Computing Equilibria in Finance Economies*

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Abstract

The general equilibrium model with incomplete asset markets provides a unified framework for many problems in finance and macroeconomics. In its simplest version with only two time periods and a single physical commodity the model is ideally suited for the study of problems in cross sectional asset pricing and portfolio theory. In this paper we develop a homotopy algorithm to approximate equilibria in these 'finance economies'. Since the algorithm is tailor made for finance economies, the number of nonlinear equations that has to be solved for, and therefore the computing time, is an order of magnitude smaller than that of existing general purpose algorithms. The algorithm is shown to be generically convergent. We implement the algorithm using HOMPACK. To illustrate its performance, we present various numerical examples and report running times.

KEYWORDS: Computational methods, asset pricing, general equilibrium, incomplete markets.

JEL CODES: C61, C62, C63, C68, D52, D58, G11, G12.

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1 Introduction

During the last two decades there has been substantial interest in the general equilibrium model with incomplete asset markets, the GEI-model. One of the important features of this model is its integrated approach to the real, financial and monetary sectors of an economy. The version of the model that is studied most, involves two time periods. There is uncertainty as to which one of several states of nature will realize in the second period. In the first time period, it is possible to trade on spot markets for commodities and on financial markets for assets that permit income to be transferred across time and states of nature. In the second period one of the states of nature realizes, which determines the pay-offs of the portfolio of assets purchased in the first period. The resulting revenues are used to buy commodities on the spot markets.

The model finds its origins in the contributions of Arrow (1953) and Radner (1972). Existence of an equilibrium turned out to be hard to prove, unless exogenously given lower bounds on trades are imposed. The latter approach, however, was put under suspicion by the influential paper of Hart (1975), where it was shown that the equilibrium may depend crucially on the arbitrarily chosen specification of the lower bounds. This caused research in the GEI-model to stagnate. A revival occurred when Duffie and Shafer (1985) succeeded in giving a generic existence proof. For almost all GEI-economies, a competitive equilibrium exists. The tools required to show existence of an equilibrium are demanding, and involve many results from differential topology, including the concept of the Grassmannian manifold.

The complications of the GEI-model imply that it is no longer possible to compute equilibria by the same methods that are used for the standard general equilibrium model. For instance, convergence of Scarf's algorithm, see Scarf (1967), or the homotopy algorithm of Eaves (1972) is not guaranteed. By using algorithms that operate on the Grassmannian manifold, Brown, DeMarzo and Eaves (1996b) and DeMarzo and Eaves (1996) have produced computational methods that converge for a generic GEI-economy. The contribution of Brown, DeMarzo and Eaves (1996a) is even more remarkable, as it develops a generically convergent algorithm by means of switching homotopies. This algorithm does not involve the Grassmannian manifold, and it is therefore the only existence proof of an equilibrium in the GEI-model that avoids this manifold. For numerical purposes, one may also want to use the homotopy algorithm of Schmedders (1998) that does not involve homotopy switching. The drawback of that algorithm is that it is an open question whether it displays generic convergence to an equilibrium.

Our interest in computing solutions in the GEI-model is derived from our desire to study the pricing of financial assets. For instance to study whether the equity premium puzzle of Mehra and Prescott (1985), the huge difference between the historical returns on stocks and the historical returns on riskless bonds, can be explained by market incompleteness. Or to study whether the lessons of the capital asset pricing model remain valid in a setting with market incompleteness, heterogeneous investors, assets whose distributions have fat tails, and so on. For all these applications, one needs to approximate a multivariate probability distribution by a finite probability distribution. In order to achieve a reasonable approximation, a big state space is required. In a companion paper, Herings and Kubler (2000), we focuses on the application to the capital asset pricing model and we need up to 32,768 states of nature for reasonable approximations to log-normal distributions.

Most existing algorithms transform the equilibrium problem into a problem that involves so-called state prices. This is essential to show convergence, but has the drawback that the number of equations increases rapidly. For instance, the homotopy proposed by Brown, DeMarzo and Eaves (1996b) involves 2S+1 non-linear equations. Another complication in applying that algorithm is that it involves closed form solutions for the demand functions for assets of the agents, but such closed form solutions are notoriously hard to obtain when asset markets are incomplete. The solution to that problem is to state the problem not in terms of demand functions themselves, but in terms of the first order conditions of agents that yield the demand function. This approach is suggested in Garcia and Zangwill (1981), and followed by Schmedders (1998). It increases the number of equations further, to $2(H+1)(S+1)+HJ+1$ non-linear equations, where H is the number of agents in the economy and J the number of financial assets traded. In the application described in this paper, $H = 3$ and $J = 8$, so the use of Schmedders' algorithm involves solving 16,417 non-linear equations.

The algorithms of Brown, DeMarzo and Eaves (1996b) and Schmedders (1998) are designed to deal with the general version of the GEI-model. In many applications, one is interested in what is known as the finance version of the GEI-model, or finance economy for short. In the finance version of the GEI-model, the modeling of the financial sector is the same as in the general version of the GEI-model. The consumption sector, however, is drastically simplified, in that in each time period, at each state of the world, there is only one commodity, called income. In this paper we develop an algorithm that is tailor made for finance economies. The restriction to finance economies leads to a great reduction in the number of equations to be solved for, and thereby to great improvements in computing times. If closed form solutions for demand functions are available, then the number of non-linear equations to be solved for by our algorithm equals $J+1$. Otherwise, the number amounts to $(H+2)(J+1)+H+1$, which is 49 in the application reported on in this paper.

Our algorithm is a homotopy algorithm, a class of algorithms introduced in Eaves (1972). We do not follow the piecewise linear approach of Eaves (1972), but exploit the differentiability that is present in the problem and choose methods from the theory of differential equations to follow the homotopy path in our implementation. We show that this is possible for almost all finance economies. For recent surveys on homotopies, the reader is referred to Judd (1998) or Eaves and Schmedders (1999).

Compared to traditional general equilibrium theory, finance economies pose a number of additional difficulties. The prices of assets are not necessarily positive, but may well be zero or negative. This rules out some of the algorithms that are used in traditional general equilibrium theory, for instance the simplicial variable dimension algorithm of Doup, van der Laan and Talman (1987) or its differentiable counterpart described in Herings (1997). Crucial to the convergence proof of homotopy methods applied to traditional general equilibrium models is the boundary behavior of the excess demand function. When prices of commodities converge to zero, demand for commodities explodes. As a price of zero has no special meaning in the case of financial assets, that boundary behavior cannot be used. The convergence proof of our homotopy algorithm builds on the approach to show existence of an equilibrium in finance economies as outlined in Hens (1991).

The paper is organized as follows. Section 2 introduces the notation and the model of a finance economy. In Section 3 we present an algorithm that is tailored to compute equilibria in finance economies. Special attention is given to the problem that closed form solutions for demand functions of assets rarely exist in finance economies. A second, related, algorithm is introduced that does not require closed form solutions. In Section 4 we show generic convergence of the algorithm, that is for an open set of finance economies with full Lebesgue measure, the algorithm converges to an equilibrium. Section 5 discusses the implementation of the algorithm, and in Section 6 we describe numerical examples. Section 7 concludes.

2 The Finance Economy

The finance version of the GEI-model describes an economy over two periods of time, $t = 0, 1$, with uncertainty over the state of nature resolving in period $t = 1$. There are $S + 1$ states in the economy; at time $t = 0$ the economy is in state $s = 0$, at time $t = 1$ one state of nature s out of S possible states of nature realizes. In each state $s = 0, \ldots, S$, there is a single nondurable consumption good, which we call income.

There are H agents, indexed by $h = 1, \ldots, H$, that participate in the economy. Agent h is characterized by the initial income stream $e^h = (e_0^h, e_1^h, \ldots, e_S^h)^\top \in \mathbb{R}^{S+1}_{++}$ and his preferences over income streams available for consumption $c^h = (c_0^h, c_1^h, \ldots, c_S^h)^\top \in \mathbb{R}^{S+1}_{++}$.

To distinguish between first period consumption and the random second period consumption, we define $\tilde{x} = (x_1, \ldots, x_S)^\top$ for any vector $x = (x_0, x_1, \ldots, x_S)^\top$. Aggregate incomes are $e = \sum_{h=1}^H e^h$. An agents' preferences are represented by a continuous, strictly quasi-concave utility function $u^h : \mathbb{R}^{S+1}_{++} \to \mathbb{R}$.

There are J financial assets, indexed $j = 1, \ldots, J$, that are used to reshuffle income across states. Asset j pays dividends at date $t = 1$ which we denote by $d^j \in \mathbb{R}^S$. The price of asset j at time $t = 0$ is q_j. We collect all assets' dividends in a pay-off matrix

$$A = (d^1, ..., d^J) \in \mathbb{R}^{S \times J}.$$

At time $t = 0$ agent h chooses an asset portfolio $\theta^h \in \mathbb{R}^J$ which uniquely defines the agents' consumption by $c_0^h = e_0^h - \theta^h \cdot q$ and $\tilde{c}^h = \tilde{e}^h + A\theta^h$. The net demand of agent h, $\tilde{c}^h - \tilde{e}^h$, therefore belongs to the marketed subspace $\langle A \rangle = \{z \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^J,\ z = A\theta\}$.

The exogenous parameters defining a finance economy $\mathcal{E} = ((u^h, e^h)_{h=1,\ldots,H}; A)$ are agents' utility functions and endowments, and the pay-off matrix. Without loss of generality, we assume throughout that there are no redundant assets, so $\text{rank}(A) = J$. If there are redundant assets, it follows from an arbitrage argument that their price is uniquely determined by the price of the other assets. Markets are incomplete when $J < S$. Prices for assets are said to be arbitrage free if it is not possible to achieve a positive income stream in all states by trading in the available assets. It is well known that a price system $q \in \mathbb{R}^J$ precludes arbitrage if and only if there exists a strictly positive state price vector $\pi \in \mathbb{R}^S_{++}$ such that $q = \pi^\top A$. We define Q to be the set of arbitrage free prices for assets.

Definition 2.1 (Competitive Equilibrium): A competitive equilibrium for an economy $\mathcal{E}$ is a collection of portfolio-holdings $\theta^* = (\theta^{1*}, \ldots, \theta^{H*}) \in \mathbb{R}^{HJ}$ and prices for assets $q^* \in \mathbb{R}^J$ that satisfy the following conditions:

(1) $\theta^{h*} \in \arg\max_{\theta^h \in \mathbb{R}^J} \; u^h(c^h)$ s.t. $c^h = e^h + \begin{pmatrix} -q^{*\top} \\ A \end{pmatrix} \theta^h$ and $c^h \in \mathbb{R}^{S+1}_{++}, \quad h = 1, \ldots, H;$

(2) $\sum_{h=1}^H \theta^{h*} = 0.$

Under an additional assumption of strictly increasing utility functions, and a condition on the utility function, the so-called boundary condition presented in Assumption A1 below, existence of an equilibrium follows from the results of Geanakoplos and Polemarchakis (1986).

3 The Algorithm

In this section we develop a globally convergent algorithm to compute equilibria in finance economies. The presentation of the algorithm, and the convergence proof, is simplified by restricting attention to an economy without first period consumption. From the arguments given in Geanakoplos and Polemarchakis (1986) it follows that this is without loss of generality. Indeed, given the pay-off matrix A of the previous section, if we define the matrix $\overline{A} \in \mathbb{R}^{(S+1)\times(J+1)}$ by $\overline{A}_{00} = 1$, $\overline{A}_{0j} = 0$, $j = 1, \ldots, J$, $\overline{A}_{s0} = 0$, $s = 1, \ldots, S$, and $\overline{A}_{sj} = A_{sj}$, $s = 1, \ldots, S$, $j = 1, \ldots, J$, then state 0 can be identified with the first period, and purchasing one unit of asset 0 corresponds to having one more unit of first period consumption. In Sections 3-5, the index of assets runs from 0 to J.

We strengthen the assumptions made so far to Assumption A below, which states the standard assumptions on finance economies that are invoked when twice differentiability of the demand for assets is required.

A1 u^h is three times continuously differentiable, $\partial u^h(c^h) \in \mathbb{R}^{S+1}_{++}$ for all $c^h \in \mathbb{R}^{S+1}_{++}$ (strong monotonicity), $y^\top \partial^2 u^h(c^h) y < 0$ for all $y \neq 0$ such that $\partial u^h(c^h) y = 0$, for all $c^h \in \mathbb{R}^{S+1}_{++}$ (negative Gaussian curvature), and $\{c^h \in \mathbb{R}^{S+1}_{++} \mid u^h(c^h) \geq u^h(\bar{c}^h)\}$ is closed in $\mathbb{R}^{S+1}$ for all $\bar{c}^h \in \mathbb{R}^{S+1}_{++}$ (boundary condition).

A2 $e^h \in \mathbb{R}^{S+1}_{++}$.

A3 $\text{rank}(\overline{A}) = J+1$ and $\overline{A}_{\cdot 0} > 0$.[1]

Usually Assumption A3 is replaced by the weaker assumption that there is $\overline{\theta} \in \mathbb{R}^{J+1}$ such that $\overline{A\theta} > 0$. Assumption A3 is without loss of generality. Indeed, if there is $\overline{\theta} \in \mathbb{R}^{J+1}$ such that $\overline{A\theta} > 0$, we take $\overline{A\theta}$ as asset 0 and we delete an asset j for which $\overline{\theta}_j \neq 0$. Equilibria of the original economy are obtained by a simple transformation of the equilibria of the thus resulting economy. Under A3 it holds that $q_0 > 0$ for all $q \in Q$, a property that is used in the convergence proof of the algorithm.

Given arbitrage free prices for assets $q \in Q$, the demand for assets by agent h, denoted $g^h(q)$, is the asset portfolio that solves the following maximization problem

$$\max_{\theta^h \in \mathbb{R}^J} u^h(c^h) \text{ s.t. } c^h = e^h + \begin{pmatrix} -q^\top \\ A \end{pmatrix} \theta^h \text{ and } c^h \in \mathbb{R}^{S+1}_{++}.$$

If prices for assets are arbitrage free, then the maximization problem is well-defined. A1 guarantees that the solution to the optimization problem is unique.

¿From the demand function for assets of agent h, $g^h : Q \to \mathbb{R}^{J+1}$, the total demand function for assets $G : Q \to \mathbb{R}^{J+1}$ follows as $G = \sum_{h=1}^{H} g^h$. Prices for assets q^* induce a competitive equilibrium for an economy $\mathcal{E}$ if and only if $G(q^*) = 0$.

The following properties are useful when showing convergence of the algorithm.

LEMMA 3.1: *If the economy $\mathcal{E}$ satisfies A1-A3, then the following properties hold.*

1. *The function $G : Q \to \mathbb{R}^{J+1}$ is twice continuously differentiable.*
2. *For all $q \in Q$, for all $\lambda > 0$, $G(\lambda q) = G(q)$.*
3. *For all $q \in Q$, $q \cdot G(q) = 0$.*
4. *If $(q^n)_{n \in \mathbb{N}}$ is a sequence in Q, $q^n \to \overline{q} \in \partial Q$,[2] $\overline{q} \neq 0$, then for all $\hat{q} \in Q$, $\hat{q} \cdot G(q^n) \to \infty$.*

PROOF. See Hens (1991). Q.E.D.

Let $g^0 : Q \to \mathbb{R}^{J+1}$ be the excess demand function for assets of some artificial agent having a utility function and initial endowments satisfying Assumptions A1 and A2. We will discuss a sensible choice for this agent later on. Since Lemma 3.1 also applies to an economy consisting of just one agent, we obtain the properties of Lemma 3.1 for g^0.

[1] The notation $x > 0$ means that all components of the vector x are non-negative and at least one component is positive.

[2] ∂Q represents the boundary of Q.

The function g^0 with component zero deleted is denoted by $\widehat{g}^0$; G with component zero deleted is denoted by $\widehat{G}$. The homogeneity of degree 0 stated in Lemma 3.1.2, implies that there is no loss of generality in normalizing prices for assets by taking $\sum_{j=0}^{J}(q_j)^2 = 1$. We propose to compute equilibria in a finance economy by means of the homotopy $\mathcal{H} : [0,1] \times Q \to \mathbb{R}^{J+1}$ defined by

$$\mathcal{H}(t,q) = \begin{cases} \sum_{j=0}^{J}(q_j)^2 - 1 \\ t\widehat{G}(q) + (1-t)\widehat{g}^0(q). \end{cases}$$

We are looking for solutions to $\mathcal{H}(t,q) = 0$. If $\mathcal{H}(\overline{t},\overline{q}) = 0$, then $\overline{q}_0$ is positive, so Lemma 3.1.3 implies $\overline{t}G_0(\overline{q}) + (1-\overline{t})g_0^0(\overline{q}) = 0$. At any solution $(\overline{t},\overline{q})$ to $\mathcal{H}(t,q) = 0$, it holds that $\overline{t}G(\overline{q}) + (1-\overline{t})g^0(\overline{q}) = 0$. In particular, if $\overline{t} = 1$, it follows that $\overline{q}$ is a competitive equilibrium price system.

4 Generic Convergence

A homotopy is in general constructed in such a way that there is a unique solution to $\mathcal{H}(0,q) = 0$, solutions to $\mathcal{H}(1,q) = 0$ are solutions to the problem of interest, and the unique solution to $\mathcal{H}(0,q) = 0$ is linked by a path of solutions to $\mathcal{H}(t,q) = 0$, for varying t, to one solution to $\mathcal{H}(1,q) = 0$. By following this path a solution to the problem of interest is found. When the unique solution to $\mathcal{H}(0,q) = 0$ is indeed linked by a path to a solution to $\mathcal{H}(1,q) = 0$, then the homotopy is said to converge. For an excellent discussion on the numerical techniques available to follow the path we refer to Allgower and Georg (1990).

It cannot always be guaranteed that our homotopy converges. There may exist economies such that the set of solutions $\mathcal{H}^{-1}(0)$ does not link the unique solution to $\mathcal{H}(0,q) = 0$ to one solution to $\mathcal{H}(1,q) = 0$. The set $\mathcal{H}^{-1}(0)$ may display bifurcations, and even higher dimensional solution sets. Nevertheless, we show that for typical economies, convergence of the homotopy takes place, and non-convergence can only happen in exceptional cases.

To make precise what typical means, we have to parameterize economies. We fix a tuple of utility functions $(u^h)_{h=1,\ldots,H}$ and an asset pay-off matrix $\overline{A}$. Then any choice of initial endowments $(e^h)_{h=1,\ldots,H} \in \mathbb{R}_{++}^{H(S+1)}$ induces an economy $\mathcal{E} = ((u^h, e^h)_{h=1,\ldots,H}; A)$. In this way, economies are parametrized by initial endowments. A property is said to be typical if it holds for a class of economies that is large in both a topological and a measure theoretic sense, that is when it holds for a set of initial endowments that is open and of full Lebesgue measure.

Theorem 4.1: *Fix utility functions $(u^h)_{h=1,\ldots,H}$ and an asset pay-off matrix $\overline{A}$ satisfying A1 and A3. Then, for all initial endowments e in an open set of initial endowments with full Lebesgue measure $E \subset \mathbb{R}_{++}^{H(S+1)}$, the homotopy $\mathcal{H}$ related to the resulting economy $\mathcal{E}$ satisfies the following.*

- *$\mathcal{H}^{-1}(\{0\})$ is a compact C^2 1-dimensional manifold with boundary, with boundary given by $\mathcal{H}^{-1}(\{0\}) \cap (\{0,1\} \times Q)$.*

- *There is an odd number of solutions in $\mathcal{H}^{-1}(\{0\}) \cap (\{1\} \times Q)$, i.e. there is an odd number of competitive equilibria.*

For any choice of initial endowments e in $\mathbb{R}^{H(S+1)}_{++}$, the homotopy $\mathcal{H}$ related to the resulting economy $\mathcal{E}$ satisfies the following.

- *There is one solution in $\mathcal{H}^{-1}(\{0\}) \cap (\{0\} \times Q)$.*

- *There is no sequence $(t^n, q^n)_{n\in\mathbb{N}}$ in $\mathcal{H}^{-1}(\{0\})$ converging to $(t,q) \in [0,1] \times \partial Q$.*

PROOF. The only solution in $\mathcal{H}^{-1}(\{0\}) \cap (\{0\} \times Q)$ is obviously given by

$$(0, q^0) = (0, \partial u^0(e^0)\overline{A}/\|\partial u^0(e^0)\overline{A}\|_2).$$

Suppose $(t^n, q^n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}^{-1}(\{0\})$ converging to $(t,q) \in [0,1] \times \partial Q$. Then, $t^n\widehat{G}(q^n) + (1-t^n)\widehat{g}^0(q^n) = 0$, so, for $\overline{q} \in Q$,

$$0 = \overline{q} \cdot (t^n G(q^n) + (1-t^n) g^0(q^n)),$$

but, by Lemma 3.1.4,

$$\overline{q} \cdot (t^n G(q^n) + (1-t^n) g^0(q^n)) \to \infty,$$

a contradiction. Solutions to the homotopy equations stay away from $[0,1] \times \partial Q$. It follows that $\mathcal{H}^{-1}(\{0\})$ is compact.

The proof is completed by showing that $\partial_q \mathcal{H}(0,q)$, and, generic in initial endowments, $\partial_q \mathcal{H}(1,q)$, and $\partial_{t,q}\mathcal{H}(t,q)$ have full rank for points in $\mathcal{H}^{-1}(\{0\})$.

It holds that $g^0(q) = \theta$ if and only if there is $\lambda \neq 0$ such that

$$\begin{aligned} &\partial u^0(e^0 + \overline{A}\theta)\overline{A} - \lambda q^\top = 0, \\ &q \cdot \theta = 0. \end{aligned}$$

By the inverse function theorem it holds that

$$\begin{pmatrix} \partial_q g^0(q) \\ \partial_q \lambda^0(q) \end{pmatrix} = \begin{bmatrix} \overline{A}^\top \partial^2 u^0(e^0 + \overline{A}\theta)\overline{A} & -q \\ q^\top & 0 \end{bmatrix}^{-1} \begin{bmatrix} -\lambda I \\ \theta^\top \end{bmatrix},$$

where I denotes the $(J+1)$-dimensional unit matrix. The first matrix on the right-hand side is indeed invertible. Suppose not, then there is $(y,z) \in (\mathbb{R}^{J+1} \times \mathbb{R}) \setminus \{0\}$ such that

$$\begin{bmatrix} \overline{A}^\top \partial^2 u^0(e^0 + \overline{A}\theta)\overline{A} & -q \\ q^\top & 0 \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0.$$

It follows that $y \neq 0$, since otherwise $y = 0$ and $\overline{A}^\top \partial^2 u^0(e^0 + \overline{A}\theta)\overline{A}y - qz = 0$ implies $z = 0$, contradicting $(y, z) \neq 0$. Since $\overline{A}$ has full column rank, $\overline{A}y \neq 0$. Moreover, $\partial u^0(e^0 + \overline{A}\theta)\overline{A}y = \lambda q \cdot y = 0$, so the non-zero Gaussian curvature of u^0 implies

$$0 = y^\top \overline{A}^\top \partial^2 u^0(e^0 + \overline{A}\theta)\overline{A}y + y^\top qz = y^\top \overline{A}^\top \partial^2 u^0(e^0 + \overline{A}\theta)\overline{A}y \neq 0,$$

a contradiction.

Consider $(0, q^0) \in \mathcal{H}^{-1}(\{0\}) \cap (\{0\} \times Q)$. Since $\lambda^0(q^0) \neq 0$, $(\partial_q g^0(q^0), \partial_q \lambda^0(q^0))$ has rank $J+1$, so $\partial_q g^0(q^0)$ has at least rank J. It follows that $\partial_q \widehat{g}^0(q^0)$ has rank J, since $q \cdot g^0(q) = 0$ for $q \in Q$ and $g^0(q^0) = 0$ imply $\partial_q g^0_0(q^0) = -\sum_{j=1}^J (q_j/q_0)\partial_q g^0_j(q^0)$. Since homogeneity of degree zero of g^0 in prices for assets implies $\partial_q \widehat{g}^0(q^0)q^0 = 0$, it follows that

$$\partial_q \mathcal{H}(0, q^0) = \begin{bmatrix} -2q^0_0 & \cdots & -2q^0_J \\ & \partial_q \widehat{g}^0(q^0) & \end{bmatrix}$$

has full rank, $J+1$.

We define $\overline{\mathcal{H}} : [0,1] \times Q \times \mathbb{R}^{S+1}_{++} \to \mathbb{R}^{J+1}$ by

$$\overline{\mathcal{H}}(t, q, e^1) = \begin{cases} \sum_{j=0}^J (q_j)^2 - 1, \\ t\widehat{G}(q, e^1) + (1-t)\widehat{g}^0(q), \end{cases}$$

where $\widehat{G}(q, e^1) = \widehat{g}^1(q, e^1) + \sum_{h=2}^H \widehat{g}^h(q)$. We show next that $\overline{\mathcal{H}} : (0,1) \times Q \times \mathbb{R}^{S+1}_{++} \to \mathbb{R}^{J+1}$ is transversal to zero, or equivalently, that $\partial_{t,q,e^1}\overline{\mathcal{H}}(\overline{t}, \overline{q}, \overline{e}^1)$ has full row rank whenever $\overline{\mathcal{H}}(\overline{t}, \overline{q}, \overline{e}^1) = 0$.

For $j' = 1, \ldots, J$, define the asset portfolio $\overline{\theta}^{j'}$ by $\overline{\theta}^{j'}_0 = -\overline{q}_{j'}$, $\overline{\theta}^{j'}_{j'} = \overline{q}_0$, and $\overline{\theta}^{j'}_j = 0$, $j \neq 0$, $j \neq j'$. Then changing the initial endowment of agent 1 to $\overline{e}^1 + \alpha \overline{A\theta}^{j'}$ with α sufficiently small, changes his asset demand to $g^1(\overline{q}, \overline{e}^1) - \alpha\overline{\theta}^{j'}$. Since the vectors $\overline{\theta}^{j'}$, $j' = 1, \ldots, J$, are independent, even with component 0 deleted, it follows that $\partial_{e^1}\widehat{G}(\overline{q}, \overline{e}^1)$ has rank J.

Homogeneity of degree zero of $\widehat{G}$ in prices for assets implies $\partial_q \widehat{G}(\overline{q}, \overline{e}^1)\overline{q} = 0$. It follows that $\partial_{t,q,e^1}\overline{\mathcal{H}}(\overline{t}, \overline{q}, \overline{e}^1)$ has rank $J+1$. By the transversal density theorem, see Mas-Colell (1985), I.2.2, page 45, the set of economies for which $\partial_{t,q}\mathcal{H}(\overline{t}, \overline{q})$ has full rank for all points in $\mathcal{H}^{-1}(\{0\})$ has full Lebesgue measure.

Exactly the same argument shows that for a set of initial endowments with full Lebesgue measure $\partial_q \mathcal{H}(1, \overline{q})$ has full rank for points in $\mathcal{H}^{-1}(\{0\}) \cap (\{1\} \times Q)$.

The transversality proofs given, show that for a set of initial endowments with full Lebesgue measure $\mathcal{H}^{-1}(\{0\})$ is a C^2 1-dimensional manifold with boundary, where the boundary is given by $\mathcal{H}^{-1}(\{0\}) \cap (\{0,1\} \times Q)$.

Using Lemma 3.1.3, it follows by a standard argument that the set of initial endowments for which transversality holds can be taken open and of full Lebesgue measure.

Concluding, for an open set of initial endowments with full Lebesgue measure, $\mathcal{H}^{-1}(\{0\})$ is a compact C^2 1-dimensional manifold with boundary, therefore a finite collection of arcs and loops.[3] Each arc has two boundary points. Since all boundary points belong to $\{0,1\} \times Q$, and there is exactly one boundary point in $\{0\} \times Q$, it follows that for an open set of initial endowments with full Lebesgue measure, there is an odd number of solutions in $\mathcal{H}^{-1}(\{0\}) \cap (\{1\} \times Q)$. Q.E.D.

Since $\mathcal{H}$ is a system of $J+1$ independent equations in $J+2$ variables, it is not surprising that $\mathcal{H}^{-1}(\{0\})$ is generically a compact 1-dimensional manifold with boundary, i.e. a finite collection of arcs and loops. There is a unique solution to $\mathcal{H}(0,q) = 0$, obtained by taking q equal to $\partial u^0(e^0)\overline{A}$. The boundary behavior of G guarantees that there is no sequence $(t^n, q^n)_{n \in \mathbb{N}}$ in $\mathcal{H}^{-1}(\{0\})$ converging to $(t,q) \in [0,1] \times \partial Q$. Therefore the unique solution to $\mathcal{H}(0,q) = 0$ is generically part of a path in $\mathcal{H}^{-1}(\{0\})$ that does not run off to the boundary, but reaches $t = 1$. The unique solution to $\mathcal{H}(0,q) = 0$ is thereby connected to exactly one point $(1, q^*) \in \mathcal{H}^{-1}(\{0\})$, a competitive equilibrium for $\mathcal{E}$, and the homotopy converges. Notice that there is no need to compute the set Q explicitly. Our homotopy is constructed in such a way that its projection on the set Q stays away from ∂Q.

COROLLARY 4.2: *Let $\mathcal{E}$ be an economy satisfying A1-A3. Then, for an open set of initial endowments with full Lebesgue measure, the homotopy $\mathcal{H}$ converges to a competitive equilibrium.*

If there are multiple equilibria, then in addition to the arc connecting q^0 and a competitive equilibrium q^*, there is a finite number of arcs, each one having two more competitive equilibria as its end points. This gives a constructive proof of the fact that there is an odd number of competitive equilibria. In fact, using the properties of a homotopy, we can get an index theorem for our economy, a result already obtained by Hens (1991), and for certain classes of economies with more than one good per state by Schmedders (1998).

The computation of the demand for assets as a function of prices for assets is not necessarily an easy problem. It is notoriously hard when the asset market is incomplete. The theoretical homotopy $\mathcal{H}$ is therefore replaced by the diffeomorphic implementable homotopy $\mathcal{H}^* : [0,1] \times Q \times \mathbb{R}^{(H+1)(J+1)} \times \mathbb{R}^{H+1} \to \mathbb{R}^{1+J+(H+1)(J+1)+(H+1)}$,

$$\mathcal{H}^*(t,q,\theta,\lambda) = \begin{cases} \sum_{j=0}^{J} (q_j)^2 - 1, \\ \sum_{h=0}^{H} \theta_j^h, \quad j = 1, \ldots, J, \\ \partial u^h(e^h + \overline{A}\theta^h)\overline{A} - \lambda^h q^\top, \quad h = 0, \ldots H, \\ q \cdot \theta^h, \quad h = 0, \ldots, H. \end{cases}$$

We have replaced the demand functions of the agents by their first order conditions, an approach proposed in Garcia and Zangwill (1981).

[3] An arc is a set homeomorphic to the unit interval and a loop a set homeomorphic to the unit circle.

THEOREM 4.3: *Let $\mathcal{E}$ be an economy satisfying A1-A3. Then ${\mathcal{H}^*}^{-1}(\{0\})$ is C^2 diffeomorphic to $\mathcal{H}^{-1}(\{0\})$.*

PROOF. It holds that $(\bar{t}, \bar{q}, \bar{\theta}, \bar{\lambda}) \in {\mathcal{H}^*}^{-1}(\{0\})$ if and only if $(\bar{t}, \bar{q}) \in \mathcal{H}^{-1}(\{0\})$, $\bar{\theta}^h = g^h(\bar{q})$, $h = 0, \ldots, H$, and $\bar{\lambda}^h = \partial u^h(e^h + \bar{A} g^h(\bar{q}))\bar{A}_{\cdot 0}/\bar{q}_0$, $h = 0, \ldots, H$. The claim follows since g^h and ∂u^h are twice continuously differentiable functions. Q.E.D.

Since ${\mathcal{H}^*}^{-1}(\{0\})$ is diffeomorphic to $\mathcal{H}^{-1}(\{0\})$, the results of Theorem 4.1 carry over to ${\mathcal{H}^*}^{-1}(\{0\})$.

COROLLARY 4.4: *Let $\mathcal{E}$ be an economy satisfying A1-A3. Then, for an open set of initial endowments with full Lebesgue measure, the homotopy $\mathcal{H}^*$ converges to a competitive equilibrium.*

The speed of homotopy algorithms depends mainly on two factors, the number of equations and the arc length of the homotopy path. A quick comparison shows the great benefits of developing a special purpose homotopy tailored to the finance GEI-model. The homotopy algorithms as reported in Brown, DeMarzo and Eaves (1996a) and Schmedders (1998) are designed to deal with the general GEI-model with multiple commodities per state, but can be applied to finance economies.

The homotopy proposed by Brown, DeMarzo and Eaves (1996a) needs closed form solutions for excess demand functions and should therefore be compared with our homotopy $\mathcal{H}$. Applied to two-period finance economies, their algorithm has $2S+1$ equations, whereas ours only has $J+1$. The algorithm of Schmedders (1998) does not require closed-form solutions for excess demand functions, and also uses the first order conditions. The number of equations of his algorithm amounts to $2(H+1)(S+1)+HJ+1$, whereas the number of equations in our algorithm $\mathcal{H}^*$ equals $(H+2)(J+1)+H+1$.

In both cases, we roughly need a fraction $J/2S$ only of the equations of alternative algorithms. This is especially favorable when S is high, which is the case for many applications, and also for the application discussed in detail in this paper, where $J = 8$ and $S = 2,048$. The high number of states is used to get a good discrete approximation of a continuously distributed multivariate random variable. On top of the great number of equations saved, our method also has the flexibility of choosing the initial price system as desired, contrary to the homotopies of Brown, DeMarzo and Eaves (1996a) or Schmedders (1998). Since it is not too hard to make a reasonable guess for an equilibrium price system using the method of the next section, our algorithm will generally substantially reduce the arc length of the homotopy path.

5 Implementation

We implemented the algorithm using HOMPACK - a suite of FORTRAN 77 subroutines designed to solve systems of non-linear homotopy equations with path-following methods. See Watson (1979) and Watson, Billups and Morgan (1987) for details on HOMPACK.

We now turn to the determination of the starting point and the specification of the artificial agent's demand function.

The demand function $g^0(q)$ should be chosen such that an a priori selected starting point $q^0 \in Q$ with $\sum_{j=0}^{J}(q_j^0)^2 = 1$ is the unique solution to $g^0(q) = 0$ and $\sum_{j=0}^{J}(q_j)^2 - 1 = 0$.

We take a Cobb-Douglas utility function for the artificial agent,

$$u^0(c^0) = \sum_{s=0}^{S} \rho_s \gamma_s \ln(c_s^0), \quad c^0 \in \mathrm{I\!R}_{++}^{S+1}.$$

Let $\pi^0 \in \mathrm{I\!R}_{++}^{S+1}$ be any state price vector such that ${\pi^0}^\top \overline{A} = q^0$. If the artificial agent is defined by

$$\begin{aligned} e_s^0 &= 1, \quad s = 0, \ldots, S, \\ \gamma_s &= \pi_s^0/\rho_s, \quad s = 0, \ldots, S, \end{aligned}$$

then the unique solution in Q to $g^0(q) = 0$ and $\sum_{j=0}^{J}(q_j)^2 - 1 = 0$ is indeed given by q^0.

In applications with first period consumption, there is usually no need to solve for $\pi^\top \overline{A} = q^0$.[4] Instead, we take π^0 equal to the weighted average over all agents of $\partial u^h(e^h)$, with weight for agent h equal to $1/\lambda^h$, where λ^h denotes the marginal utility of first period consumption at the initial endowment e^h. Next we take q^0 equal to ${\pi^0}^\top \overline{A}/\|{\pi^0}^\top \overline{A}\|_2$. Prices for assets q^0 are in general a very reasonable first guess for equilibrium prices for assets.

6 Numerical Examples

In order to illustrate the performance of our algorithm and in order to show how running times increase with the number of households, the number of assets and the number of states, we consider several examples. In all examples, households have constant relative risk aversion utility of the form

$$u^h(c^h) = v^h(c_0^h) + \delta^h \sum_{s=1}^{S} \frac{1}{S} v^h(c_s^h), \quad c^h \in \mathrm{I\!R}_{++}^{S+1},$$

[4]If $q^0 \in Q$ with $\sum_{j=0}^{J}(q_j^0)^2 = 1$ is given and there is a need to solve for $\pi^\top \overline{A} = q^0$, an easy way to achieve this for an economy where the first asset corresponds to first-period consumption is to solve the following linear program

$$\begin{aligned} \min \sum_{s=0}^{S} \pi_s \quad \text{s.t.} \quad & \overline{A}^\top \pi - \pi_0 q^0/q_0^0 = 0 \\ & \pi - 1_{S+1} \geq 0, \end{aligned}$$

and divide the solution found, say π^0, by $\|{\pi^0}^\top \overline{A}\|_2$.

with

$$v^h(c^h_s) = \frac{(c^h_s)^{1-\gamma^h}}{1-\gamma^h}, \quad c^h_s > 0,$$

where δ^h is the discount factor and γ^h the coefficient of relative risk aversion. We choose $\delta^1 = \delta^2 = \delta^3 = 0.95$, and $\gamma^1 = 6$, $\gamma^2 = 4$ and $\gamma^3 = 2$. (Varying the discount factor and gamma does not have a significant effect on running times as long as the coefficient of relative risk aversion remains below 9.)

We will consider an example with 3 households and 8 assets, an example with 2 households and 8 assets and an example with 3 households and 5 assets. For each example we consider the case of 10,000 states, of 20,000 states, of 30,000 states and of 40,000 states. This results in 12 examples for which we will report running times.

Each agent is endowed with an initial portfolio $(0, \theta^h_-)$ of the riskless bond and the available stocks, with current income, representing current labor income plus dividends from θ^h_-, $e^1_0 = 2/3$, $e^2_0 = 1$, and $e^3_0 = 4/3$ (in the case of three agents), and with stochastic future labor income given by some $l^h \in \mathbb{R}^S_{++}$. We are back in the framework of Section 2 by setting $e^1_0 = 2/3$, $e^2_0 = 1$ and $e^3_0 = 4/3$, and $\tilde{e}^h = l^h + \sum_{j=2}^J \theta^h_{-j} d^j$ for $h = 1, ..., H$. The first agent has no capital income, $\theta^1_- = 0$. For the other agents we have $\theta^2_{-j} = 1/3$, $j = 2, \ldots, J$, and $\theta^3_{-j} = 2/3$, $j = 2, \ldots, J$. In the case of only two agents, we just drop the first agent from the economy.

The assets available are given by a riskless bond and risky stocks. The dividends of risky assets $j = 2, \ldots, J$ depend on a single common factor $f \in \mathbb{R}^S$ as well as on an idiosyncratic shock $\varepsilon^j \in \mathbb{R}^S$. We denote asset j's load in the factor by c_j, varying from 0.25 to 1.75 (for the case of 8 assets) or from 0.75 to 1.5 (for the case of 5 assets) in steps of 0.25.

The standard deviation of both the factor and the idiosyncratic shock determining the dividends are 0.13 - giving an overall standard deviation of the stock market of 0.17. The standard deviation of labor income is chosen to be 0.10 and labor income constitutes 2/3 of total income. For the case of 3 households and 8 assets, there are 11 random variables - $((l^h)_{h=1,...,H}, f, (\varepsilon^j)_{j=2,...,J})$. Throughout this section we assume that all random variables are log-normally distributed, so l^h_s, f_s, and ε^j_s are drawn independently from a log-normal distribution. The log-normal distribution with mean μ and variance σ^2 is denoted by $\mathrm{LN}(\mu, \sigma^2)$. Asset 1 is the riskless bond. For $j \geq 2$, we define asset j's dividend to be

$$d^j_s = 1/3 \cdot 1/7 \cdot 1.02 \cdot f^j_s \cdot \varepsilon^j_s$$

and we choose

$$\begin{aligned} l^h_s &\sim \mathrm{LN}(2/3 \cdot 1.02, (2/3)^2 \cdot 0.01), \\ f^j_s &\sim \mathrm{LN}(1, c_j \cdot 0.0161), \\ \varepsilon^j_s &\sim \mathrm{LN}(1, 0.0161). \end{aligned}$$

The actual $(f_s^j)_{j=2}^J$ are all based on a single realization of a normal random variable $\widehat{f}_s$. For each asset j, we linearly transform the realization of this random variable in such a way that after taking the exponent a log-normally distributed random variable with mean 1 and variance $c_j \cdot 0.0161$ results. The construction of the random variables implies that all dividends themselves are log-normally distributed. To get a similar variance of the entire stock market as before, the variance of the factors and the idiosyncratic shock have to be chosen to be 0.0161 instead of 0.0169.

For the base-case $H = 3$ and $J = 8$ solving for an equilibrium involves solving a system of 49 non-linear equations. The number of equations is independent of the number of states. However, running times increase significantly with the number of states, since the time needed for a single evaluation of the first order conditions increases. For the case of two households and 8 assets we have 39 equations and for the case of 3 households and 5 assets we have 34 equations.

	10000 states	20000 states	30000 states	40000 states
3 agents, 8 assets	6.41	16.31	25.01	32.09
2 agents, 8 assets	4.09	10.35	16.14	22.10
3 agents, 5 assets	3.47	10.31	17.10	21.09

TABLE 1: Running times (mm.ss).

Table 1 shows the running times in minutes. All running times refer to an implementation in FORTRAN 77 on a 500 MHz Pentium III processor running Red Hat Linux. Note that the running times do increase significantly with the number of states. However, the increase is more or less linear in the number of states. If the number of states would increase the number of equations, running times in this order of magnitude would be impossible.

7 Conclusion

In this paper we develop a homotopy algorithm to compute equilibria in the finance version of the GEI-model that is particularly useful for cases with a large state space. The generic convergence of this algorithm is shown, where generic means that for an open set of finance economies with full Lebesgue measure convergence takes place. The implementation of the algorithm is discussed. Its effectiveness is verified by means of numerical examples. In Herings and Kubler (2000) the algorithm is used to explore asset pricing implications of the GEI model when the number of states is large.

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