Trade balance and terms of trade in U.S.: a time-scale decomposition analysis

Luca De Benedictis*and Marco Gallegati†

October 8, 2005

Abstract

The aim of this paper is to provide evidence on the nature of the relationship between the terms of trade and the trade balance for US on a scale-by-scale basis using wavelet analysis. Thus, after decomposing the two variables into their time-scale components using the *maximum overlap discrete wavelet transform (MODWT)*we analyze the time scale relationships between the terms of trade and the trade balance through the wavelet correlation analysis, and nonparametric regression models (GAMs). Wavelet correlation analysis indicates that, if the association between the trade balance and the terms of trade depends mainly on the elasticity of substitution between foreign and domestic goods, the Armington elasticities may be different across scales, and in particular, tend to get larger as the time horizon of the agents increases. Moreover, the long-run relationship between the trade balance and the terms of trade from the nonparametric fitted functions seems to provide support to the existence of the Harberger-Laursen-Metzler effect .

Keywords: Trade variables, Wavelet correlation analysis, Generalized Additive Models

JEL codes: C12, C22, E30, F10

*University of Macerata, Macerata, Italy

†DEA, Universit Politecnica delle Marche, Ancona, Italy

1 Introduction

The role of changes in the terms of trade has been widely analyzed in the trade theory literature, and most of this literature aims to test what is known as the Harberger-Laursen-Metzler effect which states that an improvement in the terms of trade improves, via country's real income, the trade balance (Harberger, 1950, Laursen and Metzler, 1950). International trade models states that the relation between the terms of trade and the trade balance depends on the elasticity of substitution between foreign and domestic goods, *i.e.* the Armington elasticity (Armington, 1969), but the two main frameworks, international real business cycle models and static applied general equilibrium models, have very different views about the value that the Armington elasticity should take (low and high values, respectively). A way to reconcile these opposite views is to consider the nature of the shocks to the terms of trade and to distinguish them according to their permanent or transitory nature, because if the agents react differently to permanent and temporary changes in the terms of trade, the values of the Armington elasticities will differ (Ruhl, 2003).

A potential shortcoming of conventional empirical analyses among trade variables and ouput may concern the fact that the nature of the relationship may change at different time horizons. Economic analysis has usually been restricted to at most two time horizons, that is the short run and the long run; this not because economic decisions can be referred only to these two scales, but mainly due to the lack of analytical tools that could decompose economic time series into more than just two time scales. International trade provides an example of markets in which, as the agents involved, firms and consumers, interact at different time horizons, the relationships among trade variables may well vary across time scales. In such a context, a useful analytical tool may be wavelet analysis. Wavelets are particular types of function $f(x)$ that are localized both in time and frequency domain and used to decompose a function $f(x)$, i.e. a signal, a surface, a series, etc..) in more elementary functions which include information about the same $f(x)$. The main advantage of wavelet analysis is its ability to decompose macroeconomic time series, and data in general, into their time scale components. Several applications of wavelet analysis in economics and finance have been recently provided by Ramsey and Lampart (1998a, 1998b), Ramsey (2002), Kim and In (2003) and Crivellini *et al.* (2005) among the others, but no attempts have been made to apply this methodology to the analysis of trade variables.

The objective of this paper is to provide evidence on the nature of the relationship between the terms of trade and the trade balance for US on a scale-by-scale basis, as it may help to isolate some key features of the data and thereby may provide building blocks for theoretical models of the dynamics of international trade. Thus, after decomposing the trade balance (defined as the natural logharitm of the ratio of exports to imports values) and the terms of trade (defined as the relative price of imports to exports) into their time-scale components using to the *maximum overlap discrete wavelet transform (MODWT)*, we analyze the relationship among these variables at the different time scales using i) wavelet correlation analysis, as it may provide a lead/lag relationship between the two processes, and ii) generalized additive model (GAM), as this framework may enable us to characterize the dynamic relationships among these variables without making any a priori explicit or implicit assumption about the shape of the relationship. The paper is organized as follows: section 2 presents the data and the methodology, while section 3 is devoted to the analysis of the relationship between the trade balance and the terms of trade using corss-correlation functions in the frequency and time scale domains. In section 4 we analyze the shape and the significance of the relationship using generalized additive model, and section 5 concludes the paper (the main properties of the wavelets as well as the method for calculating the wavelet correlation coefficient are dealt with in the Appendix).

2 Data and methodology

The series were filtered using a relatively new (at least for economists) statistical tool, the discrete wavelet transform, that, roughly speaking, decomposes a given series in orthogonal components, as in the Fourier approach, but according to scale (time components) instead of frequencies. The comparison with the Fourier analysis is useful first because wavelets use a similar strategy: find some orthogonal objects (wavelets functions instead of sines and cosines) and use them to decompose the series. Second, since Fourier analysis is a common tool in economics, it may be useful in understanding the methodology and also in the interpretation of results. Saying that, we have to stress the main difference between the two tools. Wavelet analysis does not need stationary assumption in order to decompose the series. This is because Fourier approach decomposes in frequency space that may be interpreted as events of time-period T (where T is the number of observations). Put differently, spectral decomposition methods perform a global analysis whereas, on the other hand, wavelets methods act locally in time and so do not need stationary cyclical components. Recently, to relax the stationary frequencies assumption it has been developed a windowing Fourier decomposition that essentially use, for frequencies estimation, a time-period M (the window) event less than the number of observations T. The problem with this approach is the right choice of the window and, more important, its constancy over time. Many economic and financial time series are nonstationary and, moreover, exhibits changing frequencies over time. Much of the usefulness of wavelet analysis has to do with its flexibility in handling a variety of nonstationary signals. Indeed, as wavelets are constructed over finite intervals of time and are not necessarily homogeneous over time, they are localized in both time and scale. Thus, two interesting features of wavelet time scale decomposition for economic variables will be that, i) since the base scale includes any non-stationary components, the data need not be detrended or differenced, and ii) the nonparametric nature of wavelets takes care of potential nonlinear relationships without losing detail (Schleicher, 2002).

Figure 1 about here

Figure 1 shows the time series plots of the raw time series. The data for the terms of trade and the trade balance refers to the US between 1947:1 and 2004:2 and are from the Federal Reserve Bank of St. Louis (FRED II). We measure the terms of trade as the relative price of imports to exports using implicit price deflators, and the trade balance as the natural logharitm of the ratio of exports to imports values.[1] We perform a *J-level* decomposition of the quarterly series of the terms of trade and trade balance for US using the *maximal overlap discrete wavelet transform* (*MODWT*) which is a non-orthogonal variant of the classical discrete wavelet transform that, unlike the orthogonal discrete wavelet transform, is translation invariant, as shifts in the signal do not change the pattern of coefficients. The wavelet filter used in the decomposition is the Daubechies least asymmetric (LA) wavelet filter of lenght $L = 8$, that is $LA(8)$, based on eight non-zero coefficients (Daubechies, 1992), with periodic boundary conditions. The application of the translation invariant wavelet transform with a number of scales $J = 5$ produces six wavelet and scaling filter coefficients $v_5, w_5, w_4, w_3, w_2, w_1$ which, translated back into the time domain, gives the smooth V_5 and detail W_5 , W_4 , W_3 , W_2 , W_1 signals. Since we use quarterly data scale 1 represents 2-4 quarters period dynamics, while scales 2, 3, 4 and 5 correpond to 4-8, 8-16, 16-32 and 32-64 quarters period dynamics, respectively.

3 Correlation analysis in the time scale domain

In this section we investigate the relationship between the trade balance and the terms of trade over different time scales using wavelet correlation analysis, as a scale-by-scale analysis may provide further insight into the contemporaneous and lead/lag relationship between the trade balance and the terms of trade. A useful reference for this analysis may be represented by the results in Backus *et al.* (1994). According to Backus *et al.* "the most important parameter for the trade balance - terms of trade relationship is

[1] One reason for its use is that the ratio is not sensitive to the unit of measurement and can be interpreted both as nominal or real trade balance (Bahamani-Oskooee, 1991). Other studies (Backus *et al.*, 1994, among the others) use the ratio of the net exports to output.

the elasticity of substitution between foreign and domestic goods" (Backus *et al.* 1994, p.94). Indeed, the experiments conducted to document the properties of their theoretical economy suggest that the parameter of the elasticity of substitution affects signficantly the contemporaneous correlation between net exports and the terms of trade, as this correlation is negative for small elasticities and positive for large elasticities. In figure 1 we report the wavelet correlation coefficients at lag 0 against the wavelet scales between the trade balance and the terms of trade. The correlation between the two series is negative at the finest crystals, that is from d_1 to d_3 , and positive at the coarsest crystals, that is d_4 and d_5 , that, according to Backus *et al.* interpretation, may be interpreted as evidence that the elasticity of substitution between foreign and domestic goods, *i.e.* the Armington elasticity, tends to increase as the time scale increases.

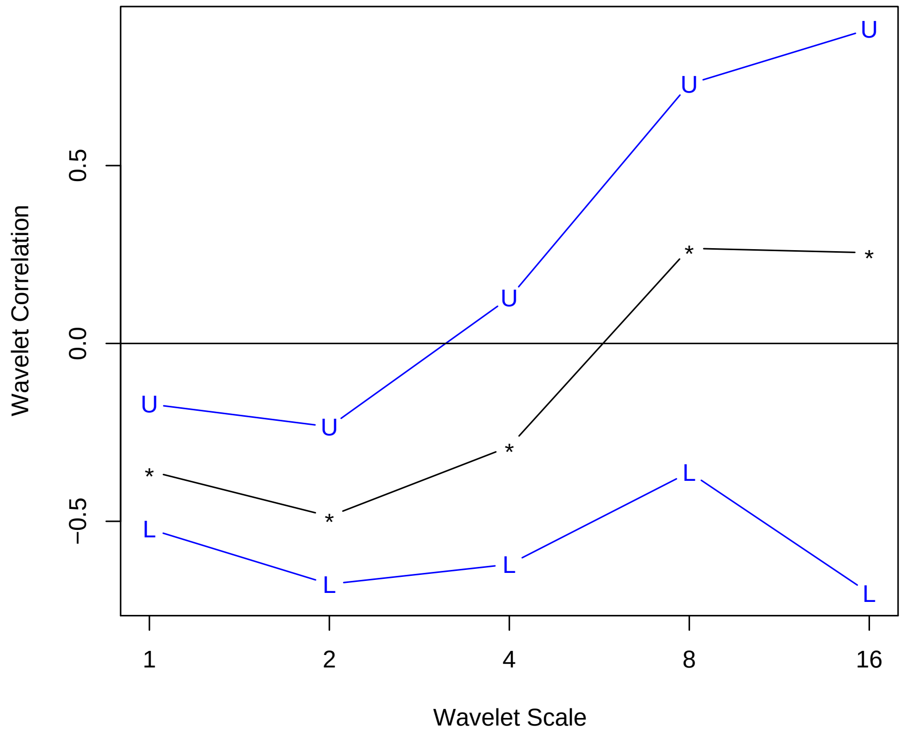

Figure 1: Wavelet correlation between terms of trade and trade balance for US

The dynamic relationship between the trade balance and the terms of trade may be examined through wavelet cross-correlation analysis. The cross-correlation function considers the relationship between two series shifted by a positive or negative time lag and plotted against the lag value. Figure 2 shows the properties of the cross-correlation functions both in the frequency and time scale domain. In the top left panel of figure 1 we graph the cross-correlation functions for the trade balance and the terms of trade for leads and lags up to three years, with both variables filtered according to the Baxter-King (1995) approximate band-pass filter based on the Fourier transform.[2] In particular, as suggested by Baxter and King, we pass frequencies between 6 and 32 periods as it corresponds to the typical business cycle frequency range with quarterly data (see Stock and Watson, 1998). The cross-correlation function between the two band-pass filtered series displays an asymmetric S-shape, a pattern that Backus *et al.* (1994) labeled as the *S-curve*.[3] Indeed, the cross-correlation function between tt_t and tb_{t+k},[4] for -12 ¡ k ¡12 is negative for negative values of k up to -4, positive for values of k between -4 and 9, and finally negative again for values of k greater than 9, thus indicating the tendency for the trade balance to be positively correlated in the short and medium run with past movements in the terms of trade, but negatively correlated in the long term. In this way a deterioration of the terms of trade (increase in our terminology) is generally associated with an increase in the trade balance in the following quarters,[5] reaching a peak after one year and then becoming negative after two years.

In Figure 3 we report the wavelet cross-correlation functions for the different time scales from d_4 , in the top left panel, to d_1 , in the bottom right panel, as they can provide an estimate of the lead/lag relationship between the trade balance and the terms of trade on a scale-by-scale basis. The horizontal S pattern evidenced first in Backus *et al.* (1994) seems to be a characteristic common to all scales, but with differences linked to the time horizons of the scales.

In the top left panel the cross-correlation function for wavelet scale d_4, corresponding to 16-32 quarters period dynamics, displays the typical asymmetric shape of the *S-curve* Backus *et al.* (1994) with the function crossing

[2]The approximate band-pass filter permits decomposition of a series between specified frequency bands corresponding to the low, business cycle and high frequencies of the spectrum. The filter, designed to make the filtered series stationary if the raw series is integrated of order one or two, employs a centered moving average method using up to 12 weighted leads and lags.

[3]In particular, our cross-correlation function displays a larger similarity with the shape of the post-72 rather than with the shape of the pre-72 Backus *et al.* cross-correlation function.

[4]Trade balance and the terms of trade are labeled *tb* and *tt*, respectively.

[5]A deterioration of the terms of trade improve competitiveness, as the price of imports increases relatively to the price of exports, and thus the balance of trade tends to increase as exports increase and imports diminish (the so-called J-curve effect).

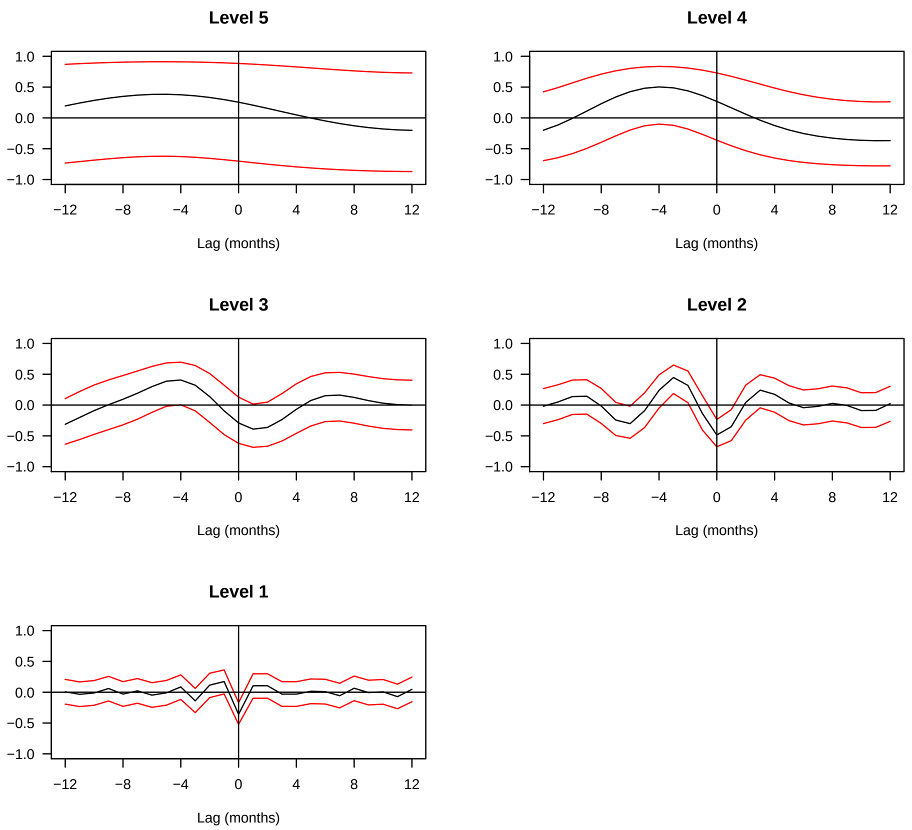

Figure 2: Wavelet cross-correlation between terms of trade and trade balance for US

the axis to the left of k=0 and a large positive peak at lag four.[6] For wavelet scales d_4 and d_3 these peaks represent the largest positive cross-correlation and thus, for these wavelet scales, we conclude that the balance of trade lags the terms of trade by four quarters. The cross-correlation function reported in the bottom panels depicts a strong contemporaneous negative relationship between the two variables at the shortest scales, that is d_2 and d_1, as the largest cross-correlation occurs at lag zero and is negative. Finally, the magnitude of the wavelet cross-correlation coefficients tends to decrease and do not exhibit cross-correlations significantly different from zero as the lags get larger when the wavelet scale decreases.

Figure 4 about here

[6] We say that a variable x lags (leads) another variable y by k quarters if the maximum correlation value verifies at y_t,x_{t+k} (y_t,x_{t-k}).

4 Nonparametric analysis

When the crystals v_5, w_5,......, w_1 are translated back into the time domain we may obtain the smooth V_5 and detail W_5,......, W_1 signals which represent the multiresolution decomposition of the original signal (or time series). As the smooth and details signals represent components of the original signal at different resolution levels we may analyze the characteristics of the relationship between the trade balance and the terms of trade at the different time horizons.

In a parametric framework the response of the trade balance to terms of trade (or exchange rate) movements may be explored within the context of vector autoregression (VAR) model through the analysis of the impulse response function, as it provides the response of the VARs variables following an i.i.d. shock. As VARs are reduced-form model, the interpretation of such shocks as structural shocks requires a clear understanding of the causal relationship among the variables of the model, [7] as these identifying assumptions are crucial for identification of the structural shocks.

In what follows we apply a methodology that allows us exploring the issues related to the relashionship between the trade balance and the terms of trade for US without making any *a priori* explicit or implicit assumption about the form of the relationship, that is nonparametric regression. Indeed, nonparametric regressions can capture the shape of a relationship between variables without us prejudging the issue, as they estimate the regression function $f(.)$ linking the dependent to the independent variables directly, without providing any parameters estimate. [8] There are several approaches available to estimate nonparametric regression models,[9] and most of these methods assume that the nonlinear functions of the independent variables to be estimated by the procedures are *smooth* continuos functions. One such model is the *generalized additive regression model* (GAM),[10]

$$y_i = \alpha + \sum_{j=1}^{k} f_j\left(x_{ij}\right) + \varepsilon_i \qquad (1)$$

where the functions $f_j(.)$ are *smooth* regression functions to be estimated from the data, and the estimates of $f_j(x_{ij})$ for every value of x_{ij}, written

[7]The ordering of the variables in the Choleski decomposition.

[8]The traditional nonlinear regression model introduce nonlinear functions of dependent variables using a limited range of transformed variables to the model (quadratic terms, cubic terms or piecewise constant function). An example of a methodology testing for nonlinearity without imposing any a priori assumption about the shape of the relationship is the smooth transition regression used in Eliasson (2001).

[9]See Fox (2000a, 2000b) for a discussion on nonparametric regression methods.

[10]GAMs were introduced by Hastie and Tibshirani (1986) and are described in detail in Hastie and Tibshirani (1990).

as $\hat{f}_j(x_{ij})$, are obtained using a fitting alghoritm known as *backfitting*.[11] Such a model allows us to gain more flexibility, as it replaces the linearity assumtpion with some univariate smooth functions in a nonparametric setting, but retain the additivity assumption. Moreover, an important advantage of GAMs is the possibility to evaluate the statistical significance of the smooth nonparametric components. Two smoothing functions are available to estimate these partial-regression functions $f_j(.)$, that is spline and locally-weighted regression smoothers. Both smoothers have similar fits with the same equivalent number of parameters, but the local regression (*loess*) method developed by Cleveland (1979) provides robust fitting when there are outliers in the data, support multiple dependent variables and computes confidence limits for predictions when the error distribution is symmetric, but not necessarily normal. In the *loess* method the regression function is evaluated at each particular value of the independent variable, x_i, using a local neighborhood of each point and the fitted values are connected in a nonparametric regression curve. In fitting such a local regression, a fixed proportion of the data is included in each given local neighborhood, called the *span* of the local regression smoother (or the smoothing parameter), and the data points are weighted by a smooth function whose weights decrease as the distance from the center of the window increases.

Thus, we estimate, for each time scale component, a simple additive model of the balance of trade on the terms of trade (and an intercept α),

$$TB_t = \alpha + lo\,(TT_t) + \varepsilon_t \tag{2}$$

where TB_t is trade balance, TT_t the terms of trade and $lo(.)$ is the locally weighted regression smoother (*loess*). The solid lines in figure 4 show the nonparametric estimate of the regression function evaluated at the longest scale, V_5, as well as at all other scales, from W_5 to W_1, for the terms of trade using a span=0.6.[12] These smooth plots are drawn by connecting the points in plots of the fitted values for each function against its regressor, while the dashed lines above and below the smooth curves are constructed, at each of the fitted values, by adding and subtracting two pointwise standard-error.[13] Thus, the plot of TT versus $\hat{lo}\,(TT)$ will reveal the nature of the estimated relationship between the dependent (trade balance) and the dependent variables (terms of trade), other things in the

[11] A full description of how the alghoritm works in GAMs is available in Hastie and Tibshirani (1990).

[12] We use different smoothing parameters in estimating equation [2], but our main findings do not show excess sensitivity to the choice of the span in the loess function within what appear to be reasonable ranges of smoothness (*i.e.* between 0.3 and 0.8).

[13] Under additional assumptions of no bias these upper and lower curves can be viewed as approximate 95% pointwise confidence intervals bands.

model being constant. Three main results emerge from the analysis of the nonparametric fitted functions in figure 4. First, at the longest scale a long-run negative relationship emerges between the trade balance and the terms of trade, a relationship that strenghtens for the terms of trade in the 2^{nd} half of the sample. Second, at the scales (with the exception of the finest scale, W_1) the nonparametric fitted functions of the independent variables exhibit very different patterns across scales. Indeed, while the relationship between the terms of trade and the trade balance is positive (negative) at the medium-time horizon (W_5, W_4 scales), it is negative (absent) at a short-time horizon (W_3, W_2 scales). In this way, unfavorable movements in terms of trade (due, for example, to a devaluation) will deteriorate the trade balance at a short-time horizon and improve it at a medium-time horizon. Thus, the time-scale evidence about the effects of the terms of trade on the trade balance seem to be compatible with a *J-curve* pattern of the trade balance with the "import value effect" prevailing at the shortest scales W_2, W_3 (worsening the trade balance first) and the "volume effect" prevailing at the coarsest scales W_4, W_5 (improving the trade balance later).[14]

Table 1 about here

Finally, in order to evaluate the statistical significance of the nonparametric fitted functions at the different time scales we report in Table 1 the results of a type of score test from the additive regression model estimated in equation (6). The column headed *"Nonpar d.f."* contains the nonparametric degrees of freedom used up by the fit. They are related to the complexity of the nonparametric curve fitted, as the more complex the curve, the higher the peanlty for complexity and the more the degrees of freedom lost (there are no exact degrees of freedom as, due to the nonparametric nature of GAMs, no parameters estimate are obtained). The column headed *"Nonpar* χ^2*"* contains an approximate χ^2-statistic (see Bowman and Azzalini, 1997, p.163) and represents a type of score test to evaluate, through the p-values reported in the last column, the nonlinear contribution of each nonparametric term in the additive regression model. The F-values for nonparametric effects reported in Table 1 indicate that the significance of the nonparametric terms gets larger as the wavelet scale increases, while the nonparametric terms of the finest scales, W_2 and W_1, are not significant at the usual signficance levels.

Table 1: F-values for nonparametric effects for $TB_t = \alpha + lo\,(TT_t) + \varepsilon_t$

[14] The terms "import value effect" and "volume effect" are from Krugman and Obstfeld (2001).

	Nonpar. d.f.	*Nonpar.* χ^2	$P(\chi^2)$
V_5	1.3	22.879	0.000
W_5	2.2	12.454	0.000
W_4	3.3	5.638	0.001
W_3	3.7	2.545	0.045
W_2	4	0.780	0.540
W_1	3.4	0.5112	0.695

The results from the analysis of the nonparametric fitted functions and of the F-values for nonparametric effects suggest that the response of trade balance to terms of trade changes is not uniform across scales: the relationship appears to be negative at the finest scales, W_2 and W_3, positive at the coarsest scales, W_4 and W_5, and negative at the longest possible scale, V_5, while as regards the significance of the nonparametric terms the results indicate that it tends to increase as the wavelet scale increases.

5 Conclusions

In this paper we apply a relatively new statistical tool, wavelet analysis, to investigate the relationship between the trade balance and the terms of trade at the different time scales. Thus, after decomposing the trade balance (defined as the ratio of net exports to output) and the terms of trade (defined as the relative price of imports to exports) into their time-scale components using to the *maximum overlap discrete wavelet transform (MODWT)*, we analyze the relationship among these variables at the different time scales using i) wavelet correlation analysis, as wavelet coefficients may provide information about the contemporaneous and the lead/lag relationship between the two processes at the various time scales; and ii) nonparametric regression models (GAMs), as this framework may enable us to evaluate the signficance of the dynamic relationships among these variables without making any a priori explicit or implicit assumption about the shape of the relationships.

There are two main findings emerging from the time scales analysis of the trems of trade - trade balance relationship. First, wavelet correlation analysis indicates that, if the association between the trade balance and the terms of trade depends mainly on the elasticity of substitution between foreign and domestic goods, the Armington elasticities may be different across scales, and in particular, tend to get larger as the time horizon of the agents increases. Second, the analysis of the nonparametric fitted functions and of the F-values for nonparametric effects suggests that the response of the trade balance to movements in the terms of trade is not uniform across scales and the significance of the nonparametric terms gets larger as the wavelet scale increases. Moreover, while no relationship seems to occur at the lowest scale, 1, the relationship appears to be negative at the finest scales, 2 and 3, positive at the coarsest scales, 4 and 5, and negative at the longest possible scale (the trend resolution level). Thus, the sign of the long-run relationship between the trade balance and the terms of trade seems to provide support to the existence of the Harberger-Laursen-Metzler effect .

6 References

Armington P.S., 1969, "A Theory of Demand for Products Distinguished by Place of Production", *IMF Staff Papers*, 16(1), 159-176.

Backus D.K., Kehoe P.J. and F.E. Kidland, 1994, Dynamics of the Trade Balance and the Terms of Trade: The J-Curve?, *The American Economic Review,* 84(1), 84-103.

Baxter M. and R.G. King, 1995, "Measuring Business Cycles: Approximate Band-pass Filters for Economic Time Series", *NBER Working Paper*, no.5022.

Bowman A. W. and A. Azzalini, 1997, *Applied Smoothing Techniques for Data Analysis*, Clarendon Press, Oxford.

Cleveland W. S., 1979, "Robust Locally-Weighted Regression and Scatterplot Smoothing", *Journal of the American Statistical Association*, 74, 829-836.

Crivellini M., Gallegati Marco, Gallegati Mauro and A. Palestrini, 2005, "Output Fluctuations in G7 Countries: a Time-scale Decomposition Analysis", *Monographs of official statistics, Papers and Proceedings of the 4^{th} Eurostat Colloquium on modern tools for business cycle analysis*, forthcoming.

Daubechies I., 1992, *Ten Lectures on Wavelets*, CBSM-NSF Regional Conference Series in Applied Mathematics, SIAM: Philadelphia.

Elliot G. and A. Fatas, 1996, "International business cycles and the dynamics of the current account", *European Economic Review*, 40, 361-387.

Fox J., 2000a, *Nonparametric Simple Regression: Smoothing Scatterplots*, Sage, Thousands Oaks CA.

Fox J., 2000b, *Multiple and Generalized Nonparametric Regression*, Sage, Thousands Oaks CA.

Harberger A.C., 1950, "Currency Depreciation, Income and The Balance of Trade", *Journal of Political Economy*, 58, 47-60.

Hastie and Tibshirani, 1986, "Generalized Additive Models", *Statistical Science*, 1, 297-318.

Hastie T. and R. Tibshirani, 1990, *Generalized Additive Models*, Chapman & Hall, London.

Kim S. and F.H. In, 2003, "The Relationship Between Financial Variables and Real Economic Activity: Evidence From Spectral and Wavelet Analyses", *Studies in Nonlinear Dynamics and Econometrics*, 7, No. 4.

Laursen S. and L.A. Metzler, 1950, "Flexible Exchange Rates and the Theory of Employment", *The Review of Economics and Statistics*, 32, 281-299.

Ramsey J.B. and C. Lampart, 1998a, “The Decomposition of Economic Relationship by Time Scale Using Wavelets: Money and Income“, *Macroeconomic Dynamics*, 2, 49-71.

Ramsey J.B. and C. Lampart, 1998b, “The Decomposition of Economic Relationship by Time Scale Using Wavelets: Expenditure and Income“, *Studies in Nonlinear Dynamics and Econometrics*, 3, No. 4, 23-42.

Ramsey J.B., 2002, “Wavelets in Economics and Finance: Past and Future”, *C.V. Starr Center for Applied Economics*, No. 2002-02.

Ruhl K.J., 2003, "Solving the Elasticity Puzzle in International Economics", *University of Minnesota and Federal Reserve Bank of Minneapolis.*

Schleicher C., 2002, "An Introduction to Wavelets for Economists", *Bank of Canada Working Paper,* No. 02-3.

A Wavelet analysis

Coming back to wavelets and going into some mathematical detail we may note that there are two basic wavelet functions: the father-wavelet and the mother-wavelet. The formal definition of the father wavelets is the function

$$\phi_{J,k} = 2^{-\frac{J}{2}} \phi \left(\frac{t - 2^J k}{2^J} \right) \tag{3}$$

defined as non-zero over a finite time length support that corresponds to given mother wavelets

$$\psi_{J,k} = 2^{-\frac{J}{2}} \psi \left(\frac{t - 2^J k}{2^J} \right) \tag{4}$$

with $j = 1, \ldots, J$ in a *J-level* wavelets decomposition. The former integrates to 1 and reconstructs the longest time-scale component of the series (trend), while the latter integrates to 0 (similarly to sine and cosine) and is used to describe all deviations from trend. The mother wavelets, as said above, play a role similar to sins and cosines in the Fourier decomposition. They are compressed or dilated, in time domain, to generate cycles fitting actual data.

For a discrete signal or function $f_1, f_2, ..., f_n$, the wavelet representation of the signal or function $f(t)$ in $L^2(R)$ can be given by

$$f(t) = \sum_k s_{J,k} \Phi_{J,k}(t) + \sum_k d_{J,k} \Psi_{J,k}(t) + ... + \sum_k d_{j,k} \Psi_{j,k}(t) + ... + \sum_k d_{1,k} \Psi_{1,k}(t) \tag{5}$$

where J is the number of multiresolution components or scales, and k ranges from 1 to the number of coefficients in the specified components. The coefficients d_{jk} and s_{Jk} of the wavelet series approximations in [3] are the details and smooth wavelet transform coefficients representing, respectively, the projections of the time series onto the basic functions generated by the chosen family of wavelets, that is

$$d_{j,k} = \int \psi_{j,k} f(t) dt$$
$$s_{J,k} = \int \phi_{J,k} f(t) dt$$

for $j = 1, 2, ..., J$. The smooth coefficients s_{Jk} mainly capture the underlying smooth behaviour of the data at the coarsest scale, while the details coefficients $d_{1k}, .., d_{jk}, .., d_{Jk}$, representing deviations from the smooth behaviour, provide progressively finer scale deviations. Each of the sets of the coefficients $s_J, d_J, d_{J-1}, ..., d_1$ is called a crystal.

The multiresolution decomposition of the original signal $f(t)$ is given by the following expression

$$f(t) = S_J + D_J + D_{J-1} + ... + D_j + ... + D_1 \tag{6}$$

where $S_J = \sum_k s_{J,k} \Phi_{J,k}(t)$ and $D_j = \sum_k d_{J,k} \Psi_{J,k}(t)$ with $j = 1, \ldots, J$. The sequence of terms $S_J, D_J, ..D_j, ..., D_1$ in (4) represent a set of signals components that provide representations of the signal at the different resolution levels 1 to J, and the detail signals $\mathbf{D}_j$ provide the increments at each individual scale, or resolution, level.

In addition to the features stated in the appendix Whitcher *et al.* (1999, 2000) have extended the notion of wavelet covariance for the maximal overlap DWT (MODWT) and defined the wavelet cross covariance and wavelet cross correlation between two processes. The maximal overlap DWT (MODWT) may be regarded as a modified version of the discrete wavelet transform (DWT), but as MODWT employs circular convolution the coefficients generated by both beginning and ending data could be spurious.[15] For a signal $f(t)$ the MODWT applying the Daubechies compactly supported wavelet produces J vectors of wavelet coefficients $\widehat{w}_1, \widehat{w}_2, ..., \widehat{w}_J$ and one vector of scaling coefficients, $\widehat{s}_J$. The wavelet variance for a signal $f(t)$ is defined to be the variance of the wavelet coefficients at scale 2^{j-1} and an unbiased estimator using the MODWT after removing all coefficients affected by the periodic boundary conditions through

$$\widehat{v}^2_{f(t)}\left(2^{j-1}\right) = \frac{1}{\widehat{N}_j} \sum_{t=L_{J-1}}^{N-1} \widehat{\mathbf{w}}^2_{j,t} \tag{7}$$

[15]If the lenght of the filter is L, there are $\left(2^j - 1\right)(L-1)$ coefficients affected for 2^{j-1}-scale wavelet and scaling coefficients, while $\left(2^j - 1\right)(L-1) - 1$ beginning and $\left(2^j - 1\right)(L-1)$ ending components in 2^{j-1} -scale details and smooth would be affected (Perival and Walden, 2000).

where $\widehat{N}_j = \frac{N}{2^j - L_j}$ with $L_j = \left[(L-2)\left(1-2^j\right)\right]$ being the lenght of the scale 2^{j-1} wavelet filter, and the vector $\widehat{\mathbf{w}}$ are *n-dimension* vectors containing the coefficients s_J, d_J, ..., d_1 of the wavelet series approximations. Thus level j wavelet variance is simply the variance of the wavelet coefficients at that level (Gencay *et al.,* 2002). Similarly, the covariance is defined to be the covariance between the scale wavelet coefficients of $f(t)$ and $g(t)$. Again, after removing all wavelet coefficietns affected by the boundary conditions, an unbiased estimator of the wavelet covariance using the MODWT may be given by:

$$\widehat{Cov}_{f(t)g(t)}\left(2^{j-1}\right) = \frac{1}{\widehat{N}_j} \sum_{t=L_{J-1}}^{N-1} \widehat{\mathbf{w}}_{j,t}^{f(t)} \widehat{\mathbf{w}}_{j,t}^{g(t)} \tag{8}$$

Analogously to the usual unconditional correlation coefficients, the MODWT estimator of the wavelet cross correlation coefficients may then be obtained making use of the wavelet covariance $\widehat{Cov}_{f(t)g(t)}$ and the square root of their wavelet variances $\widehat{v}_{f(t)}^2$ and $\widehat{v}_{g(t)}^2$ as follows:

$$\widehat{\rho}_{f(t)g(t)}\left(2^{j-1}\right) = \frac{\widehat{Cov}_{f(t)g(t)}\left(2^{j-1}\right)}{\widehat{v}_{f(t)}\left(2^{j-1}\right)\widehat{v}_{g(t)}\left(2^{j-1}\right)} \tag{9}$$