Package Sizes, Tariffs, Quantity Discount and Premium

Babu Nahata

Department of Economics, University of Louisville

Louisville, Kentucky 40292, USA.

e-mail: nahata@louisville.edu

and

Serguei Kokovin and Evgeny Zhelobodko*

Department of Economics, Novosibirsk State University,

630090 Novosibirsk, Russia

email: kokovin@math.nsc.ru and ezhel@mail.ru

* We are grateful to Victor Polterovich and Segei Guriev for many valuable comments. Kokovin and Zhelobodko gratefully acknowledge the financial assistance from the Russian Foundation for Basic Research (grants 00-15-98884 and 01-01-00896), and from the Russian Foundation for Humanitarian Research (grant 99-02-00141) and by the Integracia Program for Russian Universities (grant -274). Nahata gratefully acknowledges financial support from the University of Louisville (IRIG-RIG 277858 and PCG 277858). An earlier version of the paper was presented at the 28th meeting of the Euriopean Association for Research in Industrial Economics at Trinity College, Dublin and the 16th meeting of the European Economic Association at the University of Lausanne, Switzerland.

Package Sizes, Tariffs, Quantity Discount and Premium

Abstract: We analyze nonlinear pricing problem under monopoly using two hidden types of agents with linear demands and fully characterize all possible optimal solutions for both ordered and non-ordered demands. We show that both optimal packages can either contain Pareto-efficient quantities or one package can be undersized or oversized. All these effects are non-degenerate and are expected to hold for nonlinear demands. Surprisingly, the total output under nonlinear price discrimination with self-selection is neither unambigously realted to efficiency nor to the degree of monopoly power (demand elasticity). We also show that under limited range of parameters quantity premia can occur only when demands are ordered.

Key Words: Principal-agent, self-selection, nonlinear pricing, package pricing, Pareto efficiency

JEL Codes: D42, L10, L40

Package Sizes, Tariffs, Quantity Discount and Premium

I. Introduction

In this paper we analyze nonlinear pricing when a set of packages is used as self-selection devices to identify hidden types of consumers by a monopolistic seller. Packages are offered on take-it-or-leave-it basis. Hence, although consumers self-select the size of their choice, the quantity contained in the package is non-negotiable. Because the monopolist, *a priori,* cannot identify the hidden types of consumers, he must take into consideration the so-called 'incentive-compatibility' and the 'participation' constraints in designing the sizes and tariffs charged for different packages. Casual observation shows that packages are quite prevalent for many types of goods and are offered in many sizes, generally with differing tariffs, but sometimes the observed tariffs are the same.[1]

Broadly speaking, the pricing of packages is a nonlinear self-selection pricing problem.[2] The problem of package pricing has been analyzed in the literature, but the analysis remains mostly restricted to the case when consumer valuations (i.e., their monetary benefit functions) are *ordered.* This assumption, sometimes named as *Spence-Mirrlees-condition,* implies that the demand of one type of consumer is everywhere higher than the demand of the other type. For this case the optimal package sizes can be obtained by using the 'chain-rule' theorem (see Katz (1983)). Once the optimal sizes are determined, the tariffs can be

[1]Southwest Airlines occasionally runs a campaign, 'companion flies free.' Either a traveller can fly alone or take a companion with her. The fares for both options are the same.

[2]For a recent comprehensive treatment of nonlinear pricing, see Wilson (1993). Tirole (1988) provides some basic models of packages for both discrete and continuous types of consumers using mainly linear demands.

calculated from the respective demand curves. Chain-rule simplifies the solution structure considerably because each consumer 'almost envies' *only* his closest-lower-demand neighbor and no one else and hence the incentive-compatibility constraints related only to this neighbor's package are binding. The standard result in this case is that, normally, all low-demand consumers have too low consumption with zero consumer surplus, while the highest demand consumer enjoys a positive consumer surplus by consuming a package that is *always* Pareto-efficient (marginal valuation equals marginal cost).

Some seemingly counter-intuitive pricing strategies observed in real life cannot be explained by the ordered valuations assumption. For example, in the airlines industry in the US, frequently a round-trip ticket (a bigger package) is cheaper than one-way (a smaller package). On the other hand, some foreign airlines often times offer one-way and round-trip tickets at the s*ame* fare.[3] In Japan soft-drink containers sold through vending machines *only* are sometimes priced the same even though they contain different amounts (different size packages). Actually different consumers do buy both smaller and larger containers at the same price depending on their preferences. These observed, but not yet explained, pricing practices have provided the main impetus to analyze this problem. Our results provide some plausible economic rationale for such observed pricing practices.

We analyze package pricing in a more general setting and allow consumer-valuations to be either non-ordered or ordered. Ordered valuations mean that the indifference curves of consumers cross only once, the so-called "single-crossing property." In terms of demand it

[3]The authors have personally encountered both situations. When the intended journey was only one-way the authors bought the cheaper round-trip ticket in the US and did not use the return coupon of the ticket. In the case of foreign travel when the fares were the same the authors bought only the one-way ticket.

is called the "non-crossing condition," implying that the inverse demand functions do not cross. In reality the demand functions would intersect; for instance, when the marginal utility of consumers whose 'choking price' (the willingness-to-pay for an infinitesimal small unit) is relatively high and decreases faster than that of another consumer whose choking price is relatively smaller.[4]

In the non-ordered case, two different incentive-compatible and profit maximizing packages offered to *two* types of consumers could *simultaneously* be Pareto-efficient and first-best. This striking result shows that a monopolistic nonlinear price discrimination practiced by using packages is compatible with allocative efficiency. In contrast, two-part tariff and packages for all agents under ordered demands *cannot* contain quantities that are all Pareto-efficient, except for some degenerate cases. Further, under a special condition two different size packages containing Pareto-efficient quantities can have exactly the *same* tariffs. This seemingly strange result is consistent with rational behavior of consumers when they derive some disutility in buying additional quantity.

We also show that in a non-ordered case sometimes it may be profitable for the seller to offer packages containing quantity *exceeding* the Pareto-efficient level (oversizing). This seemingly odd result where the marginal cost exceeds the marginal payment is consistent with optimizing behavior of the seller.[5] When a package is oversized, the consumer may choose not to buy the package. Thus, for an oversized package to be optimal for

[4]This seems to be the case with business and leisure travellers. The choking price for business travellers is relatively high but their sensistivity to price is relatively low. On the other hand, the choking price for leisure travellers is relatively low but they are more sensitive to price and their satiation level of quantity is relatively higher. Thus the demand functions of these two types of consumers would interest at some positive level of quantity. In real life one can find many situations where the two demands would cross.

[5]Katz (1984) also shows that some consumers may consume more than the first-best outcomes. However, his setting is different than ours in the sense that he allows consumer to purchase multiple packages.

the seller, an additional consideration whether a consumer has disutility or not becomes quite important. Under linear pricing consumption is usually identified with the amount purchased. Conceptually, this becomes questionable when a consumer buys a package on a take-it-or-leave-it basis with some fixed quantity. For example, when a consumer's demand is satiable, (marginal valuation can either be zero or negative) but the fixed amount in the package exceeds this satiation level, then whether or not a consumer can freely dispose excess quantity bought becomes a crucial consideration. Oversizing of package happens either for sufficiently large cost, or for no-free disposal situations.

Finally, there is a boundary case when the distinction between ordered and non-ordered preferences disappears because most preferred (i.e., Pareto-efficient) quantities for both types of consumers are exactly the same. For this boundary case, a single optimal package offered to both types of consumers is Pareto-efficient. Obviously, the single tariff charged will be the smaller of the two total willingness to pay.

These two effects, namely the *efficiency effect* and the *same-tariff effect,* arise only under the non-ordered case. These new insights are contrary to the conventional wisdom as they clearly demonstrate that the total output of an industry is *neither* unambiguously related to its efficiency *nor* to its degree of monopolization. These effects are formally shown for linear demands. But we show latter they are expected to hold even for broader classes of nonlinear demand functions. The reason is that the domains of parameters yielding these effects are *solid sets.* This convinces us that the domains of demand parameters remain non-empty.

Generally, consumers who buy a larger package receive a quantity discount and pay a lower average price (total outlay divided by the total quantity). But it is also possible that consumers buying a larger package also pay a quantity premium (higher average price). Katz (1984), perhaps, was the first to show that quantity premia for ordered demands are possible under nonlinear pricing. Gerstner and Hess (1987), using an inventory theoretic approach reconfirm that quantity premia can exist, a result similar to Katz. They also assume that consumer's valuations are ordered (high and low). Can quantity premium occur for non-ordered demands? We show that when demands cross, quantity premia *cannot* exist. Moreover, even within ordered demands premia can occur only for a certain range of parameters and we completely identify the range when two demands are linear.

By restricting our analysis to two types of consumers with linear or piece-wise-linear demands, we are able to characterize completely *all* optimal solutions by deriving explicit formulas under *all* possible combinations of exogenously given parameters. It turns out that, depending on different assumptions, there could be as many as seven to ten different types of solutions to the problem of package sizes. For each type of solution we derive the corresponding domain of demand parameters.

The paper is organized as follows. Section 2 contains the problem formulation for two-consumers with linear demands. In section 3, we present the main results followed by a discussion of the results. Using graphical approach we further show that the qualitative results are expected to hold under nonlinear demands as well. Section 4 focuses on quantity discounts and premia. Section 5 summarizes the main findings. The Appendix contains outlines of the proofs.[6]

[6] Detailed proofs can be obtained at www.math.nsc.ru/~mathecon/kokovin.html

II. Model

Consider a case when a monopolist sells a homogeneous good using two packages to 2 types of hidden consumers, who self-select the package of their choice. No consumer can buy more than one package, and arbitrage is prevented. To simplify the analysis we assume all consumers of a given type choose the same package.[7] Under this setting it is sufficient for the monopolist to design exactly 2 packages, though some of them may be identical in size or contain quantity equal to zero. Let the total cost function $C(x)$ be linear (constant returns to scale) so that the marginal cost c is a constant. Let the numbers of two types be m_1 and m_2 respectively; their ratio is denoted by $\gamma = m_2/m_1$. We assume that the utility functions are quasi-linear and depend on quantity and tariff (outlay), so no income effects are present. The valuation functions $V_i(x_i)$ of the two types of consumers are quadratic (no-free disposal case) or piece-wise quadratic (free disposal case). In the first case, the demand functions are linear everywhere; for the other case they are piece-wise linear.

In the first case, valuation functions are: $V_1(x_1) = a_1x_1 - b_1x_1^2/2$, and $V_2(x_2) = a_2x_2 - b_2x_2^2/2$ (with parameters $a_i, b_i > 0$). The corresponding inverse demand functions are linear: $p_1 = a_1 - b_1x_1$ and $p_2 = a_2 - b_2x_2$. All possible situations depending on the relative sizes of demands and cost can be characterized using seven parameters, namely c, a_i, b_i, m_i. But the same complete characterization can be obtained using only four meaningful parameters as a result of simple normalization. In some cases, even only three

[7] This assumption simplifies matters, although, it may rule out some unusual optimal pricing policies. Following another common convention, we suppose that among equivalent options, an agent chooses the package preferred by the principal. It is reasonable because the principal may offer a small reward to the consumer for such a behavior.

parameters are sufficient. It is easy to see that in most cases only the triangles of the two inverse demand functions that are above marginal cost are relevant for the optimal solutions. Using this observation we normalize demands by converting one consumer's above-marginal-cost-demand-triangle into a standard simplex described below.

Denote $\alpha = (a_2 - c)/(a_1 - c)$ (when $a_2 \leq a_1$ then $\alpha \leq 1$), and $\beta = [(a_2 - c)/b_2]/[(a_1 - c)/b_1]$. Set the normalized net valuations as $v_1(x_1) = x_1 - x_1^2/2 - \bar{c}x_1$, and $v_2(x_2) = \alpha x_2 - \alpha x_2^2/(2\beta) - \bar{c}x_2$ and normalized cost $\bar{c} = 0$. If we normalize the first consumer's choking-price to be 1, then the corresponding first-best quantity for this consumer also becomes 1. Actually, the normalization (α, β), $(1, 1)$ simply changes the units of measurements. Net tariffs are denoted by t_i and are related to initial gross tariff $T_i = t_i + cx_i$. One can see that optimization in terms of normalized parameters is equivalent to the optimization under the initial terms.[8] Therefore, without any loss of generality and for notational simplicity, from the very beginning we can assume net valuations as $v_1(x_1) = x_1 - x_1^2/2$, and $v_2(x_2) = \alpha x_2 - \alpha x_2^2/(2\beta)$. Further among different cases relating (α, β) to $(1, 1)$ we consider only those cases when $\alpha \leq 1$. The opposite case when $\alpha > 1$ can be considered by simply renumbering consumers and renormalizing.

Formally, in terms of *net tariffs* the package-optimization problem for two consumers can be formulated as follows.[9]

$$\begin{aligned} \pi(x,t)/m_1 &= t_1 + \gamma t_2 \rightarrow \max_{x,t} \quad \text{s.t.} \qquad (1) \\ v_1(x_1) - t_1 &\geq 0; \qquad x_1 \geq 0 \end{aligned}$$

[8] Optimization with respect to gross and net tariffs is equivalent. For a formal proof see our paper (2001)

[9] Gross tariffs $T_i = t_i + cx_i$ and can be easily obtained from the solution.

$$
\begin{aligned}
v_2(x_2) - t_2 &\geq 0; \qquad x_2 \geq 0 \\
v_1(x_1) - t_1 - v_1(x_2) + t_2 &\geq 0; \qquad t_1 \geq 0 \\
v_2(x_2) - t_2 - v_2(x_1) + t_1 &\geq 0; \qquad t_2 \geq 0 \qquad (2)
\end{aligned}
$$

The above formulation has four constraints: two incentive-compatibility and two participation constraints. When an incentive-compatibility constraint $v_i(x_i) - t_i \geq v_k(x_k) - t_k$, is active, then, following Wilson (1993), it can be interpreted as consumer i almost 'envying' consumer k'choice ($i \to k$), implying that if her tariff were to be raised she would switch. The second type of constraints $v_i(x_i) - t_i \geq 0$, usually named as participation constraints, state that any consumer has an option not to buy any package. In another paper (2001) we prove that one consumer will always buy a package containing Pareto-efficient quantity, so at the optimal solution, *at most*, three constraints can be binding.

III. Main Results

We now present the results for both no-free and free disposal cases. In explaining some special cases our analysis assumes that there is at least an infinitesimally small transaction cost to the seller. Our immediate goal is to characterize the domain of four crucial parameters $(\alpha, \beta, \gamma, c) \geq 0$ that yield different optimal solution structures.

In presenting results, if a package contains a quantity larger than an agent's locally Pareto-efficient level (EQ) we label it *oversized* ($> EQ$). When the seller finds it optimal to offer a package to only one type of consumers and ignore the other type (i.e., $x_i = 0$), then the solution is labeled as *ignoring* solution.

For the sake of the expositional clarity, the no-free disposal case is considered first.

A. No-free Disposal

Conceptually, no-free disposal implies that a consumer derives *disutility* from buying additional quantities beyond her satiation level. For example, a consumer may not like to buy and carry a big bottle of soft-drink when she is not very thirsty. This may also be the case when a person taking a short vacation do not want to carry larger packages of toothpaste, shaving cream, etc. due to storage considerations and hence because of disutility smaller packages may be preferred.

For no-free disposal case, as well as for large c, Proposition 1 below characterizes all optimal solutions.

PROPOSITION 1. For two consumers each having a linear demand and no-free disposal or $c \geq 1$, all possible optimal solutions to the package pricing problem lie within the seven regions defined by the parameters in Table 1 and depicted in Figure 1.

[Table 1 and Figure 1 here]

Table 1. Optimal Solution Structures for No-Free Disposal Case

Regions	Active consraints	Package Sizes
A: $\frac{1}{\gamma+1} \leq \alpha \leq 1, \beta \leq 1$	1→2→0	$x_1 = 1,\ x_2 = \frac{\gamma\alpha-1+\alpha}{\gamma\alpha-\beta+\alpha}\beta < EQ$
B: $1 \leq \beta \leq \frac{2\gamma\alpha+\alpha-\gamma\alpha^2-\alpha^2}{1-\alpha+\gamma\alpha}$	1→2→0	$x_1 = 1,\ x_2 = \frac{\gamma\alpha-1+\alpha}{\gamma\alpha-\beta+\alpha}\beta > EQ$
I1: $0 < \alpha \leq \frac{1}{\gamma+1}, \beta \leq 1$	1→2=0, 1→0	$x_1 = 1,\ x_2 = 0\ (Ignoring)$
D: $\max\{1, \frac{2\gamma\alpha+\alpha-\gamma\alpha^2-\alpha^2}{1-\alpha+\gamma\alpha}\} \leq \beta,$ $\beta \leq 2-\alpha$	1→2→0, 1→0	$x_1 = 1,\ x_2 = 2\beta\frac{1-\alpha}{\beta-\alpha} > EQ$
E: $2-\beta \leq \alpha \leq \frac{\beta}{2\beta-1}$	1→0, 2→0	$x_1 = 1,\ x_2 = \beta$
F: $\frac{\beta}{2\beta-1} \leq \alpha \leq \bar{\alpha}(\gamma,\beta)$	2→1→0, 2→0	$x_1 = 2\beta\frac{1-\alpha}{\beta-\alpha} < EQ,\ x_2 = \beta$
G: $\bar{\alpha}(\gamma,\beta) \leq \alpha < 1$	2→1→0	$x_1 = \beta\frac{(\gamma+1-\gamma\alpha)}{\beta(1+\gamma)-\gamma\alpha} < EQ,\ x_2 = \beta$

Where,

$$\bar{\alpha}(\gamma,\beta) = \frac{\left(\gamma+\gamma\beta+2\beta-1-\sqrt{(\gamma^2-2\gamma^2\beta-2\gamma\beta-2\gamma+\gamma^2\beta^2+4\gamma\beta^2+4\beta^2-4\beta+1)}\right)}{2\gamma}.$$

In regions **A** and **B**, the formulas for the two net tariffs are exactly the same.[10]

$t_2 = \beta\alpha\frac{(\gamma\alpha-1+\alpha)(\gamma\alpha-2\beta+\alpha+1)}{2(\gamma\alpha-\beta+\alpha)^2}$,

$t_1 = \alpha\frac{\beta\alpha^2-2\beta\gamma^2\alpha-2\alpha\beta-\alpha\beta^2+2\gamma\alpha+\alpha-\beta-4\beta\gamma\alpha+\alpha\beta^2\gamma^2+\beta\gamma^2\alpha^2+2\beta\gamma\alpha^2+\gamma^2\alpha+2\beta^2}{2(\beta-\gamma\alpha-\alpha)^2}$.

For regions **I1**, **D**, **E** $t_1 = \frac{1}{2}$, and it is the same for all three regions. For other regions the tariffs are:

$t_{2I} = 0$, $t_{2D} = 2\beta(1-\alpha)\alpha\frac{\beta-1}{(-\beta+\alpha)^2}$, and $t_{2E} = \alpha\beta/2$

$t_{1F} = 2\beta(1-\alpha)\alpha\frac{\beta-1}{(\beta-\alpha)^2}$, and $t_{2F} = \alpha\beta/2$.

$t_{1G} = \beta\frac{(\gamma+1-\gamma\alpha)(\beta+\beta\gamma-2\gamma\alpha+\beta\gamma\alpha)}{2(-\beta-\beta\gamma+\gamma\alpha)^2}$,

$\hat{t}_{2G} = \beta\frac{\beta+\alpha-2\beta\gamma^2\alpha-4\beta\gamma\alpha-\beta\gamma^2\alpha^2+2\alpha\beta^2\gamma+\alpha\beta^2\gamma^2-2\alpha\beta+2\beta\gamma+2\gamma^2\alpha^2+\alpha\beta^2+\beta\gamma^2-\gamma^2\alpha}{2(\beta+\beta\gamma-\gamma\alpha)^2}$.

[10] Recall that gross tariffs are obtained from these net tariffs as $T_i = t_i + cx_i$.

Proof. We only give the outline of the proof in the Appendix. Detailed derivations are available (see footnote 6).

We explain Table 1, and Fig. 1. There are seven formulas for different solutions related to different sizes of (α, β) demand triangle (α is the height and β is the length). The related regions are labeled anti-clockwise **A** through **G**. Each region is identified by the active constraints, i.e., those constraints becoming equalities. For instance, notation $1 \to 2 \to 0$ means that the first consumer is almost-envying the second one, who is almost inclined not to buy, while $2 = 0$ would mean she actually does not buy. In regions **A** and **I1,** $\beta \leq 1$. Therefore these two regions represent the ordered case. The other five regions where $\beta \geq 1$ represent the non-ordered case. As identified in Table 1, all boundaries separating one region from the other are supposed to belong to both regions. In particular, the border point $[\beta = 1,\ \alpha = 1/(1+\gamma)]$ belongs to all four neighboring regions, namely **A**, **I1**, **B**, **D**.[11] Since the consumers can be renumbered and renormalized, the line $[\beta > 1,\ \alpha = 1]$ is equivalent to line $[\beta < 1,\ \alpha = 1]$, and therefore is excluded from our results. The line $[\alpha = 0]$ and points $[\alpha = 1,\ \beta = 1]$ are also excluded as pathological cases as they pertain only to one type of consumers. Within each region, we also include figures of the two demand triangles that approximate the related areas of two demand triangles for the two types of consumer. White triangle, with height 1 and length 1, relates to the first consumer, while the grey triangle with height and length α, β respectively is related to the second consumer.

[11] For the line $\alpha = \frac{1}{\gamma+1}$ and line $\beta = 1, \alpha < \frac{1}{\gamma+1}$ only one package can be expected in practice. The reason being that the seller prefers pne package over a more complicated two-package scheme, though formally both strategies are optimal, giving the same profit.

Note that the regions with respect to active constraints follow a logical order. The Pareto efficiency region **E** occupies the central position. It relates to participation constraints $1 \to 0, 2 \to 0$ which are active at the solution. None of the incentive-compatibility constraints are active in region **E**. This implies that the demand parameters are such that the two demand triangles, shown within the region **E**, are more or less equal in areas.[12] In particular, the curve $\alpha\beta = 1$, which is approximately in the middle of this region, implies that the areas of two triangles are *exactly* the same. Since incentive-compatibility constraints are not active, no consumer envies each other's package. In this case, two different size packages can be sold for the same net tariffs. Either disutility, or zero utility from larger than the desired packages ensures this result. The more different the demand triangles, the less likely would be such an outcome.

Let us move clockwise around the point $(\alpha, \beta) = (1, 1)$. By decreasing α and β we reduce the area of the grey triangle and get the region **D** which is to the left and below the Paretian region **E.** Here the incentive-compatibility constraint $1 \to 2$, the first consumer envies the second, (i.e., temptation to switch to second consumer's package becomes stronger) becomes active and the participation constraint $1 \to 0$ remains active. So a solution satisfying constraints $1 \to 2 \to 0, 1 \to 0$ results. Farther to the left, in region **B,** the incentive-compatibility constraint $1 \to 2$ becomes so strong that the other constraints become redundant, giving the solution $1 \to 2 \to 0$. Similar logic works to the right of the Paretian region. In region **F,** the area of the white triangle becomes sufficiently less than the area of the grey triangle, and for the second consumer the temptation to switch to

[12] Gross tariff T_1 includes the area of the white triangle (net tariff) and the costs cx_1. Similarly, gross tariff T_2 includes the area of grey triangle and costs cx_2.

first consumer's package becomes stronger, making the incentive-compatibility constraint $2 \to 1$ active, while the participation constraint $2 \to 0$ remains active. In region **G** the incentive-compatibility constraint $2 \to 1$ becomes so strong that it is the only one that is active. When the areas of the demand triangles are very different, as it is the case in region **G,** only one type of consumers matters. Thus in essence, the result in region **G**, is very similar to the ordered demands case **A**.

B. Free-disposal case

In many situations free disposal may be more typical.[13] Although most results shown for the no-free disposal case above also hold under free disposal, for the sake of completeness we analyze this case separately. The formulation is modeled with piece-wise linear demands with a kink at the satiation point. At the kink (zero price) the inverse demand function becomes horizontal.

In the no-free disposal case we normalized marginal cost to zero, though our formulas do work for positive cost $c > 0$ as well. However, to formally incorporate free disposal we need to make some modifications by considering costs explicitly.

From intuitive reasoning it is clear that the initial non-normalized linear demand functions for the two consumers with free disposal are:

$p_2(x) = \max\{0, (\alpha+c) - x(\alpha+c)/\beta\}$. $p_1(x) = \max\{0, (1+c) - x(1+c)\}$.[14] The related valuation functions,revealed from demands, become piece-wise quadratic.

[13] The obvious example is when a consumer does not derive any disutility by not using the return-flight coupon of a round-trip airline ticket. Many other examples from real life can be given.

[14] These statements simply states that there cannot be any disutility. Compare it with no-free disposal function $p_2(x) = (\alpha+c) - x(\alpha+c)/\beta$ where marginal utility can be negative. Note that as before, $(\alpha+c)$ is the chocking price, that is, the price at which an infinitesimally small quantity will be bought.

$$V_1(x) = \left[\begin{array}{ll} (1+c)x - x^2/2 & \text{when } x \le 1+c \\ (1+c)^2/2 & \text{when } x \ge 1+c \end{array} \right.$$

$$V_2(x) = \left[\begin{array}{ll} (\alpha+c)x - \alpha x^2/(2\beta) & \text{when } x \le \beta\frac{\alpha+c}{\alpha} \\ \frac{(\alpha+c)^2}{2\alpha}\beta & \text{when } x \ge \beta\frac{\alpha+c}{\alpha}. \end{array} \right.$$

For the statement of the proposition 2, it is convenient to reformulate the package problem in terms of net valuation functions v_i and net revenues. This reformulation simply amounts to subtracting cx from the valuation functions and tariffs: $v_i(x) := V_i(x) - cx$, $t_i(x) := T_i(x) - cx$.

We state the main results for the free disposal case in terms of $v_i(x)$, $t_i(x)$ in the following proposition. As before, all formulas are in terms of net tariffs.

PROPOSITION 2**.** Under free disposal all possible optimal solutions to package pricing problem lie in ten regions depicted in Figure 2. Four regions **A**, **I1**, **F**, **G** as stated in Table 1 remain the same, and six regions **B**, **C**, **D**, **E**, **H**, **I2** are stated in Table 2.

[Table 2, Figure 2 here]

TABLE 2. New Solution Structures for Free Disposal Case

Regions	Active constraints	Package sizes
B: $1 \leq \beta \leq \min\{\frac{2\gamma\alpha+\alpha-\gamma\alpha^2-\alpha^2}{1-\alpha+\gamma\alpha}$,	$1 \to 2 \to 0$	$x_1 = 1$, $x_2 < EQ$
$\alpha\frac{\gamma+1+c\gamma+c}{c+\gamma\alpha+\alpha}\}$		
C: $\alpha_{ign}(\beta, c, \gamma) \leq \alpha$,	$1 \to 2 \to 0$	$x_1 = 1$, $x_2 > EQ$
$\alpha\frac{\gamma+1+c\gamma+c}{c+\gamma\alpha+\alpha} \leq \beta \leq \frac{\alpha(1+c)^2}{(c^2(1-(\frac{\gamma}{1+\gamma})^2)+2\alpha c+\alpha^2)}$		
I2: $\alpha \leq \alpha_{ign}(\gamma, \beta, c)$, $\frac{(1+c)^2}{4c} \geq \beta \geq 1$	$1 \to 2 = 0$	$x_1 = 1$, $x_2 = 0$
D: $\beta \geq \frac{2\gamma\alpha+\alpha-\gamma\alpha^2-\alpha^2}{1-\alpha+\gamma\alpha}$,	$1 \to 2 \to 0, 1 \to 0$	$x_1 = 1$, $x_2 > EQ$
$\max\{\frac{2-\beta}{1}, \frac{(1+c)^2-2c\beta}{\beta}\} \geq \alpha \geq \frac{\beta(1-c)}{2\beta-1-c}$		
H: $0 < \alpha \leq \max\{2-\beta, \frac{(1+c)^2-2c\beta}{\beta}\}$,	$1 \to 2 \to 0, 1 \to 0$	$x_1 = 1$, $x_2 > EQ$
except α, β in B, C, I2, D		
E: $\frac{\beta}{2\beta-1} \geq \alpha, \beta \geq 1$	$1 \to 0$, $2 \to 0$	$x_1 = 1$, $x_2 = \beta$
$\alpha \geq \max\{2-\beta, \frac{(1+c)^2-2c\beta}{\beta}\}$		

Here $\bar{\alpha}(\gamma, \beta) := \frac{\left(\gamma+\gamma\beta+2\beta-1-\sqrt{(\gamma^2-2\gamma^2\beta-2\gamma\beta-2\gamma+\gamma^2\beta^2+4\gamma\beta^2+4\beta^2-4\beta+1)}\right)}{2\gamma}$,

$\alpha_{ign}(\gamma, \beta, c) = \frac{-2\beta c+1+2c+c^2+\sqrt{(c^2+2c-4\beta c+1)}+\sqrt{(c^2+2c-4\beta c+1)c}}{2(1+\gamma)\beta}$,

All sizes and tariffs for Table 2 are the same as in Table 1, except for the two new regions **C** and **H** which are given below.

Region **C**: $x_1 = 1$, $x_2 = \frac{\beta(\gamma\alpha-1+\alpha)}{\gamma\alpha-\beta+\alpha}$, $t_{2C} = \frac{\beta(c+\alpha+\alpha\gamma)(\alpha+\alpha\gamma-c)}{2\alpha(1+\gamma)^2}$,

$t_{1C} = \frac{4c\beta\alpha\gamma+2c\beta\alpha\gamma^2+\beta\alpha^2+c^2\beta-4\alpha c\gamma-2\alpha c-\alpha c^2+\beta\alpha^2\gamma^2+2\beta\alpha^2\gamma+2c\beta\alpha+2c^2\beta\gamma-\alpha c^2\gamma^2-2\alpha c^2\gamma-2\alpha c\gamma^2}{2\alpha(1+\gamma)^2}$,

Region **H**: $x_1 = 1, \hat{x}_2 = \frac{1}{\alpha}\beta\left(c+\alpha-\sqrt{\left(\frac{\beta c^2+2\beta c\alpha+\beta\alpha^2-\alpha-2c\alpha-\alpha c^2}{\beta}\right)}\right)$, $t_1 = 1/2$,

$\hat{t}_{2C} = \frac{(c+\alpha-\sqrt{(\beta c^2+2\beta c\alpha+\beta\alpha^2-\alpha-2c\alpha-\alpha c^2)/\beta})*(\alpha\beta-\beta c+\sqrt{\beta}\sqrt{(\beta c^2+2\beta c\alpha+\beta\alpha^2-\alpha-2c\alpha-\alpha c^2)})}{2\alpha}$.

Proof. We give the outline of the proof in Appendix. Detailed derivations are available (see footnote 6).

We briefly explain Table 2 and Figure 2. Instead of seven regions under no-free disposal case we now have ten regions. Comparison with Fig.1 shows that regions **A**, **I1**, **F**, **G** remain unchanged. The reason is that since there are no oversized packages in these regions, they are not affected by free disposal. Pareto-efficient region **E** becomes somewhat smaller; its lower bound is slightly higher, for $\beta > 1 + c$.

The logical order of regions around the Paretian region **E** remains essentially the same as in the no-free disposal case discussed earlier. It can be added that the region **D** now is split into two regions, **D** and **H,** having the *same* active constraints $1 \to 2 \to 0, 1 \to 2$, but *different* formulas for package sizes. Now instead of one region, **B,** with constraint $1 \to 2 \to 0$, there are two regions, **B** and **C**. In some sense, region **B** can also be viewed as an extension of region **A** because it has the same active constraint $1 \to 2 \to 0$ and formulas. New region **I2** is just an extension of region **I1** emerging due to similar reasons.

Few observations about how changes in parameters affect the results are worth mentioning. For large marginal cost $c \geq 1$, Fig.2 converges to Fig.1 of the no-free disposal case. In another case, as $c \to 0$, the lower border of the efficiency region **E** converges to the same-net-tariff line ($\alpha = 1/\beta$), and the ignoring solution region **I2** takes up most of the space below this line, replacing parts of regions **E, H, C, D,** and **B**. In contrast, changing the ratio γ keeps the number of regions intact but changes their areas significantly. For instance, as expected, large γ decreases the area of region **I2** (ignoring solution), while for a smaller γ this region becomes relatively larger. These observations are also true for the no-free disposal case.

C. Discussion of the results

We discuss results in terms of net valuation function $v_i(x) = (V_i(x) - cx)$ which represents the consumer surplus had the consumer paid the price equal to marginal cost c for the quantity x. This net valuation can also be viewed as the potential profit from this consumer. In the case of zero marginal cost, it is exactly the consumer's valuation.

We discuss *Pareto efficiency and same-net-tariff effect* first. The main reason why efficiency occurs is due to the equality or approximate equality of net valuations for the efficient quantities x_i^* (i.e., quantities which satisfy the equality $MRS_i = MRT$) of the two types of consumers. For linear demands, such equality holds when $x_2^* = \beta$ and $x_1^* = 1$. In the parameters space the equality of net-valuations is represented by the same-net-tariff line, $v_2(x_2^*) = \alpha\beta/2 = 1/2 = v_1(x_1^*)$, that goes through the middle of the Paretian region **E**. A consumer decides which package to buy by comparing the net tariff with her net valuation. On the 'same-net-tariff line' the net tariffs are the same for two consumers and they also equal to their net valuations. As a result switching to another package is not desirable. Although two consumers have the choice between two different packages at the same (net) tariff, each consumer will self-select her most-preferred quantity x_i^* that results in Pareto-efficiency. This provides a plausible explanation for pricing different soft-drink bottles in Japan at the same price and foreign airlines charging one-way and round-trip the same fare. In both cases, marginal costs are small, so the gross tariffs do not seriously differ from the net tariffs.

Now consider situations below the same-tariff line $\alpha\beta/2 = 1/2$ where net valuations as well as net tariffs differ in the sense that $t_2 < t_1$ but not by too much. Here the seller must take into account the possibility of switching by the first consumer (who is buying a smaller package) to a larger package . The consumer will not switch under no-free disposal assumption, because although tariffs differ slightly, disutility may be significant. On the other hand, when the costs are positive, the total tariff becomes larger because of larger quantity. Hence the costs play the same role as disutility in discouraging the first consumer from buying a larger than the desired package. Thus for situations below the same-net-tariff line $\alpha\beta/2 = 1/2$, but $\alpha \geq 2 - \beta$, either disutility or large costs, or both, are enough for Pareto-efficiency. In the absence of significant costs or disutility, the region below the same-net-tariff line cannot be Pareto-efficient. As $c \to 0$, the lower border of the efficiency region coincides with the same-tariff line .

In contrast, the existence of Pareto efficiency for the region above the same-tariff line does not depend upon costs or free disposal, because switching from the most-preferred quantity to a smaller package at the same price can never be desirable. Thus, the region of Pareto efficiency is rather broad, always having a non-zero measure; therefore, it is *not* a pathological case.

At least at the Paretian solution the seller captures the whole surplus as profit for linear demands (it is likely to hold for non-linear demands also). Thus the first-best solution is attainable in spite of the fact that the consumer types are hidden. Asymmetric information does not result in any efficiency loss!

Oversizing effect is more likely to occur when the choking price of the second consumer is sufficiently lower than that of the first consumer, and at the same time the area under her demand curve is not too large (see regions **B, C, D, H**). When this is the case, the first consumer buys a smaller package but pays a larger net tariff. For this case, to prevent the first consumer from switching to a larger and cheaper (in terms of net tariff) package, the seller must choose between two alternatives. Either ignore the second consumer altogether or make her package oversized. When there is no-free disposal or large costs, and the demand of the first consumer is steeper (it holds for linear demands, but may also hold for non-linear demands), then oversizing the package becomes more profitable. Indeed, there exists a sufficiently large quantity x_2 at which the second consumer's net valuation of the package size still remains positive, but because of disutility or costs, the net valuation of the first consumer for x_2 is reduced to zero. This prevents switching by the first consumer to the oversized package x_2, and at the same time the seller earns some positive profit from the second consumer by not ignoring her. This can be explained by Figure 3. Here the lightly shaded area $\mathbf{S}_1$ is what the first consumer gains if she switches from $x_1 = 0$ to another package $\mathbf{S}_2 \geq \beta$ while the darker area $\mathbf{S}_2$ is what she looses. The equality $\mathbf{S}_1 = \mathbf{S}_2$ represents the optimality condition for the package sizes for the seller. Note that x_2 is high enough to prevent switching, but it is not too high. The upper part of $\mathbf{S}_1$ shows the gain in consumer surplus from switching if optimal size for the second package were to be $x_2 = \beta$.

Why do no-free disposal assumption and large costs have similar effects on optimal packages? Essentially, the difference between free-disposal and no-free disposal cases lies in whether the demand function switches to zero or not at the zero-price line. When costs are high enough then the optimal quantities are relatively smaller than the corresponding quantities representing the switching points. In this case, free-disposal or not makes no difference for solutions; thus high costs work the same way as no-free disposal. In contrast, free disposal with (almost) zero cost prevents oversizing. Because in this case it is more profitable to ignore one consumer altogether. Except for the oversizing issue, free disposal assumption essentially plays no major role.

It is important to stress that the demand parameters α, β affect the qualitative features of solutions in Table 1 quite differently than parameter γ.[15] Parameters α, β are the sole determinants of the basic tree-structure of the solution as well as oversizing or undersizing. In contrast, γ *do not influence basic solution structure or oversizing*, but it adds only additional arcs to the tree in some cases. It means that basic features of solution remain *independent* by the numbers of any consumer type and do not vary when population of a particular type increases or decreases. At the same time, the size of packages, and amount of tariffs, discounts and premiums are determined by all three parameters α, β, γ.

Our results mentioned above provide some plausible economic rationale or lack of it for some pricing strategies mentioned earlier. In particular, the US airlines industry, where a round-trip ticket is cheaper than one-way, does not represent a standard optimal solution, because free disposal is possible. Such price-quantity bundles can be rational on two grounds. It seems probable that very few consumers are buying one-way ticket and they are not well informed. Then in essence this pricing policy represents the "ignoring" solution (region **I1** or **I2** in Table 2). The seller, in principle, offers only one optimal package, namely round-trip; the one-way-high-price option is exercised only by

[15] More detailed derivation of comparative statics of solution w.r.t. γ see the Appendix.

the uninformed customers.

D. Nonlinear demands

We have derived our results using linear demands. The linearity assumption was useful in deriving exact formulas that explicitly characterize all regions. However, qualitatively all these results also hold even when the demands are nonlinear. Figure 4 demonstrates this fact. Consider the quantity-tariff curves $V_i(x) - t_i = 0$ related to net valuations (or zero costs) so that the maximal point is partially efficient. On the first consumer's (thick) indifference curve $V_1(x) - t_1 = x_1 - x_1^2/2 - t_1 = 0$, the maximum point is at $x_1 = 1$, $t_1 = 1/2$. Let us draw the second consumer's (dotted) indifference curve, $V_2(x) - t_2 = \alpha x_2 - \alpha x_2^2/(2\beta) - t_2$ w.r.t (α, β) so that its maximum is at $x_2 = \beta$, $t_2 = \alpha\beta/2$. When the maximum given by the dark circle is above $t = x - x^2/2$ and below $t = x^2/(2x-1)$ then we have the Pareto efficient solution depicted in region **E**. This is so because both consumers almost envy to 0. For this case the seller can behave as if he has full information and information asymmetry does not prevent in obtaining the first-best solution. However, when the maximum $(\beta,\ \alpha\beta/2)$ goes below the thick indifference curve, such efficient solution becomes impossible because the first consumer would switch to second consumer's package $(x_2 = \beta,\ t_2 = \alpha\beta/2)$ and would derive more utility. Similarly, when the maximum $(\beta,\ \alpha\beta/2)$ happens to be above the curve $t = x^2/(2x-1)$ one can see that reverse envy occurs: the first consumer's maximum $x_1 = 1$, $t_1 = 1/2$ lies below the second consumer's indifference curve and hence the second consumer would switch to first consumer's package $(1, 1/2)$ because it gives more surplus. Following this logic we can conclude that *any two net valuation functions $V_i(x)$ would result in Pareto efficient packages if and only if each* *maximum* (x_i, t_i) *of the normalized indifference curve* $V_i(x) - t_i = 0$ *is not below the other curve.*

Similarly four other parameter zones, separated by the solid line and labeled in accordance with Fig. 1, show the cases of undersizing, oversizing and ignoring solutions. Thus qualitatively our results can be generalized for non-linear demands.

Figure 4 helps us to demonstrate that the qualitative effects namely efficiency, oversizing and undersizing are quite general and do not depend on the linearity assumption of demands. Indeed, when one transforms the V_1 and V_2 curves in such a way that both maxima remain in similar position (above or below) with respect to other curve then the qualitative effects mentioned in the paper would remain intact.

IV. Quantity Discounts and Premiums

In this section we present the results related to quantity discount or premium.

Definition. For two packages i, j with total tariffs T_i, T_j and quantities x_i, x_j quantity *premium (discount)* is defined by relation $T_i/x_i > T_j/x_j$ $(T_i/x_i < T_j/x_j)$ for $x_i > x_j$.

Under the ordered demands case, Katz (1984), in a different setting than ours, shows the existence of quantity premia.

For linear demands we characterize the entire region of parameters yielding premium. Note that the region for premium also has a non-zero measure and the existence of a premium is not a pathological case. The region of premium happens to belong to ordered demands case only.

PROPOSITION 3. For two consumers with linear demands all optimal solutions to package pricing problem giving discounts or premiums are stated in Table 3.

Table 3.

Regions	Results
in A: $\alpha < \frac{\beta-\beta\gamma+\gamma}{1+\gamma}, 1 > \alpha > \frac{1}{1+\gamma}, \beta \leq 1, \gamma > 1$	Premiums
in A: $\alpha = \frac{\beta-\beta\gamma+\gamma}{1+\gamma}$	Same price
All other cases	Discounts
I1,I2: $0 < \alpha \leq \frac{1}{\gamma+1}, \beta \leq 1$ and $\beta = 1$	One package*

*For the line $\alpha = \frac{1}{\gamma+1}$ one package actually occurs when the seller prefers it to a more complicated two-package scheme, though formally both strategies are optimal giving the same profit.

Proof Available from authors (see footnote 6).

In Fig.1 and Fig.2 the regions for premium are the dark triangles shown within the region **A.** The horizontal line is $1/(1+\gamma)$. The sloped line is $\frac{\beta-\beta\gamma+\gamma}{1+\gamma}$. Note that this domain of parameters α, β yielding premia is non-empty if and only if γ is greater than 1 ($m_2 > m_1$). Furthermore, as γ increases the area of domain of parameters α, β yielding premium also increases. Therefore, for a very large γ, the probability of premium for ordered demands is almost 1/2 under a uniform distribution.

The intuition behind only discounts for the non-ordered case can be explained graphically in terms of demand triangles. For region **E** the average price (in terms of net tariffs, which gives the same discount index as gross tariffs, see proof in the appendix) is the average height of the triangle. Consequently, the lower triangle must have a smaller average price, yielding discount. Similarly for all lower regions **D**, **B**, **C**, and **H** the average price is even less than the average height of a triangle, yielding discount. Similar arguments apply to regions **G**, and **F**. Note that for x_1 the average price is the average height of a trapezoid which is more than 1/2, while for x_2, because $\alpha < 1$, the average height of a triangle is less than 1/2.

V. Conclusions

For two types of consumers with linear-demands we have presented a complete analysis for both non-ordered, and ordered demands. Our analysis provides several new insights. First, in spite of asymmetric information, Pareto efficiency can exist under monopoly with hidden types of agents and socially optimal output could be produced . This usually never is the case with most other nonlinear pricing strategies and normally it is also not the case when demands do not cross. Second, the consumption level of one type of consumer can be inefficiently high due to oversized package. Third, under non-ordered demands consumers always enjoy quantity discounts. Quantity premia can occur only when the demands are ordered and low-demand consumers are more numerous. Finally, the qualitative effects also hold for non-linear demands.

Appendix

Outline of the proof for Proposition 1.

We provide the basic steps in deriving optimal solutions for the regions in Table 1. From a general 'tree-theorem' established in our other paper (2001), it follows that it is sufficient to study only three logically possible systems of active constraints namely $[1 \to 2 \to 0]$, $[2 \to 0, 1 \to 0]$ and $[2 \to 1 \to 0]$, including also their subcases $[2 \to 1 \to 0, 2 \to 0]$, $[1 \to 0, 1 \to 2 \to 0]$. For each system we first derive expressions for x_i, t_i from the constraints that are supposed to be active. By the same 'tree-theorem' the non-envied consumer must consume Pareto-efficient quantity, i.e., $x_1 = 1$ for $[1 \to 2 \to 0]$, and $x_2 = \beta$ for $[2 \to 1 \to 0]$. Then substitute these expressions into the remaining (non-active) constraints and into the objective function. After this substitution there remains only one variable to optimize, namely x_1 for $[2 \to 1 \to 0]$, and x_2 for $[2 \to 1 \to 0]$. The non-active constraints (or, more precisely, the constraints not included into the system) can all be expressed in terms of admissible intervals for the optimizing variable.

For example, for $[1 \to 2 \to 0]$ the objective function is $\pi = \gamma t_2 + t_1 = (\gamma x_2(\alpha - x_2\alpha/(2\beta)) + \frac{\beta - 2x_2\beta + x_2^2\beta + 2x_2\alpha\beta - x_2^2\alpha}{2\beta}$ and the admissible intervals are $[0 \leq x_2 \leq \frac{\alpha+\beta-2\alpha\beta}{\beta-\alpha}$ or $1 \leq x_2 \leq \frac{2\beta(1-\alpha)}{\beta-\alpha}]$. This domain can be shown to be non-empty only for parameters $(\alpha \leq \beta/(2\beta - 1))$ below regions **F**, and **G**. By maximizing the objective function on this domain we obtain either unconstrained solutions related to regions **A** and **B**, or border solutions related to some not-in-system constraint becoming active. For ordered demands $(\beta \leq 1)$, it turns out that this new active constraint can only be the positivity constraint (left border of the domain) resulting in region **I1** (ignoring solution). For the non-ordered demands ($\beta > 1$), it turns out that the not-in-system constraint becoming active can only be the constraint $[1 \to 0]$ (the right border of the domain). This results in region **D** and solutions of the type $[1 \to 0,\, 2 \to 1 \to 0]$.

For $[2 \to 1 \to 0]$ the steps are exactly the same, giving regions **F, G**.

For $[1 \to 0,\, 2 \to 0]$ the quantities x_1, x_2 must be Pareto-efficient because no consumer envies other consumer's package: $x_1 = 1$, $x_2 = \beta$. Substituting these values into the incentive-compatibility constraints we obtain the region **E** of parameters α, β and make sure that incentive-compatibility constraints are satisfied. In this region other bundles may also be admissible, as shown in analyzing systems $[1 \to 2 \to 0]$, $[2 \to 1 \to 0]$. However, because Paretian solution is the first-best solution, the other possible bundles need not be considered in this region. ||

Outline of the proof of Proposition 2.

The proof is long and tedious requiring symbolic-algebra software to handle high-degree polynomials. Very briefly the idea is as follows.

The initial optimization problem has two threshold functions $V_1(x_1), V_2(x_2)$ or switches applicable to all constraints. Both functions switch from a parabola to a line. First one at point $x_1 = 1 + c$, while the second switches at $x_2 = \beta\frac{\alpha+c}{\alpha}$. By normalizing the functions to net valuations and using other transformations, the problem is reduced to some other equivalent optimization problem having only one switch point $x_1 = 1 + c$. Then optima can be studied separately both to the left and to the right from this switch point for the same three systems of constraints as in the no-free disposal case. Finally, solutions to the left and to the right from $x_1 = 1 + c$ can be compared to find the highest profit solution. However, direct comparison results in excessively tedious polynomials. To compare different solutions, some indirect ways based on the concavity or convexity of the objective function on the admissible intervals (revealed from non-active constraints) are used.

Outline of the proof of Proposition 3 (for premium only).

We can focus on net revenues t_i only because actual tariffs $T_i(x) = t_i(x) + cx_i$, have the same discounts (or premiums) index: $\Delta_{[region]}(\alpha, \beta) = t_2/x_2 - t_1/x_1 < 0 \Leftrightarrow T_2/x_2 - T_1/x_1 < 0$. This index means discount when $x_2 \geq x_1$, and it means premium in the opposite case. To obtain parameters yielding either discount or premium, we just take formulas of optimal quantity-tariff packages from Tables 1 and 2 and directly substitute them into this discounts/premiums index. We do this for all described regions of parameters α, β, for both the no-free disposal and free-disposal cases.

Premium can exist only in region **A** $[1 > \alpha > \frac{1}{\gamma+1}, 1 \geq \beta\ ]$.

From Table 1 we have,

$x_2 = \frac{\gamma\alpha - 1 + \alpha}{\gamma\alpha - \beta + \alpha}\beta$ and $x_1 = 1$ so $x_2 \leq x_1$. The tariffs are

$t_2 = x_2(\alpha - x_2\alpha/(2\beta)) = \beta\alpha\frac{(\gamma\alpha - 1 + \alpha)(\gamma\alpha - 2\beta + \alpha + 1)}{2(\gamma\alpha - \beta + \alpha)^2}$ and

$t_1 = \frac{1}{2\beta}(\beta - 2x_2\beta + x_2^2\beta + 2x_2\alpha\beta - x_2^2\alpha) =$

$= \alpha\frac{\beta\alpha^2 + 2\beta\gamma\alpha^2 + \beta\gamma^2\alpha^2 + \beta^2\gamma^2\alpha - 4\gamma\alpha\beta - 2\beta\gamma^2\alpha - \beta + 2\gamma\alpha + \gamma^2\alpha + \alpha - 2\beta\alpha + 2\beta^2 - \beta^2\alpha}{2(\gamma\alpha - \beta + \alpha)^2}.$

The difference between the average price that determines the discount or premium can be written as

$\Delta_{[A]}(\alpha, \beta) = \frac{t_2}{x_2} - \frac{t_1}{x_1} =$

$= \frac{1}{2}\alpha^2\frac{-(-\gamma^2\alpha - \beta\gamma - 2\gamma\alpha + \beta - \alpha + \gamma + \beta\alpha + 2\gamma\alpha\beta + \beta\gamma^2\alpha + \beta^2\gamma^2 - 2\beta\gamma^2 + \gamma^2 - \beta^2)}{(\gamma\alpha - \beta + \alpha)^2}.$

The numerator is negative when $(\alpha(1+\gamma)-(\beta-\beta\gamma+\gamma))<0 \Leftrightarrow \alpha<\frac{\beta-\beta\gamma+\gamma}{1+\gamma}$.

We are studying interval $\alpha>\frac{1}{1+\gamma}$. These two inequalities are consistent iff $1<\beta-\beta\gamma+\gamma$ giving the solution $\{\operatorname{signum}(\gamma-1)\,\beta<\operatorname{signum}(\gamma-1)\}$.

Note that when $(\gamma-1)<0$ and $\beta\leq 1$ we have a contradiction. Thus, for the case $[\gamma<1,\alpha>\frac{1}{1+\gamma}]$ we have $\frac{t_2}{x_2}-\frac{t_1}{x_1}>0$, that means quantity *discount* everywhere. In the opposite case when $\gamma>1$ and $\beta\leq 1$, the inequality gives us the region for *premium* as $[\alpha<\frac{\beta-\beta\gamma+\gamma}{1+\gamma},\alpha>\frac{1}{1+\gamma}]$. In the remaining part of region **A** we have only discount.

All other regions of parameters can be analyzed the same way, or one can use the geometric reasoning given above. It turns out that, except for region **A**, all other regions have discount. Only for some boundary cases average price paid by the two consumers is the same (i.e., zero discount).

References

Gerstner, Eitan., and Hess, James D. 1987. Why do hot dogs come in packs of 10 and buns in 8s or 12s? A demand side investigation. *Journal of Business* 60, no. 4: 491-517.

Katz, Michael L. 1983. Non-Uniform pricing, output and welfare under monopoly. *Review of Economic Studies*, 50, no. 1: 37-56.

Katz, Michael L. 1984. Nonuniform pricing with unobservable numbers of purchases. *Review of Economic Studies*, 51, no. 3: 461-70.

Nahata, Babu., Kokovin, Serguei., and Zhelobodko, Evgeny. 2001. Self-selection, Non-ordered Valuations and Pareto Efficiency.*www.math.nsc.ru/˜mathecon/kokovin.html*

Tirole, Jean. 1989. *Theory of Industrial Organization.* Cambridge, MA: MIT Press.

Wilson, Robert B. 1993. *Nonlinear Pricing.* Oxford: Oxford University Press.

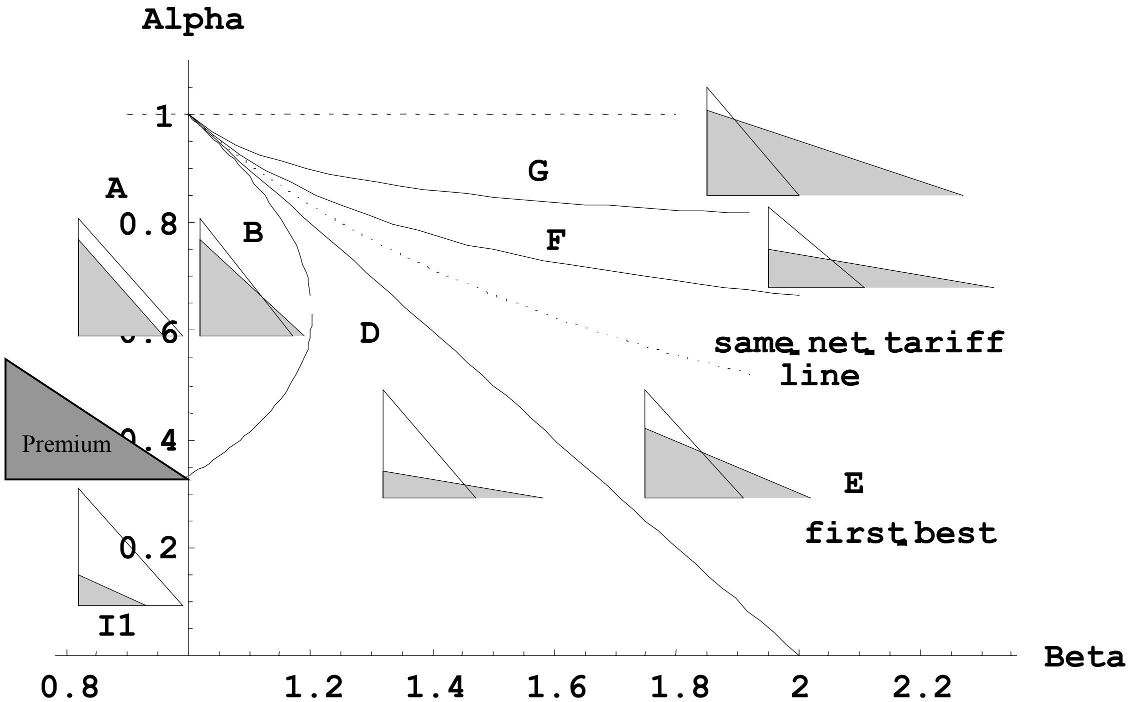

FIG.1.Regions of different solutions with large costs or *without* free disposal, $\gamma = 2$, $c > 1$.

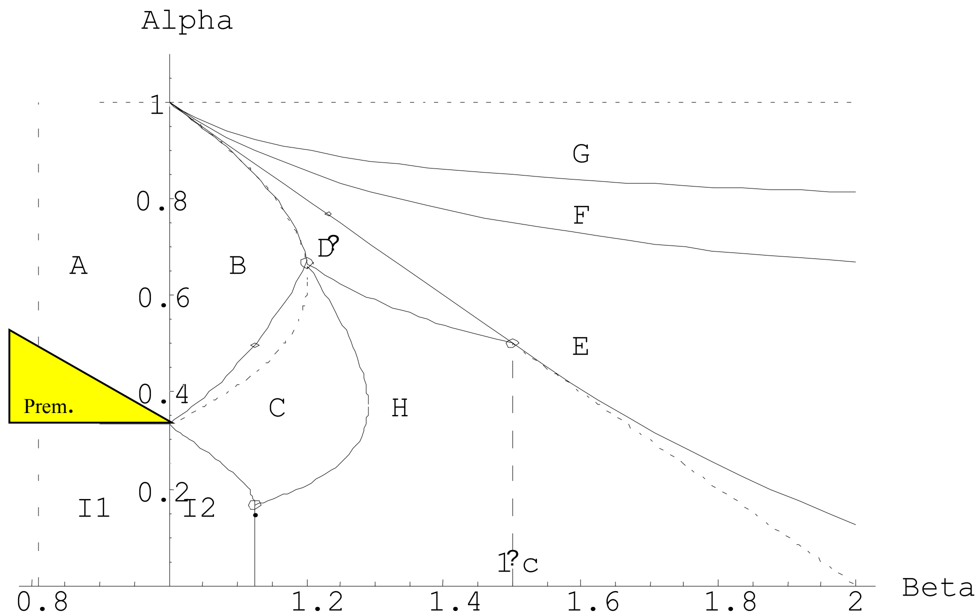

FIG.2. Regions of different solutions *with* free disposal and small costs $\gamma = 2$, $c = 0.5$.

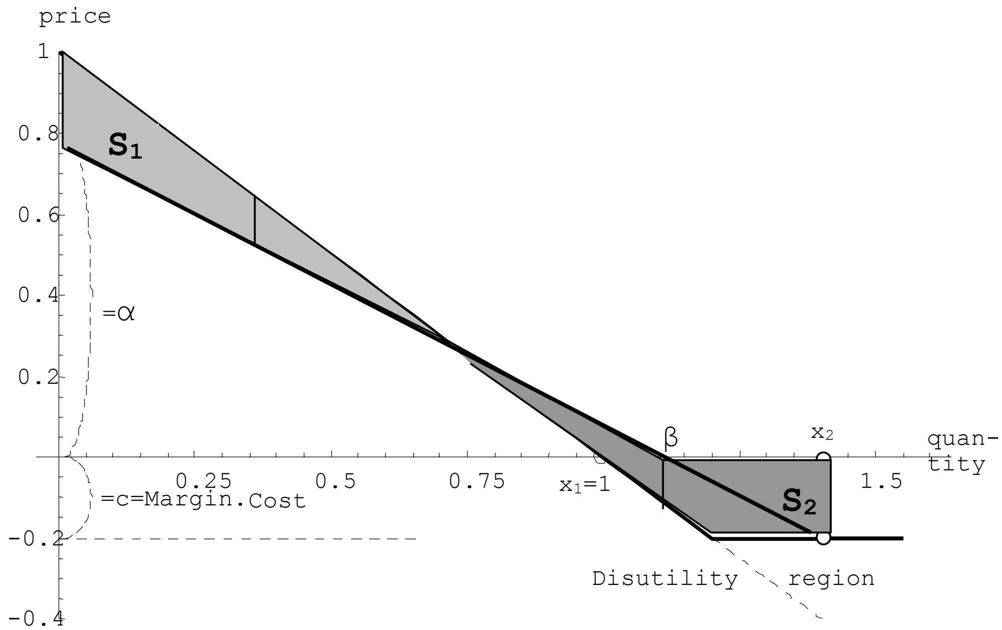

FIG 3. Oversizing effect with non-ordered (crossing) demands, with free disposal (dashed-lines extensions of demand curves), without free disposal (kinked lines).

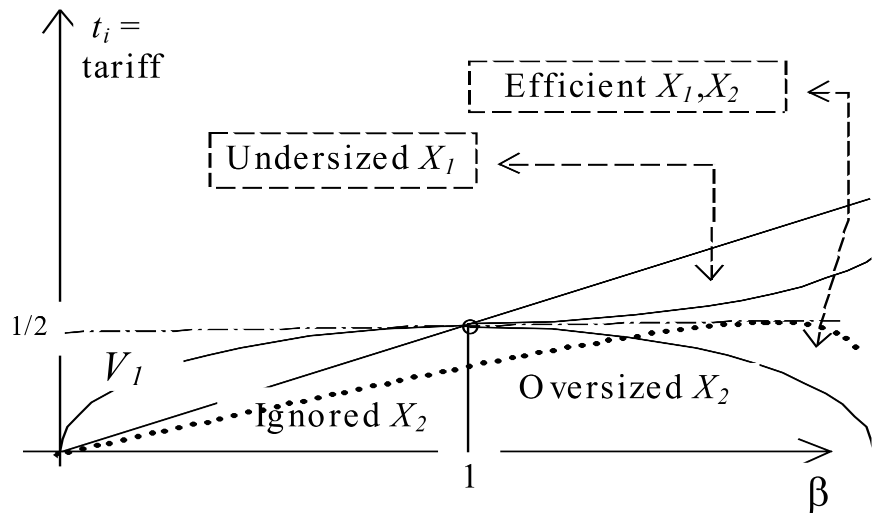

FIG 4. Efficient-quantity/tariff zones yielding different outcomes.