Lags, Convexity and the Investment-Uncertainty Relationship

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March 2005

Abstract

The effect that investment lags has on the uncertainty-investment relationship is studied by modifying the Bar-Ilan and Strange (1996) model in a manner that enables analytical solution. It turns out that: (i) If the time lag is sufficiently small, uncertainty affects investment negatively; (ii) A sufficiently large time lag engenders an inverse u-shape relationship between the degree of uncertainty and the profit level that triggers investment; (iii) When such an inverse u-shape exists, the higher is the length of the time lag (or the degree of profit convexity) the wider is the range of a positive uncertainty-investment relationship.

Keywords: Investment, Uncertainty, Time to build

JEL Classification: D81

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Introduction

Hartman (1972) has shown that output price uncertainty exerts a positive effect on investment when the profit is a convex function of the output price. Bernanke (1983), McDonald and Siegel (1986), Dixit (1989) and others have shown that despite this convexity uncertainty affects investment negatively if the firm can choose the investment timing optimally.[1,2] In the most recent swing of this pendulum Bar-Ilan and Strange (1996) have shown that introducing time-to-build (in the form of a time lag between the moment of investment and the moment in which profits starts to accrue) into Dixit's (1989) model enables a positive effect of uncertainty on investment.

The explanation for the result of Bar-Ilan and Strange (1996) is that the introduction of this time lag pulls the rug from under Bernanke's Bad News Principle that underlies the negative effect uncertainty has on investment. According to this principle "Good News" regarding the investment are irrelevant to the investment timing. Thus, increased uncertainty, which makes the "Bad News" worse and the "Good News" better, affects investment timing only via the worsening of the "Bad News". The reason why "Good News" is irrelevant for the timing of investment is that in the absence of time-to-build the proceeds attached to them can be collected the minute they are realized and therefore are collected both in the case of early investment and in the case when investment is postponed until the arrival of these "Good News". Introducing time-to-build makes it impossible for the firm to receive

[1] For detailed surveys of this literature see Pindyck (1991) or Dixit and Pindyck (1994).

[2] Caballero (1991) have shown a positive uncertainty-investment relationship in a two-period model where delaying investment is possible. His results, however, rely strongly on the assumption that the firm operates for a finite number of periods known in advance.

the proceeds of “Good News” the minute they are realized and therefore restores their relevancy to the investment timing decision.

Note that introducing an investment lag to the model is not sufficient for uncertainty to have a positive effect on investment unless the profit is a convex function of the output price. In Bar-Ilan and Strange (1996), as in Dixit (1989), the option the firm has to abandon the investment when the output price is too low generates this convexity.

Since Bar-Ilan and Strange (1996) can only solve their model numerically, their analysis is limited to showing that for some parameter values uncertainty affects investment positively. The purpose of this paper is to broaden our understanding of how time-to-build affects the uncertainty-investment relationship beyond that. In order to do that I use here a version of the Bar-Ilan and Strange (1996) model modified as follows. First, the exit option is deleted in order to enable an analytical solution. Second, in order to restore the convexity of the profit function in the output price, their assumption that the production process generates a flow of fixed quantity is replaced by the weaker assumption that production is done under a decreasing marginal labor productivity function where the labor input, and therefore output, are flexible. With these modifications, the model now yields the following results:

- For a sufficiently small time lag, uncertainty affects investment negatively.
- For a sufficiently large time lag there is an inverse u-shape relationship between the degree of uncertainty and the profit level that triggers investment.
- When such an inverse u-shape exists, the longer the time lag (or the degree of profit convexity) the wider the range of positive uncertainty-investment relationship.

The following Section presents the model and its analysis. Some of the more technical proofs where relegated to an appendix.

2. The Model

Time in the model is continuous. Consider an infinitely lived, risk-neutral firm that can enter a project in which it produces output according to:

$$(1) \qquad Q_t = AL_t^{\alpha}$$

Where Q_t and L_t are, respectively, the instantaneous output and labor input of the production process and A and α are constants satisfying $A > 0$ and $0 \leq \alpha < 1$.[3] There is no cost for adjusting the amount of labor the firm employs.

By standard optimization, if the firm enters the project its instantaneous profit (π_t), given the output price (P_t) and the labor wage (x), satisfy:

$$(2) \qquad \pi_t = CP_t^{\gamma}$$

Where C and γ are constants defined by:

$$\gamma \equiv \frac{1}{1-\alpha} \qquad\qquad C \equiv \left(\frac{A}{w^{\alpha}}\right)^{\gamma}\left(\alpha^{\alpha\gamma} - \alpha^{\gamma}\right).$$

[3]The case analyzed by Bar-Ilan and Strange (1996) and Dixit (1989) is that of fixed quantity, which corresponds to $\alpha = 0$.

Note that $\gamma \geq 1$ since $0 \leq \alpha < 1$. Also note that $\gamma'(\alpha) > 0$ implying that the higher α the higher the convexity of π in P. Finally note that $C \geq 0$ since $\gamma \geq 1$ and $0 \leq \alpha < 1$.

To enter the project the firm must incur the cost $k > 0$. A lag of length $h \geq 0$ exists between the time in which the firm pays the entry cost and the time in which the project becomes active, where the term "active" means that profits start to accrue. The firm's discount rate is denoted by ρ. After the firm enters the project it cannot exit it. ρ, k and x are constants. The uncertainty arises from the output price, P_t, which evolves exogenously over time according to the rule:

$$(3) \qquad dP_t = \mu P_t dt + \sigma P_t dz,$$

where $\sigma > 0$ and dz is the increment of a standard Wiener process, uncorrelated across time and at any one instant satisfying $E(dz) = 0$ and $E(dz^2) = dt$. This means that P_t is a geometric Brownian Motion. By Itô's lemma, π_t is a geometric Brownian motion too, with the constant parameters:

$$(4) \qquad \mu_\pi = \gamma\,[\mu + \tfrac{1}{2}\,(\gamma - 1)\sigma^2] \qquad\qquad \sigma_\pi = \gamma\sigma$$

Convergence of the firm's expected net present value requires the assumption $\mu_\pi < \rho$, which means that σ must satisfy:

$$(5) \qquad \sigma < \bar{\sigma} \equiv \sqrt{2\frac{\rho - \gamma\mu}{\gamma(\gamma - 1)}}$$

Thus constructed, the model closely resembles the model solved by Bar-Ilan and Strange (1996). The three differences between these models are: (i) Their model contains an option to exit the project by paying a fixed exit cost denoted by l. The no exit case analyzed here corresponds to their analysis of the specific case where l approaches infinity; (ii) Their model contains a flow of a production cost with constant magnitude that they denote as w. The model analyzed here corresponds with the specific case in their model where $w = 0$; (iii) In their model the instantaneous output is assumed constant at unity and therefore the instantaneous profit is $P_t - w$. Assuming $w = 0$ renders the profit flow in both models (P_t in theirs and π_t here) a geometric Brownian Motion. Since the only property of P_t relevant to their solution procedure is its being a geometric Brownian Motion, it is possible to use their analysis in pages 612 – 615 by replacing P, μ and σ by π, μ_π and σ_π, respectively, and assuming that $w = 0$ and that l approaches infinity. The results are that the optimal policy is to enter once the profit process, π_t, reaches a certain threshold level denoted by π_H^h and given by:[4]

(6) $$\pi_H^h = \frac{\beta}{\beta - 1}(\rho - \mu_\pi)e^{-\mu_\pi h}k$$

where β is the single positive root of the quadratic:

(7) $$\tfrac{1}{2}\sigma_\pi^2\beta^2 + (\mu_\pi - \tfrac{1}{2}\sigma_\pi^2)\beta - \rho = 0$$

[4] Equation (6) here is in fact equation (12) in Bar-Ilan and Strange (1996).

Applying the values of 0 and 1 in this quadratic reveals that one of its roots is negative and the other, β, exceeds unity. For brevity of notations, β' and β'' denote the first and second derivatives of β with respect to σ^2.

2.1 With no time lag

In this section the case of no time lag between paying the entry cost and the start of production exists, i.e., $h = 0$, is analyzed. Based on (6) the entry threshold in that case is:

$$\pi_H = \frac{\beta}{\beta - 1}(\rho - \mu_\pi)k \tag{8}$$

Differentiating with respect to σ^2 yields:

$$\begin{aligned} \frac{\partial \pi_H}{\partial \sigma^2} &= \left[\frac{-1}{(\beta-1)^2}\beta'(\rho - \mu_\pi) - \frac{\beta}{\beta-1}\frac{\partial \mu_\pi}{\partial \sigma^2}\right]k \\ &= \frac{\beta}{\beta-1}(\rho - \mu_\pi)\left[\frac{-\beta'}{(\beta-1)\beta} - \frac{\gamma(\gamma-1)}{2(\rho-\mu_\pi)}\right]k = \pi_H \cdot f(\mu, \sigma^2, \rho, \gamma) \end{aligned} \tag{9}$$

where:

$$f(\mu, \sigma^2, \rho, \gamma) \equiv \frac{-\beta'}{(\beta-1)\beta} - \frac{\gamma(\gamma-1)}{2(\rho-\mu_\pi)} \tag{10}$$

The following proposition presents some properties of $f(\mu, \sigma^2, \rho, \gamma)$.

Proposition 1: $f(\mu, \sigma^2, \rho, \gamma)$ satisfies:

(a) $$\frac{\partial f(\mu, \sigma^2, \rho, \gamma)}{\partial \sigma^2} < 0$$

(b) $$\lim_{\sigma \to 0} f(\mu, \sigma^2, \rho, \gamma) = \begin{cases} \dfrac{\gamma}{2\mu} & \text{If } \mu > 0 \\ -\dfrac{\gamma(\rho - \mu)}{2\mu(\rho - 2\mu)} & \text{if } \mu \leq 0 \end{cases} \equiv f^*(\mu, \rho, \gamma) > 0$$

(c) $$\lim_{\sigma \to \bar{\sigma}} f(\mu, \sigma^2, \rho, \gamma) = \frac{\gamma^3(\gamma - 1)(\rho - \mu)}{2(2\gamma\rho - \gamma^2\mu - \rho)^2} \equiv f^{**}(\mu, \rho, \gamma) \geq 0$$

(d) $$\frac{\partial f(\mu, \sigma^2, \rho, \gamma)}{\partial \gamma} < 0$$

Proof: in the appendix.

Figure 1 below depicts $f(\mu, \sigma^2, \rho, \gamma)$ based on *Proposition 1*.

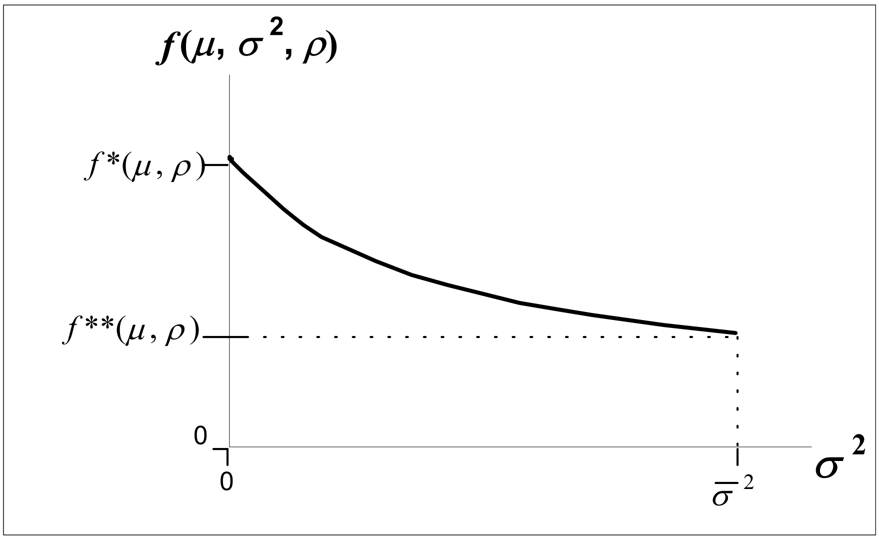

Figure 1: $\frac{\partial \pi_H}{\partial \sigma^2} \equiv f(\mu, \sigma^2, \rho, \gamma)$ as a function of σ^2.

The immediate corollary from proposition 1 is that $\frac{\partial \pi_H}{\partial \sigma^2} > 0$ throughout the relevant range. Thus, in the absence of a time lag uncertainty has a negative effect on entry, despite the "a-la Hartman" convexity of the profit function in the stochastic price.

2.2 With a time lag

Returning to the case of a time lag and applying (8) in (6) enables presenting the entry threshold as:

$$\text{(11)} \qquad \pi_H^h = e^{-\gamma\left(\mu + \frac{(\gamma-1)\sigma^2}{2}\right)h} \pi_H$$

where π_H is the value of π_H^h when $h = 0$ given by (8). Differentiating with respect to σ^2 yields:

$$\text{(12)} \qquad \frac{\partial \pi_H^h}{\partial \sigma^2} = e^{-\mu_\pi h} \frac{\partial \pi_H}{\partial \sigma^2} - h \frac{\gamma(\gamma - 1)}{2} e^{-\mu_\pi h} \pi_H$$

$$= \pi_H e^{-\mu_\pi h} \left[f\left(\mu, \sigma^2, \rho, \gamma\right) - h \frac{\gamma(\gamma - 1)}{2} \right]$$

where the second equality follows from (9). The following two propositions show that $\frac{\partial \pi_H^h}{\partial \sigma^2}$ might be negative, provided that π is convex in P, i.e., that $\gamma > 1$. They also show that the range of values of σ^2 in which $\frac{\partial \pi_H^h}{\partial \sigma^2} < 0$ is an increasing function of h and γ, i.e., that a positive relation between entry and uncertainty becomes more likely as the investment lag or the degree of convexity rise.

Proposition 2: If $\gamma = 1$ then $\frac{\partial \pi_H^h}{\partial \sigma^2} > 0$.

Proof: If $\gamma = 1$ then, by part (c) of *Proposition 1*, $\underset{\sigma \to \overline{\sigma}}{Lim}\, f\left(\mu, \sigma^2, \rho, \gamma\right) = 0$. Therefore, by part (a) of *Proposition 1*, $f(\mu, \sigma, \rho, \gamma) > 0$ for all σ in the relevant range. Thus, by (12), $\frac{\partial \pi_H^h}{\partial \sigma^2} > 0$. □

For brevity, the following proposition makes use of the definitions:

(13) $$h^{**}(\mu, \rho, \gamma) \equiv 2\frac{f^{**}(\mu, \rho, \gamma)}{\gamma(\gamma - 1)} \qquad h^{*}(\mu, \rho, \gamma) \equiv 2\frac{f^{*}(\mu, \rho, \gamma)}{\gamma(\gamma - 1)}.$$

Proposition 3: If $\gamma > 1$ then:

(a) If $h \leq h^{**}(\mu, \rho, \gamma)$ then π_H^h is increasing in σ^2 throughout the relevant range.

(b) If $h^{**}(\mu, \rho, \gamma) < h < h^{*}(\mu, \rho, \gamma)$ then there is a single value of σ, denoted by $\sigma^*(\mu, \rho, \gamma, h)$, which brings π_H^h to a maximum.

(c) If $h > h^{*}(\mu, \rho, \gamma)$ then π_H^h is decreasing in σ^2 in all the relevant range.

(d) $\sigma^*(\mu, \rho, \gamma, h)$ is decreasing in h and in γ.

Proof: follows directly from proposition 1. □

Note that (d) implies that $\sigma^*(\mu, \rho, \gamma, h) = 0$ when $h > h^*(\mu, \rho, \gamma)$. Figure 2 below shows σ^* as a function of h.

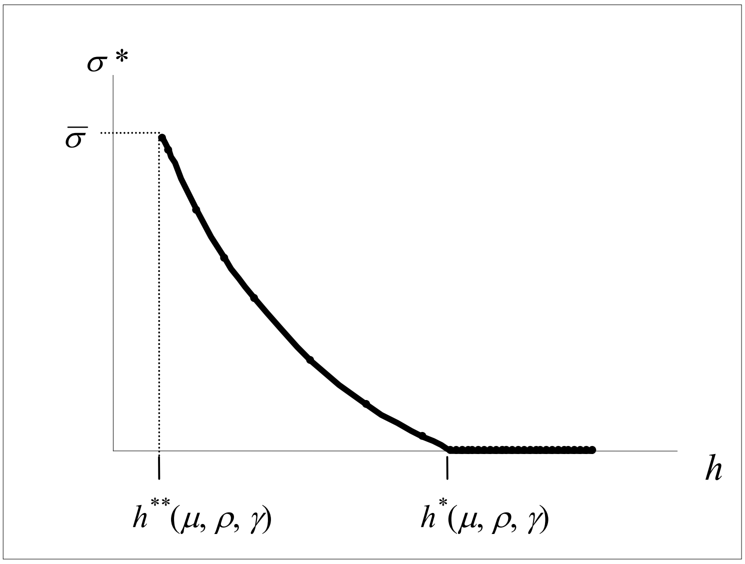

Figure 2: σ^* as a function of h. The higher the investment lag, the lower the level of σ^2 from which π_H decreases in σ^2. The larger the convexity of the profit function in output prices the more to the left this function.

Concluding Remarks

In this paper I have studied the effect of time-to-build on the uncertainty-investment relationship in a model when investment can be delayed. It was shown that if the time lag between the moment of investment and the moment when profits start to accrue is sufficiently small then uncertainty affects investment negatively, as the related literature usually shows. However, when this time lag is sufficiently long, an inverse u-shape relationship exits between uncertainty and investment.

A thorough analytical understanding of the effect of time-to-build on the uncertainty-investment relationship should be helpful to future empirical work. As this paper has shown, empirical models of investment under uncertainty should not analyze the effect of time-to-build in separation from other factors, but rather in interaction with the qualitative nature of the uncertainty-investment relationship.

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Appendix

A. Properties of β

Applying (4) in (7) yields that β is the positive root of:

(a.1) $$\tfrac{1}{2}\gamma^2\sigma^2\beta^2 + (\gamma\mu - \tfrac{1}{2}\gamma\sigma^2)\beta - \rho = 0$$

Lemma 1: β satisfies the following:

(a) $\beta' < 0$

(b) $\beta'' > 0$

(c) $\lim\limits_{\sigma\to\bar{\sigma}} \beta = 1$

(d) $$\lim_{\sigma\to 0} \beta = \begin{cases} \dfrac{\rho}{\gamma\mu} & if \;\; \mu > 0 \\ \infty & f \;\; \mu \le 0 \end{cases}$$

(e) $$\lim_{\sigma\to\bar{\sigma}} \beta' = -\frac{\gamma(\gamma-1)^2}{2\left(2\gamma\rho - \rho - \gamma^2\mu\right)}$$

(f) $$\lim_{\sigma\to 0} \beta' = \begin{cases} -\dfrac{\rho(\rho-\mu)}{2\gamma\mu^3} & if \;\; \mu > 0 \\ -\infty & f \;\; \mu \le 0 \end{cases}$$

Note that (a) and (c) imply that if $\mu > 0$ then $\beta < \dfrac{\rho}{\gamma\mu}$.

Proof: The proof of (c) and (d) follows directly from (a.1). Implicit derivation of (a.1) yields:

$$\text{(a.2)} \qquad \beta\,' = -\frac{\frac{1}{2}\gamma^2\beta^2 - \frac{1}{2}\gamma\beta}{\gamma^2\sigma^2\beta + \gamma\mu - \frac{1}{2}\gamma\sigma^2} = -\tfrac{1}{2}\gamma\beta^2\frac{\gamma\beta - 1}{2\rho - \gamma\left(\mu - \frac{1}{2}\sigma^2\right)\beta} < 0$$

where the second equality follows from (a.1). $\beta\,' < 0$ follows from the first equality for the case of $2\mu \geq \sigma^2$ and from second equality for the case of $2\mu < \sigma^2$, taking into account in both cases that $\beta > 1$. This proves (a).

Based on (a.2):

$$\text{(a.3)} \qquad \beta\,'' = -\frac{(2\gamma\beta - 1)\beta'\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right) - \beta(\gamma\beta - 1)\left[\gamma\beta + \gamma\sigma^2\beta' - \frac{1}{2}\right]}{2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)^2}$$

$$= -\beta'\frac{(2\gamma\beta - 1)\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right) - \beta(\gamma\beta - 1)\gamma\sigma^2 + 2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)\left(\gamma\beta - \frac{1}{2}\right)}{2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)^2}$$

$$= -\beta'\frac{6\rho - 2\gamma\beta\mu - 2\mu + \sigma^2}{2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)^2}$$

where the second equality follows from (a.2) and the third equality follows from tedious, yet straightforward algebra. If $\mu \leq 0$ then all terms in numerator are positive and therefore $\beta\,'' > 0$. If $\mu > 0$ then the numerator depends negatively on β and therefore, since $\beta < \frac{\rho}{\gamma\mu}$, in that case:

$$\text{(a.4)} \qquad \beta\,'' > -\beta'\frac{6r - 2\gamma\frac{\rho}{\gamma\mu}\mu - 2\mu + \sigma^2}{2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)^2} = -2\beta'\frac{4\rho - 2\mu + \sigma^2}{2\left(\gamma\sigma^2\beta + \mu - \frac{1}{2}\sigma^2\right)^2} > 0$$

where the inequality follows from $\rho > \mu_\pi$ and from $\beta' < 0$. This proves (b). The proof of (e) follows directly from (a.2) and from (c). The proof of (f) follows from the second equality of (a.2) together with (d). □

B. Proof of proposition 1

Based on (10):

(a.5) $$\frac{\partial f(\mu,\sigma^2,\rho,\gamma)}{\partial\sigma^2} = -\frac{\beta''(\beta-1)\beta-\beta'(2\beta-1)}{(\beta-1)^2\beta^2} - \frac{\gamma(\gamma-1)}{2(\rho-\mu_\pi)^2} < 0$$

where the inequality follows from parts (a) and (b) of lemma 1 together with the results that $\beta > 1$ and $\gamma \geq 1$. Thus (a) is proved. The proof of (b) for the case of $\mu > 0$ stems from applying parts (a) and (d) of lemma 1 in (10). For the case of $\mu \leq 0$ the proof of (b) requires applying (a.2) in (10) and then using L'Hôpital's rule. In order to prove (c) it is useful to present (10) as:

(a.6) $$f(\mu, \sigma^2, \rho, \gamma) \equiv \frac{\frac{-\beta'}{\beta} - \frac{\gamma(\gamma-1)}{2}\frac{\beta-1}{\rho-\mu_\pi}}{\beta-1}$$

As σ approaches $\bar{\sigma}$ both numerator and denominator of $\frac{\beta-1}{\rho-\mu_\pi}$ approach 0 since β approaches 1. Using L'Hôpital's rule yields that in that case both the numerator and the denominator of $f(\mu, \sigma^2, \rho, \gamma)$ approach 0. Thus, by using L'Hôpital's rule again:

(a.7) $$\lim_{\sigma\to\bar{\sigma}} f\left(\mu,\sigma^2,\rho,\gamma\right) =$$

$$\lim_{\sigma\to\bar{\sigma}} \frac{-\dfrac{\beta''\beta-\beta'^2}{\beta^2}-\dfrac{\gamma(\gamma-1)}{2}\dfrac{\beta'(\rho-\mu_\pi)+(\beta-1)\frac{1}{2}\gamma(\gamma-1)}{(\rho-\mu_\pi)^2}}{\beta'}$$

Since both numerator and denominator of the second term in the main numerator approach zero a repeated use of L'Hôpital's rule is needed:

(a.8) $$\lim_{\sigma\to\bar{\sigma}} f\left(\mu,\sigma^2,\rho,\gamma\right) =$$

$$\lim_{\sigma\to\bar{\sigma}} \frac{-\dfrac{\beta''\beta-\beta'^2}{\beta^2}-\dfrac{\gamma(\gamma-1)}{2}\dfrac{\beta''(\rho-\mu_\pi)-\beta'\frac{1}{2}\gamma(\gamma-1)+\beta'\frac{1}{2}\gamma(\gamma-1)}{-2(\rho-\mu_\pi)\frac{1}{2}\gamma(\gamma-1)}}{\beta'}$$

$$=\lim_{\sigma\to\bar{\sigma}}\left(\beta'-\frac{\beta''}{2\beta'}\right)=\frac{\gamma^3(\gamma-1)}{2}\frac{\rho-\mu}{\left(2\gamma\rho-\gamma^2\mu-\rho\right)^2}\geq 0$$

Where the third equality springs from (a.3) together with part (e) of lemma 1 and the inequality follows from the assumption that $\rho > 2\mu+\sigma^2$. This proves (c).

To prove (d) first note by implicit derivation of (a.1) that:

(a.9) $$\beta'(\gamma) = -\frac{\beta}{\gamma}$$

and therefore:

(a.10) $$\frac{\partial \beta'}{\partial \gamma} = \frac{\partial \beta'(\gamma)}{\partial \sigma^2} = -\frac{\beta'}{\gamma}$$

Applying (a.10) in a differentiation of $f(\mu, \sigma^2, \rho, \gamma)$, as captured by (10), with respect to γ yields after tedious, yet straightforward, simplifications:

(a.11) $$\frac{\partial f(\mu, \sigma^2, \rho, \gamma)}{\partial \gamma} = \frac{\beta'}{\gamma(\beta - 1)^2} - \frac{(\gamma - 1)\rho + \gamma(\rho - \gamma\mu)}{2(\rho - \mu_\pi)^2} < 0$$

where the inequality follows from $\beta' < 0$ and from $\rho > \mu_\pi > \gamma\mu$. □