# Hysteresis in Export Markets *

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**Abstract**

This paper develops a dynamic monopolistic competition model with heterogenous firms to analyze the effects of uncertainty on international trade. We characterize a stationary equilibrium, with $N$ symmetric countries, where firms' productivities evolve stochastically over time. Our model retains the main results of previous recent papers like Melitz (2003) and Bernard, Eaton, Jensen and Kortum (2003) and provides additional new predictions. Reentry export costs generate hysteresis in export participation creating a band of inaction within the stationary distribution of firms' productivities. The decision to export becomes history-dependent and new entrants and incumbent firms might sustain temporary negative profits before becoming profitable. Most importantly, the model is very amenable to estimation and simulation, therefore representing a useful tool for analyzing the effects of trade policies. Several moments, like average age, size and productivity of different categories of firms (exporters, entrants, exiters, incumbents), the hazard rate of exiting or of becoming an exporter as a function of age and others have closed-form solutions that are crucial for matching static and dynamic features of the data.

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## 1 Introduction

Recent empirical studies, like Bernard and Jensen (1995), Clerides et al. (1998) and Anderson (2005) suggest that successful theoretical frameworks for studying firms and the decision to export should incorporate intraindustry heterogeneity in size and productivity and take into account international trade costs. Firms are heterogeneous in many respects: firms' productivities differ widely even within narrowly defined industries; exporting firms are, on average, more productive and bigger than nonexporting counterparts. International trade costs are large, even between apparently highly integrated economies. Some costs are increasing in the amount shipped while other costs are fixed and occur every time the firm tries to enter in a foreign market.

In the past couple of years, the literature on international trade has successfully addressed some of these issues: Bernard, Eaton, Jensen and Kortum (2003)[1] develop a Ricardian model of plant-level export behavior while Melitz (2003) provides a model based on monopolistic competition. These models represent an important step in reconciling macro- and micro-level trade data but they are mainly static in nature. Therefore, they don't address dynamic features of the data like the fact that firm productivity, size and exporting status change over time; firms' decisions to export do not depend only on current productivity; exporting firms coexist with nonexporting firms even if they are observationally equivalent in terms of productivity, size or other characteristics. Moreover, they do not explain other facts like the existence of a measure of exporters that are less productive than nonexporters or the fact that entrants or even incumbents might sustain temporary negative profits before becoming profitable.

On the other side, the literature on firm dynamics has developed models which allow for heterogeneity in the characteristics of firms. A recent contribution in this respect is the work by Luttmer (2004) who proposes, in a closed economy context, an analytically tractable model of balanced growth that allows for extensive heterogeneity in the technologies used by firms.

[1] BEJK from here on.

In this paper, we propose a new model of international trade that, reconciling the approaches from these two strands of literature, represents a natural evolution of the first wave of new trade models and is consistent with more features of the data. Building upon Melitz (2003) and Luttmer (2004), we construct a dynamic general equilibrium model of trade among multiple countries that combines the following four key ingredients: 1) firms compete globally to sell their differentiated product; 2) firms have heterogeneous productivity; 3) for each firm, productivity evolves over time stochastically; 4) entry (or reentry) into the export market requires the firm to sustain a sunk cost.

In this world, an entrepreneur makes an initial investment to set up a firm and draws a productivity level from a common distribution. Production for the domestic market starts even if (unlike in Melitz (2003)) initial profits are negative (as long as they are not too negative) and continues until the sum of current profits and the value of the option to exit are high enough. If firm productivity exceeds a cutoff level it becomes profitable to sell the goods on foreign markets. In order to do that, the firm must sustain a sunk cost which can be interpreted as the cost of establishing distribution channels, learning about the foreign markets preferences and standards and adapting to them, updating old export products. If, later on, productivity falls under the level at which the firm started exporting, the entrepreneur prefers to keep exporting until the value of current exporting profits plus the value of the option to stop exporting is bigger than the value of the option of reentering the export market.

The equilibrium in an open economy is therefore characterized by the following: in every country, 1) consumers maximize their intertemporal utility by choosing a sequence of dynastic consumption of a composite good, made of available domestic and foreign varieties, subject to the intertemporal budget constraint 2) there is a closed-form stationary distribution of firms' productivities and within it a band of inaction; 3) while the distribution is stationary, new firms enter, incumbent firms become more or less productive, export or stop exporting, eventually exit; 4) labor and goods markets clear.

An important aspect of the model provided in this paper is that it is easily amenable to estimation and effectively exploits the advantages of panel dataset at the firm or plant level. Closed-form solutions allow the derivation of several static and dynamic moments that are useful for matching the model to the data: average size, productivity and age of incumbents, entrants, exiters, exporters or nonexporters, the hazard function for exiting or for becoming an exporter represent some examples.

In this model, like in Melitz (2003) and BEJK (2003), firms that are more productive are bigger, both in terms of output and revenue, and are more likely to export. However, participation into the export market is characterized by hysteresis. Hysteresis is defined as "a retardation of the effect when the forces acting upon a body are changed (as if from viscosity or internal friction)"[2] and in our context means that export participation is history-dependent. The presence of hysteresis has been documented by, among others, Roberts and Tybout (1997) and is important in understanding why trade policies might have different effects in different countries or in the same country at different stages of its evolution and why even temporary policies might have permanent effects. The presence of hysteresis implies also that, as we observe in the data[3], despite being more productive on average, some exporters are less productive than some nonexporters. Our model departs from Melitz (2003) also in predicting that new firms might sustain initial negative profits and, only later on, become profitable. As observed in the data, the stationary productivity distribution follows a Pareto density in the upper tail but it's increasing in the lower tail, implying that there are fewer small firms than would be the case if Zipf's law was satisfied.[4]

The structure of the paper is as follows. In the next section, we discuss some facts about firm productivity, profitability, entry and export participation using Chilean

[2] Quote from Webster's Seventh New Collegiate Dictionary.

[3] See Irrarazabal and Opromolla (2005) for Chile, BEJK (2003) for the USA.

[4] In terms of the distribution, this means that the probability that the size (that in our model is directly connected to productivity) of a firm is greater than some $z^*$ is proportional to $1/z^*$: $P(z > z^*) = \alpha/z^\theta$, with $\theta \simeq 1$. See Simon and Bonini (1958), Gabaix (1999) and Axtell (2001) for discussions of Zipf's law and its empirical evidence.

Table 1: Export Status Transition Matrix, Average 1990-96

| | export in $t+1$ | do not export in $t+1$ | exit in $t+1$ |
|---|---|---|---|
| export in $t$ | .84 | .12 | .04 |
| do not export in $t$ | .04 | .89 | .07 |

data. In section 3, we describe the building blocks of the economy under autarky. In section 4, we describe a multi-country trade model under two alternative cost assumptions. First, we solve for the trade equilibrium assuming that a firm can costlessly entry and exit into the export market. Then, we introduce entry (and reentry) costs and show how hysteresis arises in the context of international trade with heterogeneous producers. Finally section 5, concludes and proposes some extensions to the present framework.

## 2 Facts

We begin by presenting some empirical evidence to which we are going to refer to while developing the theoretical model. We use data from the "Encuesta Nacional Industrial Anual" (ENIA) conducted annually by Instituto Nacional de Estadistica (INE), the Chilean government statistical office. ENIA is an unbalanced panel dataset covering all Chilean manufacturing plants with ten or more workers. The dataset extends from 1979 to 1996, includes information on approximately 11,000 plants altogether, with about 4,800 plants per year. It contains detailed information on production, value added, sales, employment and wages (both white-collar and blue-collar), exports, investment, depreciation, energy consumption, balance sheet information and other plant characteristics. Data on plant-level exports were only collected after 1990.

We start by looking at persistence in export participation: Table 1 shows the average 1-year transition matrix between export and nonexport status for the period 1990-1996.

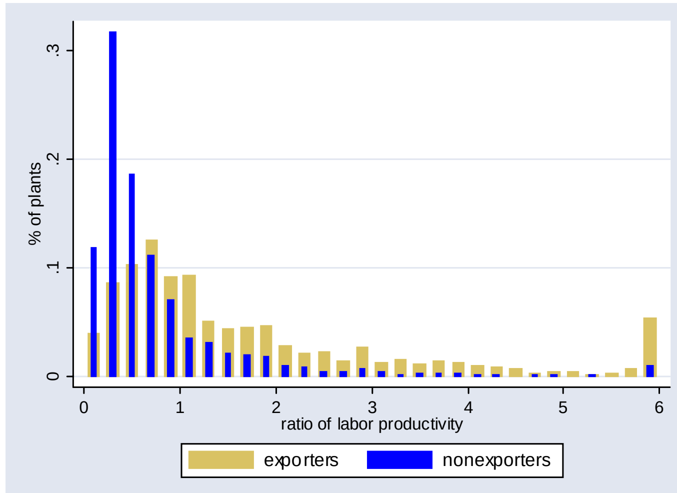


Figure 1: Ratio of Plant Labor Productivity to Overall Mean: Exporters vs. Nonexporters, 1992

Table 1 suggests the presence of a high degree of persistence in export status. Out of 100 plants that export in year $t$, 84 plants keep exporting the next year, 12 plants stop exporting but keep producing for the domestic market and 4 plants becomes inactive. If plants were not exporting to start with, 4 plants start exporting in $t+1$, 89 plants keep selling their product on the domestic market only and 7 plants shut down. We use this matrix to perform a simple test to check if the data are consistent with ergodicity and find that the average ratio between the number of exporters and the number of nonexporters is about 3.8, it's declining over time and is quite close to the ergodic ratio of 2.8.

A recurrent feature of the data is that exporters are on average more productive than nonexporters. Figure (1), from Irrarazabal and Opromolla (2005), shows the histogram of productivity by export status. Productivity is measured as value added per worker and is normalized using the same overall (exporters and nonexporters) mean productivity. The distribution of exporters productivity is shifted to the right with respect to the productivity distribution of non-exporting plants. Value added per worker at the average exporting Chilean plant is 85 percent higher than at the

Table 2: Productivity Difference When Plants Start and Stop Exporting

| Year | Ln Productivity | | | | |
|---|---|---|---|---|---|
| | When Start Exporting | When Stop Exporting | Productivity Ratio$^{(2)}$ | p-value$^{(1)}$ ($H_a$: $diff = 0$) | p-value$^{(1)}$ ($H_a$: $diff > 0$) |
| 1991 | 8.54 | 8.39 | 1.16 | .28 | .14 |
| 1992 | 8.70 | 8.61 | 1.09 | .51 | .26 |
| 1993 | 8.94 | 8.64 | 1.35 | .04 | .02 |
| 1994 | 8.97 | 8.71 | 1.30 | .05 | .02 |
| 1995 | 9.14 | 9.00 | 1.15 | .34 | .17 |
| 1996 | 9.32 | 9.23 | 1.09 | .50 | .25 |

Note: (1) Two-sample t test with equal variances; $H_0$ : the difference between ln(productivity) when start exporting and ln(productivity) when stop exporting is equal to zero; (2) exp(ln prod$_{start}$-ln prod$_{stop}$);

average plant that does not export.

This result surpasses previous findings for U.S. and French data: BEJK show that the productivity advantage of U.S. plants is about 33 percent overall while Eaton et al. (2004) find that the French exporting firms' value added per worker is 12.5 percent higher than nonexporting counterparts. It's interesting to note that even though average productivity is higher for exporters there is a consistent measure of exporters that are less productive than nonexporters.

The third fact that we present is related to entry and exit from the export market. Tables (2) and (3) compare the average productivity and size of new exporters and old exporters, that is, of plants that just started exporting and of plants that just stopped exporting. We find some evidence in support of the hypothesis that plants decision to enter the export market and to exit the export market does not rely on the same reference productivity level. Average productivity of new exporters is always higher than average productivity of old exporters, even though the difference passes a two-sample t-test only during part of the sample period.

Figure (2) shows the size distribution of plants, measured in terms of employment, in 1992. The size distribution follows a Power Law with exponent equal to one (Zipf's' Law) if the plot of the natural logarithm of the number of plants above some size level $s$ against the natural logarithm of plants labor force results in a straight line with

Table 3: Size Difference When Plants Start and Stop Exporting

| Year | Ln Sales When Start Exporting | Ln Sales When Stop Exporting | Sales Ratio(2) | p-value(1) ($H_a$: $diff = 0$) | p-value(1) ($H_a$: $diff > 0$) |
|---|---|---|---|---|---|
| 1991 | 13.90 | 13.55 | 1.42 | .09 | .05 |
| 1992 | 13.77 | 13.71 | 1.06 | .80 | .40 |
| 1993 | 14.00 | 13.36 | 1.90 | .01 | .00 |
| 1994 | 14.06 | 13.32 | 2.10 | .00 | .00 |
| 1995 | 14.28 | 14.12 | 1.17 | .49 | .25 |
| 1996 | 14.23 | 14.01 | 1.25 | .27 | .14 |

Note: (1) Two-sample t test with equal variances; $H_0$ : the difference between ln(sales) when start exporting and ln(sales) when stop exporting is equal to zero; (2) exp(ln sales$_{start}$-ln sales$_{stop}$);

slope equal to minus one.[5] The data follow Zipf's' law quite closely for most of the size range with two exceptions: there are fewer very small and very big firms than what would be consistent with Zipf's law. Keeping in mind these and other features of the data, we now proceed to outline the model.

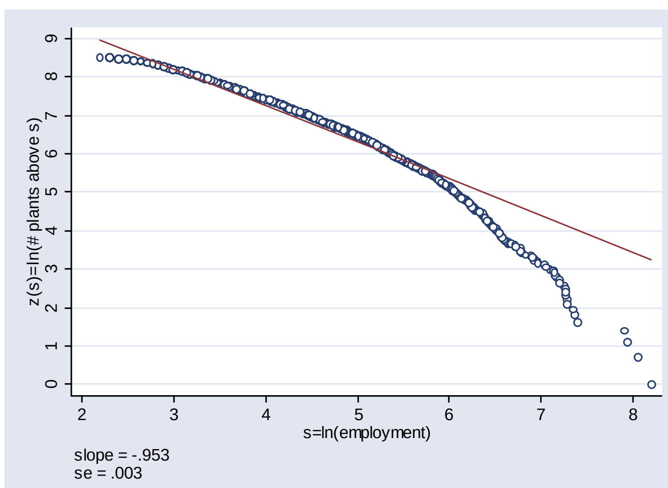


Figure 2: Firm Size Distribution, 1992

[5]The size $s$ of a plant is distributed as a power law with exponent $\alpha$ and minimum size $s_0$ if the density of $s$ is $f(s) = \alpha s_0^{\alpha} s^{-\alpha-1}$ ($s_i \geq s_0$, $\alpha > 0$). Suppose that $N$ is the number of plants. The rank of all firms in the sample is $r(s_i) = N\left(\frac{s_0}{s_i}\right)^{\alpha}$ where the rank is decreasing in the size of the plant. Taking natural logs leads to $\ln r(s_i) = c - \alpha \ln(s_i)$ where $c = \ln N + \alpha \ln(s_0)$.

# 3 Set Up of the Model

## 3.1 Preferences

Time is continuous. Let $\Omega_h$ be the set, of measure $L$, of infinitely-lived consumers alive at time $t$. The economy consists of one sector that produces differentiated products. Each consumer is endowed with one unit of labor at every point in time that is supplied inelastically. The wage is normalized to one. Goods are perishable and hence can only be used for consumption. The representative consumer has preferences over sequences $\{C_t\}_{t\geq 0}$ of a composite goods given by:

$$U = E\left[\int_0^{\infty} e^{-\rho t} C_t dt\right] \tag{1}$$

where $\rho$ is the time-discounting rate.

In the equilibrium to be defined later there will be a measure $\Omega_t$ of differentiated goods available at time $t$ and indexed by $u$. The composite goods is defined as

$$C_t = \left(\int c_t(u)^{\frac{\sigma-1}{\sigma}} du\right)^{\frac{\sigma}{\sigma-1}}$$

where $\sigma > 1$ is the elasticity of substitution between any two differentiated goods.

The representative consumer chooses $c_t$ to minimize the cost of acquiring $C_t$ given a standard present-value budget constraint. Wealth consists of claims to firms and labor income. As a result,

$$c_t(u) = C_t \left[\frac{p_t(u)}{P_t}\right]^{-\sigma} \tag{2}$$

with the corresponding consumption-based price index

$$P_t = \left[\int p_t(u)^{1-\sigma} du\right]^{\frac{1}{1-\sigma}} \tag{3}$$

### 3.2 Production

There is a continuum of firms, of measure $M$, each choosing to produce a distinct variety of the goods using labor as input.[6] At age $a$, a firm employs $l_a$ units of labor to produce $e^{z_a} l_a$ units of the goods. For simplicity, we will refer to $z_a$ as the productivity of the firm.

All firms share the same overhead per period fixed cost $f_d > 0$, constant over time and age. A firm must exit if this fixed cost is not paid and exit is irreversible. Productivity evolves, independently across firms, according to a Brownian motion with drift $\alpha$ and diffusion coefficient $\xi$,

$$z_a = \bar{z} + \alpha a + \xi B_a \tag{4}$$

where $\{B_a\}_{a \geq 0}$ is a Wiener process and $\bar{z}$ is the initial productivity of a firm. The initial productivity $\bar{z}$ is drawn from a time-invariant probability density $g(\bar{z})$. After entry, the trend of log productivity is determined by $\alpha$.

A monopolistic competitive producer, with productivity $z_a$, offers the product according to the following pricing rule

$$p(z_a) = \frac{\bar{m}}{e^{z_a}} \tag{5}$$

where $\bar{m} = \sigma/(\sigma - 1)$ is the fixed markup. Firm revenue is

$$r(z_a) = R \left( \frac{p(z_a)}{P} \right)^{1-\sigma} \tag{6}$$

where $R$ is aggregate expenditure on the composite good. Firm profits are then

$$\pi(z_a) = r(z_a) - l(z_a) - f_d = \frac{r(z_a)}{\sigma} - f_d$$

where $\frac{r(z)}{\sigma}$ represents variable profits. Using (5) and (6) we can see that profits depend also on aggregate price and revenue:

$$\pi(z_a) = \frac{R}{\sigma} \left( \bar{m}^{-1} P e^{z_a} \right)^{\sigma - 1} - f_d \tag{7}$$

[6] Since we will consider a stationary equilibrium where all aggregate variables are constant, from now on we drop the time subscript in order to simplify the notation.

A more productive firm charges a lower price (since marginal costs are lower and the markup is constant), is bigger both in terms of output and revenue (since lower price means higher demand and demand is elastic), and earns higher profits than a less productive firm (since variable profits are a constant fraction of firm's revenue).

Note that the productivity cutoff $z_0$ at which current profits are zero is

$$e^{z_0} = \frac{\bar{m}}{P}\left(\frac{\sigma f_d}{R}\right)^{\frac{1}{\sigma-1}}$$

This cutoff is decreasing in the price level $P$ (since wage and the markup are the same for every firm, the price index is really a measure of the degree of competition that the firm has to face in order to sell its product) and in the level of expenditure on differentiated goods $R$. It is increasing in the fixed cost $f_d$ and in the elasticity of substitution $\sigma$ (provided that $R > \sigma f_d$, when goods become more substitutable, price deviations from the general price have bigger effects).

Using the price rule (5), the demand equation (2) and the production function we can find the labor demand of a firm,

$$l_a = C(e^{z_a})^{\sigma-1}\bar{m}^{-\sigma}P^{\sigma} \tag{8}$$

which turns out to be a fraction $\frac{1}{\bar{m}}$ of the firm revenue

$$l_a = \frac{1}{\bar{m}}r(z_a)$$

### 3.3 Entry, Exit and the Stationary Distribution

In this section we derive the stationary distribution of firms' productivities. We depart from previous models used in international trade (see Melitz (2003), Chaney (2005), and Eaton and Kortum (2002)) by using a model of industry equilibrium with dynamic stochastic productivities similar to the one of Luttmer (2004). Contrary to Melitz (2003), firms are subject to both ex-ante and ex-post uncertainty. Entry requires the entrepreneur to sustain a sunk cost before being able to know the initial productivity level. If the initial productivity is high enough the firm starts producing. After that, productivity evolves over time according to (4) and the firm remains active until its value, given by the solution of an ordinary differential equation, is positive. A positive drift coupled with a positive diffusion coefficient in (4) makes it worthwhile to keep producing even if current profits are negative. However, if profits become too negative or if the firm receives a bad shock uncorrelated to productivity, exit takes place. We begin by deriving the value of a firm as a function of its productivity level and we then proceed by deriving the ergodic productivity distribution.

### 3.3.1 Entry and Exit

Incumbent firms, indexed by $i \in \Omega_I$, exit the industry when their productivity falls below some threshold.[7] Exit is irreversible. Firms exit because of productivity reasons or because of bad shocks uncorrelated with productivity: these occur each period with an exogenous probability $\delta$. This allows the model to capture exit uncorrelated to productivity and better match the data. A positive $\delta$, in absence of population growth, also allows for a stationary productivity distribution compatible with either a positive or a negative drift in the stochastic process for productivity. The present value of revenues, and therefore of variable profits, is finite if the combined discount factor, given by the sum of the interest rate and the exogenous probability of exit $\delta$, is smaller than the drift of variable profits. The following assumption guarantees that this is the case (see the Appendix).

**Assumption 1**: *The preferences and productivity parameter satisfy*

$$\rho + \delta > \alpha(\sigma - 1) + \xi^2(\sigma - 1)^2/2$$

The value function of a firm with productivity $z$ can be expressed as the sum of operating profits over the interval $(t, t + dt)$ and the continuation value beyond $t + dt$. The value for a firm that discount flows at the interest rate $r$ is

$$V_d(z) = \left\{ \pi(z)dt + e^{-(r+\delta)dt} E\left[V(z + dz|z)\right] \right\}$$

[7] Recall that firms must sustain a positive fixed operating cost $f_d$.

Using Ito's lemma and equation (7) for firm's profits, we obtain an ordinary differential equation in the range of $z$ where a firm is not shut down:

$$(r+\delta)V_d(z) = \left(\frac{R}{\sigma}\left(\bar{m}^{-1}Pe^{z}\right)^{\sigma-1} - f_d\right) + \alpha V'(z) + \frac{1}{2}\xi^2 V''(z)$$

Since a firm has to pay a positive fixed operating cost, it is optimal to shut down when productivity falls below some threshold $z_d$. The value of a firm must be zero at that point implying the following value matching and smooth-pasting conditions

$$V_d(z_d) = 0$$

$$V_d'(z_d) = 0$$

These conditions provide a complete characterization of the optimal policy of an active firm, the associated value function and the critical value $z_d$.[8] With these boundary conditions, the value of a firm with productivity $z$ is (see the Appendix)

$$V_d(z) = \frac{f_d}{(r+\delta)}\frac{\beta_2}{\beta_2-(\sigma-1)}\left[e^{(\sigma-1)(z-z_d)} - \frac{\beta_2-(\sigma-1)}{\beta_2} - \frac{\sigma-1}{\beta_2}e^{\beta_2(z-z_d)}\right] \qquad (9)$$

for $z > z_d$, where $\beta_2$ is the negative root of the characteristic polynomial (see the Appendix). The value of the firm is increasing in $z$ and can be nicely interpreted as the sum of two components: the first two terms in $V_d(z)$ reflect the expected discounted present value of the profits flow while the third term represents the value of the option to suspend the operation when the productivity falls below $z_d$ (see the Appendix).

The exit productivity threshold $z_d$ is

$$e^{z_d} = \frac{\bar{m}}{P}\left(\frac{\sigma f_d}{R}\right)^{\frac{1}{(\sigma-1)}}\gamma_d \qquad (10)$$

where $\gamma_d = \left[\frac{-\beta_2}{r+\delta}\frac{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)}{\beta_2-(\sigma-1)}\right]^{\frac{1}{(\sigma-1)}}$ is less than one when the productivity drift $\alpha$ is positive (see the Appendix). This implies that $z_d < z_0$, that is, incumbents and new entrants remain active even if profits are currently negative, as long as they

[8] See Stokey (2005) "Brownian Models in Economics" lectures notes for more details.

are not too negative. The intuition is that when $z \in (z_d, z_0)$ the value of the firm is still positive because the stochastic process for profits has a positive drift and because of the option value to exit. By inspecting the above expression we see that the barrier is inversely related to the aggregate price level. This will result critical when we open the economy to international trade. A multilateral trade opening will decrease the aggregate price level, therefore pushing the lower threshold $z_d$ up. This will be interpreted as a selection effect associated with international competition.

Prior to entry firms are identical. To enter, they must make an initial investment $f_e > 0$ (measured in units of labor) which is thereafter sunk. Firms then draw their productivity from a common distribution $g(\bar{z})$ with continuous cdf $G(\bar{z})$. Upon entry a firm may decide to exit immediately and not produce (and not pay the fixed cost $f_d$). New firms will keep trying to enter the industry until the expected value of a firm net of entry cost is zero, that is until the following free-entry condition is satisfied,

$$f_e = \int_{z_d}^{\infty} V_d(\bar{z}) dG(\bar{z}) \tag{11}$$

### 3.3.2 The Stationary Distribution $\mu(z)$

As in Melitz (2003), in order to determine aggregate variables we need an expression for the equilibrium distribution of productivities. We are going to characterize a time invariant distribution of productivities with finite mean. In the stationary equilibrium there is a measure of firms defined over the set of possible ages, initial and current productivities. The density of this measure is $M\mu(a, z, \bar{z})$ where $\mu(a, z, \bar{z})$ is a probability density. The density $M\mu(a, z, \bar{z})$ must satisfy the following version of the Kolmogorov forward equation[9]

$$\frac{\partial \mu(a, z, \bar{z})}{\partial a} = \frac{1}{2}\xi^2 \frac{\partial^2 \mu(a, z, \bar{z})}{\partial^2 z} - \alpha \frac{\partial \mu(a, z, \bar{z})}{\partial z} - \delta \mu(a, z, \bar{z}) \tag{12}$$

[9] See Dixit and Pindyck (1994) for a derivation of the Kolmogorov forward equation using a discrete random walk approximation.

Potential entrants are sampling the distribution of initial productivity $g(\bar{z})$. Successful entry attempts are those for which $\bar{z} > z_d$. This implies that the first boundary condition of the equation above is

$$\mu(0, z, \bar{z}) = \begin{cases} g(\bar{z})\frac{M_a}{M} & z > z_d \\ 0 & z \leq z_d \end{cases}$$

where $M_a$ is the number of (attempting) entrants. Another condition is given by the presence of the lower barrier $z_d$: since firms exit at $z_d$ and no new firm enter with a productivity level inferior to $z_d$, we have that $\mu(a, z_d, \bar{z}) = 0$ for all $a > 0$. Finally, to ensure that $\mu$ is a density distribution we also require $\mu(a, z, \bar{z})$ goes to zero for large values of $(a, z, \bar{z})$. The solution to the above equation is then given by[10]

$$\mu(a, z, \bar{z}) = e^{-\delta a}\psi(a, z|\bar{z})g(\bar{z})\frac{M_a}{M} \tag{13}$$

for all $a > 0$ and $z > z_d$, where

$$\psi(a, z|\bar{z}) = \frac{1}{\xi\sqrt{a}}\left[\phi\left(\frac{z - \bar{z} - \alpha a}{\xi\sqrt{a}}\right) - e^{-\frac{2\alpha(z-\bar{z})}{\xi^2}}\phi\left(\frac{z + \bar{z} - 2z_d - \alpha a}{\xi\sqrt{a}}\right)\right]$$

After lengthy derivations (see the Appendix) we find that the probability density of productivity conditional on a particular initial level $\bar{z}$ is

$$\mu(z|\bar{z}) = \frac{\theta}{\theta + \theta_*}\left[\frac{\min\left\{e^{(\theta_*+\theta)(z-z_d)}, e^{(\theta+\theta_*)(\bar{z}-z_d)}\right\} - 1}{e^{\theta(z-z_d)}\left(\frac{e^{\theta_*(\bar{z}-z_d)}-1}{\theta_*}\right)}\right] \tag{14}$$

while the following is the joint productivity and age probability density conditional on a particular initial productivity level $\bar{z}$

$$\mu(a, z|\bar{z}) = \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right]^{-1} e^{-\delta a}\psi(a, z|\bar{z}) \tag{15}$$

where $\theta$ and $\theta_*$ are the roots, both non-negative, of the characteristic polynomial defined by (12).

In order to have a stationary distribution with a finite mean we need to impose the following assumption

[10] See Luttmer (2004) and Harrison (1985) for a similar derivation.

**Assumption 2**: *The productivity parameters satisfy*

$$\delta > \alpha + \xi^2/2$$

This condition guarantees that $\theta > 1$ and the mean of $e^{z-z_d}$ is finite and can be expressed as

$$e^{\tilde{z}-z_d} = \int_{z_d}^{\infty} e^{z-z_d}\mu(z|\bar{z})dz = \frac{\theta}{(\theta-1)}\left[\frac{\frac{1-e^{-(1+\theta_*)(\bar{z}-z_d)}}{(\theta_*+1)}}{\frac{1-e^{-\theta_*(\bar{z}-z_d)}}{\theta_*}}\right]e^{\bar{z}-z_d} \tag{16}$$

where the right hand-side is greater than $e^{\bar{z}-z_d}$ if $\alpha + \xi^2/2 > 0$. This means that the average firm is more productive than new entrants.

Recall that the solution of the stationary problem is for a particular rate of entry. Therefore we can use (13) and the fact that (15) is a density to find the rate of attempted entry consistent with the stationary equilibrium

$$\frac{M_a}{M} = \left[\int\limits_{z_d}^{\infty}\left[\frac{1-e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right]g(\bar{z})d\bar{z}\right]^{-1} \tag{17}$$

Note that now we can interpret the solution of the Kolmogorov equation under a different light,

$$\mu(a, z, \bar{z}) = \mu(a, z|\bar{z})\mu(\bar{z})$$

where

$$\mu(\bar{z}) = \frac{\frac{1-e^{-\theta_*(\bar{z}-z_d)}}{\delta}g(\bar{z})}{\left[\int\limits_{z_d}^{\infty}\left[\frac{1-e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right]g(\bar{z})d\bar{z}\right]}$$

is the marginal probability density for the initial productivity $\bar{z}$ in the stationary equilibrium. As Luttmer (2004) points out, this probability density is different from $g(\bar{z})$, the density of initial productivity among potential entrants, because of the pre- and post-selection process. Pre-selection implies requires firms to enter into the market only if their initial productivity is bigger than $z_d$, so that $g(\bar{z})$ is truncated at $z_d$. Post-selection implies that firms with initial productivity close to the cutoff $z_d$

have a higher probability of exiting due to a negative firm-specific shock and therefore they are downweighted in $\mu(\bar{z})$: the term $1 - e^{-\theta_*(\bar{z}-z_d)}$ is increasing in $\bar{z}$. The effect of post-selection is stronger when exit uncorrelated to productivity is relatively less important (lower $\delta$).

Finally, we are ready to determine an expression for the stationary distribution of productivities. In this ergodic equilibrium the distribution of productivities is determined by the process of entry, exit, and selection. In particular, selection determines how the initial distribution from where the potential firms draw their productivities generates the stationary distribution of active firms. In the appendix we show that the stationary distribution of productivity is given by

$$\mu(z) = \int_{z_d}^{\infty} \mu(z|\bar{z})\mu(\bar{z})d\bar{z} = \frac{\int_{z_d}^{\infty} \mu(z|\bar{z}) \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right] g(\bar{z})d\bar{z}}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right] g(\bar{z})d\bar{z}\right]} \tag{18}$$

Note that the probability density of productivity is a weighted average of the conditional probability densities $\mu(z|\bar{z})$. If $g(.)$ is a degenerate distribution with all the mass at some point $\bar{z}$, then $\mu(z|\bar{z})$ itself is the equilibrium firm productivity density. Each $\mu(z|\bar{z})$ is proportional to $e^{-\theta z}$ when $z > \bar{z}$ and therefore behaves, in the upper tail, as a Power law with exponent $\theta$ for $e^z$, our productivity coefficient. When $z_d < z < \bar{z}$, the probability density is increasing in $z$ (see Figure (3)).

This is consistent with the productivity distribution that we showed in Section 2. This expression has a structure similar to the distribution shown in Melitz(2003) in equation (8). In our setting, though, the equilibrium productivity distribution $\mu(z)$ differs from the productivity distribution of potential entrants because of both pre-selection and post-selection.

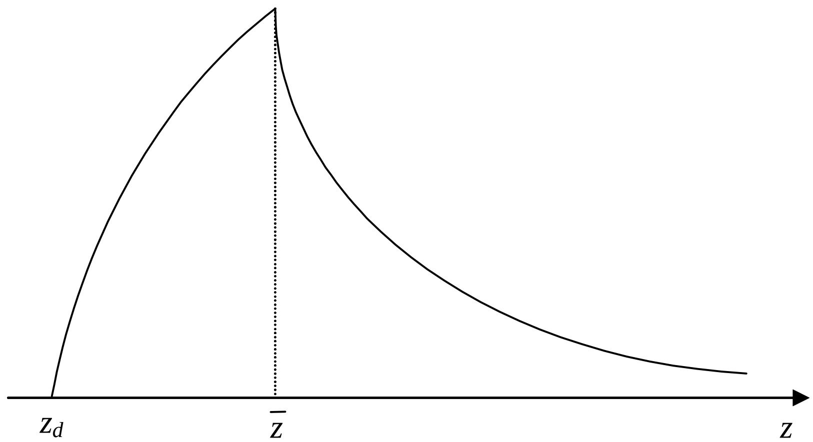


Figure 3: Equilibrium Probability Density of Productivity conditional on Initial Productivity $\bar{z}$

# 4 Equilibrium in a Closed Economy

A closed-economy stationary equilibrium is defined by

$$\left\{ \left[ \left( C_t^h \right)_{t \geq 0} \right]_{h \in \Omega_h}, \left[ \left( p^i(z_a) \right)_{a \geq 0}, \left( l^i(z_a) \right)_{a \geq 0}, D_{a \geq 0}^i \right]_{i \in \Omega_I}, \right\}$$

where (i) each consumer $h$ chooses optimally sequences $\{C_t\}_{t \geq 0}$ of a composite goods to maximize the intertemporal utility function of equation (1) subject to an intertemporal budget constraint; (ii) incumbent firms choose, at every age $a$, price $p(z_a)$ and variable labor $l(z_a)$ in order to maximize profits, given by equation (7), taking the price index of the economy as given; (iii) incumbent firms, at every age $a$, decide if to keep being active ($D_a = 1$) or to exit ($D_a = 0$) knowing that productivity evolves according to the Brownian motion of equation (4); (iv) firms enter, if the productivity draw is bigger than $z_d$, until the free-entry condition (11) is satisfied; (v) labor and goods markets clear.

With an expression for the stationary distribution of productivity, we can proceed to solve for the stationary equilibrium under the autarky situation. First, notice that by the intertemporal problem of the consumer we pin down the interest rate as

$$r = \rho$$

Now we can determine the root of the characteristic polynomial $\beta_2$ and hence the value function $V(z)$ from equation (9).

We can use the free entry condition (11) and the threshold equation (10) to solve for $\{z_d, P\}$. Furthermore, using the good market clearing condition we have that

$$C = \frac{R}{P} = \frac{L_g \bar{m}}{P}$$

where we have used the fact that aggregate revenue must be equal to a markup on payments to production workers.

Recall the optimal variable labor choice of the firm. Summing over all ages we find that

$$L_g = M \int_{z_d}^{\infty} C(e^z)^{\sigma-1} \bar{m}^{-\sigma} P^{\sigma} \mu(z) dz$$

The aggregated labor required to set up a firm and to operate (overhead fixed cost) a firm equals

$$\begin{aligned} L_e &= f_e M_a \\ L_f &= f_d M \int_{z_d}^{\infty} \mu(z) dz \end{aligned}$$

Recall that the equilibrium entry rate is given by (17). Hence, from the following labor market clearing condition we can pin down the number of firms

$$L_p + L_f + L_e = L \tag{19}$$

Finally, to find the number of entrants we use again the rate of entry condition (17). Notice that an increase in $M$ does not affect the lower barrier, the average level of productivity (or profitability), and therefore the average profit level and revenue.

# 5 The Open Economy

Consider the world economy consisting of $N$ countries all with the structure described in section 3. We are going to analyze a symmetric countries equilibrium. Each country is assumed to be endowed with $L$ units of labor and labor is not mobile across countries. Therefore, the nominal wage rate is common and we normalize it to one. Preferences are common to all countries and given by equation (1). In each country, potential entrants need to pay a fixed entry cost $f_e$ to enter the industry. They draw an initial productivity from an exogenous distribution $G(\bar{z})$. Productivity thereafter evolves according to a Brownian motion with parameters $(\alpha, \xi)$. A firm decides to enter when the expected profits are high enough to cover the entry cost. If the firms decide to enter it also has to pay a fixed operating cost $f_d$. The fixed costs $f_e$ and $f_d$ are incurred for both exporting and non exporting firms. Given the structure of the shock and the fixed cost of production, in each country a stationary firm size distribution will emerge.

In what follows we analyze two kinds of international equilibrium. In the first case, that we develop as a benchmark, firms can costlessly enter and exit from the export market. In the second case, we assume that there is a positive sunk entry cost $f_{hy}$ that the firm has to sustain every time the firm wants to enter the exporting sector. In both cases, we suppose that a firm in country $i$ that exports to country $j$ bears a fixed operating export cost $f_x$ per foreign market. Goods that are exported are then subject, like in Melitz (2003) and BEJK (2003) to a melting transportation cost $\tau > 1$. That is, we assume that country $i$ needs to ship $\tau$ units of the goods for one unit to arrive in country $j$.

## 5.1 The Case of No Entry Cost

In this section, we describe the trade equilibrium when there is no entry cost. We consider the economy in its stationary equilibrium and we drop the index for age and time. Like in the closed-economy equilibrium, a firm with productivity $z$ operating in the domestic market will obtain profits

$$\pi_d(z) = \frac{R}{\sigma}\left(\bar{m}^{-1}Pe^{z}\right)^{\sigma-1} - f_d$$

If the firm decides to export to any other single country, the firm will earn some *additional* profits

$$\pi_{ex}(z) = \tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}Pe^{z}\right)^{\sigma-1} - f_x \tag{20}$$

Assume that each firm can enter and exit from the export sector costlessly. This implies that period profits from exporting to another country are

$$\pi_{ex}(z) = \max\left[\tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}Pe^{z}\right)^{\sigma-1} - f_x, 0\right]$$

We are going to derive the solution of the following non-homogeneous ordinary differential equation

$$-(r+\delta)V_{ex}(z) + \frac{1}{2}\xi^2 V_{ex}''(z) + \alpha V_{ex}'(z) + \pi_{ex}(z) = 0$$

Since the forcing function is defined differently when current profits are positive or negative we need to solve the equation separately for the two cases and then stitch together the two solutions at the point where profits from exporting are zero,

$$\pi_{ex}(z) = 0 \iff e^{z_{ex}} = \frac{\tau\bar{m}}{P}\left(\frac{\sigma f_x}{R}\right)^{\frac{1}{\sigma-1}}$$

Since, in the data we observe that some firms don't export we will assume that $\left(\frac{\tau}{\gamma_d}\right)^{\sigma-1} > \frac{f_d}{f_x}$ which guarantees that $z_{ex} > z_d$. Note that this also implies that the least productive exporter is more productive than the most productive nonexporter (see Figure (4)). This is something that we do not usually observe in the data and that we will take into account in the cost-of-entry case.

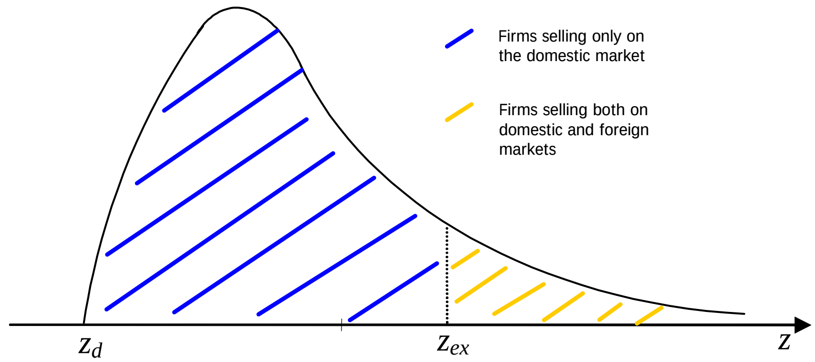


Figure 4: Firms' Productivity Distribution, No Cost of Entry in Foreign Markets

The value of the firm is $V_{>ex}(z)$ for the region where $z > z_{ex}$,

$$\begin{aligned} V_{>ex}(z) &= \frac{f_x\left[e^{\beta_2(z-z_{ex})}\left(\frac{\sigma-1-\beta_1}{\beta_2-\beta_1\beta_2}\right)-e^{(\sigma-1)(z-z_{ex})}\right]}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)} \\ &\quad -\frac{f_x}{(r+\delta)}\left[\frac{\beta_1}{\beta_2-\beta_1\beta_2}e^{\beta_2(z-z_{ex})}+1\right] \end{aligned} \tag{21}$$

and $V_{<ex}(z)$ for the region where $z < z_{ex}$,

$$V_{<ex}(z) = \left\{\begin{array}{c} \frac{f_x}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)} \\ \frac{\beta_2(\sigma-2)}{\beta_1} - \beta_2\frac{f_x}{(r+\delta)} \end{array}\right\}\frac{e^{\beta_1(z-z_{ex})}}{\beta_2-\beta_1\beta_2} \tag{22}$$

Note that in $V_{>ex}(z)$, the second and fourth terms represents the expected discounted profit flows from exporting while the remaining two terms represent the value of the option to stop exporting should productivity fall below $z_{ex}$. In the region $(z_d, z_{ex})$, $V_{<ex}(z)$ represents instead the value of the option to resume exporting should productivity rise above $z_{ex}$.

Having determined the cutoff productivity value for exporting $z_{ex}$, we can now draw on the stationary distribution analysis to derive some expressions for the average

age ($\bar{a}_{ex}$), productivity ($\bar{z}_{ex}$), and size ($\bar{r}_{ex}$, in terms of revenues).

$$\bar{a}_{ex} = \int_0^\infty a \int_{z_{ex}}^\infty \mu(a, z|z > z_{ex}) dz da; \qquad \bar{z}_{ex} = \int_{z_{ex}}^\infty z \int_0^\infty \mu(a, z|z > z_{ex}) da dz;$$

$$\bar{r}_{ex} = R\left(\bar{m}^{-1}P\right)^{\sigma-1} \int_{z_{ex}}^\infty e^{(\sigma-1)z} \mu(a, z|z > z_{ex}) da dz$$

where $\mu(a, z|z > z_{ex})$ is the probability density over age and productivity conditional on being an exporter. Moreover, knowledge of the productivity distribution and relevant cutoff values allows us to derive the probability that plants won't exit before $t$ years from now as a function of age, that is the following survivor function,

$$P(T(z_d) > t|a) = \int_{z_d}^\infty P(T(z_d) > t|x)\mu(x|a) dx$$

where

$$P(T(z_d) > t|x) = 1 - \Phi\left(\frac{z_d + x + \alpha\xi t}{\xi\sqrt{t}}\right) e^{\frac{2}{\xi}\alpha(z_d - x)} - \Phi\left(\frac{z_d - x - \alpha\xi t}{\xi\sqrt{t}}\right)$$

is the probability that the firm will exit no sooner than $t$ years from now given its current productivity $x$. These are the model equivalent of some of the facts that we showed in Section 2 and can be used to match the model to the data.

**Trade Equilibrium**

To determine the equilibrium we assume that all fixed costs, the distribution $g(\bar{z})$ and the stationary distribution $\mu(z)$ are identical for all countries. Once countries are allowed to trade, new firms will keep trying to enter the industry until the expected value of setting up a firm net of entry cost is zero, that is

$$f_e = \int_{z_d}^{z_{ex}} \left[V_d(\bar{z}) + (N-1)V_{<ex}(z)\right] dG(\bar{z}) + (N-1)\int_{z_{ex}}^{+\infty} \left[V_{>ex}(z) - V_{<ex}(z)\right] dG(\bar{z}) \tag{23}$$

where the equilibrium condition for the cutoffs is given by (10) and (**??**). Note that the expected value of a firm include both the possibility that the firm will produce only for the domestic market and that the firm will also export to all the other $N-1$ countries. An open-economy, no-entry-cost, stationary equilibrium is defined by

$$\left\{ \left[ \left( C_t^h \right)_{t \geq 0} \right]_{h \in \Omega_h}, \left[ \left( p^i(z_a) \right)_{a \geq 0}, \left( l^i(z_a) \right)_{a \geq 0}, D^i_{a \geq 0}, E^i_{a \geq 0} \right]_{i \in \Omega_I}, \right\}$$

where (i) each consumer $h$ chooses optimally sequences $\{C_t\}_{t \geq 0}$ of a composite goods to maximize the intertemporal utility function of equation (1) subject to an intertemporal budget constraint; (ii) incumbent firms choose, at every age $a$, price $p(z_a)$ and variable labor $l(z_a)$ in order to maximize domestic profits, given by equation (7), and eventual exporting profits, given by equation (20), taking the price index of the economy as given; (iii) incumbent firms, at every age $a$, decides if to keep being active ($D_a = 1$) or to exit ($D_a = 0$) and if to export ($E_a = 1$) or not ($E_a = 0$) knowing that productivity evolves according to the Brownian motion of equation (4); (iv) firms enter, if the productivity draw is bigger than $z_d$, until the free-entry condition (23) is satisfied; (v) labor and goods markets clear.

The free-entry condition, together with the thresholds $z_d$ and $z_{ex}$

$$\begin{aligned} e^{z_d} &= \frac{\bar{m}}{P} \left( \frac{\sigma f_d}{R} \right)^{\frac{1}{(\sigma-1)}} \gamma_d \\ e^{z_{ex}} &= \frac{\tau \bar{m}}{P} \left( \frac{\sigma f_x}{R} \right)^{\frac{1}{\sigma-1}} \end{aligned}$$

form a system of three equations in the three unknowns $z_d$, $z_{ex}$ and $P$.

Having solved for the thresholds $z_d$ and $z_{ex}$ and the price level we can determine the equilibrium number of firms. To determine the price index observe that all brands that are produced in a country, by domestic or by foreign firms, have a consumer price $1/\bar{m}e^z$ and all imported brands have a consumer price of $\tau/\bar{m}e^z$ when the exporter productivity is $z$. The equation for the price index

$$P^{1-\sigma} = \int_0^{M^*} p(u)^{1-\sigma} du = M \int_{z_d}^{\infty} \left( \frac{1}{\bar{m}e^z} \right)^{1-\sigma} \mu(z)dz + (N-1)M \int_{z_{ex}}^{\infty} \left( \frac{\tau}{\bar{m}e^z} \right)^{1-\sigma} \mu(z)dz$$

can be used to determine the number of firms $M$.[11]

[11] $M^*$ represents the number of firms supplying the market or the number of varieties offered.

A multilateral trade opening in the model explained above reduces the price level $P$ and therefore induces an increase in the cutoff productivity level $z_d$.This result resembles similar outcomes in previous trade models with heterogenous firms. After trade opening, firms who want to export have to pay a fixed cost $f_x$, which self-select high productivity draw firms into the export market. Domestic firms with high enough productivity to survive pay higher wages. Since the labor is constant, competition in the local labor market pushes up the real wage and forces low productivity firms to leave the industry.

### 5.2 The Case of Entry Cost: Hysteresis

We know analyze the conditions that determines the equilibrium with reentry costs. When firms' productivities are stochastic the introduction of an entry cost gives rise to hysteresis. This phenomenon has been studied by Dixit (1989), Baldwin (1989) and suggested by Clerides et-al (1998) in a context similar to ours. Our framework differentiates from previous attempts to model hysteresis in export markets in that we embed the problem in a general equilibrium framework.

As before we have that firms need to pay a fixed per-period export cost to sell in each foreign market. However, now assume that each time a firm decides to enter or reenter the export market, it has to pay a sunk cost $f_{hy}$. This cost can be interpreted as the cost of establishing distribution channels, learning about the foreign markets preferences and standards and adapting to them and updating old export products.

The next proposition summarizes the optimal policy of a firm subject to entry and reentry costs.

**Proposition 1** *An optimal strategy is characterized by three thresholds* $\{z_d, z_{low}, z_{hi}\}$ *with* $z_d < z_{low} < z_{hi}$ *such that (i) a firm is active if* $z > z_d$*, (ii) a nonexporting firm will stay as a nonexporter as long as* $z < z_{hi}$*, and (iii) an exporting firm will keep exporting as long as* $z > z_{low}$*. Furthermore, there is a band of inaction* $z_{low} < z < z_{hi}$

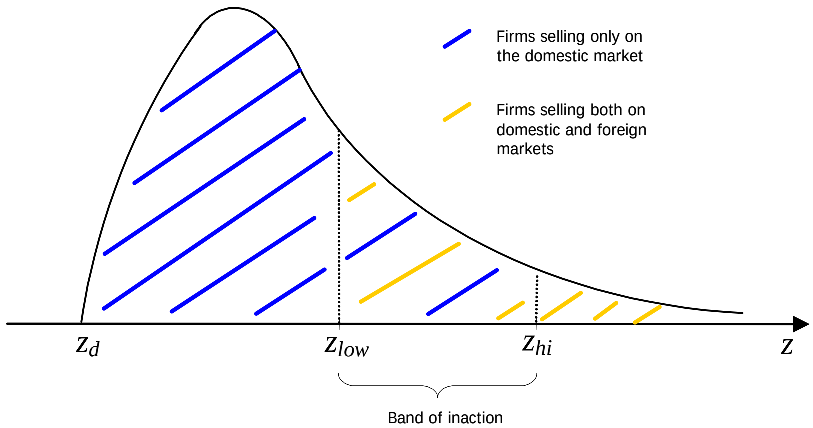


Figure 5: Firms' Productivity Distribution with Cost of Entry in Foreign Markets

*where an exporting plant will keep exporting and a nonexporting plant will decide not to enter the export sector.*

The previous proposition allows us also to state the following implication of the model which we observe in the data.

**Proposition 2** *There is a positive measure of nonexporting firms that are more productive than some exporting firms*

These two propositions are graphically illustrated in Figure (5).

To prove the previous propositions we need to show the calculation of the relevant thresholds. As in the case of no entry-cost we have that the value of a firm that only sells in the domestic marker is given by:

$$V_d(z) = \left\{ \pi_d(z) \triangle t + e^{-(r+\delta)\triangle t} E\left[V_d(z + \triangle z|z)\right] \right\}$$

On the other hand, if the firms is also exporting we must adjust our analysis with respect to the previous case. The value of the firm is now a function of two state variables, the productivity level $z$ and a discrete state variable which indicates whether the firm is currently exporting or not. For a non exporting firm, in the region $(z_d, z_{hi})$ we have

$$(r+\delta)V_{ex,low}(z) = \alpha V'_{ex,low}(z) + \frac{1}{2}\xi^2 V''_{ex,low}(z)$$

which has the general solution

$$V_{ex,low}(z) = a_1 e^{\beta_1 z} + a_2 e^{\beta_2 z}$$

where $a_1$ and $a_2$ are constant to be determined and $\beta_1$ and $\beta_2$ are the roots of the quadratic equation determined earlier.

Since the option to export gets very far out of the money as $z$ becomes lower and lower, the coefficient $a_2$ corresponding to the negative root $\beta_2$ should be set to zero. This leaves,

$$V_{ex,low}(z) = a_1 e^{\beta_1 z}$$

Next let's consider the value of an exporting firm in the region $(z_{low}, \infty)$. The ordinary differential equation is

$$(r+\delta)V_{ex,hi}(z) = \pi_{ex}(z) + \alpha V'_{ex,hi}(z) + \frac{1}{2}\xi^2 V''_{ex,hi}(z)$$

where $\pi_{ex}(z)$ is given by equation (20).

The general solution to this equation (after setting the coefficient corresponding to the positive root to zero) is

$$V_{ex,hi}(z) = b_2 e^{\beta_2 z} - \frac{\tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}P\right)^{\sigma-1}}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z} - \frac{f_x}{(r+\delta)}$$

The boundary conditions are

$$\begin{aligned}
V_{ex,hi}(z_{hi}) &= V_{ex,low}(z_{hi}) + f_{hy} \\
V'_{ex,hi}(z_{hi}) &= V'_{ex,low}(z_{hi}) \\
V_{ex,hi}(z_{low}) &= V_{ex,low}(z_{low}) \\
V'_{ex,hi}(z_{low}) &= V'_{ex,low}(z_{low})
\end{aligned}$$

which after replacing the expressions for $V_{ex,low}(.)$ and $V_{ex,hi}(.)$ give a system of four equations in the four unknowns $z_{low}$, $z_{hi}$, $a_1$ and $b_2$. The first and third conditions guarantee continuity in the function expressing the value of the firm. Note that, unlike in the previous case without cost of entry, the value of an exporting firm and the value of a nonexporting firm are interlinked and must be determined simultaneously. The equations forming the system are very nonlinear in the thresholds so that it's difficult to get an analytical solution. A numerical solution can obtained but we start here by illustrating some properties about the thresholds (see the Appendix for derivations). First, the thresholds satisfy $0 < z_{low} < z_{hi} < \infty$ and the coefficients $a_1$ and $b_2$ are positive. Second, suppose that the firm is not an exporter and that it believes that $z$ will persist unchanged forever. The firm will decide to become an exporter if $\pi_{ex}(z) > (r + \delta)f_{hy}$. This is the exporting cutoff when there is no uncertainty and $z$ is constant over time. In our case instead, $\pi_{ex}(z_{hi}) > (r + \delta)f_{hy} > 0$ which means that $z_{hi}$, the exporting cutoff, is larger than the productivity level at which the firm decides to become an exporter when there is no uncertainty and $z$ is constant over time. This also implies that $z_{hi}$ is bigger than $z_{ex}$, the productivity cutoff for exporting in the no-entry-cost case. When domestic producers take into account the uncertainty over future profits, they are more reluctant to become exporters and when they are already exporters they are more reluctant to abandon. This is consistent with the difference between the average productivity for new and old exporters that we showed in Section2. Third, the width of the band of inaction $(z_{low}, z_{hi})$ is an increasing function of the sunk cost $f_{hy}$. As the sunk cost $f_{hy}$ increases the lower cutoff, $z_{low}$, is decreasing while the upper cutoff, $z_{hi}$, is increasing.

**Trade Equilibrium**

To determine the equilibrium with the assumption of positive reentry cost we proceed as in the previous case. However, we need to be careful to account for the export status within the band of inaction and we need to state the following,

**Conjecture** *There exists a probability density $\mu_e(z)$ with positive support over $(z_{low}, z_{hi})$ and such that $\mu_e(z_{low}) = 0$ and $\mu_e(z_{hi}) = \mu(z_{hi})$, representing the distribu-*

*tion of exporters over the band of inaction.*[12]

We can therefore determine the equilibrium price index in the economy using the expressions for the productivity cutoffs $z_d$, $z_{low}$ and $z_{hi}$ together with the following equation,

$$P^{1-\sigma} = M \int_{z_d}^{\infty} \left(\frac{1}{\bar{m}e^z}\right)^{1-\sigma} \mu(z)dz + (N-1)M \left[\int_{z_{low}}^{z_{hi}} \left(\frac{\tau}{\bar{m}e^z}\right)^{1-\sigma} \mu_e(z)dz + \int_{z_{hi}}^{\infty} \left(\frac{\tau}{\bar{m}e^z}\right)^{1-\sigma} \mu(z)dz\right]$$

As before new firms are trying to enter the industry until the expected value of the firm net of entry cost is zero, that is

$$f_e = \int_{z_d}^{z_{hi}} \left[V_d(\bar{z}) + (N-1)V_{low}(\bar{z})\right] dG(\bar{z}) + (N-1) \int_{z_{hi}}^{+\infty} \left[V_{hi}(\bar{z}) - V_{low}(\bar{z})\right] dG(\bar{z}) \tag{24}$$

where the equilibrium condition for the cutoffs are given by (10) and by the system showed earlier. The remaining steps are similar to those explained earlier for the no-cost of entry case. We just need to formally define the equilibrium concept for this case. An open-economy, entry-cost, stationary equilibrium is defined by

$$\left\{ \left[\left(C_t^h\right)_{t\geq 0}\right]_{h\in\Omega_h}, \left[\left(p^i(z_a)\right)_{a\geq 0}, \left(l^i(z_a)\right)_{a\geq 0}, D^i_{a\geq 0}, E^i_{a\geq 0}\right]_{i\in\Omega_I}, \right\}$$

where (i) each consumer $h$ chooses optimally sequences $\{C_t\}_{t\geq 0}$ of a composite goods to maximize the intertemporal utility function of equation (1) subject to an intertemporal budget constraint; (ii) incumbent firms choose, at every age $a$, price $p(z_a)$ and variable labor $l(z_a)$ in order to maximize domestic profits, given by equation (7), and eventual exporting profits, given by equation (**??**), taking the price index of the economy as given; (iii) incumbent firms, at every age $a$, decides if to keep being active ($D_a = 1$) or to exit ($D_a = 0$) and if to export ($E_a = 1$) or not ($E_a = 0$) knowing that productivity evolves according to the Brownian motion of equation (4); (iv) firms enter, if the productivity draw is bigger than $z_d$, until the free-entry condition (24) is satisfied; (v) labor and goods markets clear.

[12] Determination of a closed-form solution for $\mu_e(.)$ is the subject of ongoing research.

# 6 Conclusions and Extensions

This paper presents a model of international trade with heterogenous producers subject to uncertain productivity. Our innovation is to introduce firms specific permanent productivity shocks and derive a stationary industry equilibrium in a multi-country competition model.

We first embed firms subject to ex-post uncertainty into a monopolistic competition model. We derive the stationary distribution of firm characteristics and establish the conditions for the equilibrium of the economy under autarky. We then determine an equilibrium for an integrated world market with symmetric countries. Several results of previous trade models with heterogenous producers are derived. We then show how uncertainty alters in a nontrivial way some of the conclusions of previous studies. In particular, we show that introducing positive entry costs in a framework where productivity is evolving stochastically changes the well know partition of firms by exporting status. We derive explicit conditions to determine the factors that affect the band of inaction in which domestic firms continue to sell domestically and exporting firms continue to export. Our model retains the prediction that exporters are more productive than nonexporters but also allows for the natural fact that some nonexporters are more efficient than some exporters. Finally, in our framework, both entrants and incumbent firms might sustain temporary negative profits because of the expectation of becoming profitable later on. An important feature of the model is that it is amenable to simulation and estimation and can be used as an effective tool to better understand the consequences of trade opening and trade policies. We derived closed-form solutions for several static and dynamic moments that can be used to match the model to the data. The model can then be easily extended to analyze the effects of trade policies in a context of multiple asymmetric countries. The study of transition dynamics is a more complex but natural extension. All these are subjects of ongoing research.

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# A Appendix

## A.1 Value of the Firm

### A.1.1 Domestic

In order to derive the value of a firm that is selling only on the domestic market (equation (9)) and the corresponding productivity cutoff (equation (10)) we need to solve the following non-homogeneous ordinary differential equation

$$-(r+\delta)V_d(z)+\frac{1}{2}\xi^2 V_d''(z)+\alpha V_d'(z)+\pi_d(z)=0$$

where $\pi_d(z)=\frac{R}{\sigma}\left(\bar{m}^{-1}Pe^z\right)^{\sigma-1}-f_d=A_d e^{(\sigma-1)z}+B_d$ where $A_d=\frac{R}{\sigma}\left(\bar{m}^{-1}P\right)^{\sigma-1}$ and $B_d=-f_d$. The general solution of the differential equation is

$$V_d(z)=V_d^p(z)+V_d^h(z)$$

where $V_d^p(z)$ is a particular solution of the non-homogeneous ode and $V_d^h(z)$ is the general solution of the homogeneous ode. The latter has the form

$$V_d^h(z)=c_1 e^{\beta_1 z}+c_2 e^{\beta_2 z}$$

where $c_1$ and $c_2$ are two constants to be determined and $\beta_1$ and $\beta_2$ are the roots of the characteristic equation associated to the homogeneous ode,

$$\frac{1}{2}\xi^2\beta^2+\alpha\beta-(r+\delta)=0$$

that is,

$$\begin{aligned}\beta_1 &= -\frac{\alpha}{\xi^2}+\sqrt{\left(\frac{\alpha}{\xi^2}\right)^2+\frac{r+\delta}{\xi^2/2}}>0\\ \beta_2 &= -\frac{\alpha}{\xi^2}-\sqrt{\left(\frac{\alpha}{\xi^2}\right)^2+\frac{r+\delta}{\xi^2/2}}<0\end{aligned}$$

The general solution of the homogeneous equation represents the value of the option to exit. The likelihood of abandonment in the not-to-distant future becomes extremely small as $z$ goes to $\infty$, so the value of the abandonment option should go to zero as $z$ becomes very large. Hence the coefficient $c_1$ corresponding to the positive root $\beta_1$ should be zero. This leaves

$$V_d^h(z) = c_2 e^{\beta_2 z}$$

We need to find the particular solution to the non-homogeneous ode. Using the "undetermined coefficients" method, the particular solution when the forcing term has the form $A_d e^{(\sigma-1)z} + B_d$ is $V_d^p(z) = Ce^{(\sigma-1)z} + D$. Hence, we just need to plug this into the non-homogeneous ode and find $C$ and $D$ that makes $V_d^p(z)$ a particular solution. This delivers

$$V_d^p(z) = -\frac{A_d}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z} + \frac{B_d}{(r+\delta)}$$

and

$$V_d(z) = c_2 e^{\beta_2 z} - \frac{A_d}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z} + \frac{B_d}{(r+\delta)}$$

Now using the boundary conditions $V_d(z_d) = 0$ and $dV_d(z_d)/dz = 0$ we can determine $c_2$ and $z_d$. Let's start with $V_d(z_d) = 0$ to determine $c_2$.

$$\begin{aligned} V_d(z_d) &= 0 \;\;\Leftrightarrow\;\; c_2 e^{\beta_2 z_d} - \frac{A_d}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z_d} + \frac{B_d}{(r+\delta)} = 0 \\ c_2 &= \left( \frac{A_d}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z_d} - \frac{B_d}{(r+\delta)} \right) e^{-\beta_2 z_d} \end{aligned}$$

Now we can use $dV_d(z_d)/dz = 0$ and the expression for $c_2$ to determine $z_d$,

$$dV_d(z_d)/dz = 0 \quad \Leftrightarrow \quad c_2 \beta_2 e^{\beta_2 z_d} - \frac{A_d(\sigma-1)}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z_d} = 0$$

so that

$$e^{z_d} = \frac{\bar{m}}{P} \left( \frac{f_d \sigma}{R} \right)^{\frac{1}{(\sigma-1)}} \gamma_d$$

where $\gamma_d = \left[ \frac{-\beta_2}{(r+\delta)} \frac{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)}{\beta_2 - (\sigma-1)} \right]^{\frac{1}{(\sigma-1)}}$.

We can now find the general solution to the non-homogeneous ode (equation (9)),

$$V_d(z) = \frac{f_d}{(r+\delta)} \frac{\beta_2}{\beta_2 - (\sigma-1)} \left[ e^{(\sigma-1)(z-z_d)} - 1 - \frac{(\sigma-1)}{\beta_2} \left( e^{\beta_2(z-z_d)} - 1 \right) \right]$$

Note that

$$dV_d(z)/dz = \frac{f_d}{(r+\delta)}\frac{\beta_2(\sigma-1)}{\beta_2-(\sigma-1)}\left[e^{(\sigma-1)(z-z_d)} - e^{\beta_2(z-z_d)}\right] > 0$$

since $\sigma > 1$ and $\beta_2 < 0$.

**Interpretation of $V_d(z)$**

1. The two components of the general solution of the ode $V_d(z)$ have a straightforward interpretation. Using Ito's Lemma and recalling that $dz = \alpha da + \xi dB$, we can derive the stochastic process for the domestic variable profits of a firm $\pi_d^v(z) = \frac{R}{\sigma}\left(\bar{m}^{-1}Pe^z\right)^{\sigma-1}$.

$$d\pi_d^v(z) = \left[\alpha(\sigma-1)\pi_d^v(z) + 1/2\xi^2(\sigma-1)^2\pi_d^v(z)\right]da + \xi(\sigma-1)\pi_d^v(z)dB$$

that is, variable profits behave like a geometric Brownian motion with drift $\left[\alpha(\sigma-1)+1/2\xi^2(\sigma-1)^2\right]\pi_d^v(z)$ and diffusion $\xi(\sigma-1)\pi_d^v(z)$. Hence, if we denote today's variable profits by $\pi_d^v(z_a)$, the expected value and variance of variable profits $a^*$ years from now are

$$\begin{aligned} E\left[\pi_{d,a+a^*}^v\right] &= \pi_d^v(z_a)e^{\left[\alpha(\sigma-1)+1/2\xi^2(\sigma-1)^2\right]a^*} \\ V\left[\pi_{d,a+a^*}^v\right] &= \pi_d^v(z_a)e^{2\left[\alpha(\sigma-1)+1/2\xi^2(\sigma-1)^2\right]a^*}(e^{\xi^2(\sigma-1)^2a^*}-1) \end{aligned}$$

so that the expected present discounted value of variable profits over some period of time is

$$\begin{aligned} E\left[\int_0^\infty \pi_{d,a+a^*}^v e^{-(r+\delta)a^*}da^*\right] &= \int_0^\infty \pi_d^v(z_a)e^{\left[\alpha(\sigma-1)+1/2\xi^2(\sigma-1)^2\right]-(r+\delta)a^*}da^* \\ &= \frac{\pi_d^v(z_a)}{(r+\delta)-\alpha(\sigma-1)-1/2\xi^2(\sigma-1)^2} \end{aligned}$$

which represents the value of a firm without fixed costs $f_d$. Since $f_d$ is constant over time, the expected present discounted value of total profits over some period of time is

$$\frac{\pi_d^v(z_a)}{(r+\delta)-\alpha(\sigma-1)-1/2\xi^2(\sigma-1)^2} - \frac{f_d}{(r+\delta)} = V_d^p(z)$$

so that the other component of the general solution of the ode, $V_d^h(z)$, represents the value of the option to exit.

2. The exit cutoff $z_d$ is smaller than the zero-profit cutoff $z_0$. Note that $z_0 - z_d = (1-\gamma_d)z_0$. So we just need to prove that $\gamma_d < 1$. This condition can be expressed as

$$(r+\delta) > \alpha\beta_2 + 1/2\xi^2(\sigma-1)\beta_2$$

which is satisfied when $\alpha > 0$ since $\beta_2 < 0$, $\sigma > 1$.

### A.1.2 Export

In order to derive the component of the value of the firm due to export we need to solve the following non-homogeneous ordinary differential equation

$$-(r+\delta)V_{ex}(z) + \frac{1}{2}\xi^2 V''_{ex}(z) + \alpha V'_{ex}(z) + \pi_{ex}(z) = 0$$

where $\pi_{ex}(z) = \max\left[\tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}Pe^z\right)^{\sigma-1} - f_x, 0\right] = \max\left[A_{ex}e^{(\sigma-1)z} + B_{ex}, 0\right]$ and $A_{ex} = \tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}P\right)^{\sigma-1}$ and $B_{ex} = -f_x$. Since the forcing function is defined differently when current profits are positive or negative we need to solve the equation separately for the two cases and then stitch together the two solutions at the point where $\pi_{ex}(z) = 0$, that is at

$$e^{z_{ex}} = \frac{\tau\bar{m}}{P}\left(\frac{\sigma f_x}{R}\right)^{\frac{1}{\sigma-1}}$$

In the region $z < z_{ex}$, we have that $\pi_{ex}(z) = 0$ and only the homogeneous part of the equation remains. Therefore the general solution is

$$V_{ex}^{h0}(z) = k_1 e^{\beta_1 z} + k_2 e^{\beta_2 z}$$

where $k_1$ and $k_2$ are two constants to be determined and $\beta_1$ and $\beta_2$ are the roots, derived earlier, of the characteristic equation associated to the homogeneous ode.

In the region $z > z_{ex}$, we take another linear combination of the exponential solutions of the homogeneous part, and add on any particular solution of the full equation. Using the "undetermined coefficients" method, the particular solution is

$$V_{ex}^{p}(z) = -\frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z} + \frac{B_{ex}}{(r+\delta)}$$

and the general solution for the case $z > z_{ex}$ is

$$V_{ex}^{h1}(z) = b_1 e^{\beta_1 z} + b_2 e^{\beta_2 z} - \frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z} + \frac{B_{ex}}{(r+\delta)}$$

Now note that, in the region $z < z_{ex}$, operation is suspended but there is a positive probability that the productivity process will at some future time move into the region $z > z_{ex}$, when operation will resume and profits from exporting accrue. The value $V_{ex}^{h0}(z)$ when $z < z_{ex}$, is just the expected present value of such future flows. As $z$ becomes very small the event of its rising above $z_{ex}$ is very unlikely and so the value $V_{ex}^{h0}(z)$ should go to zero. We can therefore set the constant $k_2$, corresponding to the negative root $\beta_2$, to zero.

Now let's consider the region $z > z_{ex}$. The part of $V_{ex}^{h1}(z)$ different from the particular solution represents the additional value of the option to suspend operations in the future should $z$ fall below $z_{ex}$. As $z$ becomes very large the value of this option should go to zero and so we can set to zero $b_1$, the constant associated to the positive root $\beta_1$. We have then

$$V_{ex}(z) = \begin{cases} k_1 e^{\beta_1 z} & z < z_{ex} \\ b_2 e^{\beta_2 z} - \frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)} e^{(\sigma-1)z} + \frac{B_{ex}}{(r+\delta)} & z > z_{ex} \end{cases}$$

Note that since the Brownian motion can diffuse freely across the $z_{ex}$ boundary, the value function cannot change abruptly across it. The solution must be continuously differentiable across $z_{ex}$. We therefore have the following two conditions

$$\begin{aligned} k_1 e^{\beta_1 z_{ex}} &= b_2 e^{\beta_2 z_{ex}} - \frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z_{ex}} + \frac{B_{ex}}{(r+\delta)} \\ k_1 \beta_1 e^{\beta_1 z_{ex}} &= b_2 \beta_2 e^{\beta_2 z_{ex}} - \frac{A_{ex}(\sigma-1)}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z_{ex}} \end{aligned}$$

which is a system of two equations in two unknowns, $k_1$ and $b_2$. The solutions are

$$b_2 = \frac{\beta_1}{\beta_2 - \beta_1\beta_2}\left[\begin{array}{c} \frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)} e^{(\sigma-1-\beta_2)z_{ex}}\left(\frac{\sigma-1-\beta_1}{\beta_1}\right) \\ +\frac{B_{ex}}{(r+\delta)}e^{-\beta_2 z_{ex}} \end{array}\right] > 0$$

and

$$k_1 = \left\{\begin{array}{c} \frac{\beta_2}{\beta_2-\beta_1\beta_2}\left[\frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)}e^{(\sigma-1)z_{ex}}\left(\frac{\sigma-1-\beta_1}{\beta_1}\right)+\frac{B_{ex}}{(r+\delta)}\right] \\ -\frac{A_{ex}}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)}\frac{1}{\beta_1}e^{(\sigma-1)z_{ex}} \end{array}\right\} e^{-\beta_1 z_{ex}} > 0$$

We can now find the final expression for $V_{ex}(z)$. For the region where $z > z_{ex}$ we have equation (21),

$$\begin{aligned} V_{ex}(z) &= \frac{f_x}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)}\left[e^{\beta_2(z-z_{ex})}\left(\frac{\sigma-1-\beta_1}{\beta_2-\beta_1\beta_2}\right)-e^{(\sigma-1)(z-z_{ex})}\right] \\ &\quad -\frac{f_x}{(r+\delta)}\left[\frac{\beta_1}{\beta_2-\beta_1\beta_2}e^{\beta_2(z-z_{ex})}+1\right] \end{aligned}$$

while for the region where $z < z_{ex}$ we have equation (22),

$$V_{ex}(z) = \left\{\frac{f_x}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)}\frac{\beta_2(\sigma-2)}{\beta_1}-\beta_2\frac{f_x}{(r+\delta)}\right\}\frac{e^{\beta_1(z-z_{ex})}}{\beta_2-\beta_1\beta_2}$$

## A.2 Stationary Distribution

### A.2.1 The Density $m(z|\bar{z})$

In order to derive the conditional density $m(z|\bar{z})$ start from

$$m(a,z|\bar{z}) = e^{-\delta a}\psi(a,z|\bar{z})$$

where

$$\psi(a,z|\bar{z}) = \frac{1}{\xi\sqrt{a}}\left[\phi\left(\frac{z-\bar{z}-\alpha a}{\xi\sqrt{a}}\right)-e^{-\frac{2\alpha}{\xi^2}(\bar{z}-z_d)}\phi\left(\frac{z+\bar{z}-2z_d-\alpha a}{\xi\sqrt{a}}\right)\right]$$

The roots of the characteristic polynomial of the Kolmogorov equations are non-negative and equal to

$$\begin{aligned} \theta &= \frac{1}{\xi^2}\left(-\alpha+\sqrt{\alpha^2+2\xi^2\delta}\right) \\ \theta_* &= \frac{1}{\xi^2}\left(\alpha+\sqrt{\alpha^2+2\xi^2\delta}\right) \end{aligned}$$

Integrating out $a$ we obtain an expression for the marginal probability of productivities given an initial starting level. We are going to consider only $\bar{z} > z_d$, so that, since $z > z_d$, we have $z + \bar{z} - 2z_d > 0$.

$$
\begin{aligned}
m(z|\bar{z}) &= \int_0^\infty m(a, z|\bar{z}) da \\
&= \int_0^\infty e^{-\delta a} \frac{1}{\xi\sqrt{a}} \phi\left(\frac{z - \bar{z} - \alpha a}{\xi\sqrt{a}}\right) da - e^{-\frac{2\alpha}{\xi^2}(\bar{z} - z_d)} \int_0^\infty e^{-\delta a} \phi\left(\frac{z + \bar{z} - 2z_d - \alpha a}{\xi\sqrt{a}}\right) da \\
&= m_1 - e^{-\frac{2\alpha}{\xi^2}(\bar{z} - z_d)} m_2
\end{aligned}
$$

where the solutions of the two integrals $m_1$ and $m_2$ is,

$$
\begin{aligned}
m_1 &= \frac{\min\left\{e^{\theta_*(z-\bar{z})}, e^{-\theta(z-\bar{z})}\right\}}{\sqrt{\alpha^2 + 2\xi^2\delta}} \\
m_2 &= \frac{\exp\left(-\theta\left[z + \bar{z} - 2z_d\right]\right)}{\sqrt{\alpha^2 + 2\xi^2\delta}}
\end{aligned}
$$

Now we can combine the results to get

$$
m(z|\bar{z}) = -\frac{1}{\alpha}\frac{\theta - \theta_*}{\theta + \theta_*}\frac{\left[\min\left\{e^{(\theta+\theta_*)(z-z_d)}, e^{(\theta+\theta_*)(\bar{z}-z_d)}\right\} - 1\right]}{e^{\theta_*(\bar{z}-z_d)}e^{\theta(z-z_d)}}
$$

The last step is to convert $m(z|\bar{z})$ into a probability density. Integrating $m(z|\bar{z})$ over $z$, we find

$$
\int_{z_d}^\infty m(z|\bar{z}) dz = -\frac{1}{\alpha}\frac{\theta - \theta_*}{\theta\theta_*}\left[1 - e^{-\theta_*(\bar{z}-z_d)}\right] = \frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}
$$

that can be used to transform $m(z|\bar{z})$ and $m(a, z|\bar{z})$ into the probability densities (14) and (15)

$$
\mu(z|\bar{z}) = \frac{m(z|\bar{z})}{\int_{z_d}^\infty m(z|\bar{z}) dz} = \frac{\theta}{\theta + \theta_*}\left[\frac{\min\left\{e^{(\theta_*+\theta)(z-z_d)}, e^{(\theta+\theta_*)(\bar{z}-z_d)}\right\} - 1}{e^{\theta(z-z_d)}\left(\frac{e^{\theta_*(\bar{z}-z_d)} - 1}{\theta_*}\right)}\right]
$$

and

$$
\mu(a, z|\bar{z}) = \frac{m(a, z|\bar{z})}{\int_{z_d}^\infty m(z|\bar{z}) dz} = \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right]^{-1} e^{-\delta a}\psi(a, z|\bar{z})
$$

### A.2.2 Mean Value of Productivity

We need to calculate the mean of the stationary distribution as

$$
\begin{aligned}
& \int_{z_d}^{\infty} e^{z-z_d}\mu(z|\bar{z})dz \\
= & \int_{z_d}^{\infty} e^{z-z_d}\frac{\theta}{\theta+\theta_*}\left[\frac{\min\left\{e^{(\theta_*+\theta)(z-z_d)}, e^{(\theta+\theta_*)(\bar{z}-z_d)}\right\}-1}{e^{\theta(z-z_d)}\left(\frac{e^{\theta_*(\bar{z}-z_d)}-1}{\theta_*}\right)}\right]dz \\
= & \frac{\theta\theta_*}{\theta+\theta_*}\frac{1}{e^{\theta_*(\bar{z}-z_d)}-1}\left\{\begin{array}{c} \int_{z_d}^{\bar{z}}\left[e^{(\theta_*+1)(z-z_d)}-e^{(1-\theta)(z-z_d)}\right]dz \\ +\left(e^{(\theta+\theta_*)(\bar{z}-z_d)}-1\right)e^{(1-\theta)(-z_d)}\int_{\bar{z}}^{\infty}e^{(1-\theta)z}dz \end{array}\right\}
\end{aligned}
$$

Assumption 2 guarantees that $\theta > 1$ which is necessary for the last integral, in the previous line, to be finite. We can then go on with,

$$
\begin{aligned}
= & \frac{\theta\theta_*}{\theta+\theta_*}\frac{1}{e^{\theta_*(\bar{z}-z_d)}-1}\left\{\begin{array}{c} \int_{z_d}^{\bar{z}}\left[e^{(\theta_*+1)(z-z_d)}-e^{(1-\theta)(z-z_d)}\right]dz \\ +\left(e^{(\theta+\theta_*)(\bar{z}-z_d)}-1\right)e^{(1-\theta)(-z_d)}\frac{e^{(1-\theta)\bar{z}}}{(\theta-1)} \end{array}\right\} \\
= & \frac{\theta}{(\theta-1)}\left[\frac{\frac{1-e^{-(1+\theta_*)(\bar{z}-z_d)}}{(\theta_*+1)}}{\frac{1-e^{-\theta_*(\bar{z}-z_d)}}{\theta_*}}\right]e^{\bar{z}-z_d}
\end{aligned}
$$

Hence, we derive equation (16),

$$
e^{\tilde{z}} = \int_{z_d}^{\infty} e^{z-z_d}\mu(z|\bar{z})dz = \frac{\theta}{(\theta-1)}\left[\frac{\frac{1-e^{-(1+\theta_*)(\bar{z}-z_d)}}{(\theta_*+1)}}{\frac{1-e^{-\theta_*(\bar{z}-z_d)}}{\theta_*}}\right]e^{\bar{z}-z_d}
$$

### A.2.3 Equilibrium Rate of Entry

Consider the stationary distribution

$$
\mu(a,z,\bar{z}) = e^{-\delta a}\psi(a,z|\bar{z})g(\bar{z})\frac{M_a}{M}
$$

The latter is a probability density for a particular value of $M_a/M$. This is used to determine the amount of entry that must take place relative to the number of existing firms. We will use the fact that $p(a, z|\bar{z})$ is a density function.

$$1 = \int_{z_d}^{\infty}\int_{z_d}^{\infty}\int_{0}^{\infty} \mu(a, z, \bar{z}) da dz d\bar{z} = \int_{z_d}^{\infty}\int_{z_d}^{\infty}\int_{0}^{\infty} e^{-\delta a}\psi(a, z|\bar{z})g(\bar{z})\frac{M_a}{M} da dz d\bar{z}$$

Replace the definition of $p(a, z|\bar{z})$,

$$1 = \int_{z_d}^{\infty}\int_{z_d}^{\infty}\int_{0}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right] \mu(a, z|\bar{z})g(\bar{z})\frac{M_a}{M} da dz d\bar{z}$$

so that the equilibrium rate of entry is

$$\frac{M_a}{M} = \left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right] g(\bar{z}) d\bar{z}\right]^{-1}$$

### A.2.4 The Stationary Distribution of Productivity

Use the attempted entry rate equation (17) to get the probability density

$$\mu(a, z, \bar{z}) = \frac{e^{-(\varepsilon+\eta)a}\psi(a, z \mid \bar{z})g(\bar{z})}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right] g(\bar{z}) d\bar{z}\right]}$$

and replace the equation for $p(a, z|\bar{z})$ into the above to get

$$\mu(a, z, \bar{z}) = \mu(a, z|\bar{z}) \frac{\left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right] g(\bar{z})}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right] g(\bar{z}) d\bar{z}\right]}$$

Now integrate with respect to $a$ and over $\bar{z}$ in the region $(z_d, \infty)$ to get

$$\mu(z) = \frac{\int_{z_d}^{\infty} \mu(z|\bar{z}) \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right] g(\bar{z}) d\bar{z}}{\left[\int_{z_d}^{\infty} \left[\frac{1 - e^{-\theta_*(\bar{z}-z_d)}}{\delta}\right] g(\bar{z}) d\bar{z}\right]}$$

### A.2.5 Average Age, Productivity and Size of Exporters

For the case with no hysteresis, we need to derive the marginal probability density of age and the marginal probability density of productivity over the interval $(z_{ex}, \infty)$. For age, we start by integrating $\mu(a, z, \bar{z})$ over $\bar{z}$ to derive $\mu(a, z)$. Then we get the conditional probability density $\mu(a, z|z > z_{ex})$ as the ratio of $\mu(a, z)$ and the probability that $z > z_{ex}$,

$$\begin{aligned} \mu(a, z|z > z_{ex}) &= \frac{\mu(a, z)}{\Pr_{\mu(z)}[z > z_{ex}]} \\ &= \frac{\int_{z_d}^{\infty} \mu(a, z|\bar{z})\mu(\bar{z})d\bar{z}}{\int_{z_{ex}}^{\infty}\int_{z_d}^{\infty} \mu(z|\bar{z})\mu(\bar{z})d\bar{z}dz} \end{aligned}$$

so that the average age of exporters is given by

$$\begin{aligned} \bar{a}_{ex} &= \int_{0}^{\infty} a\mu(a|z > z_{ex})da \\ &= \int_{0}^{\infty} a \int_{z_{ex}}^{\infty} \frac{\int_{z_d}^{\infty} \mu(a, z|\bar{z})\mu(\bar{z})d\bar{z}}{\int_{z_{ex}}^{\infty}\int_{z_d}^{\infty} \mu(z|\bar{z})\mu(\bar{z})d\bar{z}dz} dzda \end{aligned}$$

Let's now consider productivity. We just need to derive $\mu(z|z > z_{ex})$ starting from $\mu(a, z|z > z_{ex})$,

$$\mu(z|z > z_{ex}) = \int_{0}^{\infty} \frac{\int_{z_d}^{\infty} \mu(a, z|\bar{z})\mu(\bar{z})d\bar{z}}{\int_{z_{ex}}^{\infty}\int_{z_d}^{\infty} \mu(z|\bar{z})\mu(\bar{z})d\bar{z}dz} da$$

and the average productivity of exporters is

$$\begin{aligned}
\bar{z}_{ex} &= \int_{z_{ex}}^{\infty} z\mu(z|z > z_{ex})dz \\
&= \int_{z_{ex}}^{\infty} z \int_{0}^{\infty} \frac{\int_{z_d}^{\infty} \mu(a, z|\bar{z})\mu(\bar{z})d\bar{z}}{\int_{z_{ex}}^{\infty}\int_{z_d}^{\infty} \mu(z|\bar{z})\mu(\bar{z})d\bar{z}dz} dadz
\end{aligned}$$

while the average size, in terms of revenues, is

$$\begin{aligned}
\bar{r}_{ex} &= \int_{z_{ex}}^{\infty} R\left(\bar{m}^{-1}Pe^{z}\right)^{\sigma-1} \mu(z|z > z_{ex})dz \\
&= R\left(\bar{m}^{-1}P\right)^{\sigma-1} \int_{z_{ex}}^{\infty} e^{(\sigma-1)z} \int_{0}^{\infty} \frac{\int_{z_d}^{\infty} \mu(a, z|\bar{z})\mu(\bar{z})d\bar{z}}{\int_{z_{ex}}^{\infty}\int_{z_d}^{\infty} \mu(z|\bar{z})\mu(\bar{z})d\bar{z}dz} dadz
\end{aligned}$$

### A.2.6 Survivor Function

We derive the survivor function for firms of age $a$ in two steps[13]: first we derive the survivor function for firms whose current productivity is $x$, $P(T(z_d) > t|x)$. Then we derive the probability density of productivity conditional on age, $\mu(x|a)$. Finally we combine the two to find $P(T(z_d) > t|a)$, the probability of exiting no sooner than $t$ years from now given the current age of the firm $a$.

$$P(T(z_d) > t|x) = 1 - \Phi\left(\frac{z_d + x + \alpha\xi t}{\xi\sqrt{t}}\right) e^{\frac{2}{\xi}\alpha(z_d - x)} - \Phi\left(\frac{z_d - x - \alpha\xi t}{\xi\sqrt{t}}\right)$$

[13] Derivation is based on Harrison (1985) and is available upon request.

$$\mu(x|a) = \frac{\mu(a,x)}{\mu(a)} = \frac{\int_{z_d}^{\infty} \mu(a,x,\bar{z})d\bar{z}}{\int_{z_d}^{\infty}\int_{z_d}^{\infty} \mu(a,x,\bar{z})d\bar{z}dx}$$

so that

$$P(T(z_d) > t|a) = \int_{z_d}^{\infty} P(T(z_d) > t|x)\mu(x|a)dx$$

## A.3 Hysteresis Derivations

### A.3.1 System of Equations

The system of equations that implicitly defines the threshold values $z_{low}$ and $z_{hi}$ and the constants $a_1$ and $b_2$ is

$$\begin{aligned}
b_2 e^{\beta_2 z_{hi}} - \frac{\tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}P\right)^{\sigma-1}}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)} e^{(\sigma-1)z_{hi}} - \frac{f_x}{(r+\delta)} &= a_1 e^{\beta_1 z_{hi}} + f_{hy} \\
\beta_2 b_2 e^{\beta_2 z_{hi}} - \frac{\tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}P\right)^{\sigma-1}(\sigma-1)}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)} e^{(\sigma-1)z_{hi}} &= \beta_1 a_1 e^{\beta_1 z_{hi}} \\
b_2 e^{\beta_2 z_{low}} - \frac{\tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}P\right)^{\sigma-1}}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)} e^{(\sigma-1)z_{low}} - \frac{f_x}{(r+\delta)} &= a_1 e^{\beta_1 z_{low}} \\
\beta_2 b_2 e^{\beta_2 z_{low}} - \frac{\tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}P\right)^{\sigma-1}(\sigma-1)}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)} e^{(\sigma-1)z_{low}} &= \beta_1 a_1 e^{\beta_1 z_{low}}
\end{aligned}$$

### A.3.2 Properties of $z_{low}$ and $z_{hi}$

**Exporting Cutoff ($z_{hi}$)** Define the function,

$$\begin{aligned}
D(z) &\equiv V_{ex,hi}(z) - V_{ex,low}(z) \\
&= b_2 e^{\beta_2 z} - a_1 e^{\beta_1 z} - \frac{\tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}P\right)^{\sigma-1}}{\frac{1}{2}\xi^2(\sigma-1)^2+\alpha(\sigma-1)-(r+\delta)} e^{(\sigma-1)z} - \frac{f_x}{(r+\delta)}
\end{aligned}$$

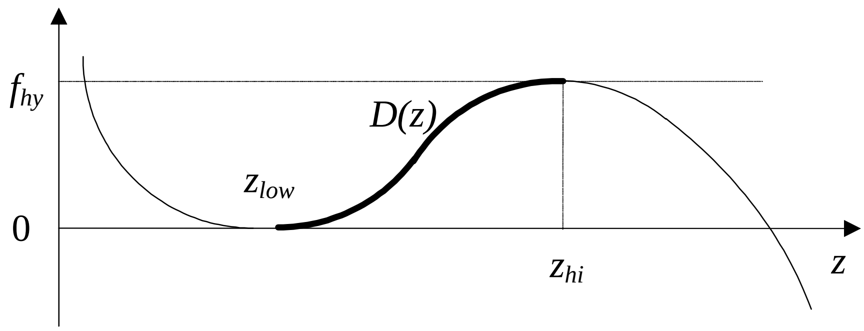


Figure 6: Firm's Incremental Value of Becoming an Exporter over $(z_{low}, z_{hi})$

that can be interpreted as the firm's incremental value of becoming an exporter, over the range $(z_{low}, z_{hi})$.[14] When $z$ is small, the dominant term in $D(z)$ is the one with the negative root $\beta_2$. It is decreasing and convex in $z$. When $z$ is large the dominant term is the one with the positive root $\beta_1$.[15] This term is negative, decreasing and concave. For intermediate values, the third term contributes to the increasing portion of $D(z)$ (see Figure (6)).

Consider the upper threshold $z_{hi}$. Subtracting the differential equation for $V_{low}$ from the one for $V_{hi}$, we have

$$(r+\delta)D(z) = \pi_{ex}(z) + \alpha D'(z) + \frac{1}{2}\xi^2 D''(z)$$

Evaluating at $z_{hi}$ and using the boundary conditions that must hold at $z_{hi}$, we get

$$-(r+\delta)f_{hy} + \pi_{ex}(z_{hi}) = -\frac{1}{2}\xi^2 D''(z_{hi}) > 0$$

or $\pi_{ex}(z_{hi}) > (r+\delta)f_{hy} > 0$ which means that $z_{hi}$ is larger than the productivity level at which the firm decides to become an exporter when there is no uncertainty and $z$ is constant over time.

[14] Here we follow a similar case illustrated in Dixit and Pindyck (1994).

[15] This is guaranteed by Assumption 1, which by stating that $\rho + \delta > \alpha(\sigma-1) + \xi^2(\sigma-1)^2/2$, actually impose that $(\sigma - 1) < \beta_1$. To prove it just solve the inequality for $(\sigma-1)$ and find that the solutions gives $|\sigma - 1| < \beta_1$. Recall also that demand is elastic so that $\sigma > 1$.

**Width of the Band of Inaction** Define, over the range $(z_{low}, z_{hi})$,

$$\begin{aligned} D(z) &= V_{ex,hi}(z) - V_{ex,low}(z) \\ &= b_2 e^{\beta_2 z} - a_1 e^{\beta_1 z} - \frac{\tau^{1-\sigma}\frac{R}{\sigma}\left(\bar{m}^{-1}P\right)^{\sigma-1}}{\frac{1}{2}\xi^2(\sigma-1)^2 + \alpha(\sigma-1) - (r+\delta)} e^{(\sigma-1)z} - \frac{f_x}{(r+\delta)} \end{aligned}$$

Write the value-matching and smooth-pasting conditions in terms of $D(z)$,

$$\begin{aligned} D(z_{hi}, a_1, b_2) &= f_{hy} \\ D_z(z_{hi}, a_1, b_2) &= 0 \\ D(z_{low}, a_1, b_2) &= 0 \\ D_z(z_{low}, a_1, b_2) &= 0 \end{aligned}$$

We are going to find out what is the effect of a small change in $f_{hy}$ on the cutoff thresholds $z_{low}$ and $z_{hi}$. First, totally differentiate the value-matching conditions

$$D_z(z_{hi}, a_1, b_2)dz_{hi} + D_{a_1}(z_{hi}, a_1, b_2)da_1 + D_{b_2}(z_{hi}, a_1, b_2)db_2 = df_{hy}$$

$$D_z(z_{low}, a_1, b_2)dz_{low} + D_{a_1}(z_{low}, a_1, b_2)da_1 + D_{b_2}(z_{low}, a_1, b_2)db_2 = 0$$

which, using the smooth-pasting condition, simplify to

$$D_{a_1}(z_{hi}, a_1, b_2)da_1 + D_{b_2}(z_{hi}, a_1, b_2)db_2 = df_{hy}$$

$$D_{a_1}(z_{low}, a_1, b_2)da_1 + D_{b_2}(z_{low}, a_1, b_2)db_2 = 0$$

and

$$-e^{\beta_1 z_{hi}} da_1 + e^{\beta_2 z_{hi}} db_2 = df_{hy}$$

$$-e^{\beta_1 z_{low}} da_1 + e^{\beta_2 z_{low}} db_2 = 0$$

Solving the system we find that $db_2 = e^{(\beta_1-\beta_2)z_{low}} da_1$ and that $da_1 = \chi df_{hy}$ where $\chi = \left(e^{\beta_2(z_{hi}-z_{low})} e^{\beta_1 z_{low}} - e^{\beta_1 z_{hi}}\right)^{-1} < 0$ since $e^{\beta_2(z_{hi}-z_{low})} < 1$, $\beta_1 > 0$ and $z_{hi} > z_{low}$. Now differentiate the first smooth-pasting condition at $z_{hi}$

$$D_{zz}(z_{hi}, a_1, b_2)dz_{hi} + D_{za_1}(z_{hi}, a_1, b_2)da_1 + D_{zb_2}(z_{hi}, a_1, b_2)db_2 = 0$$

that, after using the expressions for $da_1$ and $db_2$ yields

$$\begin{aligned}
D_{zz}(z_{hi}, a_1, b_2)dz_{hi} &= \beta_1 e^{\beta_1 z_{hi}} da_1 - \beta_2 e^{\beta_2 z_{hi}} db_2 \\
&= \frac{\left[\beta_1 e^{\beta_1 z_{hi}} - \beta_2 e^{\beta_2 z_{hi} + (\beta_1 - \beta_2) z_{low}}\right]}{\left(e^{\beta_2 (z_{hi} - z_{low})} e^{\beta_1 z_{low}} - e^{\beta_1 z_{hi}}\right)} df_{hy}
\end{aligned}$$

Recall that $D(z)$ is concave at $z_{hi}$ so that $D_{zz}(z_{hi}, a_1, b_2) < 0$. Note that the term in front of $df_{hy}$ is negative as well so that $dz_{hi}/df_{hy} > 0$. When the cost of entering the export market is higher, the export cutoff is also higher.

Now differentiate the second smooth-pasting condition at $z_{low}$

$$D_{zz}(z_{low}, a_1, b_2)dz_{low} + D_{za_1}(z_{low}, a_1, b_2)da_1 + D_{zb_2}(z_{low}, a_1, b_2)db_2 = 0$$

which yields

$$\begin{aligned}
D_{zz}(z_{low}, a_1, b_2)dz_{low} &= \beta_1 e^{\beta_1 z_{low}} da_1 - \beta_2 e^{\beta_2 z_{low}} db_2 \\
&= \frac{\left[\beta_1 e^{\beta_1 z_{low}} - \beta_2 e^{\beta_2 z_{low} + (\beta_1 - \beta_2) z_{low}}\right]}{\left(e^{\beta_2 (z_{hi} - z_{low})} e^{\beta_1 z_{low}} - e^{\beta_1 z_{hi}}\right)} df_{hy}
\end{aligned}$$

Recall that $D(z)$ is convex at $z_{low}$ so that $D_{zz}(z_{low}, a_1, b_2) > 0$. Note that the term in front of $df_{hy}$ is still negative so that $dz_{low}/df_{hy} < 0$. When the cost of entering the export market is higher, the export-abandon cutoff is lower. This proves that the width of the band of inaction is an increasing function of $f_{hy}$.