Akademie věd České republiky
Ústav teorie informace a automatizace

Academy of Sciences of the Czech Republic
Institute of Information Theory and Automation

RESEARCH REPORT

Martin Šmíd
mailto:martin@klec.cz, http://www.klec.cz/martin

Forecasting in Continuous Double Auction

No. 2128 rev.2. May 2005 revised Dec 2005

ÚTIA AV ČR, P. O. Box 18, 182 08 Prague, Czech Republic
Telex: 122018 atom c, Fax: (+420) 266 053 111
E-mail: utia@utia.cas.cz

Abstract

Recently, the continuous double auction, i.e. the trading mechanism used in the majority of the financial markets, is the subject of an extensive study. In the present paper, a model of the continuous double auction with the completely random flow of the limit orders is studied. The main result of the paper is an approximate formula for the distribution of the market price and the traded volume at the time τ given the information available at $t < \tau$.[1]

Keywords: limit order markets, continuous double auction, price and volume, forecasting, market microstructure

AMS classification: 91B26,
JEL classification: C51,G10

1 Introduction

In the present paper, the trading mechanism with the following rules is studied:

1. At any time instant, each agent may place a *buy (limit) order* or a *sell (limit) order*, each limit order containing a *limit price* and an *order size* (i.e. the required/offered amount of the commodity). For simplicity, we assume the order size to be unit.[2]

2. If a newly arrived limit order matches with the best waiting limit order of the opposite type (i.e. the one with the most favorable limit price, let us call it *best counterpart*) then a trade is made for the limit price of the best counterpart (if there is more then one counterpart with the best limit price then the oldest one, i.e. the one with the earliest placement date, is executed).

3. If a newly arrived limit order finds no counterpart then it remains waiting until it is executed or canceled by its submitter.

The trading mechanism, described here, is usually called *continuous double auction* (CDA), the list of all the currently waiting buy orders is called *buy limit order book*, the highest limit price of the orders contained in the buy limit order book is called *(best) bid*, the list of all the currently waiting sell orders is called *sell limit order book* and the lowest limit price of the orders contained in the sell limit order book is called *(best) ask*.

In reality, many markets possess the structure described above: many financial markets, first of all, various marketplaces, real estate markets, trading made by means of the advertising in newspapers etc.

In the present paper, the complete randomness of the agents' actions is assumed. In particular, the times of the arrivals of the limit orders are assumed to follow a Poisson process, their limit prices are regarded as i.i.d. random variables independent of the arrival times and the lifetimes of the limit orders are supposed to be exponentially distributed independent both of the arrival times and of the limit prices.

The model introduced by the present paper is a generalization of the model of Smith et al. [2003]; the generalizations consist in possibly non-uniform density of the limit prices and in possibly continuous price space (the lack of ticks). The main result of the present paper is an approximate formula for the future distribution of the market price and of the total traded volume.

The paper is organized as follows: in Section 2, the model of the CDA with complete randomness is defined, in Section 3, the forecast of the price and the volume is constructed. Section 4 concludes the paper.

[1]This work was supported by grant no. 402/04/1294 and by grant no. 402/03/H057 of the Czech Science Foundation and by grant no. 454/2004/AEK/FSV of the Grant Agency of the Charles University.

[2]Cf. Smith et al. [2003] for a partial justification of the assumption of the unit order size.

2 Continuous Double Auction with Complete Randomness

2.1 Definitions

Denote $\tau^1_{b\star} < \tau^2_{b\star} < \dots$ the times of the arrivals of the buy orders. For each $i \in \mathbb{N}$, denote x^i the limit price of the i-th buy order[3] and denote $\tau^i_{b\dagger}$ the time at which the i-th buy order is canceled provided that it is not executed until $\tau^i_{b\dagger}$ ($\tau^i_{b\dagger}$ will be called *cancelation time*). Analogously, denote $\tau^1_{s\star} < \tau^2_{s\star} < \dots$ the arrival times of the sell orders, y^i the limit price of the i-th sell order and $\tau^i_{s\dagger}$ its cancelation time for each $i \in \mathbb{N}$.

According to the informal description of the CDA, given in Introduction, a buy order may find itself in four possible states: *prenatal* (not yet arrived), *waiting*, *executed* and *canceled*.

Denote $X^i_t, Y^i_t \in \{prenatal, waiting, executed, canceled\}$, $t \in \mathbb{R}^+$, the state of the i-th buy order, sell order respectively, at the time t for each $i \in \mathbb{N}$, $t \in \mathbb{R}^+$. Further, denote $\mathcal{N}_{\mathbb{R}}$ the space of all the counting measures[4] on $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$. The *buy limit order book* may be mathematically described as

$$B_t \in \mathcal{N}_{\mathbb{R}}, \qquad B_t(A) \triangleq |\{i : x^i \in A, X^i_t = waiting\}|, \qquad A \in \mathbb{B}(\mathbb{R})$$

(the symbol $|\bullet|$ denotes the number of elements of the set, $\mathbb{B}(\Xi)$ stands for the Borel σ-algebra of a metric space Ξ). Symmetrically, we describe the *sell limit order book* as

$$S_t \in \mathcal{N}_{\mathbb{R}}, \qquad S_t(A) \triangleq |\{i : y^i \in A, Y^i_t = waiting\}|, \qquad A \in \mathbb{B}(\mathbb{R}).$$

Finally, define the *i-th (best) bid* as $b^i_t = \max\{p \in \mathbb{R} : B_t[p, \infty) \geq i\}$ and the *i-th (best) ask* as $a^i_t = \min\{p \in \mathbb{R} : S_t(-\infty, p] \geq i\}$ for each $i \in \mathbb{N}$ (it is understood that $\max \emptyset = -\infty$ and $\min \emptyset = \infty$).

2.2 The Dynamics of the System

Assume, throughout the present subsection, that no pair of the events (i.e. the arrivals of the orders and their cancelations) happens at the same time. According to the informal definition, the process $(X^1_t, Y^1_t, X^2_t, Y^2_t, \dots)$ evolves according to the following rules:

- The i-th buy order is in the state *prenatal* at the time 0 for each $i \in \mathbb{N}$.
- For each $i \in \mathbb{N}$, X^i_t may jump only at the times $\tau^i_{b\star}, \tau^i_{b\dagger}, \tau^1_{s\star}, \tau^2_{s\star}, \dots$ as follows:
  - *When $t = \tau^i_{b\star}$:* If $x^i < a^1_{t-}$ then $X^i_t = waiting$, otherwise $X^i_t = executed$.
  - *When $t = \tau^i_{b\dagger}$:* If $X^i_{t-} = waiting$ then $X^i_t = canceled$, otherwise $X^i_t = X^i_{t-}$ ($= executed$).
  - *When $t = \tau^j_{s\star}$ for some $j \in \mathbb{N}$:* If the i-th buy order is currently the best buy order (i.e. the oldest of all the waiting buy orders with the limit price b^1_{t-}) and $x^j \leq b^1_{t-}$ then $X^i_t = executed$, otherwise $X^i_t = X^i_{t-}$.
- The symmetric rules hold for the processes Y^i, $i \in \mathbb{N}$.

2.3 The Market Price and the Traded Volume

We naturally define the *market price* p_t and the *traded volume* q_t as follows:

- $p_0 = undefined$,[5] $q_0 = 0$.

---

[3] i.e. the one with the arrival time $\tau^i_{b\star}$

[4] The measure is *counting* if its values on the measurable sets are nonnegative integers.

[5] We take p_t as a process taking values in the space $\mathbb{R}^{undefined} \triangleq \mathbb{R} \cup \{undefined\}$.

- The process (p_t, q_t) is piecewise constant right continuous and it they may jump only at the times $\tau_{b\star}^1$, $\tau_{s\star}^1$, $\tau_{b\star}^2$, $\tau_{s\star}^2$, ... as follows:
  - *When* $t = \tau_{b\star}^i$ *for some* $i \in \mathbb{N}$*:* If the newly buy arrived order was executed at t then $p_t = a_{t-}^1$ and $q_t = q_{t-} + 1$, otherwise $(p_t, q_t) = (p_{t-}, q_{t-})$.
  - *When* $t = \tau_{s\star}^i$ *for some* $i \in \mathbb{N}$*:* If the newly arrived sell order was executed at t then $p_t = b_{t-}^1$ and $q_t = q_{t-} + 1$, otherwise $(p_t, q_t) = (p_{t-}, q_{t-})$.

2.4 The Stochastic Properties of the Order Flow

Assume that there exist a probability measure π and positive constants $u, c \in \mathbb{R}^+$ such that $x^i \sim \pi$, $\Delta\tau_{b\star}^i \triangleq (\tau_{b\star}^i - \tau_{b\star}^{i-1}) \sim \mathrm{Exp}(u)$ and $\Delta\tau_{b\dagger}^i \triangleq (\tau_{b\dagger}^i - \tau_{b\star}^i) \sim \mathrm{Exp}(c)$ for each $i \in \mathbb{N}$ (the symbol $\mathrm{Exp}(m)$ denotes the exponential distribution with mean $1/m$).

Analogously, suppose that there exist a probability measure ρ and positive constants $v, d \in \mathbb{R}^+$ such that $y^i \sim \rho$, $\Delta\tau_{s\star}^i \triangleq (\tau_{s\star}^i - \tau_{s\star}^{i-1}) \sim \mathrm{Exp}(v)$ and $\Delta\tau_{s\dagger}^i \triangleq (\tau_{s\dagger}^i - \tau_{s\star}^i) \sim \mathrm{Exp}(d)$ for each $i \in \mathbb{N}$.

Finally, assume that $\Delta\tau_{b\star}^1, \Delta\tau_{b\dagger}^1, x^1, \Delta\tau_{s\star}^1, \Delta\tau_{s\dagger}^1, y^1, \Delta\tau_{b\star}^2, \Delta\tau_{b\dagger}^2, x^2, \Delta\tau_{s\star}^2, \Delta\tau_{s\dagger}^2, y^2, \dots$ are mutually independent.

Since both the arrivals of the buy orders and the arrivals and the sell orders follow the time-spatial Poisson processes [Šmíd, 2005a] and since, in both the cases, the cancelation times are independent on the arrivals, we call our setting *complete random arrival of the orders*.

Denote $(\tau^i)_{i=1}^\infty$ the increasing sequence of the elements of the set $\{\tau_{b\star}^i : i \in \mathbb{N}\} \cup \{\tau_{b\dagger}^i : i \in \mathbb{N}\} \cup \{\tau_{s\star}^i : i \in \mathbb{N}\} \cup \{\tau_{s\dagger}^i : i \in \mathbb{N}\}$ and put $\tau^0 = 0$. From the absolute continuity of the exponential distribution it follows that the times $\tau^0, \tau^1, \dots$ mutually differ almost sure hence the dynamics of our system is well defined by subsection 2.2.

3 Forecasts of the Market Price and the Traded Volume

As it was already mentioned, our aim is a forecast of (p_τ, q_τ) given the state of the system up to the time $t < \tau$. Since $\Xi_t \triangleq (B_t, S_t, p_t, q_t)$ is a continuous time Markov chain (Appendix, Theorem 1.), the forecast can be based solely on the state of the system at the time t. Moreover, when we modify our definition so that Ξ_0 may take other values then $(0, 0, undefined, 0)$, we may assume that $t = 0$.

Fix $\tau \geq 0$ and $\Xi_0 = (B_0, S_0, p_0, q_0) \in \mathcal{N}_\mathbb{R} \times \mathcal{N}_\mathbb{R} \times \mathbb{R}^{undefined} \times \mathbb{Z}^+$, $\mathbb{R}^{undefined} \triangleq \mathbb{R} \cup \{undefined\}$. To construct the forecast for the time τ, we shall use the usual technique, i.e. the expansion according to the number of the events:

$$\mathbb{P}((p_\tau, q_\tau) \in A) = \sum_{k=0}^{\infty} \mathbb{P}((p_\tau, q_\tau) \in A | n_\tau = k)\mathbb{P}(n_\tau = k) \tag{1}$$

where n_τ is the number of the jumps of Ξ until τ. However, since the inter-event times of the process Ξ are dependent on the state of the process, the evaluation of (1) could be quite complicated. Hence, we have to modify the process Ξ so that its inter-event times are i.i.d. first.

3.1 Uniformization

Let $N \in \mathbb{N}$ be a sufficiently large constant and let $(\theta_v^i)_{i=1}^\infty$ be i.i.d. exponential variables with mean one independent of Ξ. Let v be a counting process[6] starting from zero. Denote $(\bar{\tau}^i)_{i=1}^\infty$ the times of the jumps of the process $\bar{\Xi} \triangleq (\Xi, v) = (B, S, p, q, v)$ defined by the following rules:

[6] i.e. piecewise constant right continuous with unit increments

- $\bar{\tau}^0 = 0$,
- if there is no event of Ξ between $\bar{\tau}^{i-1}$ and $\bar{\tau}^{i-1} + \Delta\tau_v^i$, where $\Delta\tau_v^i = [(N - |B_{\bar{\tau}^{i-1}}|)c + (N - |S_{\bar{\tau}^{i-1}}|)d]^{-1}\theta_v^i$, then let the process v jump at $\bar{\tau}^{i-1} + \Delta\tau_v^i$, otherwise leave v unchanged at $\bar{\tau}^{i-1} + \Delta\tau_v^i$.

It could be shown (Appendix, Theorem 2) that $\bar{\Xi}_t$ is a Markov chain with

$$\Delta\bar{\tau}^i \triangleq \bar{\tau}^i - \bar{\tau}^{i-1} \sim \mathrm{Exp}(\bar{r}), \quad \bar{r} = u + v + N(c+d), \quad \text{for each } i \leq \tilde{N} \tag{2}$$

where $\tilde{N} = \tilde{N}(N) \triangleq N - \max(|B_0|, |S_0|) + 1$ such that

$$(\Delta\bar{\tau}^i)_{i=1}^{\tilde{N}} \text{ are i.i.d and independent of } \bar{\Xi}_{(k)} = (B_{(k)}, S_{(k)}, p_{(k)}, q_{(k)}, v_{(k)}) \triangleq \bar{\Xi}_{\bar{\tau}^k}.^7 \tag{3}$$

3.2 The Expansion and a Truncation

Denote $\bar{n}_\tau$ the number of the jumps of $\bar{\Xi}$ up to the time τ and let $A \in \mathbb{R}^{undefined} \times \mathbb{Z}^+$. Clearly, we may write

$$\mathbb{P}((p_\tau, q_\tau) \in A) = \sum_{k=0}^{\infty} \mathbb{P}((p_{(k)}, q_{(k)}) \in A | \bar{n}_\tau = k)\mathbb{P}(\bar{n}_\tau = k) = U_{A,\tilde{N}} + \eta_1$$

where

$$U_{A,\tilde{N}} = \sum_{k=0}^{\tilde{N}} \mathbb{P}((p_{(k)}, q_{(k)}) \in A | \bar{n}_\tau = k)\mathbb{P}(\bar{n}_\tau = k) \overset{(2),(3)}{=} \sum_{k=0}^{\tilde{N}} \mathbb{P}((p_{(k)}, q_{(k)}) \in A)\mathbb{P}(\mathrm{Poisson}(\bar{r}) = k)$$

and where

$$\eta_1 = \eta_1(\tilde{N}) \leq \sum_{k=\tilde{N}+1}^{\infty} \mathbb{P}(\bar{n}_\tau = k) = 1 - \sum_{k=1}^{\tilde{N}} \mathbb{P}(\bar{n}_\tau = k) = \mathbb{P}(\mathrm{Poisson}(\bar{r}) \geq \tilde{N} + 1).$$

It is straightforward that, to compute $\mathbb{P}((p_\tau, q_\tau) \in A)$ with a required accuracy, it suffices to choose N sufficiently large and to evaluate $U_{A,\tilde{N}}$. We deal with the latter task in the next subsection.

3.3 Forecasting of the Embedded Chain

Fix $k \leq \tilde{N}$ all through the present subsection. It follows from the basic probability theory that, to specify the distribution of $(p_{(k)}, q_{(k)})$, it suffices to evaluate $\mathbb{P}((p_{(k)}, q_{(k)}) \in A)$ for all the sets $A = I \times \{0, 1, \ldots, \hat{q}\}$ where $\hat{q} \in \mathbb{N}$ and where $I = (-\infty, \hat{p})$ for some $\hat{p} \in \mathbb{R}$.

Hence, fix $A = I \times \{0, 1, \ldots, \hat{q}\}$. Further, choose a disjoint partition $\mathcal{M} = \{M^1, M^2, \ldots, M^m\}$ of $\mathbb{R}$ containing all the points of $B_{(0)}$ and all the points of $S_{(0)}$[8] such that $I = M^1 \cup M^2 \cup \cdots \cup M^{m'}$ for some $m' \leq m$. Clearly,

$$\mathbb{P}((p_{(k)}, q_{(k)}) \in A) = \mathbb{P}((\phi_{(k)}, q_{(k)}) \in B)$$

where $B = B(A) \triangleq \{1, 2, \ldots, m'\} \times \{1, 2, \ldots, \hat{q}\}$ and

$$\phi_{(k)} = \begin{cases} 0 & \text{if } p_{(k)} = undefined \\ 1 & \text{if } p_{(k)} \in M^1 \\ \ldots. & \end{cases}$$

[7]The process $(\bar{\Xi}_{(k)})_{k=1}^{\infty}$ is usually called *embedded chain*).

[8]i.e. $\mathcal{M} \supseteq \{\{y\} : B_{(0)}\{y\} > 0 \vee S_{(0)}\{y\} > 0\}$

Further, define the random elements $e_{(i)} \in E \triangleq \{v, b^1_\star, s^1_\star, b^1_\dagger, s^1_\dagger, b^2_\star, s^2_\star, b^2_\dagger, s^2_\dagger, \dots, b^m_\star, s^m_\star, b^m_\dagger, s^m_\dagger\}$, $i = 1, 2, \dots, k$, such that $e_{(i)} = v$ if a void event happened, i.e. v was incremented, at the "time" i and, for each $1 \le j \le m$,

$$e_{(i)} = \begin{cases} b^j_\star & \text{if a new buy order whose limit price lies in } M^j \text{ arrived at } i \\ s^j_\star & \text{if a new sell order whose limit price lies in } M^j \text{ arrived at } i \\ b^j_\dagger & \text{if one of the waiting buy orders whose limit price lies in } M^j \text{ was canceled at } i \\ s^j_\dagger & \text{if one of the waiting sell orders whose limit price lies in } M^j \text{ was canceled at } i. \end{cases}$$

After denoting $e_{(1,k)} \triangleq (e_{(1)}, e_{(2)}, \dots, e_{(k)})$ and putting $\mathcal{E}^k \triangleq \otimes_{\nu=1}^k E^\nu$, we may write

$$\mathbb{P}((\phi_{(k)}, q_{(k)}) \in B) = \sum_{\epsilon \in \mathcal{E}^k} \mathbb{P}((\phi_{(k)}, q_{(k)}) \in B | e_{(1,k)} = \epsilon) \mathbb{P}(e_{(1,k)} = \epsilon).$$

Unfortunately, some of the summands of the latter formula are not easy to compute: Suppose, for instance, that $k = 4$, $p_{(0)} = b^1_{(0)}$, $\mathcal{M} = \{(-\infty, b^1_{(0)}), \{b^1_{(0)}\}, (b^1_{(0)}, a^1_{(0)}), \{a^1_{(0)}\}, (a^1_{(0)}, \infty)\}$ and $\epsilon = (b^3_\star, s^3_\star, b^3_\star, s^3_\star)$. Here, $\mathbb{P}(\phi_{(k)} = 3 | e_{(1,k)} = \epsilon) = \mathbb{P}(\text{at least one pair of the new orders matches})$ which is not trivial but computable. However, after we add $s^3_\dagger$ or $b^3_\dagger$ into $e_{(1,k)}$, the evaluation starts to lead to untractable combinatorial problems.

Fortunately, by the refinement of the partition $\mathcal{M}$, the total probability of the "scenarios" $e_{(1,k)}$ for which the conditional distribution is problematic to compute may be arbitrarily decreased: Denote $\beta^j_\epsilon = |\{i \in \mathbb{N} : i \le k : \epsilon^i = b^j_\star\}|$ and $\sigma^j_\epsilon = |\{i \in \mathbb{N} : i \le k : \epsilon^i = s^j_\star\}|$ for each $1 \le j \le m$ (ϵ^i denotes the i-th component of ϵ) and put

$$\tilde{\mathcal{E}}^k \triangleq \{\epsilon \in \mathcal{E}^k : \beta^j_\epsilon = 0 \vee \sigma^j_\epsilon = 0 \text{ for each } j \in C\}, \qquad C \triangleq \{1 \le j \le m : |M^j| > 1\}.$$

Clearly,

$$\mathbb{P}((\phi_{(k)}, q_{(k)}) \in B) = \sum_{\epsilon \in \tilde{\mathcal{E}}^k} \mathbb{P}((\phi_{(k)}, q_{(k)}) \in B | e_{(1,k)} = \epsilon) \mathbb{P}(e_{(1,k)} = \epsilon) + \eta^k_2 \tag{4}$$

where $\eta^k_2 = \eta^k_2(\mathcal{M}) \le \mathbb{P}(e_{(1,k)} \in \mathcal{E}^k - \tilde{\mathcal{E}}^k)$. It is relatively easy to compute both the conditional and the unconditional probabilities in (4): For any real measure μ, denote $\mu^\mathcal{M}$ the restriction of μ to $\sigma(\mathcal{M})$. Since $e_{(i)}$ is conditionally independent of $e_{(1,i-1)}$ given $(B^\mathcal{M}_{(i-1)}, S^\mathcal{M}_{(i-1)})$ for each $1 \le i \le k$,[9] since

$$\mathbb{P}(e_{(i)} = \varepsilon | B^\mathcal{M}_{(i-1)} = B, S^\mathcal{M}_{(i-1)} = S) = \gamma(\varepsilon, B, S), \qquad \gamma(\varepsilon, B, S) = \begin{cases} \frac{u\pi(M^j)}{\bar{r}} & \text{if } \varepsilon = b^j_\star \text{ for some } j \\ \frac{v\rho(M^j)}{\bar{r}} & \text{if } \varepsilon = s^j_\star \text{ for some } j \\ \frac{cB(M_j)}{\bar{r}} & \text{if } \varepsilon = b^j_\dagger \text{ for some } j \\ \frac{dS(M_j)}{\bar{r}} & \text{if } \varepsilon = s^j_\dagger \text{ for some } j \\ \frac{(N-|B|)c+(N-|S|)d}{\bar{r}} & \text{if } \varepsilon = v \end{cases}$$

for each pair of counting measures B, S defined on $(\mathbb{R}, \sigma(\mathcal{M}))$, each $\varepsilon \in E$ and each $i \le k$ and since (by Appendix, Theorem 3) there exist (easily computable) mappings $G_1, G_2, \dots, G_{k-1}$ such that, on the set $[e_{(1,k)} \in \tilde{\mathcal{E}}^k]$, $(B^\mathcal{M}_{(\nu)}, S^\mathcal{M}_{(\nu)}) = G_\nu(e_{(1,\nu)})$ for each $1 \le \nu < k$, we have

$$\mathbb{P}(e_{(i)} = \varepsilon | e_{(1,i-1)}) = \mathbb{E}\left[\mathbb{P}(e_{(i)} = \varepsilon | B^\mathcal{M}_{(i-1)}, S^\mathcal{M}_{(i-1)}, e_{(1,i-1)}) \middle| e_{(1,i-1)}\right]$$
$$\mathbb{E}\left[\mathbb{P}(e_{(i)} = \varepsilon | B^\mathcal{M}_{(i-1)}, S^\mathcal{M}_{(i-1)}) \middle| e_{(1,i-1)}\right] = \mathbb{P}(e_{(i)} = \varepsilon | B^\mathcal{M}_{(i-1)}, S^\mathcal{M}_{(i-1)})$$

[9] The conditional independence could be shown similarly to the proof of the Markov property of the embedded chain.

for each $\varepsilon \in E$ (the last "=" follows from the fact that $\sigma(B^{\mathcal{M}}_{(i-1)}, S^{\mathcal{M}}_{(i-1)}) \subseteq \sigma(e_{(1,i-1)})$) which yields

$$\mathbb{P}(e_{(1,k)} = \epsilon) = \prod_{i=1}^{k} \mathbb{P}(e_{(k)} = \epsilon^i | e_{(1)} = \epsilon^1, \dots, e_{(i-1)} = \epsilon^{i-1}) = \prod_{i=1}^{k} \gamma(\epsilon^i, G_{i-1}(\epsilon^1, \dots, \epsilon^{i-1}))$$

(we take $G_0 = (B^{\mathcal{M}}_{(0)}, S^{\mathcal{M}}_{(0)})$).

The conditional probabilities in (4) are also easy to compute provided that $\epsilon \in \tilde{\mathcal{E}}^k$: Since $(\phi_{(k)}, q_{(k)}) = F(e_{(1,k)})$ on the set $[e_{(1,k)} \in \tilde{\mathcal{E}}^k]$ for some (easily computable) mapping F (Appendix, Theorem 3), we have

$$\mathbb{P}((\phi_{(k)}, q_{(k)}) \in B | e_{(1,k)} = \epsilon) = \mathbf{1}_B(F(\epsilon)).$$

It remains to show that η_2^k may be made arbitrarily small by the refinement of $\mathcal{M}$: Indeed,

$$\mathbb{P}(e_{(1,k)} \in \mathcal{E}^k - \tilde{\mathcal{E}}^k) = \mathbb{P}(\beta^j_{e_{(1,k)}} > 0 \wedge \sigma^j_{e_{(1,k)}} > 0 \text{ for some } j \in C) \leq \sum_{j \in C} \zeta^j \tag{5}$$

where

$$\begin{aligned}\zeta^j =&\mathbb{P}(\beta^j_{e_{(1,k)}} > 0 \wedge \sigma^j_{e_{(1,k)}} > 0) = 1 - \mathbb{P}(\beta^j_{e_{(1,k)}} = 0) - \mathbb{P}(\sigma^j_{e_{(1,k)}} = 0) + \mathbb{P}(\beta^j_{e_{(1,k)}} = 0, \sigma^j_{e_{(1,k)}} = 0) \\ =&1 - (1-\mu_j)^k - (1-\nu_j)^k + (1 - (\mu_j + \nu_j))^k, \qquad \mu_j = u\pi(M^j)/\bar{r}, \qquad \nu_j = v\rho(M^j)/\bar{r}.\end{aligned}$$

Since, by an easy calculation,

$$\zeta^j = \sum_{i=2}^{k} \binom{k}{i} (-1)^i [(\mu_j + \nu_j)^i - (\mu_j^i + \nu_j^i)] \leq \sum_{i=2}^{k} \binom{k}{i} [2 \max_{j \in C}(\mu_j \vee \nu_j)]^i = o(\max_{j \in C}(\mu_j \vee \nu_j)) \tag{6}$$

as $\max_{j \in C}(\mu_j \vee \nu_j) \to 0$ and since the partition $\mathcal{M}$ may be constructed so that

$$\max_{j \in C}(\mu_j \vee \nu_j) \leq \frac{24 \max(u, v)}{\bar{r}(m - 2(|B_{(0)}| + |S_{(0)}|) - 1)} = O(m^{-1})$$

as $m \to \infty$ for each $j \in C$ (Appendix, Theorem 4), we get from (5) and (6) that

$$\mathbb{P}(e_{(1,k)} \in \mathcal{E}^k - \tilde{\mathcal{E}}^k) \leq m \cdot o(O(m^{-1})) = \frac{O(m^{-1})}{m^{-1}} \frac{o(O(m^{-1}))}{O(m^{-1})} \overset{m \to \infty}{\longrightarrow} 0$$

given a suitable choice of the partitions.

3.4 The Forecast

By summarizing the previous paragraphs, we are getting

$$\mathbb{P}((p_\tau, q_\tau) \in A) = e^{\bar{r}} \sum_{k=0}^{\tilde{N}} \frac{\bar{r}^k}{k!} \sum_{\epsilon \in \tilde{\mathcal{E}}^k} \mathbf{1}_B(F(\epsilon)) \prod_{i=1}^{k} \Gamma_i(\epsilon^1, \dots, \epsilon^i) + \eta_1 + \eta_2$$

for some easily computable mappings $F, \Gamma_1, \dots, \Gamma_k$. Moreover, η_1 may be made arbitrarily small by increasing $\tilde{N}$ while $\eta_2 \triangleq \sum_{k=1}^{\tilde{N}} \eta_2^k$ may be arbitrarily decreased by a suitable choice of partition $\mathcal{M}$.

3.5 Possible Further Refinements

Let us mention two ways of reducing the (possibly very large) computational complexity of the algorithm designed in the previous paragraphs.

First, not all the conditional distributions of $(p_{(k)}, q_{(k)})$ given $e_{(1,k)} = \epsilon$, $\epsilon \in \mathcal{E}^k - \tilde{\mathcal{E}}^k$, are non-computable. As it was already mentioned, some of the distributions corresponding to the scenarios from $\mathcal{E}^k - \tilde{\mathcal{E}}^k$ may be computed after a more detailed analysis, some of them are even Dirac: when we modify our example (subsection 3.3) so that $\epsilon = (v, v, b^5_\star, s^5_\star)$, then $p_{(k)} = 4$ independently of the limit prices of the new orders. Hence, we may increase the number of the scenarios we take into account to decrease the errors η^k_2.

Second, it follows from the definition of the CDA that the quantities $p_{(i)}$, $q_{(i)}$, $b^\kappa_{(i)}$, $\dots$, $b^1_{(i)}$, $a^1_{(i)}$, $\dots$, $a^\kappa_{(i)}$, depend only on $p_{(i-1)}, q_{(i-1)}, b^{\kappa+1}_{(i-1)}, \dots, b^1_{(i-1)}, a^1_{(i-1)}, \dots, a^{\kappa+1}_{(i-1}, \dots$ and on independent random variables for each $i, \kappa \in \mathbb{N}$ which implies that only the quantities

$$p_{(k-1)}, q_{(k-1)}, b^1_{(k-1)}, a^1_{(k-1)}, p_{(k-2)}, q_{(k-2)}, b^2_{(k-2)}, b^1_{(k-2)}, a^1_{(k-2)}, a^2_{(k-2)}, \dots, b^k_{(0)}, \dots, a^k_{(0)}$$

are relevant for the distribution of $p_{(k)}, q_{(k)}$. Hence, to reduce the number of the branches of our computation, we may accumulate the scenarios having equal impact on the relevant quantities.

4 Conclusion

In the present paper, a way of computing the forecasts of the market price and of the traded volume in the model of continuous double auction with complete randomness was suggested. With a "sufficiently efficient" computer, the "future" distribution of the forecasted quantities may be evaluated with an arbitrarily accuracy.

Appendix

Assume, throughout the Appendix, that all the random elements are defined on an underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Theorem 1 *$\Xi_t \triangleq (B_t, S_t, p_t, q_t)$ is a continuous time Markov taking values in $\mathbb{X} \triangleq \mathcal{N}_\mathbb{R} \times \mathcal{N}_R \times \mathbb{R}^{undefined} \times \mathbb{Z}^+$.*

The proof of the Theorem is a bit tedious exercise in the conditional probability calculus. We refer the reader to **?**, especially sections 6.7. and 6.8. or **?**, chp. 6, for more on conditioning. We will work both with the conditional probabilities (expectations) defined as random variables [**?**, (6.7.1.), (6.7.2.)] and with the conditional probabilities (expectations) given fixed value [**?**, (6.7.6.))]. The following Lemma formulates some auxiliary results concerning conditioning, not explicitly listed in the textbooks, which we will use later on.

Lemma 1 *Assume that regular conditional probabilities exist on Ω. Let X, Y, Z be random elements taking values in measurable spaces $(\mathcal{X}, \mathcal{B}), (\mathcal{Y}, \mathcal{C})$ and $(\mathcal{Z}, \mathcal{D})$ respectively*

(i) Let $S_1, S_2, \dots$ be a partition of Ω such that $S_i \in \mathcal{C}$ for each $i \in \mathbb{N}$. Let $\xi, \xi_1, \xi_2, \dots$ be real random variables such that $\xi|_{S_i} = \xi_i|_{S_i}$ for each $i \in \mathbb{N}$ and let $Y_1, Y_2, \dots$ be random elements taking values in $(\mathcal{Y}, \mathcal{C})$ such that $Y|_{S_i} = Y_i|_{S_i}$ and $S_i \in \sigma(Y_i)$ for each $i \in \mathbb{N}$. Then $\mathbb{E}(\xi|Y) = \sum_{i\in\mathbb{N}} \mathbf{1}_{S_i} \mathbb{E}(\xi_i|\mathbf{1}_{S_i}, Y_i)$.

(ii) For any $A, B \in \mathcal{A}$ it holds that $\mathbb{P}(A, B|Y = y) = P(A|\mathbf{1}_B = 1, Y = y)\mathbb{P}(B|Y = y)$ for $\mathbb{P}_Y$-almost every each $y \in \mathcal{Y}$.

(iii) If X is independent of (Y,Z), then $\mathbb{P}(X > Y|Z) = \mathbb{P}(X > Y)$.

(iv) If X is independent of (Y,Z), then

$$\mathbb{P}(X \in A, Y \in B|Z) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B|Z)$$

(v) Let (X,Y) be independent of Z, let $f : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ be a measurable function and let $g : \mathcal{Y} \times \mathcal{Z} \to \Gamma$ be a measurable mapping taking values in a measurable space $(\Gamma, \mathcal{G})$. Denote

$$\phi(B, \gamma, z) = \mathbb{P}(f(X,z) \in B|g(Y,z) = \gamma).$$

Then

$$\mathbb{P}(f(X,Z) \in B|g(Y,Z) = \gamma, Z = z) = \phi(B, \gamma, z)$$

for $\mathbb{P}_{g(Y,Z),Z}$-almost every $(\gamma, z) \in (\Gamma, \mathcal{Z})$.

Proof. (i) The assertion is a variant of the well known local property of conditional expectations [**?**, Lemma 6.2].

Let $C \in \mathcal{C}$. Denote $A = Y^{-1}(C)$. First, we show that $A \cap S_i \in \sigma(\mathbf{1}_{S_i}, Y_i)$ for each $i \in \mathbb{N}$: indeed, $A \cap S_i = \{\omega \in S_i : Y(\omega) \in C\} = \{\omega \in S_i : Y_i(\omega) \in C\} = Y_i^{-1}(C) \cap S_i \in \sigma(\mathbf{1}_{S_i}, Y_i)$.

Now,

$$\begin{aligned}\int_A \mathbb{E}(\xi|Y)d\mathbb{P} &= \int_A \xi d\mathbb{P} = \sum_{i\in\mathbb{N}} \int_{A\cap S_i} \xi d\mathbb{P} = \sum_{i\in\mathbb{N}} \int_{A\cap S_i} \xi_i d\mathbb{P} \\ &\overset{A\cap S_i \in \sigma(\mathbf{1}_{S_i}, Y_i)}{=} \sum_{i\in\mathbb{N}} \int_{A\cap S_i} \mathbb{E}(\xi_i|\mathbf{1}_{S_i}, Y_i)d\mathbb{P} = \int_A \left(\sum_{i\in\mathbb{N}} \mathbf{1}_{S_i}\mathbb{E}(\xi_i|\mathbf{1}_{S_i}, Y_i)\right) d\mathbb{P}\end{aligned}$$

which suffices to prove (i).

(ii) Let $P^y = P^y(\bullet, \bullet)$ be the conditional distribution of $\mathbf{1}_A, \mathbf{1}_B$ given $Y = y$ and let $P^y_{\mathbf{1}_B} = P^y_{\mathbf{1}_B}(\bullet)$ be conditional distribution of $\mathbf{1}_B$ given $Y = y$ (which is simultaneously the second marginal distribution of P^y). Clearly

$$\begin{aligned}\mathbb{P}(A, B|Y = y) = P^y(1,1) = \int_{\{1\}} P^y(1|b)dP^y_{\mathbf{1}_B}(b) &= P^y(1|\mathbf{1}_B = 1)P^y_{\mathbf{1}_B}(1) \\ &= \mathbb{P}(A|\mathbf{1}_B = 1, Y = y)\mathbb{P}(B|Y = y)\end{aligned}$$

(the latter "=" is easy to show using the definition of conditional probability given fixed value).

(iii) The assertion follows from **?**, (6.8.14).

(iv) Let $B \in \mathcal{B}$, $C \in \mathcal{C}$ and $D \in \sigma(Z)$. Then

$$\begin{aligned}\int_D \mathbb{P}(X \in B, Y \in C|Z)d\mathbb{P} &= \int_D \mathbf{1}_C(Y)\mathbf{1}_B(X)d\mathbb{P} = \int_{D\cap Y^{-1}(C)} \mathbf{1}_B(X)d\mathbb{P} \\ &= \int_{D\cap Y^{-1}(C)} \mathbb{P}(X \in B|Y,Z)d\mathbb{P} = \mathbb{P}(X \in B)\int_{D\cap Y^{-1}(C)} d\mathbb{P} \\ &= \mathbb{P}(X \in B)\int_D \mathbb{P}(Y \in C|Z)d\mathbb{P} = \int_D \mathbb{P}(X \in B)\mathbb{P}(Y \in C|Z)d\mathbb{P}.\end{aligned}$$

(v) We show that $\phi(\bullet, \gamma, z)$ fulfils the definition of the conditional probability given fixed value (it suffices to assume $\Omega = \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ for the sake of the proof): Let $B \in \mathbb{B}(\mathbb{R})$, let $G \in \mathcal{G}$ and let

$C \in \mathcal{D}$. Then

$$\int_{G\times C} \phi(B,\gamma,z)\mathbb{P}_{g(Y,Z),Z}(\gamma,z) = \int \mathbf{1}_G(\gamma)\mathbf{1}_C(z)\phi(B,\gamma,z)d\mathbb{P}_{g(Y,Z),Z}(\gamma,z)$$

$$\overset{\text{H-J (3.15.1)}}{=} \int \mathbf{1}_G(g(y,z))\mathbf{1}_C(z)\phi(B,g(y,z),z)d\mathbb{P}_{Y,Z}(y,z)$$

$$\overset{\text{H-J (4.5.2)}}{=} \int \left(\int \mathbf{1}_G(g(y,z))\mathbf{1}_C(z)\phi(B,g(y,z),z)d\mathbb{P}_Y(y)\right) d\mathbb{P}_Z(z)$$

$$\overset{\text{H-J (3.15.1)}}{=} \int_C \left(\int_G \phi(B,\gamma,z)d\mathbb{P}_{g(Y,z)}(\gamma)\right) d\mathbb{P}_Z(z)$$

$$\overset{\text{H-J (6.7.6)}}{=} \int_C \left(\int_{\{(y,z):g(y,z)\in G\}} \mathbf{1}_B(f(x,z))d\mathbb{P}_{X,Y}(x,y)\right) d\mathbb{P}_Z(z)$$

$$= \int \left(\int \mathbf{1}_C(z)\mathbf{1}_G(g(y,z))\mathbf{1}_B(f(x,z))d\mathbb{P}_{X,Y}(x,y)\right) d\mathbb{P}_Z(z)$$

$$\overset{\text{H-J (4.5.2)}}{=} \int_{\{(x,y,z):x\in\mathcal{X},z\in C,g(y,z)\in G\}} \mathbf{1}_B(f(x,z))d\mathbb{P}_{X,Y,Z}(x,y,z)$$

$$= \int_{(\gamma,I)^{-1}(G\times C)} \mathbf{1}_B(f(x,z))d\mathbb{P}_{X,Y,Z}(x,y,z)$$

where I is the identity mapping, which suffices for (v) (H-J stands for **?**).

□

The following Lemma is the core of the proof of the Theorem. Before we formulate it, however, let us agree that $Z_{(i)}$ will stand for Z_{τ^i} for any continuous time process Z.

Lemma 2 *Denote j_b^ν the index of the ν-th best buy order waiting at the time τ^{i-1} and j_s^ν the index of the ν-th best sell order waiting at the time τ^{i-1} (if two orders have identical limit prices then the one with lesser index is better), define*

$$\varphi^i = \begin{cases} -\infty & \text{if } \tau^i = \tau^j_{b\star} \text{ for some } j \in \mathbb{N} \\ \infty & \text{if } \tau^i = \tau^j_{s\star} \text{ for some } j \in \mathbb{N} \\ -1 & \text{if } \tau^i = \tau^{j_b^1}_{b\dagger} \\ 1 & \text{if } \tau^i = \tau^{j_s^1}_{b\dagger} \\ -2 & \text{if } \tau^i = \tau^{j_b^2}_{b\dagger} \\ \dots \end{cases}$$

and

$$\theta^i \triangleq \gamma(|B_{(i-1)}|,|S_{(i-1)}|)\Delta\tau^i, \qquad \gamma(b,s) = u + v + bc + bd$$

for each $i \in \mathbb{Z}^+$. Denote

$$\Upsilon \triangleq (x^1, y^1, x^2, y^2, \dots).$$

Then

$$\mathbb{P}(\theta^i > t, \varphi^i = k|\theta^{i-1},\varphi^{i-1},\dots,\theta^1,\varphi^1,\Upsilon) = \exp\{-t\}\alpha(|B_{(i-1)}|,|S_{(i-1)}|,k) \tag{7}$$

for each $i \in \mathbb{N}$ where

$$\alpha(b,s,k) \triangleq \begin{cases} \gamma(b,s)^{-1}u & \text{if } k = -\infty \\ \gamma(b,s)^{-1}c & \text{if } -b \le k \le -1 \\ \gamma(b,s)^{-1}d & \text{if } 1 \le k \le s \\ \gamma(b,s)^{-1}v & \text{if } k = \infty \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We proceed as follows: We set Υ to be non-random, we enrich the conditioning σ-field and we find a suitable partition of the probability space so that the conditional probability (7) will be easy to evaluate on each of the partitioning sets. The formula (7) will then follow from the local property (Lemma 1 (i)) and from Lemma 1 (v).

Denote

$$\Psi_t \triangleq (p_t, q_t, X_t^1, Y_t^1, X_t^2, Y_t^2, \dots).$$

Clearly, Ξ_t may be devised from (Ψ_t, Υ), i.e. there exists a measurable mapping f such that $\Xi_t = f(\Psi_t, \Upsilon)$ for each $t \in \mathbb{R}_0^+$ (the measurability may be easily proved) and, in particular,

$$\Xi_{(i)} = f(\Psi_{(i)}, \Upsilon). \tag{8}$$

Further, introduce a random element

$$\eta^i \in E, \qquad E \triangleq \{b, s\} \times \{\star, \dagger\} \times \mathbb{N}, \qquad i \in \mathbb{N}$$

where $b, s, \star, \dagger$ are some labels[10] such that

$$\eta^i = \begin{cases} (b, \star, 1) & \text{when } \tau^i = \tau^1_{b\star} \\ (b, \dagger, 1) & \text{when } \tau^i = \tau^1_{b\dagger} \\ (s, \star, 1) & \text{when } \tau^i = \tau^1_{s\star} \\ (s, \dagger, 1) & \text{when } \tau^i = \tau^1_{b\star} \\ (b, \star, 2) & \text{when } \tau^i = \tau^2_{b\star} \\ \dots \end{cases}$$

for each $i \in \mathbb{N}$.

It follows from our definitions (Subsections 2.2 and 2.3) that $\Psi_{(i)}$ may be devised only from $\Psi_{(i-1)}$, η^i and Υ, i.e. there exists a (non-random measurable) mapping g such that

$$\Psi_{(i)} = g(\Psi_{(i-1)}, \eta^i, \Upsilon) \tag{9}$$

for each $i \in \mathbb{N}$. Further, when we denote

$$\begin{aligned} I^i = I(\Psi_{(i-1)}) = \{(b, \star, k_b^i)\} \cup \{(s, \star, k_s^i)\} \\ \cup \{(b, \dagger, \nu) : X^\nu_{(i-1)} = \mathit{waiting}\} \cup \{(s, \dagger, \nu) : Y^\nu_{(i-1)} = \mathit{waiting}\} \end{aligned} \tag{10}$$

$$k_b^i = \min\{\nu : X^\nu_{(i-1)} = \mathit{prenatal}\}, \qquad k_s^i = \min\{\nu : Y^\nu_{(i-1)} = \mathit{prenatal}\},$$

it can be easily seen that φ^i may be devised only from η^i, I^i and Υ[11], i.e. there exists a (non-random measurable) mapping h such that

$$\varphi^i = h(\eta^i, I^i, \Upsilon). \tag{11}$$

Moreover, $h_{I^i,\Upsilon)}(\bullet) \triangleq h(\bullet, I^i, \Upsilon)$ is bijection for each realization of I^i and Υ.[12]

When we summarize (9), (10) and (11), we are getting, by induction, that a (non-random measurable) mappings h_i and H_i exist for each $i \in \mathbb{N}$ such that

$$\varphi^i = h_i(\eta^1, \dots, \eta^i, \Upsilon), \tag{12}$$

[10] An orthodox mathematician may think that $b = 0, s = 1, \star = 0, \dagger = 1$

[11] From η^i, we get the index and the type of the forthcoming event. If the event is a new buy/sell order, then $\varphi^i = -\infty/+\infty$. If the event is a cancelation of a buy/sell order then we determine φ^i by means of the list of all the waiting buy/sell orders, obtained from I^i, and of the list of the limit prices, obtained from Υ.

[12] If $\varphi^i = -\infty/+\infty$ then $\eta^i = (b, \star, k_b^i)/(s, \star, k_s^i)$ (we get k_b^i/k_s^i from I^i). If $\varphi^i \in -\mathbb{N}/\mathbb{N}$ then φ^i uniquely determine the index of the canceled order (we get it from from I^i and from Υ).

and

$$\Psi^i = H_i(\eta^1, \ldots, \eta^i, \Upsilon). \tag{13}$$

From (12), it follows that, to prove (7), it suffices to show that

$$\mathbb{P}(\theta^i > t, \varphi^i = k | \theta^{i-1}, \eta^{i-1}, \ldots, \theta^1, \eta^1, \Upsilon) = \exp\{-t\}\alpha(|B_{(i-1)}|, |S_{(i-1)}|, k). \tag{14}$$

Let us do it: When we denote $T \triangleq \{\tau^1_{b\star}, \tau^1_{b\dagger}, \tau^1_{s\star}, \tau^1_{s\dagger}, \tau^2_{b\star}, \ldots\}$ and index the set T naturally by the elements of E, we may write, for each $i \in \mathbb{N}$,

$$\tau^i = \min_{j \in I^i} T^j, \qquad \eta^i = \operatorname*{argmin}_{j \in I^i} T^j. \tag{15}$$

Moreover, when we fix $\zeta \subset E$ and $i \in \mathbb{N}$ and agree that $T^\zeta = (T^\eta)_{\eta \in \zeta}$ (similarly for the other subsets of E), we are getting

$$\tau^j|_{[T^\zeta > \tau^{i-1}]} = \tau^j_\zeta|_{[T^\zeta > \tau^{i-1}]}, \qquad \tau^j_\zeta = \min_{\nu \in I^j \setminus \zeta} T^\nu \tag{16}$$

and

$$\eta^j|_{[T^\zeta > \tau^{i-1}]} = \eta^j_\zeta|_{[T^\zeta > \tau^{i-1}]}, \qquad \eta^j_\zeta = \operatorname*{argmin}_{\nu \in I^j \setminus \zeta} T^\nu \tag{17}$$

for each $j < i$. In other words, (16) and (17) say that the random element $(\tau^j_\zeta, \eta^j_\zeta)$ may be expressed as a function of $T^{I^j \setminus \zeta}$ and $\Psi_{(j-1)}$ given that $T^\zeta > \tau^{i-1}$.

Further, denote $\Theta = \{\Delta\tau^1_{b\star}, \Delta\tau^1_{b\dagger}, \Delta\tau^1_{s\star}, \Delta\tau^1_{s\dagger}, \Delta\tau^2_{b\star}, \ldots\}$. Since

$$e \in I^j \Rightarrow \left[\begin{array}{ll} T^e = \Theta^e + \sum_{p=1}^k \Theta^{(b,\star,p)} & \text{for some } k < k^j_b \\ \text{or} & \\ T^e = \Theta^e + \sum_{p=1}^k \Theta^{(s,\star,p)} & \text{for some } k < k^j_s \end{array}\right. \tag{18}$$

for each $j \in \mathbb{N}$, since $k^j_b \le k^i_b$ and $k^j_s \le k^i_s$ for each $j < i$ and since

$$T^e > \tau^{i-1} \Rightarrow e \notin \{(b, \star, k) : k < k^i_b\} \cup \{(s, \star, k) : k < k^i_s\}$$

we can express each T^e, $e \in I^j \setminus \zeta$, $j < i$, as a function of $\Theta^{E \setminus \zeta}$ and $\Psi_{(j-1)}$ given that $T^\zeta > \tau^{i-1}$. Therefore and thanks to (16) and (17), $(\tau^j_\zeta, \eta^j_\zeta)$ may be expressed as a function of $\Theta^{E\setminus\zeta}$ and $\Psi_{(j-1)}$ given that $T^\zeta > \tau^{i-1}$ for each $j < i$. Moreover, by (9) and by induction, $(\tau^j_\zeta, \eta^j_\zeta)$ may be expressed as a function of $\Theta^{E\setminus\zeta}$ and Υ for each $j < i$ which implies that $(\tau^j_\zeta, \eta^j_\zeta)$ is independent of Θ^ζ for each $j < i$.

Put

$$S_\zeta \triangleq [I(\Psi_{(i-1)}) = \zeta].$$

Since $I(\Psi_{(i-1)}) = \zeta$ implies $T^\zeta > \tau^{i-1}$, we have, from (16) and (17), that

$$\tau^j|_{S_\zeta} = \tau^j_\zeta|_{S_\zeta}, \qquad \eta^j|_{S_\zeta} = \eta^j_\zeta|_{S_\zeta}, \tag{19}$$

for each $k < i$ which yields

$$\Psi_{(j-1)}|_{S_\zeta} = \Psi_{\zeta,(j-1)}|_{S_\zeta}, \qquad \Psi_{\zeta,(j-1)} = H_{j-1}(\eta^1_\zeta, \ldots, \eta^{j-1}_\zeta, \Upsilon) \tag{20}$$

for each $j < i$ and

$$\theta^j|_{S_\zeta} = \theta^j_\zeta|_{S_\zeta}, \qquad \theta^j_\zeta = \gamma(G(\Psi_{\zeta,(j-1)}))[\tau^j_\zeta - \tau^{j-1}_\zeta] \tag{21}$$

for each $i < j$, where G is some mapping. Moreover, since $\{x : x \in A, F(x) \in B\} = \{x : x \in A, F|_A(x) \in B\}$ for any sets A, B and a mapping F and since $T^\zeta > \tau^{i-1} = \min_{\nu \in I^j} T^\nu$ is equivalent to $T^\zeta > \tau_\zeta^{i-1} = \min_{\nu \in I^j \setminus \zeta} T^\nu$ we have

$$S_\zeta = [T^\zeta > \tau_\zeta^{i-1}] \cap [I(\Psi_{\zeta,(i-1)}) = \zeta] \tag{22}$$

Further, when we put $\Delta T_\zeta^i \stackrel{\triangle}{=} T^\zeta - \tau_\zeta^{i-1}$, we may write

$$\theta^i|_{S_\zeta} = \theta_\zeta^i|_{S_\zeta} \qquad \theta_\zeta^i|_{S_\zeta} = \gamma_\zeta \min \Delta T_\zeta^i, \qquad \gamma_\zeta = \gamma(b_\zeta, s_\zeta), \tag{23}$$
$$b_\zeta = |\{e \in \zeta, e = (b, \dagger, \nu) \text{ for some } \nu \in \mathbb{N}\}|,$$
$$s_\zeta = |\{e \in \zeta, e = (s, \dagger, \nu) \text{ for some } \nu \in \mathbb{N}\}|$$

and

$$\eta^i|_{S_\zeta} = \eta_\zeta^i|_{S_\zeta}, \qquad \eta_\zeta^i = \operatorname{argmin} \Delta T_\zeta^i \tag{24}$$

which implies, together with (11), that

$$\varphi^i|_{S_\zeta} = \varphi_\zeta^i|_{S_\zeta}, \qquad \varphi_\zeta^i = h(\eta_\zeta^i, \zeta, \Upsilon). \tag{25}$$

Since, for any positive real vector w, we have

$$\begin{aligned}
&\mathbb{P}\left(\Delta T_\zeta^i > w | \mathbf{1}_{S_\zeta}, \theta_\zeta^{i-1}, \eta_\zeta^{i-1}, \ldots, \theta_\zeta^1, \eta_\zeta^1, \Upsilon\right) \\
&\qquad \stackrel{(22)}{=} \mathbb{P}\left(\Delta T_\zeta^i > w | \mathbf{1}_{[T^\zeta > \tau_\zeta^{i-1}]}, \mathbf{1}_{[I(\Psi_{\zeta,(i-1)}) = \zeta]}, \theta_\zeta^{i-1}, \eta_\zeta^{i-1}, \ldots, \theta_\zeta^1, \eta_\zeta^1, \Upsilon\right) \\
&\qquad = \mathbb{P}\left(\Delta T_\zeta^i > w | \mathbf{1}_{[T^\zeta > \tau_\zeta^{i-1}]}, \Phi_\zeta\right)
\end{aligned}$$

where Φ_ζ is a random element independent of Θ^ζ and since $\Delta T_\zeta^i = \Theta^\zeta - V_\zeta$ for some vector V_ζ independent of Θ^ζ, we are getting

$$\begin{aligned}
&\mathbb{P}\left(\Delta T_\zeta^i > w | \mathbf{1}_{[T^\zeta > \tau_\zeta^{i-1}]} = 1, \Phi_\zeta = \varphi\right) \\
&\qquad \stackrel{\text{Lemma 1 (ii), (21)}}{=} \frac{\mathbb{P}\left(\Theta^\zeta > w + V_\zeta | \Phi_\zeta = \varphi\right)}{\mathbb{P}\left(\Theta^\zeta > V_\zeta | \Phi_\zeta = \varphi\right)} \stackrel{\text{Lemma 1 (iii)}}{=} \frac{\mathbb{P}\left(\Theta^\zeta > w + V_\zeta\right)}{\mathbb{P}\left(\Theta^\zeta > V_\zeta\right)} \\
&\qquad = \frac{\prod_{j=1}^{|\zeta|} \mathbb{P}((\Theta^\zeta)^j > w + V_\zeta^j)}{\prod_{j=1}^{|\zeta|} \mathbb{P}((\Theta^\zeta)^j > V_\zeta^j)} = \prod_{j=1}^{|\zeta|} \mathbb{P}((\Theta^\zeta)^j > w^j)
\end{aligned}$$

i.e. the conditional distribution of ΔT_ζ^i given $\mathbf{1}_{S_\zeta}, \theta_\zeta^{i-1}, \eta_\zeta^{i-1}, \ldots, \theta_\zeta^1, \eta_\zeta^1, \Upsilon$ is the same as the unconditional distribution of Θ^ζ on the set S_ζ.

Since, for any vector $(s_1, s_2, \ldots, s_n)$ of independent exponential variables with parameters $r_1, r_2, \ldots, r_n$ it is well known that

$$\mathbb{P}(\min_{i=1,2,\ldots,n} > z, \operatorname*{argmin}_{i=1,2,\ldots,n} s_i = i) = \exp\{-(r_1 + \cdots + r_n)z\} \frac{r_i}{r_1 + \cdots + r_n},$$

we are getting

$$\begin{aligned}
&\mathbb{P}(\theta_\zeta^i > t, \varphi_\zeta^i = k | \theta_\zeta^{i-1}, \eta_\zeta^{i-1}, \ldots, \theta_\zeta^1, \eta_\zeta^1, \Upsilon)|_{S_\zeta} \\
&\qquad = \mathbb{P}(\min \Delta T_\zeta^i > \gamma_\zeta^{-1} t, \operatorname{argmin} \Delta T_\zeta^i = h_{\zeta,\Upsilon}^{-1}(k) | \theta_\zeta^{i-1}, \eta_\zeta^{i-1}, \ldots, \theta_\zeta^1, \eta_\zeta^1, \Upsilon))|_{S_\zeta} \\
&\qquad = \exp\{-t\} \alpha(b_\zeta, s_\zeta, k)
\end{aligned} \tag{26}$$

because

$$h^{-1}_{\zeta,\Upsilon}(k)=\begin{cases}(b,\star,k^i_b) & \text{if } k=-\infty\\ (s,\star,k^i_b) & \text{if } k=\infty\\ (b,\dagger,j) \text{ for some } j\in\mathbb{N} & \text{if } -b_\zeta\le k<-1\\ (s,\dagger,j) \text{ for some } j\in\mathbb{N} & \text{if } 1\le k\le s_\zeta\end{cases}$$

and

$$\text{The parameter of } \Theta^e \text{ is}\begin{cases}u & \text{if } (b,\star,j) \text{ for some } j\in\mathbb{N}\\ v & \text{if } (s,\star,j) \text{ for some } j\in\mathbb{N}\\ c & \text{if } (b,\dagger,j) \text{ for some } j\in\mathbb{N}\\ d & \text{if } (s,\dagger,j) \text{ for some } j\in\mathbb{N}.\end{cases}$$

Thanks to (19), (20), (21), (23), (24) and (25), and (i) of Lemma 1 we are getting (14) (for the case of non-random Υ). Finally, when we allow Υ to be random, the formula (14) keeps holding by Lemma 1 (v).

□

Proof of the Theorem. Denote $\Delta\tau^i=\tau^i-\tau^{i-1}$. According to **?**, Lemma 12.18, it suffices to show that

(i) $(\Xi_{(i)})_{i=1}^\infty$ is a discrete time Markov chain

(ii) There exists a function $\gamma:\mathbb{X}\to\mathbb{R}$ such that

$$\gamma(\Xi_{(0)})\Delta\tau^1,\quad \gamma(\Xi_{(1)})\Delta\tau^2,\quad \dots$$

are i.i.d. exponential with unit mean independent of $(\Xi_{(i)})_{i=1}$.

(iii) $\sum_{i=1}^\infty \Delta\tau^i=\infty$ a.s.

Before we deal with (i)-(iii), note that $\Xi_{(i)}$ may be devised from $\Xi_{(i-1)}$, φ^i and Υ for each $i\in\mathbb{N}$, i.e. there exists a (non-random measurable) mapping F such that

$$\Xi_{(i)}=F(\Xi_{(i-1)},\varphi^i,\Upsilon)$$

for each $i\in\mathbb{N}$ which yields, by induction, that

$$\Xi_{(i)}=F_i(\varphi^1,\dots,\varphi^i,\Upsilon), \tag{27}$$

where F_i is a (non-random) mapping, for each $i\in\mathbb{N}$.
(i) Surely, a function $\rho_{b,s}:[0,1]\to\mathbb{Z}\cup\{-\infty,\infty\}$ may be constructed for each $b,s\in\mathbb{Z}^+$ such that

$$\mathbb{P}(\rho_{b,s}(U)=k)=\alpha(b,s,k). \tag{28}$$

for $U\sim \mathrm{U}[0,1]$.

Let $U^1,U^2,\dots$ be a sequence of i.i.d. uniform variables independent of Θ. When we put $\tilde\varphi^i\stackrel{\triangle}{=}\rho_{|B_{(i-1)}|,|S_{(i-1)}|}(U^i)$ for each $i\in\mathbb{N}$, and denote $J_{b,s}\stackrel{\triangle}{=}[|B_{(i-1)}|=b,|S_{(i-1)}|=s]$, we have

$$\begin{aligned}
&\mathbb{P}\left(\theta^i>t,\tilde\varphi^i=k|\theta^{i-1},\varphi^{i-1},\dots,\theta^1,\varphi^1\right)\big|_{J_{b,s}}\\
&\stackrel{Lemma 1,(i)}{=}\mathbb{P}\left(\theta^i>t,\rho_{b,s}(U^i)=k|\mathbf{1}_{J_{b,s}},\theta^{i-1},\varphi^{i-1},\dots,\theta^1,\varphi^1\right)\big|_{J_{b,s}}\\
&\stackrel{\text{Lemma 1 (iv)}}{=}\mathbb{P}(\rho_{b,s}(U^i)=k)\mathbb{P}\left(\theta^i>t|\mathbf{1}_{J_{b,s}},\theta^{i-1},\varphi^{i-1},\dots,\theta^1,\varphi^1\right)\big|_{J_{b,s}}\\
&\stackrel{(27),J_{b,s}\in\sigma(\Xi_{(i-1)})}{=}\mathbb{P}(\rho_{b,s}(U^i)=k)\mathbb{P}\left(\theta^i>t|\theta^{i-1},\varphi^{i-1},\dots,\theta^1,\varphi^1\right)|_{J_{b,s}}\\
&\stackrel{(26),(28)}{=}\exp\{-t\}\alpha(b,s,k)
\end{aligned}$$

for each $t \in \mathbb{R}^+$, $k \in \mathbb{Z} \cup \{-\infty, \infty\}$, $\zeta \subset E$ and $i \in \mathbb{N}$, i.e.

$$\mathcal{L}\left(\theta^i, \tilde{\varphi}^i | \theta^{i-1}, \varphi^{i-1}, \ldots, \theta^1, \varphi^1\right) = \mathcal{L}\left(\theta^i, \varphi^i | \theta^{i-1}, \varphi^{i-1}, \ldots, \theta^1, \varphi^1\right)$$

for each $i \in \mathbb{N}$ which yields, by induction,

$$\mathcal{L}(\theta^1, \varphi^1, \theta^2, \varphi^2, \ldots) = \mathcal{L}(\theta^1, \tilde{\varphi}^1, \theta^2, \tilde{\varphi}^2, \ldots)$$

hence, by (27),

$$\mathcal{L}(\theta^1, \Xi_{(1)}, \ldots) = \mathcal{L}(\theta^1, \tilde{\Xi}_{(1)}, \ldots), \qquad \tilde{\Xi}_{(i)} = F_i(\tilde{\varphi}^1, \ldots, \tilde{\varphi}^i, \Upsilon), \tag{29}$$

so that it suffices to prove (ii) for $\tilde{\Xi}$ instead of Ξ, which is easy: Since

$$\tilde{\Xi}_{(i)} = F(\tilde{\Xi}_{(i-1)}, \rho_{|B_{(i-1)}|,|S_{(i-1)}|}(U^i), \Upsilon)$$

there exists a mapping F'_Υ for each realization of Υ such that

$$\tilde{\Xi}_{(i)} = F'_\Upsilon(\tilde{\Xi}_{(i-1)}, U^i).$$

When Υ is non-random, $\tilde{\Xi}_{(1)}, \tilde{\Xi}_{(2)}, \ldots$ is Markov by **?**, Proposition 8.6. i.e.

$$\mathbb{P}(\Xi_{(i)} \in A | \Xi_{(i-1)} = \xi^{i-1}, \ldots, \Xi_{(1)} = \xi^1) \tag{30}$$

for each $\xi^{i-1}, \ldots, \xi^i \in \mathbb{X}$ and each measurable $A \in \mathbb{X}$ given that Υ is non-random. Because the relation (30) keeps holding even for random Υ by Lemma 1 (v) (we naturally assume that Υ is independent of $U^1, U^2, \ldots$), the part (i) is proved.
(ii): Denote $\Gamma^i = (\theta^{i-1}, \varphi^{i-1}, \ldots, \theta^1, \varphi^1, \Upsilon)$. Since

$$\mathbb{P}(\theta^i > t | \Gamma^i) = \sum_{k \in \mathbb{Z} \cup \{-\infty, \infty\}} \mathbb{P}(\theta^i > t, \varphi^i = k | \Gamma^i) \overset{(7)}{=} \exp\{-t\} \tag{31}$$

for each $i \in \mathbb{N}$, the variable θ^i is surely exponential with unit mean for each $i \in \mathbb{N}$. Further, since

$$\mathbb{P}(\theta^i > t | \theta^j) = \mathbb{E}(\mathbb{P}(\theta^i > t | \Gamma^i) | \theta^j) = \exp\{-t\} = \mathbb{P}(\theta^i > t)$$

for each $j < i$, we are getting the mutual independence of $\theta^1, \theta^2, \ldots$. Finally, the independence of $\theta^1, \theta^2, \ldots$ and the embedded chain follows from (27) and from the independence of $\theta^1, \theta^2, \ldots$ of $\Upsilon, \varphi^1, \varphi^2, \ldots$ guaranteed by (31).
(iii): Since

$$\sum_{i=1}^{\infty} \Delta\tau^i \text{ a.s.} \Leftrightarrow \sum_{i=0}^{\infty} \frac{1}{\gamma(\Xi_{(i)})} \text{ a.s.}$$

(see the proof of the first part of **?**, Proposition 12.19.) and since

$$\sum_{i=1}^{\infty} \frac{1}{\gamma(\Xi_{(i-1)})} \geq \sum_{i=1}^{\infty} \frac{1}{u + v + i(c+d)} = \infty$$

the part (iii) is proved as well.
□

Theorem 2 *$\bar{\Xi}_t$ is a Markov chain, $\Delta\bar{\tau}^i \sim \mathrm{Exp}(\bar{r})$ such that $(\Delta\bar{\tau}^i)_{i=1}^{\tilde{N}}$ are i.i.d and independent of $\bar{\Xi}_{(k)}$ (see Section (3.1) for the notation).*

Proof. Denote $\tilde{\gamma}(b,s) = \min(N, u+v+bd+sc)$ and $\tilde{\theta}^i = \tilde{\gamma}(|B^{(i-1)}|, |B^{(i-1)}|)\Delta\bar{\tau}^i$. Analogously to the proof of Theorem 1, it may be shown that $\tilde{\theta}^1, \tilde{\theta}^2, \dots$ are i.i.d. exponential with unit mean independent of $\bar{\Xi}_{(1)}, \bar{\Xi}_{(2)}, \dots$ which itself is a Markov chain and that $\sum_{i=1}^{\infty} \tilde{\gamma}(|B_{(i-1)}|, |B_{(i-1)}|) = \infty$ which proves the Markov property of $\bar{\Xi}$.

Further, since $\max(|B_{(i)}, S_{(i)}|) \leq N$ for each $i \leq \tilde{N} - 1$ (the number of the orders in the limit order book may jump at most by one), we have $\tilde{\gamma}(|B^{(i-1)}|, |B^{(i-1)}|) = \bar{r}$ for each $i \leq \tilde{N}$ which yields $\bar{r}\Delta\bar{\tau}^i \sim \mathrm{Exp}(1)$ i.e. $\Delta\bar{\tau}^i \sim \mathrm{Exp}(\bar{r})$, $i \leq \tilde{N}$. Both the mutual independence of $(\Delta\bar{\tau}^i)_{i=1}^{\infty}$ and their independence of the embedded chain follows from the fact that, for any random variable X and random element Y independent of X, also cX and Y are independent.

□

Theorem 3 *There exist mappings F, $G_1, G_2, \dots, G_k$ such that, on the set $[e_{(1,k)} \in \tilde{\mathcal{E}}^k]$,*

$$(\phi_{(k)}, q_{(k)}) = F(e_{(1,k)}) \tag{32}$$

and that

$$(B^{\mathcal{M}}_{(i)}, S^{\mathcal{M}}_{(i)}) = G_i(e_{(1,i)}) \tag{33}$$

for each $1 \leq i \leq k$ (see Section 3.3 for the notation).

Proof. First we show that there exists a mapping Φ such that

$$(\phi_{(i)}, q_{(i)}, B^{\mathcal{M}}_{(i)}, S^{\mathcal{M}}_{(i)}) = \Phi(e_{(i)}, \phi_{(i-1)}, q_{(i-1)}, B^{\mathcal{M}}_{(i-1)}, S^{\mathcal{M}}_{(i-1)}) \tag{34}$$

for each $i \leq k$. We shall proceed case by case.

When $e_{(i)} = v$ then $\Phi(e_{(i)}, \phi_{(i-1)}, q_{(i-1)}, B^{\mathcal{M}}_{(i-1)}, S^{\mathcal{M}}_{(i-1)}) = (\phi_{(i-1)}, q_{(i-1)}, B^{\mathcal{M}}_{(i-1)}, S^{\mathcal{M}}_{(i-1)})$ i.e. the resulting values equal to the arguments.

When $e_{(i)} = b^j_{\dagger}$ for some j then

$$\Phi(e_{(i)}, \phi_{(i-1)}, q_{(i-1)}, B^{\mathcal{M}}_{(i-1)}, S^{\mathcal{M}}_{(i-1)}) = (\phi_{(i-1)}, q_{(i-1)}, B', S^{\mathcal{M}}_{(i-1)})$$

where

$$B' = (B_{(i-1)}(M^1), \dots, B_{(i-1)}(M^j) - 1, \dots, B_{(i-1)}(M^m))$$

i.e. the resulting values equal to the arguments except for $B_{(i-1)}(M^j)$ which is decreased by one.

Similarly when $e_{(i)} = s^j_{\dagger}$ for some j.

When $e_{(i)} = b^j_{\star}$ and the best ask lied in M^{ν} for some $\nu > j$ at the previous step then the resulting values equal to the arguments except for $B_{(i-1)}(M^j)$ which is increased by one.

When $e_{(i)} = b^j_{\star}$ and the best ask lied in M^{ν} for some $\nu < j$ at the previous step then the resulting values equal to the arguments except for $S_{(i-1)}(M^{\nu})$, $q_{(i-1)}$ and $\phi_{(i-1)}$ where $S_{(i-1)}(M^{\nu})$ is decreased by one, $q_{(i-1)}$ is increased by one and $\phi_{(i-1)}$ is set to ν.

When $e_{(i)} = b^j_{\star}$ and the best ask lied in M^j at the previous step then necessarily $|M^j| = 1$ (otherwise the best ask is a limit price of a newly arrived sell limit order which would violate the definition of $\tilde{\mathcal{E}}^k$) so the newly arrived order matches with the best ask and the arguments are transformed into the resulting values the same way as in the previous paragraph.

The situation is completely symmetric in the case of a newly arrived sell order.

Since we have exhausted all the possibilities, the relation (34) is proved. The formula (32) then follows easily by induction (recall that we have fixed $(B_{(0)}, S_{(0)})$).

The proof of (33) is analogous.

□

Theorem 4 *The partition $\mathcal{M}$ may be constructed so that*

$$\max(\mu_j, \nu_j) \leq \frac{24 \max(u, v)}{\bar{r}(m - 2(|B_{(0)}| + |S_{(0)}|) - 1)}$$

(see Section 3.3 for the notation).

Lemma 3 *Let μ be a finite real measure and let $n \in \mathbb{N}$. Then a partition $(A^j)_{j=1}^m$ of $\mathbb{R}$ may be found such that $m \leq n$ and*

$$\mu(A^j) \leq |\mu| \frac{12}{n} \tag{35}$$

for each $|A^j| > 1$.

Proof. Without loss of generality, we may assume that $|\mu| = 1$ (if the measure is non-probability then it suffices to apply the Lemma to $\mu/|\mu|$ and multiply both sides of (35) by $|\mu|$).

Let $x^1, \ldots, x^\alpha$ be all the points such that $\mu\{x^i\} > 3/n$, $1 \leq i \leq \alpha$. Further, denote

$$\mu' = \mu - \sum_{i=1}^{\alpha} \mu(\{x^i\}) \delta_{x^i}$$

(δ_x denotes the Dirac measure concentrated in x). Let F be the (left-continuous) distribution function of μ'. Define $(C^j)_{j=1}^{n-2\alpha} = ([c^{j-1}, c^j))_{j=1}^{n-2\alpha}$ where

$$c^j = F^{-1}\left(|\mu'| \frac{j}{n - 2\alpha}\right)$$

for each $1 \leq j \leq n - 2\alpha$ (we define $F^{-1}(\alpha) = \inf\{x : F(x) \geq \alpha\}$, we take $[a, a) = \emptyset$). Since $F(F^{-1}(\alpha)) \leq \alpha$ and $F(F^{-1}(\alpha)+) \geq \alpha$ for each $\alpha \in [0, |\mu'|]$ (both the relations may be easily proved by contradiction), we have

$$\mu'[(c^{j-1}, c^j)] = F(F^{-1}(c^j)) - F(F^{-1}(c^{j-1})+) \leq \frac{|\mu|'}{n - 2\alpha}.$$

Moreover, since it has to be $|\mu'| \leq 1 - (3/n)\alpha$ and since $\mu'(\{x\}) \leq 3/n$ for each $x \in \mathbb{R}$, we may estimate

$$\begin{aligned} \mu'([c^{j-1}, c^j)) \leq \frac{|\mu|'}{n - 2\alpha} + \frac{3}{n} \leq \frac{1 - (3/n)\alpha}{n - 2\alpha} + \frac{3}{n} &= \frac{1 - (3/n)\alpha + (3/n)(n - 2\alpha)}{n - 2\alpha} \\ &= \frac{4 - (9/n)\alpha}{n - 2\alpha} \leq \frac{4}{n - 2\alpha} \overset{(3/n)\alpha \leq 1}{\leq} \frac{4}{n - 2(n/3)} = \frac{12}{n}. \end{aligned} \tag{36}$$

Put $(X^i)_{i=1}^{2\alpha+1} = ((-\infty, x^1), \{x^1\}, (x^1, x^2), \{x^2\}, \ldots, \{x^\alpha\}, (x^\alpha, \infty))$ and define

$$\mathbb{A} = \{A \neq \emptyset : A = X^i \cap C^j \text{ for some } i \leq 2\alpha + 1 \text{ and } j \leq n - 2\alpha\}.$$

Since any non-singleton from $\mathbb{A}$ is contained in some non-singleton from (C^j) and since the measures μ and μ' coincide on $\mathbb{R} \setminus \{x^1, \ldots, x^\alpha\}$, we have $\mu(A) = \mu'(A) \leq 12/n$ for each $A \in \mathbb{A}$, $A \notin \{\{x^1\}, \ldots, \{x^A\}\}$ by (36). It remains to prove that the partition $\mathbb{A}$ has at most n points: By the definitions, the partition (X^i) contains α singletons while no singleton is contained in (C^j). Since all the non-degenerated intervals from both (X^i) and (C^j) are open on the right-hand side, no singleton in $\mathbb{A}$ may have risen as an intersection of two non-singletons, hence $\mathbb{A}$ has to contain exactly α singletons. Further, it is clear from the definitions that the set of all the upper boundary points of the sets from (X^i) has $\alpha + 1$ elements while (C^j) possesses no more than $n - 2\alpha$ upper boundaries, hence $\mathbb{A}$ has to have no more than $(n-2\alpha)+(\alpha+1)-1=n-\alpha$ upper boundaries (we could subtract one because both (X^i) and (C^j) have $+\infty$ among their boundaries). Finally, since the number of the elements of each finite partition of $\mathbb{R}$ consisting of intervals is equal to the number of its upper boundaries plus the number of its singletons, the Lemma is proved.

□

Proof of the Theorem. Denote $\varsigma(M)=u\pi(M)/\bar{r}$ and $\sigma(M)=v\rho(M)/\bar{r}$ for each $M\in\mathbb{B}(\mathbb{R})$. By Lemma 3, there exists a partition $\mathcal{A}\triangleq(A^j)_{j=1}^{n_b}$, $n_b\leq\lfloor[m-2(|B_{(0)}|+|S_{(0)}|)]/2\rfloor$ such that

$$\varsigma(A)\leq|\varsigma|\cdot\frac{12}{\lfloor[m-2(|B_{(0)}|+|S_{(0)}|)]/2\rfloor}\leq|\varsigma|\frac{24}{m-2(|B_{(0)}|+|S_{(0)}|)-1} \tag{37}$$

for each non-singleton $A\in\mathcal{A}$ (the symbol $\lfloor x\rfloor$ denotes the greatest integer less or equal to x). Similarly, there exists a partition $\mathcal{B}=(B^j)_{j=1}^{n_s}$, $n_s\leq\lfloor[m-2(|B_{(0)}|+|S_{(0)}|)]/2\rfloor$ such that

$$\sigma(B)\leq|\sigma|\cdot 24/(m-2(|B_{(0)}|+|S_{(0)}|)-1) \tag{38}$$

for each non-singleton $B\in\mathcal{B}$.

Clearly, any disjoint partition of $\mathbb{R}$ consisting of (possibly degenerated) intervals may be represented by the set of (open or closed) intervals such that each interval from the original partition is represented by the interval with the same upper bound of the same type (open or closed) and with the infinite lower bound. Let $\mathcal{A}'$ and $\mathcal{B}'$ be such representations of $\mathcal{A}$, $\mathcal{B}$ respectively.

Denote $Z=(z_1,\dots,z_p)$, $z_1<\dots<z_p=\infty$ the set containing all the (upper) bounds of the intervals from $\mathcal{A}'$, all the (upper) bounds of the intervals from $\mathcal{B}'$, all the points of $B_{(0)}$ and all the points of $S_{(0)}$. Further, denote $\mathcal{Z}=\{(-\infty,z_i):1\leq i\leq p\}\cup\{(-\infty,z_i]:1\leq i\leq p\}$ and construct a one to one mapping $G:\mathcal{Z}\to\mathbb{N}/2\triangleq\{1/2,1,3/2,\dots\}$ such that

$$G(I)=\begin{cases} i & \text{if } I=(-\infty,z_i] \\ i-1/2 & \text{if } I=(-\infty,z_i) \end{cases}$$

Denote $s\in R^q$ the vector of all the points of $B_{(0)}$ and $S_{(0)}$ and define

$$T=\bigcup_{i=1}^{q}\{G((-\infty,s^i)),G((-\infty,s^i])\}.$$

Obviously $|T|\leq 2(|B_{(0)}|+|S_{(0)}|)$, hence $|T\cup G(\mathcal{A}'\cup\mathcal{B}')|\leq n_b+n_s+2(|B_{(0)}|+|S_{(0)}|)\leq m$, i.e. the partition $\mathcal{M}$ represented by $\mathcal{M}'=G^{-1}(T\cup G(\mathcal{A}'\cup\mathcal{B}'))$ has no more than m elements.

Thanks to the construction of the set T, all the points of the initial order books find themselves in $\mathcal{M}$. It remains to show that for each $M\in\mathcal{M}$ there exists $A\in\mathcal{A}$, $B\in\mathcal{B}$, such that $M\subseteq A$ and $M\subseteq B$ which would guarantee

$$\varsigma(M)\leq\varsigma(A) \tag{39}$$

and

$$\sigma(M)\leq\sigma(B): \tag{40}$$

Let $\overline{k}=G(I)$ where I is the interval with the same upper bound as M and let $\underline{k}=G(J)$ where J is the interval with the same upper bound as the lower bound of M. Since no image of a set from $\mathcal{A}'$ may lie between $\underline{k}$ and $\overline{k}$ (it is because $\mathcal{M}$ is disjoint), the set $A\in\mathcal{A}$ represented by

$$\max\{G(A),A\in\mathcal{A}':G(A)\leq\underline{k}\},\quad \min\{G(A),A\in\mathcal{A}':G(A)\geq\overline{k}\}$$

has to contain M. The situation with $\mathcal{B}$ is symmetric.

The assertion of the Theorem now follows from (37), (38), (39) and (40).

□

References

M. Šmíd. Conditional distribution of the limit order book given the history of the best quote process, 2005a. Research report no. 2126, Institute of Information Theory and Automation, February 2005, http://econwpa.wustl.edu:80/eps/em/papers/0503/0503015.pd,.

M. Šmíd. Forecasting in continuous double auction, 2005b. Research report no. 2128, Institute of Information Theory and Automation, May 2005.

E. Smith, J. D. Farmer, L. Gillemot, and S. Krishnamurthy. Statistical theory of the continuous double auction. *Quantitative finance*, 3(6):481–514, 2003.

Revision history

Rev.2

- The note dropped that my model generalizes Smith et al. [2003] by allowing continuous time domain - in fact, Smith et al. [2003] also assume the continuity of the time.
- Definition of price and volume corrected (wrong: $p_t = a_t^1$, right: $p_t = a_{t-}^1$, similarly b_t^1
- Added a missing $e^{\bar{r}}$ in the "final" formula.
- The definition of $B(A)$ corrected (the set where ϕ should fall should not contain zero).
- In Lemma (1) (i), $S_i \in \mathcal{C}$ is required
- In the same Lemma (v), misprint was corrected.