18]). In this paper we shall prove a version of the EAPT for Bewley preferences.

1.3. **Aim of the Paper.** The aim of the paper is to generalize the previous literature in a number of directions. First of all, we shall not assume that the preferences are complete or transitive. One motivation for this is that many agents in asset markets are not single investors but rather corporate bodies. Hence, most investment decisions are collective decisions. If markets were complete, then all group members would have the same preferences over investments. If markets are incomplete, then it is not possible to evaluate market values of all feasible investment decisions from the available price system. As a result, even if the competitive conditions prevail, generically, investors will disagree over the choice of corporate investment plans (see Duffie and Shafer [7] and Haller [9]). In such cases, a corporate investment decision will be the outcome of a collective decision process. Social choice theory implies that such a collective decision process will be incomplete or intransitive, if they are non-dictatorial. We show that the EAPT obtained here will be robust even without complete or transitive preferences.

Second, we analyze the implications of incomplete or intransitive preferences for an asset exchange economy. Under weak conditions on the strict preference relation, the existence result will be extended to economies in which unrestricted short selling is allowed and hence the portfolio space is not necessarily bounded below (see Milne [14], Page and Wooders [16], and Werner [23]). In the present paper, existence is not standard since the portfolio space is potentially unbounded. We shall note that previous proofs allow either incompleteness or unboundedness. Our proof of existence allows both at the same time. Finally, we present the *first fundamental theorem of welfare economics* in such a framework.

1.4. **Outline of the Paper.** In the following section, we derive a numerical representation for a preference relation without assuming transitivity or completeness. In Section 3, we prove the existence of a competitive equilibrium for a class of asset exchange economies and show that the resulting equilibrium allocation is in the constrained core. In Section 4, we derive the EAPT for the underlying preference relation. In Section 5, we shall extend the earlier result to include Bewley preferences. Finally, the concluding section discusses some of the implications of these results.

2. Preferences

The subject matter of this section is the representation of preferences which may be incomplete or intransitive. Consider an individual who has preferences among alternatives. These preferences are described by a strict preference relation $\succ$. Here we shall demonstrate the idea that a relation $\succ$ defined on a set X can be represented by a two-argument functional $\phi : X \times X \to \Re$ for which $x \succ y$ if and only if $\phi(x, y) > 0$ for all $x, y \in X$.

Let there be a finite number of individuals indexed by $h \in \mathcal{H} = \{1, ..., H\}$. The consumption set of individual h is given by $X_h \subset \Re^\ell$, where ℓ denotes a finite number of commodities. Given a strict preference relation $\succ_h$ defined on $X_h \times X_h$, let $P_h(x^h) = \{y \in X_h : y \succ_h x^h\}$ and $P_h^{-1}(x^h) = \{y \in X_h : x^h \succ_h y\}$ be the strict upper contour set and strict lower contour set, respectively. We make the following assumptions on preferences[1]:

Assumption 1. (i) *Continuity. The strict upper and lower contour sets are open subsets of* X_h, $\forall x^h \in X_h$; (ii) *Irreflexivity.* $x^h \notin P_h(x^h)$, $\forall x^h \in X_h$; (iii) *Convexity.* $x^h \notin con(P_h(x^h))$ $\forall x^h \in X_h$, *where* $con(P)$ *stands for the convex hull of* P; (iv) *Non-satiation. For each* $\forall x^h \in X_h$, $P_h(x^h) \neq \emptyset$.

Definition 1. Let $\succ_h$ be a preference relation defined on X_h, then the graph of $\succ_h$ is given by

$$\Gamma(\succ_h) = \left\{ \left(x^h, y\right) \in X_h \times X_h : y \succ_h x^h \right\}.$$

[1]Notice that we dispense with transitivity and strong convexity assumptions. However, a mild convexity assumption is imposed on strict upper contour sets (see Mas-Colell [13]).

Furthermore, $\succ_h$ has an open graph if $\Gamma(\succ_h)$ is an open subset of $X_h \times X_h$.

Our first result is due to Bergstrom *et al.* [4], Theorem 4, and is stated without proof.

Lemma 1. *If the preference relation $\succ_h$ is asymmetric, then it has open graph if and only if there exists a real continuous function $\phi : X_h \times X_h \to \Re$ such that $\phi\left(x^h, y\right) > 0$ if and only if $x^h \in P_h(y)$.*

3. The Asset Exchange Economy

In this section, we analyze the properties of competitive equilibrium in a finite asset exchange economy under uncertainty. Suppose that economic activity occurs over two time periods, $t = 0, 1$. Uncertainty is characterized by a set of states of the world, indexed by $s \in \mathbf{S} = \{1, ..., S\}$, and is resolved all in the second period. We shall assume that there is only one physical commodity. The first period commodity space is $\Re$ and the second period contingent commodity space is $\Re^S$ making the total commodity space $\Re^{S+1}$. In the sequel we shall consider an exchange economy where second period actions by consumers are restricted to trades in assets. Therefore, we shall treat assets to be the objects of choice rather than examining the contingent commodities explicitly.

Each consumer $h \in \mathcal{H}$ has a consumption set $X_h \subset \Re^{S+1}$ defined by $X_h = \{x^h \in \Re^{S+1}_+ : x^h \geq 0\}$, and an initial endowment $e^h \in \Re^{S+1}$. Let each consumer h be described by a preference relation $\succ_h$ defined over state contingent consumption set X_h. We make the following assumptions on X_h and e^h.

Assumption 2. (*i*) *For every $h \in \mathcal{H}$, the feasible set X_h is non-empty, closed, convex, and bounded below*; (*ii*) *For every $h \in \mathcal{H}$, the initial endowment is in the interior of the consumption set, i.e., $e^h \in int(X_h)$.*

3.1. Induced Preferences.

The basic preferences over consumption generates induced preferences over asset holdings.

Let there be J assets indexed by $j \in \mathcal{J} = \{1, ..., J\}$. Define the commodity space in the asset economy to be the space $\Re^{J+1}$, that is, J assets and the first period commodity. In order to achieve consumption, consumer h holds assets $a^h \in \Re^J$ which yields returns $\sum_{j\in\mathcal{J}} Z_j a^h_j$, where

$Z_j \in \Re^S$ is the asset return matrix since

$$\begin{bmatrix} Z_{11} & ... & Z_{1J} \\ . & ... & . \\ Z_{S1} & ... & Z_{SJ} \end{bmatrix} \begin{bmatrix} a_1^h \\ . \\ a_J^h \end{bmatrix} = \begin{bmatrix} \sum_{j\in\mathcal{J}} Z_{1j} a_j^h \\ . \\ \sum_{j\in\mathcal{J}} Z_{Sj} a_j^h \end{bmatrix}.$$

In order to derive consumer preferences over assets, define a correspondence $\Lambda : \Re^{J+1} \to \Re^{S+1}$ by $Z\beta = \alpha$, where $\beta \in \Re^{J+1}$, $\alpha \in \Re^{S+1}$, and Z is the $(S+1)(J+1)$ semi-positive matrix, that is,

$$\begin{bmatrix} 1 & 0 \\ 0 & Z \end{bmatrix}.$$

The correspondence Λ is linear and onto the range Q, which is a vector subspace of dimension $(J+1)$. Define $V_h = Q \cap X_h$, where $V_h \neq \emptyset$ since $\{0\} \subset Q \cap X_h$. Define consumer h's feasible asset trade set A_h by $\Lambda^{-1} : V_h \to \Re^{J+1}$ such that $A_h = \Lambda^{-1}(V_h)$.

Remark 1. We shall assume that the set of returns is linearly dependent with rank $J' < J$. In particular if $\mathcal{K}$ is the non-trivial kernel of Z, where $\dim(\mathcal{K}) = J - J'$ and for any $\widehat{a}^h \in \mathcal{K}$ and $\widetilde{a}^h \in A_h$, then $\left(\widehat{a}^h + \widetilde{a}^h\right) \in A_h$. Hence A_h contains linear manifold $\mathcal{K}$ of portfolios all of which produce the same returns.

Therefore, induced preferences $\succ_h^a$ over assets can be derived from basic preferences by way of the correspondence Λ^{-1}. In other words, assets are desired solely for their returns so that preferences over assets are derived preferences.

The intimate connection between basic preferences defined over commodity space and of derived preferences defined over portfolio space can be summarized in the following result due to Milne [14], in Lemma 1. His result was given for the weak preference relation. In the present paper, this involves a trivial modification for the strict preference relation and hence we shall omit the proof.

Lemma 2. *Let X_h satisfy Assumption 2, associated preferences satisfy Assumption 1, and for any $x^h \in V_h$, there exists $y^h \in V_h$ such that $y^h \succ_h x^h$. Then (i) A_h is closed, convex and $0 \in A_h$; (ii) The graph of $P_h\left(Za^h\right)$ is open in $A_h \times A_h$; (iii) For every $a^h \in A_h$, $Za^h \notin con\left(P_h\left(Za^h\right)\right)$; (iv) For every $a^h \in A_h$, $P_h\left(Za^h\right) \neq \emptyset$; (v) For every $a^h \in A_h$, $Za^h \notin P_h\left(Za^h\right)$ and $Za^h \notin cl\left(P_h\left(Za^h\right)\right)$, where $cl\,(P)$ stands for the closure of P.*

3.2. **Equilibria in Unbounded Economies.** Suppose that the consumer can go arbitrarily short in asset trading. In the presence of short selling assets, one has to work with asset trade sets without a prior lower bound. As before, a portfolio of assets will be described by a vector $a^h \in \Re^J$ with a_j^h indicating the number of the jth asset held by consumer h. We shall assume that a_j^h may be positive or negative. Each consumer h is assumed to have an endowment of assets $\overline{a}^h \in \Re^J$. For each $h \in \mathcal{H}$, let $sp\,(Z_j)_{j\in\mathcal{J}}$ be the span of $(Z_j)_{j\in\mathcal{J}}$.

When the structure of an asset market is incomplete, the attainable consumption set of each consumer can be specified as follows:

$$\widehat{X}_h = X_h \cap \left\{ x^h \in \Re_+^{S+1} : x^h - e^h \in sp\,(Z_j)_{j\in\mathcal{J}} \right\},$$

that is, the allocations are attainable by way of exchange of assets. Asset markets so constructed may be incomplete in the sense that the available assets do not span X_h. Define the asset trade set of each consumer as follows:

$$A_h = \left\{ a^h \in \Re^{J+1} : \sum_{j\in\mathcal{J}} Z_j a_j^h \in \widehat{X}_h \right\}.$$

Notice that A_h is assumed to have no lower bound. Let $A = \times_{h\in\mathcal{H}} A_h$.

Lemma 3. *Let $\{k_r\}$ be a sequence of real numbers in $\Re$ such that $k_r \geq k_{r+1}$ and* $\lim_{r\to\infty} k_r = 0$. *For any $k_r > 0$, define*

$$A_h^r = \left\{ a^h \in A_h : \left\| a^h \right\| \leq 1/k_r \right\}$$

for $r = 1, \ldots$, then $\lim_{r\to\infty} A_h^r = A_h$.

We shall omit the proof of Lemma 3 as it is straightforward.

Assumption 3. *For each $h \in \mathcal{H}$, $\overline{a}^h \in int\,(A_h)$.*

Let $\mathcal{E} = \left(A_h, \succ_h^a, \overline{a}^h, x^h\right)_{h\in\mathcal{H}}$ denote the unbounded asset exchange economy. For each $a^h \in A_h$, let $P_h\left(Za^h\right) = \left\{ Za' \in \widehat{X}_h : Za' \succ_h^a Za^h \right\}$ be the consumer h's "preferred" set. Let $\mathcal{P} = \left\{ q \in \Re^J : \|q\| \leq 1 \right\}$ and $B_h(q) = \left\{ Z'a^h \in \widehat{X}_h : q \cdot a^h \leq q \cdot \overline{a}^h \right\}$ be the set of relative prices and the consumer h's "budget set" in commodity space for a given price system $q \in \Re^J$, respectively. Let $x^{*h} = Za^{*h}$.

Definition 2. An equilibrium for $\mathcal{E}$ is an $(H+1)$ tuple of vectors $\left(a^{*1}, \ldots, a^{*H}, q^*\right)$ such that *(i)* $a^{*h} \in B_h(q^*)$; *(ii)* $q^* \in \mathcal{P}$; *(iii)* $P_h\left(x^{*h}\right) \cap B_h(q^*) = \emptyset$; *(iv)* $\sum_{h\in\mathcal{H}} a^h = \sum_{h\in\mathcal{H}} \overline{a}^h$.

THEOREM 1. *Assume that X_h satisfies Assumption 2, A_h and associated preferences satisfy the conditions of Lemma 2 and Assumption 3, then the economy $\mathcal{E}$ has an equilibrium (q^*, a^*, x^*).*

Proof. Let $\mathcal{E}^r = \left(A_h^r, P_h^r, \overline{a}^h, x_r^h\right)_{h\in\mathcal{H}}$ be an r-bounded economy constructed in the following way. Consumer h has a consumption set

$$A_h^r = \left\{a^h \in A_h : \left\|a^h\right\| \leq r\right\},$$

where $r > 0$ is large enough so that $\overline{a}^h \in int\left(A_h^r\right)$. Given Assumption 1 and the fact that $\overline{a}^h \in int\left(A_h^r\right)$, $\mathcal{E}^r$ has an equilibrium. Since the graph of P_h^r is open in $A_h^r \times A_h^r$ and $Za^h \in bd\left(P_h^r\left(Za^h\right)\right)$, where bd stands for the boundary of P, the economy satisfies the assumptions of Theorem 2 in Shafer [21]. This implies that an equilibrium exists for the bounded economy.

We shall now prove the existence of equilibrium for the unbounded economy. Let q_r, a_r, and x_r denote sequences of the prices, allocation of assets, and allocation of consumption goods in the r-bounded economy, respectively. Notice first that q_r is an element of a compact price simplex, i.e., $q_r \in \Delta$. Since the total amount of consumption good is finite, x_r is also an element of a compact set. Consider the limit as $r \to \infty$. We may assume without loss of generality that $x_r \to x^*$ and $q_r \to q^*$. Let $\widehat{a}^h$ satisfy the equation $Z\widehat{a}^h = x^{*h}$ for $1 \leq h \leq H$. Therefore, $\widehat{a}^h \in Z^{-1}x^{*h}$. Let $a^{*h} = \widehat{a}^h$ for $1 \leq h \leq H-1$ and

$$a^{*H} = \widehat{a}^H + \sum_{h=1}^{H} \overline{a}^h - \sum_{h=1}^{H} \widehat{a}^h.$$

This implies that

$$Za^{*h} = Z\widehat{a}^h + \sum_{h=1}^{H-1} Z\overline{a}^h - \sum_{h=1}^{H-1} Z\widehat{a}^h.$$

By construction one has $Z\sum \widehat{a}^h = Z\sum \overline{a}^h$.

Now suppose, if possible, that there exists $\widehat{a}^h \in P_h\left(Za^{*h}\right) \cap B_h\left(q^*\right)$. Since graph of P_h is open, there exists $\varepsilon > 0$ such that

$$\widehat{a}^h - \varepsilon \in P_h\left(Za^{*h}\right) \cap B_h\left(q^*\right).$$

Let $\widehat{a}^h - \varepsilon = \widetilde{a}^h$. This implies $\widetilde{a}^h \in P_h\left(Za^{*h}\right)$ and for sufficiently large r, $\widetilde{a}^h \in P_h\left(Za_r^{*h}\right)$. This in turn implies that

$$q_r\widetilde{a}^h = q_r\widehat{a}^h - \varepsilon < q_r\widehat{a}^h = q_r\overline{a}^h.$$

Since $q_r\widetilde{a}^h = q_r\widehat{a}^h - \varepsilon$, then $q_r\widetilde{a}^h < q_r\widehat{a}^h < q_r\overline{a}^h$. But this is a contradiction since a_r is an equilibrium for the r-bounded economy, which implies that $P_h\left(Za_r^h\right) \cap B_h\left(q_r\right) = \emptyset$. The result follows. ■

3.3. **Optimality of Competitive Allocations.** In general, an equilibrium allocation is Pareto optimal only if the market structure is essentially complete. If markets are incomplete there is no reason to expect an equilibrium allocation to be Pareto optimal. However, we can show that the equilibrium is constrained Pareto optimal. This means that it is not possible to achieve a Pareto improvement by reallocating the existing assets. More generally, we can show that the equilibrium will be in the constrained core.

Definition 3. A constrained core with respect to the preference relations $\succ^a$ is an allocation $\left(a^{*1}, ..., a^{*H}\right)$ of one portfolio for each consumer such that there does not exist an allocation $\left(a'^1, ..., a'^H\right)$ and a non-empty subset $\mathcal{G} \subset \mathcal{H}$ for which $Za'^h \in P_h\left(Za^{*h}\right)$, $\forall h \in \mathcal{G}$ such that $\sum_{h\in\mathcal{G}} a'^h = \sum_{h\in\mathcal{G}} \overline{a}^h$.

Theorem 2. *In an asset exchange economy* $\mathcal{E} = \left(A_h, \succ_h^a, \overline{a}^h, x^h\right)_{h\in\mathcal{H}}$ *every competitive equilibrium* (q^*, a^*, x^*) *is in the constrained core.*

Proof. Let (q^*, a^*, x^*) be the equilibrium allocations and price system. Suppose that there exists a non-empty subset $\mathcal{G} \subset \mathcal{H}$ and $a'^h \in A_h$ for $h \in \mathcal{G}$ such that $\sum_{h\in\mathcal{G}} a'^h = \sum_{h\in\mathcal{G}} \overline{a}^h$ and $Za'^h \in P_h\left(Za^{*h}\right)$. Since $P_h\left(Za^{*h}\right) \cap B_h\left(q^*\right) = \emptyset$, one obtains $q^*a'^h > q^*\overline{a}^h$ $\forall h \in \mathcal{G}$. Summing over $h \in \mathcal{G}$, one has $q^*\sum_{h\in\mathcal{G}} a'^h > q^*\sum_{h\in\mathcal{G}} \overline{a}^h$. But this contradicts the fact that

$$q^*\sum_{h\in\mathcal{G}} \overline{a}^h = q^*\sum_{h\in\mathcal{G}} a'^h = q^*\sum_{h\in\mathcal{G}} a^{*h}$$

obtained from summing over the budget constraint. This completes the proof. ■

Hence, an asset allocation is in the constrained core if there does not exist a way to reallocate assets which make each individual better off. Note that, in particular, this implies a version of the *first fundamental theorem of welfare economics.* Namely, every competitive equilibrium is constrained Pareto optimal.

4. Arbitrage Pricing

In this section we shall give a proof of the APT for the underlying economy without complete or transitive preferences. The intuition behind the APT is that, in equilibrium, the market rewards only undiversifiable risk and that the idiosyncratic risk is not relevant for pricing the assets. It is common to refer to undiversifiable risk as *factor risk*.

Definition 4. A factor structure on $\Re^J$ is a linear transformation $F : \Re^J \to M$, where $M \subset \Re^J$.

We shall assume that the range of F is all of M, that is, F is onto. Let $\ker(F)$ denote the set of $a \in \Re^J$ such that $F(a) = 0$. The elements of $\ker(F)$ will be called *idiosyncratic risk*.

Definition 5. Two portfolios a^h and b are said to be factor equivalent if $F\left(a^h\right) = F(b)$. Similarly, two allocations are factor equivalent if every consumer has factor equivalent portfolios.

Let $\mathcal{D} \subset \Re^J$ denote the subspace of diversified portfolios.

Definition 6. Consumer h is said to be a diversifier if whenever $F\left(a^h\right) = F(b)$ with $a^h \in \mathcal{D}$ and $b \in \Re^J \backslash \mathcal{D}$, then $a^h \succ_h^a b$.

Definition 6 says that any diversified portfolio must be strictly preferred to a factor equivalent portfolio which is not diversified.[2] In the following, we shall provide some conditions under which all consumers in a factor economy will be able to diversify away idiosyncratic risk.

Definition 7. An allocation is said to be insured if it consists entirely of diversified portfolios.

Assumption 4. *An asset economy is said to be insurable if $F(\mathcal{D}) = M$ and the aggregate endowment lies in $\mathcal{D}$, i.e., $\overline{a} \in \mathcal{D}$.*

Insurability allows a consumer to eliminate idiosyncratic risk by holding a diversified portfolio. The following proposition asserts that in a competitive equilibrium, all consumers choose diversified portfolios.

Proposition 1. *In an insurable asset exchange economy in which all consumers are diversifiers, any competitive allocation is diversified.*

Proof. The proof of Proposition 1 can be found in Kelsey and Milne [10]. ■

Let $\phi^h : A_h \times A_h \to \Re$ be a real valued function defined by $\phi^h(a, a')$ representing the preference relation $\succ_h^a$.

Assumption 5 (Differentiability). $\forall a \in A_h$*,* $\phi^h(\cdot, a) : A_h \times A_h \to \Re$ *is differentiable.*

[2]Less technically, if consumer h is a diversifier then idiosyncratic risk is a "bad" for her.

Since preferences are assumed to be incomplete or intransitive they cannot be represented by a utility function, but by a function of two arguments. Assuming ϕ^h is differentiable is analogous to assuming that an expected utility maximizer has a differentiable utility function. Differentiability of preferences implies that if a consumer is satiated in idiosyncratic risk with a diversified portfolio, then in equilibrium, the price of idiosyncratic risk is zero. This is proved in the following proposition:

PROPOSITION 2. *Assume that all consumers are diversifiers and at least one consumer has differentiable preferences. Let* $q \in \Re^J$ *be the price system and* $a' \in \ker(F)$*, then* $q \cdot a' = 0$.

Proof. Since a^* is consumer h's competitive equilibrium allocation and q is competitive equilibrium price system, $a \in P(a^*) \Rightarrow q \cdot a > q \cdot a^*$, that is,

$$q \cdot a \leqslant q \cdot a^* \Rightarrow \phi(a, a^*) \leqslant 0 \tag{1}$$

By local nonsatiation, $a^* \in cl(P(a^*))$. Hence, $\phi(a, a^*) > 0 \; \forall a \in P(a^*)$ and continuity of $\phi(\cdot, a^*)$ imply $\phi(a^*, a^*) \geqslant 0$ (Upper semi-continuity of $\phi(\cdot, a^*)$ would suffice here).

But, since the preference relation $\succ$ is irreflexive then $\phi(a^*, a^*) \not> 0$.

$$\therefore \phi(a^*, a^*) = 0. \tag{2}$$

Thus, (1) and (2) implies that a^* solves

$$\max_a \phi(a, a^*) \text{ such that } q \cdot a \leqslant q \cdot a^*.$$

By Kuhn-Tucker Theorem, since $\phi(\cdot, a^*)$ is differentiable, $\exists t > 0$ such that

$$\phi_1(a^*, a^*) = tq \tag{3}$$

Suppose $a' \in \ker(F)$, so a^* is factor equivalent to $a^* + \lambda a' \; \forall \lambda$. Let $f(\lambda) := \phi(a^* + \lambda a', a^*)$. Therefore f is differentiable at $\lambda = 0$ and

$$f'(0) = \phi_1(a^*, a^*) \cdot a' \tag{4}$$

Since $a^* + \lambda a' \in A \setminus \mathcal{D}$, $\forall \lambda \neq 0$, and since consumer h is a diversifier, $\lambda = 0$ maximizes $f(\lambda)$, so $f'(0) = 0$. Hence, by (3) and (4), $tq \cdot a' = 0$.

$$\therefore t > 0 \Rightarrow q \cdot a' = 0.$$

This completes the proof. ∎

Remark 2. Notice that if the preferences of all consumers are not differentiable, then it may be the case that q is non-unique and contains non-zero prices. However, it will still be the case that there is an equilibrium in which the price of idiosyncratic risk is zero.

Theorem 3 (APT). *Let q be the competitive equilibrium price system of an insurable asset exchange economy. Assume that all consumers are diversifiers and at least one consumer has differentiable preferences, then there exists a linear functional $\mu : M \to \Re$ on M such that for all $a \in \Re^J$,*

$$q \cdot a = \mu \cdot F(a).$$

Proof. Define $\mu : M \to \Re$ by $\mu(f) = q \cdot a$, where a has a factor structure $f = F(a)$. Assume that $F(a) = F(a')$, then $a - a' \in \ker(F)$ consists only of idiosyncratic risk. Hence, by Proposition 1, we must have $q(a - a') = 0$. This implies that $q \cdot a = q \cdot a'$ and therefore $\mu : M \to \Re$ is well defined. Since μ is a linear function defined on a vector space M with $\mu(f) \in \Re$, then it must satisfy the following condition:

$$\mu(\eta f_1 + \kappa f_2) = \eta\mu(f_1) + \kappa\mu(f_2).$$

Thus, $q \cdot a = \mu \cdot F(a)$. This completes the proof. ∎

Theorem 3 implies that the asset prices may be written as a linear combination of the factor prices.

5. Bewley Preferences

In this section we shall propose a representation for Bewley preferences. It will be assumed that incompleteness, which is reflected by multiplicity of beliefs, applies only to contingent consumption over events of unknown probabilities. We thus show that our APT also applies to Bewley preferences.

Bewley preferences are intended to model Knightian uncertainty, that is, uncertainty which cannot be represented by conventional probabilities. When faced with a decision problem involving uncertainty, individuals are sometimes unable to assign probabilities to relevant events. In particular, if a preference ordering does not satisfy completeness, individuals are not necessarily able to compute a unique expected utility for each payoff.

It is often argued that when individuals are faced with uncertainty they have beliefs, which take the form of ranges or intervals for an event. However, this approach is questionable since it is no longer the case that the probabilities of a number of mutually exclusive events will sum to one. This problem can be circumvented by assuming that individuals' beliefs should be represented by a convex set of probability distributions. The intuition is that the probabilities in a given distribution from that convex set will sum to one, while for any given event, the probabilities assigned to that event will form an interval.

Bewley [5] shows that Knightian uncertainty may give rise to incomplete preferences. In other words, individuals are modeled as having incomplete preferences, but otherwise obey the axioms of Anscombe and Aumann [1]. In this case we have a set of subjective probabilities rather than a single one. According to Bewley, an option a is preferred b if its expected value is higher with respect to *all* other probability distributions. The intuition is that if a previously unavailable alternative arises, an individual will put to use it only if doing so would put her in an unambiguously preferred position.

Let Ω be a finite set of states of nature, indexed by $s = 1, ..., S$ and Π be a closed convex set of subjective probability distributions on Ω. Let $a = a_1, ..., a_S$ and $b = b_1, ..., b_S$ be two contingent consumption vectors. When preferences are incomplete, Bewley [5] showed that there exists a set Π of probabilities on Ω such that if $a \succ b$, if and only if

$$E_\pi \left[u\left(a\right)\right] > E_\pi \left[u\left(b\right)\right]$$

$\forall \pi \in \Pi$, where $u\left(\cdot\right)$ is the decision maker's Von Neumann-Morgenstern (VNM) utility function. Bewley [5] argues that the above expression is a possible formulation of Knight's [11] distinction between risk and uncertainty. In other words, payoffs are risky if Π has only one element and uncertain otherwise. Since the state set of states of nature is finite, the above expression can be reduced to

$$a \succ b \;\Leftrightarrow\; \sum_{s=1}^{S} \pi_s E_\pi[u(a_s)] > \sum_{s=1}^{S} \pi_s E_\pi[u(b_s)]$$

$\forall \pi \in \Pi$.

Assumption 6. *The VNM utility function u is strictly increasing, concave and continuous.*

Let Z be the space of all functions from the set of all subsets Σ of $\mathbf{S}$ to $\Re$. Define $\{f^i : \Sigma \to \Re\}$ to be I linearly independent members of Z. Hence, returns on asset $j \in \mathcal{J}$ can be written as

$$Z_j = \sum_{i \in \mathcal{J}} \beta^i_j f^i + \epsilon_j,$$

where ϵ_j is the idiosyncratic risk.

Assumption 7. *ϵ_j is assumed to be state independent.*

Assumption 7 suggests that the idiosyncratic risk is defined only by objective probabilities. There are two ramifications of this assertion. First, all individuals will have the same subjective probabilities for the idiosyncratic risk. This is important because if individuals do not agree on the probability distribution of the idiosyncratic risk, then they would not agree which portfolio is to be diversified. Hence, the requirement that the price of a portfolio consisting of solely idiosyncratic risk is zero cannot be obtained. Second, each individual's beliefs over idiosyncratic risk may be represented by a single subjective probability distribution.

With the above construction M is the vector subspace of Z spanned by $\{f^i\}_{i \in \mathbf{I}}$ and $\mathcal{F}$ is the function defined by

$$\mathcal{F}(Z) = \sum_{i \in \mathbf{I}} \beta^i_j f^i.$$

The space $\mathcal{D}$ of diversified portfolios is defined to be the linear span of $\{f^i\}_{i \in \mathbf{I}}$. Throughout, we shall assume that there is a linear dependency between idiosyncratic risks. By holding a linear combination of assets an individual can eliminate idiosyncratic risk from his or her portfolio.

Definition 8. Define the derived utility function V_h on the choice set A_h by

$$V_h(a, b) = \min_{\pi \in \Pi} \left[\sum_{s=1}^{S} \pi_s E_\pi [u(Za)] - \sum_{s=1}^{S} \pi_s E_\pi [u(Zb)] \right]$$

such that $V_h(a, b) > 0$ iff $a \succ b$.

Definition 8 implies that the decision rule is incomplete. That is, if a gives higher expected utility with respect to some probability distribution, while at other times b gives higher expected utility then, a and b will not be comparable.

Proposition 3. *An individual with Bewley subjective probabilities will strictly prefer a portfolio which contains zero idiosyncratic risk to a factor equivalent portfolio which contains non-zero idiosyncratic risk.*

Proof. Suppose that a and b are factor equivalent portfolios where a is diversified and b is not. Then, by Lemma 3.1 in Kelsey and Milne [10], $(b-a)$ is "mean preserving increase in risk" on a. Let Za be the return from a portfolio with zero idiosyncratic risk and Zb be the return from a factor equivalent portfolio with non-zero idiosyncratic risk. Let $\widehat{p} = \arg\min E_p\left[u\left(Za\right)\right]$. Then

$$\begin{aligned}\min_{p\in\Delta} E_p\left[u\left(Zb\right)\right] &\leq E_{\widehat{p}}\left[u\left(Zb\right)\right] = E_{\widehat{p}}\left[u\left(Za+\left(Zb-Za\right)\right)\right] \\ &< E_{\widehat{p}}\left[u\left(Za\right)\right] = \min_{p\in\Delta} E_p\left[u\left(Za\right)\right].\end{aligned}$$

Since, by Assumption 6, $u(\cdot)$ is concave and a and b are factor equivalent, $(Zb-Za)$ is a mean preserving increase in risk on a.[3] It follows from the representation of preferences in Definition 8 that the consumers prefer a portfolio with zero idiosyncratic risk. ■

Let e_j be the portfolio that consists of one share of asset j and nothing else. Then,

$$\frac{d}{d\lambda} V\left(a+\lambda e_j,\ a\right)$$

is the partial derivative in the direction of idiosyncratic risk.

Remark 3. We only use the representation in Definition 8 when differentiability becomes an issue, and more fundamental requirement is smoothness of indifference surfaces (or boundaries of upper contour sets). The problem with Bewley preferences for EAPT is that boundary of upper contour sets are non-smooth in general. This raises the possibility of multiple equilibrium prices. Hence it is possible that the equilibrium price of idiosyncratic risk may not be zero. Consider the case that there are two states and $Za = (3,1)$, so that portfolio a delivers 3 units of the consumption good in state 1, and 1 unit in state 2. Because risk aversion may depend on wealth, it is possible that λ units of some idiosyncratic asset will have a larger utility impact in state 2 than in state 1. Note that

$$u\left(Z\left(a+\lambda e_j\right)\right) - u\left(Za\right)$$

is the zero vector at $\lambda = 0$. If e_j is a valuable asset, and hence adds (subtracts, resp.) utility when $\lambda > 0$ ($\lambda < 0$, resp.), then the utility ranking of the states changes as λ passes through zero. In particular, state 2 is ranked above state 1 when $\lambda > 0$, since utility accumulates faster in state 2. Conversely, state 1 is ranked above state 2 when $\lambda < 0$, since utility decrease more quickly in state 2. The minimizing probability in Π may therefore differ depending on

[3] Notice that an expected utility maximizer with concave utility function is a diversifier since the addition of the idiosyncratic risk is a mean preserving increase in risk and will therefore lower expected utility.

whether $\lambda > 0$ or $\lambda < 0$. This will cause the one-sided directional derivatives to differ, violating differentiability.

In order to obtain Bewley preferences, we have assumed that all idiosyncratic risk is state independent, that is, objective in Assumption 7. Moreover, in Proposition 3, we have also assumed that all idiosyncratic risk involves a mean preserving increase in risk. Hence,

$$u\left(Z\left(a+\lambda e_j\right)\right) - u\left(Za\right) \leq 0$$

for all λ, so that state utility ranking will be independent of λ, even though $u\left(Z\left(a+\lambda e_j\right)\right) - u\left(Za\right)$ may differ from the zero vector when $\lambda \neq 0$. This suggests the required smoothness and will certainly provide it in the one-state world.

Since the idiosyncratic risk is state independent, this implies that the idiosyncratic risk does not lie in the direction of the kink in the preferences. In the following we shall use the differentiability of $u(\centerdot)$ to show that the directional derivative in the direction to the idiosyncratic risk exists.

LEMMA 4. *Let $\mathcal{T} = \{f(x,\lambda) : \lambda \in \Lambda\}$ be a family of functions continuous in both arguments with the following properties*

(1) *For all $\lambda \in \Lambda$, $\mathbf{0}$ minimizes $f(x,\lambda)$;*
(2) *For all $\lambda \in \Lambda$, $f(\centerdot,\lambda)$ is differentiable at $\mathbf{0}$.*

Assume Λ is a compact set. Define $\xi(x) = \min_{\lambda\in\Lambda} f(x,\lambda)$. Then, ξ is differentiable at $\mathbf{0}$ and $\xi'(0) = 0$.

Proof. By compactness, there exists $\hat{\lambda}$ such that $\xi(0) = f(0,\hat{\lambda})$. Consider $h > 0$. Then

$$\frac{f(h,\hat{\lambda}) - f(0,\hat{\lambda})}{h} \geq \frac{\xi(h) - f(0,\hat{\lambda})}{h} = \frac{\xi(h) - \xi(0)}{h} \geq 0.$$

(The second inequality follows from the fact that $\mathbf{0}$ minimizes $f(x,\lambda)$). Since by assumption

$$\lim_{h\to 0} \frac{f(h,\hat{\lambda}) - f(0,\hat{\lambda})}{h} = 0,$$

this establishes that

$$\lim_{h\to 0^+} \frac{\xi(h) - \xi(0)}{h} = 0.$$

Note

$$\frac{\xi(-h) - \xi(0)}{-h} = \frac{\xi(0) - \xi(-h)}{h}.$$

By similar reasoning to above,

$$0 \geq \frac{\xi(0) - \xi(-h)}{h} = \frac{f(0, \hat{\lambda}) - \xi(-h)}{h} \geq \frac{f(0, \hat{\lambda}) - f(h, \hat{\lambda})}{h}.$$

Since

$$\lim_{h \to 0} \frac{f(0, \hat{\lambda}) - f(h, \hat{\lambda})}{h} = 0,$$

this establishes that

$$\lim_{h \to 0^-} \frac{\xi(h) - \xi(0)}{h} = 0$$

and hence ξ is differentiable at $\mathbf{0}$. This completes the proof. ■

Corollary 1. *Suppose that for all $a \in \mathcal{D}$, u is differentiable at Za. Then, V_h has the one-sided directional derivative at a in the direction of idiosyncratic risk.*

We shall note that the assumptions we imposed are sufficient to ensure smoothness in the direction of the idiosyncratic risk, which is all that is needed to prove the APT result. Hence, Proposition 3 and Corollary 1 enable us to prove the APT result for Bewley preferences.

Corollary 2 (APT). *Let q be the competitive equilibrium price system for an insurable asset economy. Assume that all consumers have Bewley preferences and at least one consumer's preferences satisfy Corollary 1, then there exists a linear functional μ on M such that*

$$q_j = \sum_{i \in \mathcal{I}} \beta_j^i \mu\left(f^i\right).$$

6. Conclusion

In this paper, we have demonstrated the existence of an equilibrium in an asset exchange economy when investors preferences do not obey the basic axioms of expected utility. Namely, the result in this paper is mainly directed at preferences, where beliefs cannot be represented as a single probability distribution, rather than non-expected utility preferences which are non-linear in probabilities. Throughout it has been assumed that individual preferences were given by an irreflexive binary relation with open graph that were possibly incomplete or intransitive, and the asset trade set was non-compact.

Experimental evidence from decision making shows that complex decisions, regarding demand for and pricing of assets, are made in situations where preference orderings do not confirm to Savage's expected utility theory. In such situations, it is important to examine whether equilibrium exists in markets, whether the Pareto optimality holds, and finally, whether the APT is still robust. Therefore, our study generalize various results in the existing literature of economic theory.

Possibly, the most noteworthy aspect of this paper is the part that concerns the APT. It was shown that the EAPT remains robust when preferences may be incomplete or intransitive, and when uncertainty cannot be represented by unique subjective probabilities. One of the results of incompleteness and uncertainty aversion is that uncertainty can make simple plans of actions be undominated. It appears that simple economic behavior becomes rational when it is seen from a Knightian point of view. We showed that our model implies diversification. We believe that Knightian behavior can explain many ambiguous economic phenomena. Alternatively, incompleteness or intransitivity can be motivated by the difficulties arising from constructing an aggregate preference for a group of individuals. Therefore, the extensions analyzed in this paper have useful implications for the pure theory of finance.

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