Some Reflections on Trend-Cycle Decompositions with Correlated Components

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Abstract

This paper discusses a few interpretative issues arising from trend-cycle decompositions with correlated components. We determine the conditions under which correlated components may originate from: underestimation of the cyclical component; a cycle in growth rates, rather than in the levels; the hysteresis phenomenon; permanent-transitory decompositions, where the permanent component has richer dynamics than a pure random walk. Moreover, the consequences for smoothing and signal extraction are discussed: in particular, we establish that a negative correlation implies that future observations carry most of the information needed to assess cyclical stance. As a result, the components will be subject to high revisions. The overall conclusion is that the characterisation of economic fluctuations in macroeconomic time series largely remains an open issue.

Keywords: Kalman Filter and Smoother; Signal Extraction; Frequency Domain Estimation; Hysteresis; Permanent-Transitory Decomposition; Revisions.

JEL classification: C22, C5, E32.

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1 Introduction

Measuring trends and cycles in macroeconomic time series is the topic of a very rich and presumably endless debate. Unobserved components (UC) models assume orthogonal disturbances (see, for instance, Harvey and Jäger, 1994) or perfectly correlated ones (Snyder, 1981); these restrictions are often enforced to produce just-identified decompositions, but in some cases they are over-identifying. We will be concerned with these occurrences, in which the correlation can be estimated from the available time series.

Morley, Nelson and Zivot (2002, MNZ henceforth) have recently made a significant contribution to the topic: they consider a class of UC decompositions of U.S. real gross domestic product (GDP) into a random walk trend and a purely AR(2) cycle, that depends on the identifiable correlation between the trend and cycle disturbances and that produces an ARIMA(2,1,2) reduced form. Within this class, MNZ compare the fit and the components arising from the UC model assuming orthogonal disturbances and the Beveridge-Nelson (BN, 1981) decomposition of the unrestricted ARIMA model, that features perfectly and negatively correlated disturbances. The resulting decompositions produce different stylised facts, and in particular the BN cycle is characterised by a much smaller amplitude and a shorter periodicity.

Since a degree of freedom is allowed from the fact that the UC model has one parameter less than the ARIMA reduced form, they estimate the correlation between the trend and cycle disturbances and find that the estimated value is negative, about -0.92, and significantly away from zero. The resulting *real time*, or concurrent, estimates of the trend and cycle in U.S. GDP closely resemble the BN components, which allows us to reconcile the UC with the unrestricted reduced form.

The MNZ interpretation of the negative disturbance correlation is summarised in their concluding remarks:

> *If we accept the implication that innovations to trend are strongly negatively correlated with innovations to the cycle, then the case for the importance of real shocks in the macro economy is strengthened. ...For example, a positive productivity shock, such as the invention of the Internet, will immediately shift the long run path of output upward, leaving actual output below trend until it catches up. This implies a negative contemporaneous correlation since this positive trend shock is associated with a negative shock to the transitory component of output. By contrast, a positive nominal shock, say a shift in Fed policy towards stimulus will be a positive innovation to the cycle without any impact on trend.*

This paper is concerned with interpreting decomposition with correlated trend and cycle disturbances. We argue that several observationally equivalent interpretations arise. In the first place, for certain parameter combinations, correlated disturbances can just imply that the cyclical component is underestimated. This is always the case when the correlation is positive, in which case the covariation between the trend and the cycle can be attributed to the latter, allowing it to display a moving average feature, but also arises under certain conditions (a small trend-cycle disturbance variance ratio) for negative correlation values.

However, the orthogonal decomposition imposes restrictions on the spectral density of the series, that are not supported for U.S. GDP.

Section 2 sets up the basic framework, introducing the unobserved components model that nests two special cases of interest: the UC model with correlated disturbances and the orthogonal decomposition with ARMA(2,1) cyclical component. Section 3 derives the BN decomposition in terms of the reduced form parameters. A standard tool to establish the conditions under which alternative models are equivalent is the autocovariance generating function (or the related spectral generating function); this is derived in section 4. We then establish the conditions under which they are observationally equivalent (section 5).

Strongly negatively correlated disturbances imply that the spectral density of the first differences of the series is not a global minimum at the long run frequency. We therefore investigate whether a different UC model, the cyclical growth model (section 6), which extracts a cycle from growth rates, and has similar implications on the spectral density, can be rewritten in terms of a trend-cycle decomposition with correlated disturbances. The conditions under which this is possible are established in section 7, where we also show that we have to expect a negative correlation.

A reparameterisation of the cyclical trend model gives rise to the hysteresis model of Jäger and Parkinson (1994), which postulates that the cycle modifies permanently the trend and thus provides yet another interpretation.

In section 8 we show that the MNZ interpretation is consistent with the permanent-transitory rather than trend-cycle decomposition of output; the former differs from the latter in that the permanent component has richer dynamics than a simple random walk; moreover the propagation mechanism of permanent and transitory disturbances features a common autoregressive polynomial.

We provide two illustrations (section 9): the first, concerning U.S. GDP, shows that for that series no such identifiability issues arise, as the MNZ model with correlated disturbances is the only representation consistent with the unrestricted ARIMA(2,1,2) model, but we also show that the components are grossly underestimated in real time, since a peculiar property of highly negatively correlated trend and cycle disturbances is that the future is more informative than the past for signal extraction. As a result the cycle estimates will be subject to large revisions and the final ones will display greater amplitude than the real time ones.

The second deals with Italian real GDP and serves to illustrate that alternative explanations of the nature of macroeconomic fluctuations arise with exactly the same likelihood.

In the concluding remarks we point out that all the results are conditional on a particular reduced form, that is itself a source of uncertainty in empirical applications.

2 Trend-Cycle decomposition with Correlated Components

Let us consider the following unobserved components model for decomposing output into a random walk trend component, μ_t, and a stationary ARMA(2,1) stochastic cycle, denoted

ψ_t:

$$
\begin{array}{rcll}
y_t & = & \mu_t + \psi_t & t = 1, 2, \ldots, T, \\
\\
\mu_t & = & \mu_{t-1} + \beta + \eta_t, & \\
\psi_t & = & \phi_1 \psi_{t-1} + \phi_2 \psi_{t-2} + \kappa_t + \theta \kappa_{t-1}, & \\
\left(\begin{array}{c} \eta_t \\ \kappa_t \end{array} \right) & \sim & \mathrm{WN} \left[\left(\begin{array}{c} 0 \\ 0 \end{array} \right), \left(\begin{array}{cc} \sigma_\eta^2 & \sigma_{\eta\kappa} \\ \sigma_{\eta\kappa} & \sigma_\kappa^2 \end{array} \right) \right], & \sigma_{\eta\kappa} = r \sigma_\eta \sigma_\kappa
\end{array}
\tag{1}
$$

Here WN denotes serially uncorrelated disturbances - white noise. The trend and cycle disturbances are allowed to be contemporaneously correlated, with r being the correlation coefficient. Complex stationary autoregressive roots can be imposed expressing $\phi_1 = 2\rho \cos \lambda_c$ and $\phi_2 = -\rho^2$, where ρ and λ_c (representing the modulus and the phase of the roots of the AR characteristic equation), lie respectively in $[0, 1)$ and $[0, \pi]$.

Model (1) will be labelled UC(r, θ) to stress the dependence on the two "conflicting" parameters. Its reduced form is the ARIMA(2,1,2) process:

$$
\Delta y_t = \beta + \frac{\theta(L)}{\phi(L)} \xi_t, \quad \xi_t \sim \mathrm{WN}(0, \sigma^2), \qquad t = 2, ..., T, \tag{2}
$$

where $\theta(L) = 1 + \theta_1 L + \theta_2 L^2$ and $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$ are respectively the MA and AR polynomials in the lag operator, L, and $\Delta = 1 - L$.

The reduced form has six parameters, whereas UC(r, θ) has seven. Hence, the latter is not identified and one has to restrict either r or θ. The orthogonal trend cycle decomposition considered by Clark (1987) imposes $r = \theta = 0$, and thus will be denoted UC(0,0). The MNZ paper is based on a comparison of UC$(r, 0)$ with UC$(0, 0)$. Harvey and Jäger (HJ), although they entertain I(2) - local linear - trends, consider UC$(0, \theta)$, with a restricted θ, since, as we illustrate immediately this is a function of ρ and λ_c.

As a matter of fact, HJ use the cyclic model:

$$
\begin{array}{rcl}
\psi_t & = & \rho \cos \lambda_c \psi_{t-1} + \rho \sin \lambda_c \psi^*_{t-1} + \kappa_t, \\
\psi^*_t & = & -\rho \sin \lambda_c \psi_{t-1} + \rho \cos \lambda_c \psi^*_{t-1} + \kappa^*_t,
\end{array}
\tag{3}
$$

such that the single equation representation for ψ_t is now the ARMA(2,1) process:

$$
(1 - \phi_1 L - \phi_2 L^2) \psi_t = (1 - \rho \cos \lambda_c L) \kappa_t + \rho \sin \lambda_c \kappa^*_{t-1},
$$

$\phi_1 = \rho \cos \lambda_c, \phi_2 = -\rho^2$. Harvey and Jäger assume that the cycle disturbances are uncorrelated with the trend disturbances and:

$$
\left(\begin{array}{c} \kappa_t \\ \kappa^*_t \end{array} \right) \sim \mathrm{WN} \left[\left(\begin{array}{c} 0 \\ 0 \end{array} \right), \left(\begin{array}{cc} \sigma_\kappa^2 & 0 \\ 0 & \sigma_\kappa^2 \end{array} \right) \right],
$$

so that the MA coefficient in $\psi_t \sim$ARMA(2,1) is the unique invertible solution of the quadratic equation $\theta/(1 + \theta^2) = -0.5\rho \cos \lambda_c$; let us denote this by $\tilde{\theta}$. When $\lambda_c < \pi/2$, as occurs quarterly time series, the implied value for $\tilde{\theta}$ is negative (this *cœteris paribus* produces a slightly more noisy cycle with respect to the pure AR(2) representation adopted for UC$(0, 0)$).

The UC$(0, \tilde{\theta})$ model has again an ARIMA(2,1,2) reduced form, but features five parameters, which leaves one degree of freedom that can be used either for estimating the correlation between κ_t and κ_t^*, or for allowing different variances. Or we may estimate the trend-cycle correlation, perhaps assuming that κ_t and κ_t^* are homoscedastic and equally correlated with η_t.

3 The Beveridge-Nelson Decomposition

The Beveridge and Nelson decomposition of (2) is

$$y_t = m_t + c_t, \qquad t = 1, ..., T. \tag{4}$$

where the trend, m_t, has the random walk representation:

$$m_t = m_{t-1} + \beta + \frac{\theta(1)}{\phi(1)}\xi_t. \tag{5}$$

and the cycle, c_t, has the ARMA(2,1) representation:

$$\phi(L)c_t = (1 + \vartheta^* L)\left[1 - \frac{\theta(1)}{\phi(1)}\right]\xi_t, \qquad \vartheta^* = -\frac{\phi_2\theta(1) + \theta_2\phi(1)}{\phi(1) - \theta(1)}. \tag{6}$$

The latter can be shown using results from Proietti and Harvey (2000).

The following comments are in order:

1. The two components are driven by the innovations, ξ_t; the fraction $\theta(1)/\phi(1)$, known as *persistence*, is integrated in the trend, and its complement to 1 drives the cycle. The sign of the correlation between the trend and the cycle disturbances is provided by the sign of $\phi(1) - \theta(1)$. This is negative for the ARIMA model of U.S. GDP fitted by MNZ, but when persistence is less than one then trend and cycle disturbances are positively and perfectly correlated.

2. The BN cycle has always an MA feature, unless $\phi_2\theta(1) + \theta_2\phi(1) = 0$. The MA polynomial can be non invertible, i.e. $|\vartheta_1^*|$ can be greater than 1, as it occurs for the for the ARIMA model estimated for U.S. GDP.

3. The BN components, defined on the reduced form of UC models, are always coincident with the filtered, or real time time, estimates arising from the UC(r, θ) model, whatever restriction we impose to make it identifiable (see Watson, 1989).

4. The filtered components of *identified* UC(r, θ) models are however estimated with non zero mean square error even in the case $r = -1, \theta = 0$. Hence, it would not be correct to regard the BN trend and cycle as the estimates of the components arising from UC(-1,0), as future observations reduce the estimation error. Section 9.1 investigates further this issue, where we show that the only case in which ψ_t is actually an observed component in real time arises for $r = 1$.

5. When the BN decomposition is interpreted *as a model*, the components are estimated in real time with zero mean square error, after processing a suitable number of observations so that the effect of initial conditions is marginalised, this being the only source of uncertainty (assuming known parameters).

4 The Autocovariance Generating functions of UC models

In this section we provide expressions for the autocovariance generating function (ACGF) of the UC$(r, 0)$ model considered by MNZ and that of UC$(0, \theta)$ with θ unrestricted.

The ACGF of Δy_t, denoted $g(L)$, is $g(L) = \sigma^2|\theta(L)|^2/|\phi(L)|^2$, where $|\theta(L)|^2 = \theta(L)\theta(L^{-1})$ and $|\phi(L)|^2 = \phi(L)\phi(L^{-1})$. That implied by the UC(r,0) model considered by MNZ is (see Whittle, 1983):

$$g_r(L) = g_{\Delta\mu}(L) + g_{\Delta\psi}(L) + g_{\Delta\mu,\Delta\psi}(L) + g_{\Delta\psi,\Delta\mu}(L),$$

with

$$g_{\Delta\mu}(L) = \sigma^2_\eta, \;\; g_{\Delta\psi}(L) = \frac{|1-L|^2}{|\phi(L)|^2}\sigma^2_\kappa, \quad g_{\Delta\mu,\Delta\psi}(L) = \frac{1-L}{\phi(L^{-1})}r\sigma_\eta\sigma_\kappa,$$

and $g_{\Delta\psi,\Delta\mu}(L) = g_{\Delta\mu,\Delta\psi}(L^{-1}) = r\sigma_\eta\sigma_\kappa(1-L^{-1})/\phi(L)$, where the latter is the cross-covariance generating function of $(\Delta\psi_t, \Delta\mu_t)$.

Replacing L with the complex exponential $e^{-\imath\lambda} = \cos\lambda - \imath\sin\lambda$, where $\imath$ is the imaginary unit, gives the spectral generating function, that provides a decomposition of the variance of Δy_t into the contribution of changes in the trend, in the cycle and the covariation (cross spectral density) between the two.

Equating $g(L)$ to $g_r(L)$ provides the way of deriving the reduced form parameters $(\theta_1, \theta_2, \sigma^2)$ from $(\sigma_\eta, \sigma_\kappa, r)$ and of assessing the restrictions imposed by the UC model on the reduced form.

Simple manipulations (see appendix A) show that $g_r(L)$ can be written as follows:

$$|\phi(L)|^2 g_r(L) = |\phi(L)|^2\sigma^2_\eta + |1-L|^2[\sigma^2_\kappa + r\sigma_\eta\sigma_\kappa(1+\phi_1+\phi_2+\phi_2(L+L^{-1}))]. \quad (7)$$

For the UC$(0, \theta)$ model we have

$$|\phi(L)|^2 g_\theta(L) = |\phi(L)|^2\sigma^2_{\eta^*} + |1-L|^2|1+\theta L|^2\sigma^2_{\kappa^*} \quad (8)$$

where, with a slight change of notation, $\sigma^2_{\eta^*}$ and $\sigma^2_{\kappa^*}$ denote the variance of the trend and cycle disturbances when we assume in (1) that they are mutually uncorrelated at all leads and lags.

5 Can orthogonal decompositions generate correlated trends and cycles?

The question just posed seems a contradiction in terms, but it summarises the content of this section, in which we show the conditions under which UC$(0, \theta)$ models can generate UC$(r, 0)$ models and viceversa.

These can be derived equating the corresponding ACGFs, i.e.

$$g_\theta(L) = g_r(L),$$

which amounts to equalling the right hand sides in (7) and (8). The solution gives the system of three nonlinear equations:

$$\begin{array}{rcl} \sigma^2_{\eta^*} & = & \sigma^2_\eta \\ \theta\sigma^2_{\kappa^*} & = & \phi_2 r \sigma_\eta \sigma_\kappa \\ (1+\theta^2)\sigma^2_{\kappa^*} & = & \sigma^2_\kappa + r\sigma_\eta\sigma_\kappa(1+\phi_1+\phi_2) \end{array} \tag{9}$$

The first equation follows immediately from $g_\theta(1) = g_r(1)$ and states that the size of the RW trend for both models must equal the spectral generating function at the zero frequency (the long run variance). In terms of the reduced form parameters, $\sigma^2_{\eta^*} = \sigma^2_\eta = [\theta(1)/\phi(1)]^2\sigma^2$.

Moreover, the ratio of the second and the third provide the quadratic equation:

$$\frac{\theta}{1+\theta^2} = \phi_2 r \frac{\sigma_\eta}{\sigma_\kappa}\left[1 + r\frac{\sigma_\eta}{\sigma_\kappa}(1+\phi_1+\phi_2)\right]^{-1}, \tag{10}$$

which can be solved for θ, and when real solutions are available, we select the invertible one to get the UC$(0,\theta)$ model observationally equivalent to UC$(r,0)$.

Figure 1 shows the values of θ as a function of the ratio $\sigma_\eta/\sigma_\kappa$ (in abscissa) and various values of r. The plot is conditional on the AR parameters estimated by MNZ for U.S. GDP, that is $\phi_1 = 1.34$ and $\phi_2 = -0.71$.

Denoting with A the right hand side of (10), an admissible solution for θ is available when $1 - 4A^2 > 0$, i.e. $-0.5 < A < 0.5$; this is exactly the same condition under which $g_r(e^{-\imath\lambda})$ is a global minimum at the zero frequency: as a matter of fact, the first and second order conditions for a minimum at that frequency,

$$\left.\frac{d}{d\lambda}g_r(e^{-\imath\lambda})\right|_{\lambda=0} = 0, \qquad \left.\frac{d^2}{d\lambda^2}g_r(e^{-\imath\lambda})\right|_{\lambda=0} > 0,$$

require $A > -0.5$, whereas $g_r(e^{-\imath\pi}) > g_r(0)$ requires $A < 0.5$. This states the fundamental fact that the orthogonal decomposition UC$(0,\theta)$ imposes that the spectral density of Δy_t is a minimum at zero, a result already established in Lippi and Reichlin (1992).

As figure 1 shows, any UC$(r,0)$ model with positively correlated disturbances can be reinterpreted in terms of an orthogonal UC$(0,\theta)$ model with negative θ. Heuristically, the covariation between the trend and the cycle (the cross-spectrum) is attributed to the cycle, making it more variable by the addition of an MA feature.

We can allow for negative correlation provided that the ratio $\sigma_\eta/\sigma_\kappa$ is small, i.e. the trend disturbance is a minor source of variation. In conclusion, when the spectral density of Δy_t is a minimum at zero, the cross spectrum between the components absorbs part of the cyclical variability; this can be reallocated to the cyclical component, which is underestimated by the UC(r,0) model.

In this perspective, when (10) has an admissible solution, the test of $H_0 : r = 0$ within the UC$(r, 0)$ model amounts to testing $H_0 : \theta = 0$ in the UC$(0, \theta)$ model, i.e. the (orthogonal) cycle is a pure AR(2) process devoid of the MA feature.

The UC$(r, 0)$ model adapted by MNZ to U.S. GDP has $\sigma_\eta/\sigma_\kappa = 1.24/0.75 = 1.65$ and $r = -0.92$. Figure 1 makes clear that there is no admissible UC$(0, \theta)$ model that is observationally equivalent to theirs. This can be taken as evidence that the spectral density of the changes in U.S. GDP is not a global minimum at the zero frequency.

6 Cyclical Growth and Hysteresis

Consider now an UC model that postulates that Δy_t can be additively decomposed into a cyclical component and orthogonal noise:

$$\begin{array}{rcll} \Delta y_t & = & \beta + \psi_t + \eta_t^*, & \eta_t^* \sim \mathrm{WN}(0, \sigma^2_{\eta^*}), \\ \psi_t & = & \phi_1\psi_{t-1} + \phi_2\psi_{t-2} + \kappa_t^* + \theta\kappa_{t-1}^*, & \kappa_t^* \sim \mathrm{WN}(0, \sigma^2_{\kappa^*}), \\ \mathrm{E}(\eta_t^*\kappa_t^*) & = & 0. & \end{array} \tag{11}$$

The idea is that of representing underlying growth as a smooth cyclical process. This decomposition is at the basis of the dynamic factor model for a coincident index by Stock and Watson (1991). There, given a set of N coincident indicators, $\Delta y_{it}, i = 1, \ldots, N$, that could be written as $\Delta y_{it} = \delta_i\psi_t + \eta^*_{it}$, where δ_i is the loading of the i-th indicator on the shared cycle, η^*_{it} is the idiosyncratic component, uncorrelated with ψ_t and across i, and we ignore the drift term for simplicity, the coincident index is obtained integrating the smoothed estimates of ψ_t.

Model (11) is consistent with the classical cycle definition of recessions as periods of absolute declines in economic activity; these would show up as periods of negative underlying growth, where the latter is defined as $\beta + \psi_t$. The Kalman filter and smoother can then be used to evaluate the probability of a recessionary event, $\mathrm{Prob}(\psi_t < -\beta|Y_T)$, i.e. a negative growth at time t, where Y_t denotes the available information up to and including time t. More complex events, such as the probability of a trough, $\mathrm{Prob}\{\psi_t < -\beta) \bigcap (\psi_{t-1} > \psi_t < \psi_{t+1})|Y_T\})$, can be evaluated using the simulation smoother, see De Jong and Shephard (1995), which generates draws from the joint distribution $\psi_1, \ldots, \psi_t, \ldots, \psi_T|Y_T$. The resulting business cycle chronology would produce recessions that have shorter duration than expansions, without the need for entertaining a nonlinear model to account for this particular asymmetry.

Model (11) has again an ARIMA(2,1,2) reduced form, and six parameters, but different implications. In its original specification, it simply produces estimates of underlying growth that are smoother than the original observations; it can also be interpreted as a cyclical trend model, as in Harvey (1989, p. 46), such that the trend is coincident with the observations, i.e. $y_t = \mu_t$ and $\mu_t = \mu_{t-1} + \beta + \psi_t + \eta_t^*$.

It is also observationally equivalent to the Jäger and Parkinson (1994) hysteresis model, which is such that a deviation cycle can still be defined, but the cycle modifies also permanently the trend. The hysteresis model is specified as follows:

$$\begin{array}{rcll} y_t & = & \mu_t + \psi_t, & t = 1, 2, \ldots, T, \\ \mu_t & = & \mu_{t-1} + (1+\theta)\psi^*_{t-1} + \eta^*_t, & \eta^*_t \sim \mathrm{WN}(0, \sigma^2_{\eta^*}), \\ \psi^*_t & = & \phi_1 \psi^*_{t-1} + \phi_2 \psi^*_{t-2} + \kappa^*_t, & \kappa^*_t \sim \mathrm{WN}(0, \sigma^2_{\kappa^*}) \end{array} \tag{12}$$

and $\mathrm{E}(\eta^*_t \kappa^*_t) = 0$. Notice that the cycle, ψ^*_t, is redefined as a pure second order AR process; $(1+\theta)$ represents the hysteresis parameter, i.e. the fraction of the cycle that is integrated in the trend. Obviously, $\theta = -1$ yields again the additive decomposition into orthogonal trend and cycle that corresponds to the Clark model UC$(0, 0)$. A test of no hysteresis, $H_0 : \theta = -1$, would be non standard, since the parameter lies on the boundary of the parameter space under the null.

That (12) is observationally equivalent to the cyclical growth model (11) can be seen on writing

$$\psi_t = \frac{1+\theta L}{\phi(L)}\kappa^*_t = \frac{1+\theta}{\phi(L)}\kappa^*_{t-1} + \frac{\Delta}{\phi(L)}\kappa^*_t = (1+\theta)\psi^*_{t-1} + \Delta\psi^*_t,$$

replacing in (11) and defining the components appropriately to give (12). The additive cycle emerges since the MA feature in ψ_t enables to disentangle Δ. The hysteresis model was proposed for unemployment by Jäger and Parkinson (1994) to formalise the idea that a rise in cyclical unemployment can lead to a permanent increase in the natural rate[1].

7 Can Cyclical Growth generate Correlated Trends and Cycles?

Using the same expedient of equating the ACGFs we establish a set of conditions under which (11) can provide a trend - cycle decomposition with correlated disturbances, i.e. can be written as an UC(r, 0) process.

Denoting the ACGF of Δy_t implied by (11) with

$$g_c(L) = \frac{|1+\theta L|^2}{|\phi(L)|^2}\sigma^2_{\kappa^*} + \sigma^2_{\eta^*},$$

the identity $g_c(1) = g_r(1)$, arising from equating the long run variances, yields:

$$\sigma^2_{\eta^*} = \sigma^2_\eta - \frac{(1+\theta)^2}{\phi(1)^2}\sigma^2_{\kappa^*} \tag{13}$$

Replacing in the equation $g_c(L) = g_r(L)$, we show in appendix B that we can uniquely determine the cycle MA parameter θ in (11) for given values of the correlation parameter

[1]The authors, however, failed to recognise that the model is just identified and used an additional series to support estimation of the hysteresis parameter.

r and the ratio $\sigma_\eta/\sigma_\kappa$ in UC$(r, 0)$, as the admissible invertible solution of the quadratic equation:

$$\frac{(1+\theta)^2}{(1+\theta)^2\left[\phi_1(1-\phi_2)+2\phi_2\right]+\theta\phi(1)^2}=\frac{r(\sigma_\eta/\sigma_\kappa)}{1+r(\sigma_\eta/\sigma_\kappa)(1+\phi_1+\phi_2)}. \tag{14}$$

Figure 2 shows the values of θ as a function of the ratio $\sigma_\eta/\sigma_\kappa$ (in abscissa) and various values of r. The plot is conditional on the AR parameters estimated by MNZ for U.S. GDP, that is $\phi_1 = 1.34$ and $\phi_2 = -0.71$.

Only negative values of r are considered, with $r = -0.92$ representing the value estimated by MNZ for U.S. GDP. When $r = 0$ the solution $\theta = -1$ arises for any value of the ratio $\sigma_\eta/\sigma_\kappa$, in which case the hysteresis parameter is zero and the model can be orthogonally decomposed into a RW trend and a purely AR(2) cycle. This is easily seen from (14). No admissible solutions exists for a positive r and in general an UC trend-cycle decomposition with positively correlated disturbances cannot be isomorphic to a cyclical growth model or a model with hysteresis effects. This is so since model (11) implies a spectral density for Δy_t that has a local, but not a global, minimum at the zero frequency.

Comparing figure 2 with figure 1 we notice that observationally equivalent models arise when r is negative and the ratio $\sigma_\eta/\sigma_\kappa$ is small; the value implied for θ is high and negative. For high values of $\sigma_\eta/\sigma_\kappa$ and a negative correlation greater than -0.5 there need not exist an equivalent cyclical growth model. This occurs for instance, when $\sigma_\eta/\sigma_\kappa = 1.65$ and $r = -0.92$: we need either a lower correlation or standard deviation ratio to have a model interpretable as (11).

8 Permanent-Transitory Decomposition

Let us return to the UC(r,0) model. MNZ interpret a negative r in terms of $\eta_t \to \kappa_t$, i.e. positive trend disturbances induce negative cyclical shocks; however, correlated trends and cycle disturbances can also be interpreted reversing the MNZ casuation: a positive cycle disturbance reduces permanently potential output. This is a particular form of hysteresis, with a fraction, say h, of the cyclical disturbance being integrated in the trend, working in the opposite direction of (12), for which the hysteresis parameter, $(1+\theta)$, is non negative, although the latter integrates ψ^*_{t-1} rather than $h\kappa_t$. As a matter of fact, using the orthogonalisation $\eta_t = \eta^*_t + h\kappa_t$, where the second addend is the linear projection of η_t on κ_t, so that $\mathrm{E}(\eta^*_t, \kappa_t) = 0$ and $h = r\sigma_\eta/\sigma_\kappa$, we could write $\Delta\mu_t = \beta + \eta^*_t + h\kappa_t$, which supports the interpretation that κ_t modifies permanently the trend $(\kappa_t \to \eta_t)$.

Another interpretation corresponds to the orthogonalisation

$$\kappa_t = \kappa^*_t + \omega\eta_t, \quad \omega = r\frac{\sigma_\kappa}{\sigma_\eta}, \quad \mathrm{E}(\kappa^*_t, \eta_t) = 0;$$

replacing into (1), with $\theta = 0$, and rearranging, we achieve the following orthogonal decomposition of GDP into a permanent component, $y^{(\mathsf{P})}_t$, and a transitory component,

$y_t^{(\mathsf{T})}$:

$$\begin{array}{rcll} y_t & = & y_t^{(\mathsf{P})} + y_t^{(\mathsf{T})}, & t = 1, 2, \ldots, T, \\ \phi(L)\Delta y_t^{(\mathsf{P})} & = & b + [\phi(L) + \omega\Delta]\eta_t & \eta_t \sim \mathrm{WN}(0, \sigma^2_\eta) \\ \phi(L) y_t^{(\mathsf{T})} & = & \kappa_t^*, & \kappa^*_t \sim \mathrm{WN}(0, \sigma^2_{\kappa^*}) \end{array} \tag{15}$$

with $\sigma^2_{\kappa^*} = \sigma^2_\kappa(1-\omega^2)$, $b = \phi(1)\beta$. The permanent component is generated by an ARIMA(2,1,2) process, since the term in square brackets on the right hand side is an MA(2) polynomial.

Actually, MNZ seem to refer to this decomposition when they speak of nominal shocks that do not affect the trend (κ_t^*) and of a new economy shock that induces a negative output gap: the latter can be associated to the transitory effects of η_t, that amount $\omega\eta_t/\phi(L)$ (notice that $r < 0$ implies $\omega < 0$).

We labelled (15) as a permanent-transitory decomposition because it is very close to the spirit of the Blanchard and Quah (1989) decomposition: effectively, (15) is characterised by the fact that the transmission mechanism of permanent and transitory disturbances has a common feature, represented by the polynomial $\phi(L)$, which is present in both the components. Another similar feature is the high order for the MA order of the permanent component.

To show that these features have to be expected from a Structural VAR decomposition, let us focus on the simple case where Δy_t is modelled in conjunction with another ancillary stationary variable x_t, and that the model is a first order bivariate autoregression:

$$(\boldsymbol{I} - \boldsymbol{\Phi}L)(\Delta y_t, x_t)' = \boldsymbol{\xi}_t, \quad \boldsymbol{\xi}_t \sim \mathrm{WN}(\boldsymbol{0}, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\Phi} = \{\phi_{ij}\}$ and $\boldsymbol{\Sigma}$ are 2×2 matrices, and we ignore for simplicity the presence of a drift. Inverting the AR matrix polynomial, denoting $\phi(L) = \det(\boldsymbol{I} - \boldsymbol{\Phi}L)$, and defining orthogonal standardised permanent and transitory disturbances by means of the matrix $\boldsymbol{\Gamma} = \{\gamma_{ij}\}$, such that

$$\left(\begin{array}{c} \eta_t/\sigma_\eta \\ \kappa_t^*/\sigma_{\kappa^*} \end{array} \right) = \boldsymbol{\Gamma}^{-1}\boldsymbol{\xi}_t, \quad \boldsymbol{\Sigma} = \boldsymbol{\Gamma}\boldsymbol{\Gamma}',$$

the resulting permanent-transitory decomposition of y_t, $y_t = y_t^{(\mathsf{P})} + y_t^{(\mathsf{T})}$, is as follows7:

$$\begin{array}{rcl} \phi(L)\Delta y_t^{(\mathsf{P})} & = & [(1-\phi_{22}L)\gamma_{11} + \phi_{12}\gamma_{21}L]\frac{\eta_t}{\sigma_\eta}, \\ \phi(L)\Delta y_t^{(\mathsf{T})} & = & [(1-\phi_{22}L)\gamma_{12} + \phi_{12}\gamma_{22}L]\frac{\kappa_t^*}{\sigma_{\kappa^*}} = \gamma^*\Delta\frac{\kappa_t^*}{\sigma_{\kappa^*}} \end{array} \tag{16}$$

In deriving (16) we have enforced the restriction that κ_t^* has only transitory effects on output, which, implying $[(1-\phi_{22})\gamma_{12} + \phi_{12}\gamma_{22}] = 0$, allows to extract the common factor Δ in the representation for $\Delta y_t^{(\mathsf{T})}$, for a suitable γ^*. This, together with three bilinear restrictions imposed by $\boldsymbol{\Sigma} = \boldsymbol{\Gamma}\boldsymbol{\Gamma}'$, exactly identifies the decomposition.

The relevant feature of (16) is the presence of the common AR(2) polynomial $\phi(L) = (1-\phi_{11}L)(1-\phi_{22}L) - \phi_{12}\phi_{21}L^2$. In general, apart from cancellation effects, $y_t^{(\mathsf{P})} \sim$ARIMA$(Np, 1, p)$ and $y_t^{(\mathsf{T})} \sim$ ARMA$(Np, p-1)$ with a common AR polynomial, where N and p denote the number of time series and p the VAR order, respectively.

The permanent component arising from the structural VAR(1) has an MA(1) polynomial, whereas that in (15) has a second order MA feature. However, we can think of decomposing a bivariate ARMA(1,1) model along the same lines and this would produce $y_t^{(\mathsf{P})}$ $\sim$ARIMA$(2,1,2)$ and $y_t^{(\mathsf{T})} \sim$ ARMA$(2,1)$, which now differs from (15) for the MA order for the transitory component, but we can constrain the AR and MA coefficients so as to yield a purely AR(2) transitory component.

9 Illustrative Examples

In this section we illustrate the interpretative issues arising from the trend cycle decomposition with correlated disturbances with respect to U.S. GDP and the Italian GDP. We start by comparing the fit produced by the unrestricted ARIMA(2,1,2) model, the UC$(r,0)$ model, UC$(0,\theta)$, and the cyclical growth (CG) model.

Model estimation is carried out in the frequency domain. The likelihood is defined in terms of the stationary representation of the various models, that is in terms of $\Delta y_t, t = 1,\dots,T^*$. The appealing feature of estimation in the frequency domain is that we can control one of the sources of variability of estimation results, consisting of initial conditions in the presence of nonstationary effects. The assumptions under which the likelihood is derived are the same for all the models and this guarantees that observationally equivalent models give exactly the same inferences, as we will see below. We refer to Nerlove, Grether and Carvalho (1995) and Harvey (1989, sec. 4.3) for a comprehensive treatment on frequency domain estimation.

While the time domain likelihood of UC models is based on a recursive orthogonalisation, known as the prediction error decomposition, performed by the Kalman filter, the frequency domain one is based on an alternative orthogonalisation, achieved through a Fourier transform. Denoting the Fourier frequencies by $\lambda_j = \frac{2\pi j}{T^*}$, $j = 0, 1, \dots, (T^* - 1)$, the likelihood function is defined as follows:

$$\text{loglik} = -\frac{1}{2}\left\{T^* \log 2\pi + \sum_{j=0}^{T^*-1}\left[\log g_m(\lambda_j) + 2\pi\frac{I(\lambda_j)}{g_m(\lambda_j)}\right]\right\}$$

where $g_m(\lambda_j) = g_m(e^{-\imath\lambda_j})$ denote the spectral generating function of the m-th model evaluated at frequency λ_j, and $I(\lambda_j)$ is the periodogram:

$$I(\lambda_j) = \frac{1}{2\pi}\left[c_0 + 2\sum_{\tau=1}^{T^*-1} c_\tau \cos(\lambda_j \tau)\right]$$

where c_τ denotes the sample autocovariance at lag τ,

$$c_\tau = \frac{1}{T^*}\sum_{t=1}^{T-\tau}(\Delta y_t - \bar{\Delta y})(\Delta y_{t-\tau} - \bar{\Delta y}), \quad \bar{\Delta y} = \frac{1}{T^*}\sum_{t=1}^{T^*}\Delta y_t.$$

The index m refers alternatively the ARIMA model, UC$(r,0)$, UC$(0,\theta)$, and the cyclical growth model given in (11). The corresponding spectral generating functions are straightforwardly derived from the ACGFs presented in sections 4 and 7.

All the computations were performed in Ox 3.0 (Doornik, 2001) and programmes are available upon request. Signal extraction was performed by the Kalman filter and smoother using the library of state space function SsfPack 2.3 by Koopman et al. (2000), linked to Ox 3.0. Appendix C briefly reviews the state space representation and the main methods and algorithms used in the discussion. For a thorough exposition of the state space methodology we refer to Harvey (1989) and Durbin and Koopman (2001).

9.1 U.S. GDP

We consider the U.S. GDP series for the sample period 1947.1-2001.4 in chained 1996 dollars, made available electronically by the U.S. Bureau of Economic Analysis. The series is an extension of that considered by MNZ, but it leads to the same conclusions. Table 1 presents the main estimation results along with some diagnostics. $Q(12)$ is the Ljung-Box portmanteau test statistic for residual autocorrelation based on the first 12 autocorrelations, and we also present the Doornik and Hansen (1994) test of normality. Both are computed on the standardised Kalman filter innovations (see appendix C).

Table 1: Parameter estimates and diagnostics for models of quarterly U.S. real GDP, 1947.1-2001.4; (r) denotes a restricted parameter.

	ARIMA	UC$(r,0)$	UC$(0,\theta)$	CG
ϕ_1	1.39	1.39	1.50	1.37
ϕ_2	-0.79	-0.79	-0.58	-0.71
θ_1	-1.14			
θ_2	0.63			
σ^2	0.8717			
r		-0.97	0(r)	
σ^2_η $(\sigma^2_{\eta^*})$		1.3397	0.5512	0.6063
σ^2_κ $(\sigma^2_{\kappa^*})$		0.3254	0.1005	0.0211
θ		0(r)	1.00	1.00
loglik	-295.71	-295.71	-297.18	-296.00
$Q(12)$	8.94	8.94	11.56	7.59
Normality	17.96	17.96	24.81	20.71

The ARIMA model and UC$(r,0)$ provide exactly the same likelihood inferences. Hence the reduced form of the latter coincides with the unrestricted ARIMA(2,1,2) model fitted to the series. The correlation parameter is high and negative (-.97), and the ratio $\sigma_\eta/\sigma_\kappa$ is about 2. The persistence parameter implied by the two models is 1.24: for UC$(r,0)$ it is computed from the steady state Kalman filter as the first element of the vector $\boldsymbol{Pz}f^{-1}$, corresponding to the weight assigned to the current innovation ξ_t in the real time estimates of the trend; see equation (21).

UC$(0,\theta)$ and the cyclical growth model yield a lower likelihood and the former slightly worse diagnostics; with hindsight we interpret the richer residual autocorrelation pattern as a consequence of underestimation of the zero frequency variance component. It is also noticeable that the estimated MA cycle parameter θ lies in both cases on the boundary of the parameter space; this interesting result is a clear expression of the fact that those models encounter some difficulty in interpolating the sample periodogram.

The fit of the periodogram (raw sample spectrum) can be seen from figure 3, which presents $I(\lambda_j)$ along with the estimated spectral density functions $g_m(\lambda_j)$. Notice that CG implies the same estimate at the zero frequency as the ARIMA model and UC$(r, 0)$, and thus the same persistence. On the other hand, the spectral density fitted by UC(0,θ) is characterised by a spectral peak taking place at a lower frequency, and therefore the resulting cycle estimates are characterised by a larger period; also, $\theta = 1$ implies $g_\theta(0) = g_\theta(\pi)$ as the cycle is strictly non invertible at the π frequency.

The results hence confirm MNZ findings, pointing out that among the unobserved components models considered, the UC$(r, 0)$ model is the only one that can be reconciled with the unrestricted ARIMA(2,1,2) model of GDP. However the interpretative issues raised in section 8 still hold.

This is not quite the end of the story, as we proceed to assess the estimates of the signals resulting from the UC(r,0) model. The real time estimates of the trend, $\tilde{\mu}_{t|t}$, and the cycle, $\tilde{\psi}_{t|t}$, arising from UC(r, 0) will be coincident to the BN components. They are characterised by a perfect negative correlation and the real time estimates of the cycle will have a non invertible ARMA(2,1) representation: as a matter of fact, the parameter values reported in table 1 imply a value for the ϑ^* coefficient in (6) that is equal to -1.46.

If the BN decomposition is estimated as a model, that is we set up a state space model consisting of equations (4)-(6), after processing a suitable small number of observations the real time and final estimates are fully coincident. On the other hand, the estimates arising from UC(r,0) are subject to large revisions as new observations become available. This can be clearly seen from figure 4, which displays the the real time and final estimates of the cycle, $\tilde{\psi}_{t|T}$. The final estimates contradict the assertion that the cycle has a small amplitude, their range going from about -5% to +5%.

The reason for this apparently puzzling phenomenon will be now discussed. Loosely speaking it relates to the fact that the real time estimates provide a gross underestimation of the signals as compared with the smoothed estimates, which depend heavily on future observations. Derivation and discussion of the weighting patterns for filtering and smoothing is presented in Harvey and Koopman (2000). Here we concentrate on showing that the percentage reduction in estimation uncertainty, or equivalently the gains in reliability, due to the availability of future observations is inversely related to the correlation coefficient between the trend and cycle disturbances.

For this purpose we consider a sequence of UC(r,0) with r taking values in [-1,1], fixing the remaining parameters, $\phi_1, \phi_2, \sigma_\eta$ and σ_κ, at their estimated values as they appear in table 1; changing the ratio $\sigma_\eta/\sigma_\kappa$ leaves the results of this analysis unaffected. In table 2 we report the implied persistence parameter, the MA parameter of the ARMA(2,1) model generating the real time estimates, ϑ^*, and the increase in reliability of the cycle estimates when we use a doubly infinite sample. The latter is defined as the percentage reduction in the estimation error variance when we compare the real time estimates with the final ones:

$$100\frac{\bar{P}^{(\psi)}_{t|t} - P^{(\psi)}_{t|\infty}}{\bar{P}^{(\psi)}_{t|t}},$$

Table 2: Implied persistence, BN cycle moving average coefficient, and percentage reduction in estimation error variance of the cyclical component for UC(r,0) models (1) with $\sigma_\eta/\sigma_\zeta = 2.03$ and AR parameters $\phi_1 = 1.39$ and $\phi_2 = -0.79$ and different trend-cycle correlation, r.

r	$\theta(1)/\phi(1)$	ϑ^*	% Reduction
-1.00	1.27	-1.35	100.00
-0.90	1.18	-1.79	84.11
-0.75	1.08	-3.33	68.41
-0.25	0.89	1.37	38.90
0.00	0.82	0.65	29.02
0.25	0.77	0.35	20.58
0.50	0.73	0.19	13.09
0.75	0.70	0.08	6.29
0.90	0.68	0.03	2.46
1.00	0.67	0.00	0.00

where, using results presented in appendix C, and in particular (22), $\bar{P}^{(\psi)}_{t|t}$ is the steady state estimation error variance of the real time cyclical component and $P^{(\psi)}_{t|\infty}$ is that of the corresponding final estimates, using a doubly infinite sample.

Notice that when $r = 1$ the BN or real time estimates have the same AR(2) representation as the true component ($\vartheta^* = 0$); the process generating them is coincident with the maintained model for the unobserved component, so that current and past (i.e. real time) information is all we need to form this estimate. On the contrary, when r is negative the distribution of the weights for extraction of the cycle, based on a doubly infinite sample, are highly skewed towards the future. As a matter of fact, when $r = -1$ the cycle is estimated with zero mean square error using a doubly infinite sample, but the real time estimates are characterised by high uncertainty. This is why we get a 100% increase in reliability from processing future observations.

It is also noticeable that a small negative correlation such as $r = -0.25$ yields a persistence that is less than one and, therefore, real time or BN components with perfectly and positively correlated disturbances. It should be recalled from section 3 that the correlation sign depends on persistence.

In conclusion, if we accept that trend and cyclical disturbances are negatively correlated as implied by MNZ, then we must be willing to accept also that essential information for assessing the cyclical pattern lies in future observations and that our signals are prone to high revisions.

9.2 Italian GDP

The second example, concerning Italian real quarterly GDP at 1995 prices, available for the sample period 1970.1-2001.2, provides a case in which the cyclical growth model and the trend cycle decomposition with correlated disturbances provide exactly the same inferences, that are in turn coincident with those arising for the unrestricted ARIMA(2,1,2) model. As a result, alternative explanations of the nature of macroeconomic fluctuations arise with exactly the same likelihood.

The estimated parameters, along with diagnostics, are presented in table 3. It is clear that ARIMA, UC(r,0) and CG are observationally equivalent and produce a good fit in terms of residual autocorrelation and normality. The orthogonal decomposition with

Table 3: Parameter estimates and diagnostics for models of quarterly Italian real GDP, 1970.1-2000.2; (r) denotes a restricted parameter.

	ARIMA	UC$(r,0)$	UC$(0,\theta)$	Cycl. Growth
ϕ_1	1.43	1.43	1.54	1.43
ϕ_2	-0.74	-0.74	-0.80	-0.74
θ_1	-1.04			
θ_2	0.41			
σ^2	0.5073			
r		-0.90	0(r)	
σ^2_η		0.7454	0.3556	0.2821
σ^2_κ		0.3567	0.0332	0.1102
θ		0(r)	1.00	-0.37
loglik	-133.87	-133.87	-136.46	-133.87
$Q(12)$	5.11	5.11	15.22	5.11
Normality	2.70	2.70	3.44	2.70

ARMA(2,1) cycle, UC(0,θ), yields a lower likelihood and there is significant autocorrelation left in the residuals. As in the previous example this is the likely consequence of underestimation of the spectral density at the long run frequencies. It is also noticeable that also for the Italian GDP the parameter θ is forced to the boundary: hence, there is less support for the additive decomposition of the series into orthogonal components.

Figure 5 compares the parametric fit of the periodogram ordinates $I(\lambda_j)$ provided by the various models: the estimated spectral density functions $g_m(\lambda_j)/(2\pi)$ are of course coincident for ARIMA, UC(r,0) and CG, whereas for UC(0,θ) the considerations are the same as those we made for U.S. GDP.

Several interesting features arise from the plot of the smoothed estimates of the components, $\tilde{\mu}_{t|T}$ and $\tilde{\psi}_{t|T}$, available in figure 6. As far as UC(r,0) is concerned, the cyclical component is estimated with a relatively low mean square error but the confidence bounds increase sharply at the end of the sample, this being a consequence of the peculiar weighting pattern, that gives more weight to future observations with respect to orthogonal decompositions. The confidence intervals for the very last data point give an idea of the amount of uncertainty that affects real time estimates.

The amplitude of the fluctuations and the cycle chronology are roughly the same as that implied by the orthogonal decomposition, UC$(0,\theta)$; however, the UC(r,0) cycle features a major recessionary pattern in 1971-1972, when the trend is well above y_t (see the first panel).

Finally, for the cyclical growth model we present the estimated negative growth probabilities Prob$(\psi_t+\beta < 0|Y_t)$, computed from the smoothing distribution $\psi_t|Y_T \sim N(\tilde{\psi}_{t|T}, P^{(\psi)}_{t|T})$. We could use these estimates to establish a perhaps naïve, but quite simple, business cycle chronology, e.g. focussing attention on probabilities greater than 0.5 and affecting two or more consecutive quarters. According to this criterion three major recessions would emerge, the first two associated with the two oil shock (around 1973-1974 and 1981-1983) and the third in 1991-1992. The other events from the second half of the nineties would hardly be considered as recessions, as they lack both depth and duration.

The cycle chronology arising from CG leads that arising form the orthogonal deviation cycle extracted by $\mathrm{UC}(0, \theta)$, in accordance with the the fact that the former is defined in terms of the first differences of the series, rather than the levels.

10 Concluding Remarks

This paper has addressed some of the subtleties and interpretative issues that arise from trend-cycle decompositions with correlated components. Conditional on an ARIMA(2,1,2) reduced form representation, we have considered several observationally equivalent possible explanations: alternative representations for the cyclical component, accounting for richer dynamics, permanent-transitory decompositions, a cycle in growth rates, consistent with the classical definition of a business cycle, the hysteresis model, according to which the cycle modifies permanently the trend.

We also investigated the consequences of having highly and negatively correlated disturbances for signal extraction. The conclusion was that the role of future observations in reducing the uncertainty increases with the size of the correlation coefficient. Hence, large revisions have to be expected.

All this statements were made maintaining a particular ARIMA reduced form, but in real life this is itself an additional source of uncertainty. For instance, Harvey and Jäger (1994) entertained an orthogonal trend-cycle decomposition to the U.S. real GDP series, allowing for a stochastic slope in the trend, so that the latter is an $I(2)$ process. The same model provides a data-coherent decomposition of the Italian GDP, see Proietti (2002). Discriminating among UC models unconditionally, i.e. without assuming a particular reduced form, is a far more complex issue, since, despite the recent advances in testing and model specification in the unobserved components framework (see Harvey, 2001), it is impossible to get analytic results, due to the unavailability of a common estimable reduced form.

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A Proof of results (7)

The ACGF of UC$(r,0)$ can be rewritten as:

$$|\phi(L)|^2 g_r(L) = |\phi(L)|^2\sigma_\eta^2 + |1-L|^2\sigma_\kappa^2 + [(1-L)\phi(L^{-1}) + (1-L^{-1})\phi(L)]\sigma_{\eta\kappa}.$$

Replacing the polynomial decompositions

$$\begin{array}{lclll} \phi(L) & = & \phi(1) + (1-L)\phi^*(L), & & \phi^*(L) = \phi_0^* - \phi_1^* L, \\ \phi(L^{-1}) & = & \phi(1) + (1-L^{-1})\phi^*(L^{-1}), & & \phi^*(L^{-1}) = \phi_0^* - \phi_1^* L^{-1}, \\ \phi_0^* & = & \phi_1 + \phi_2, & & \phi_1^* = -\phi_2, \end{array}$$

in the term in square brackets, and using the identities

$$|1-L|^2 = (1-L)(1-L^{-1}) = 2 - L - L^{-1} = (1-L) + (1-L^{-1}),$$

we get

$$\begin{aligned} (1-L)\phi(L^{-1}) + (1-L^{-1})\phi(L) &= |1-L|^2[\phi(1) + \phi^*(L) + \phi^*(L^{-1})] \\ &= |1-L|^2[1 + \phi_1 + \phi_2 + \phi_2(L + L^{-1})]. \end{aligned}$$

B Proof of results (14)

Replacing $|1+\theta L|^2 = (1+\theta)^2 - \theta|1-L|^2$ in the ACGF of (11)

$$|\phi(L)|^2 g_c(L) = |\phi(L)|^2\sigma_{\eta^*}^2 + |1+\theta L|^2\sigma_{\kappa^*}$$

and equating $g_c(1) = g_r(1)$ we get result (13). Substituting into the $g_c(L) = g_r(L)$ and gathering terms we get:

$$\left[(1+\theta^2)\frac{\phi(1)^2 - |\phi(L)|^2}{\phi(1)^2} - \theta|1-L|^2\right]\sigma_{\kappa^*}^2 = |1-L|^2[\sigma_\kappa^2 + r\sigma_\eta\sigma_\kappa(1+\phi_1+\phi_2+\phi_2(L+L^{-1}))]. \tag{17}$$

Now, the polynomial $|\phi(L)|^2$ on the left hand side can be written:

$$|\phi(L)|^2 = \phi(1)^2 + |1-L|^2\left\{|\phi^*(L)|^2 + \phi(1)\left[\phi_0^* + \phi_1^*(1 - |1+L|^2)\right]\right\}$$

where the polynomial $\phi^*(L) = \phi_0^* - \phi_1^* L$ is given in appendix A. The right hand side of (17) then becomes:

$$-|1-L|^2\left[\theta + \frac{(1+\theta)^2}{\phi(1)^2}\left\{|\phi^*(L)|^2 + \phi(1)\left[\phi_0^* + \phi_1^*(1 - |1+L|^2)\right]\right\}\right]\sigma_{\kappa^*}^2.$$

or, equivalently,

$$-|1-L|^2\left[\frac{(1+\theta)^2}{\phi(1)^2}\Big(\phi_1(1-\phi_2) + 2\phi_2 + \phi_2(L+L^{-1})\Big) + \theta\right]\sigma_{\kappa^*}^2.$$

The factor $|1-L|^2$ thus cancels out and, matching the coefficients associated to the same powers of L, we get the two equations:

$$\begin{array}{rcl} r\sigma_\eta\sigma_\kappa & = & -\frac{(1+\theta)^2}{\phi(1)^2}\sigma^2_{\kappa^*} \\ \sigma^2_\kappa + r\sigma_\eta\sigma_\kappa(1+\phi_1+\phi_2) & = & -\left[\frac{(1+\theta)^2}{\phi(1)^2}\left(\phi_1(1-\phi_2)+2\phi_2\right)+\theta\right]\sigma^2_{\kappa^*}. \end{array} \quad (18)$$

which, together with (13) allow to determine the parameters of UC$(r,0)$ from those of the cyclical growth model (11) and viceversa, when a solution is admissible. Dividing the two equations in (18) gives the result stated in (14).

C Estimation of unobserved components

State Space representation The UC models considered in the main text admit the time-invariant state space representation:

$$\begin{array}{rcl} y_t & = & \boldsymbol{z}'\boldsymbol{\alpha}_t, \qquad t=1,2,\dots,T, \\ \boldsymbol{\alpha}_t & = & \boldsymbol{T}\boldsymbol{\alpha}_{t-1}+\boldsymbol{c}+\boldsymbol{R}\boldsymbol{\epsilon}_t, \end{array} \quad (19)$$

with $\boldsymbol{\epsilon}_t \sim \mathrm{NID}(\boldsymbol{0},\boldsymbol{Q})$ and $\boldsymbol{\alpha}_1 \sim \mathrm{NID}(\tilde{\boldsymbol{\alpha}}_{1|0},\boldsymbol{P}_{1|0})$, independently of $\boldsymbol{\epsilon}_t, \forall t$. Initialisation of the state vector when nonstationary state components are present is discussed in Koopman (1997).

For instance, the system matrices for the UC$(r,0)$ model are:

$$\boldsymbol{z}=\begin{bmatrix}1\\1\\0\end{bmatrix},\ \boldsymbol{T}=\begin{bmatrix}1&0&0\\0&\phi_1&1\\0&\phi_2&0\end{bmatrix},\ \boldsymbol{c}=\begin{bmatrix}\beta\\0\\0\end{bmatrix},\ \boldsymbol{R}=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix},\ \boldsymbol{Q}=\begin{bmatrix}\sigma^2_\eta&\sigma_{\eta\kappa}\\\sigma_{\eta\kappa}&\sigma^2_\kappa\end{bmatrix}$$

For the Beveridge-Nelson decomposition, considered *as a model*, the system matrices are the same except for $\boldsymbol{R}$ and $\boldsymbol{Q}$, which are 3×1 and scalar, respectively:

$$\boldsymbol{R}=\begin{bmatrix}\varrho\\1-\varrho\\-(\theta_2+\phi_2\varrho)\end{bmatrix},\ \boldsymbol{Q}=\sigma^2,$$

where $\varrho=\theta(1)/\phi(1)$ is the persistence parameter. On the other hand, for UC$(0,\theta)$ we need to replace $\boldsymbol{R}$ and $\boldsymbol{Q}$ by:

$$\boldsymbol{R}=\begin{bmatrix}1&0\\0&1\\0&\theta\end{bmatrix},\ \boldsymbol{Q}=\begin{bmatrix}\sigma^2_{\eta^*}&0\\0&\sigma^2_{\kappa^*}\end{bmatrix};$$

finally, the state space representation for the cyclical growth model is obtained also replacing $\boldsymbol{z}$ and $\boldsymbol{T}$ by:

$$\boldsymbol{z}=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \boldsymbol{T}=\begin{bmatrix}1&1&0\\0&\phi_1&1\\0&\phi_2&0\end{bmatrix}.$$

Kalman Filter The Kalman filter (Anderson and Moore, 1979), is the well-known recursive algorithm for computing the minimum mean square estimator of $\boldsymbol{\alpha}_t$ and its mean square error (MSE) matrix conditional on $Y_{t-1} = \{y_1, y_2, \ldots, y_{t-1}\}$. Defining $\tilde{\boldsymbol{\alpha}}_{t|t-1} = \mathrm{E}(\boldsymbol{\alpha}_t|Y_{t-1})$, $\boldsymbol{P}_{t|t-1} = \mathrm{E}[(\boldsymbol{\alpha}_t - \tilde{\boldsymbol{\alpha}}_{t|t-1})(\boldsymbol{\alpha}_t - \tilde{\boldsymbol{\alpha}}_{t|t-1})'|Y_{t-1}]$, it is given by the set of recursions:

$$\begin{array}{rclrcl} \xi_t &=& y_t - \boldsymbol{z}'\tilde{\boldsymbol{\alpha}}_{t|t-1}, & f_t &=& \boldsymbol{z}'\boldsymbol{P}_{t|t-1}\boldsymbol{z} \\ & & & \boldsymbol{k}_t &=& \boldsymbol{T}\boldsymbol{P}_{t|t-1}\boldsymbol{z}f_t^{-1} \\ \tilde{\boldsymbol{\alpha}}_{t+1|t} &=& \boldsymbol{T}\tilde{\boldsymbol{\alpha}}_{t|t-1} + \boldsymbol{c} + \boldsymbol{k}_t\xi_t, & \boldsymbol{P}_{t+1|t} &=& \boldsymbol{T}\boldsymbol{P}_{t|t-1}\boldsymbol{T}' + \boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}' - \boldsymbol{k}_t\boldsymbol{k}_t'f_t \end{array} \tag{20}$$

$\xi_t = y_t - \mathrm{E}(y_t|Y_{t-1})$ are the filter innovations or one-step-ahead prediction errors, with variance matrix f_t.

Steady State The innovations and the state one-step-ahead prediction error, $\boldsymbol{x}_t = \boldsymbol{\alpha}_t - \tilde{\boldsymbol{\alpha}}_{t|t-1}$, can be written as

$$\xi_t = \boldsymbol{z}'\boldsymbol{x}_t, \quad \boldsymbol{x}_{t+1} = \boldsymbol{L}_t\boldsymbol{x}_t + \boldsymbol{R}\boldsymbol{\epsilon}_t,$$

where $\boldsymbol{L}_t = \boldsymbol{T} - \boldsymbol{k}_t\boldsymbol{z}'$. Thus, $\boldsymbol{x}_t$ follows a VAR(1) process that is (asymptotically) stationary if the autoregressive matrix $\boldsymbol{L}_t$, known as the *closed loop matrix* in system theory, converges to a matrix $\boldsymbol{L} = \boldsymbol{T} - \boldsymbol{k}\boldsymbol{z}'$, whose eigenvalues lie all inside the unit circle.

The basic properties that ensure convergence to such stabilising solution are *detectability* and *stabilisability* (see Burridge and Wallis, 1988). They imply that, independently of initial conditions, $\boldsymbol{P}_{t+1|t}$ converges at an exponential rate to a steady state solution $\boldsymbol{P}$, satisfying the Riccati equation $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{P}\boldsymbol{T}' + \boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}' - \boldsymbol{k}\boldsymbol{k}'f$, with $\boldsymbol{k} = \boldsymbol{T}\boldsymbol{P}\boldsymbol{z}f^{-1}$ and $f = \boldsymbol{z}'\boldsymbol{P}\boldsymbol{z}$, and the Kalman gain vector $\boldsymbol{k}$ is such that $\boldsymbol{L}$ has all its eigenvalues inside the unit circle.

Real time estimates The real time or concurrent estimates of the states and the estimation error covariance matrix are given respectively by:

$$\tilde{\boldsymbol{\alpha}}_{t|t} = \tilde{\boldsymbol{\alpha}}_{t|t-1} + \boldsymbol{P}_{t|t-1}\boldsymbol{z}f_t^{-1}\xi_t, \quad \boldsymbol{P}_{t|t} = \boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1}\boldsymbol{z}\boldsymbol{z}'\boldsymbol{P}_{t|t-1}f_t^{-1}. \tag{21}$$

The estimated unobserved components in $\tilde{\boldsymbol{\alpha}}_{t|t}$ are the same as those arising from the BN decomposition of the implied ARIMA reduced form representation. The MA parameters of the reduced form representation can be uniquely derived from the steady state using $\boldsymbol{P}\boldsymbol{z}f^{-1}$. Notice, however, that in the steady state we need $\boldsymbol{z}\boldsymbol{z}'f^{-1}$ to be equal to the pseudo-inverse of $\boldsymbol{P}$ for the components to be estimated with zero error, i.e. observable with respect to current and past information. For the BN model $f = \sigma^2 = \boldsymbol{Q}$, $\boldsymbol{P}\boldsymbol{z}f^{-1} = \boldsymbol{R}$ and $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}'$, which ensures that when the system has reached a steady state, the components are estimated in real time with zero mean square error.

Final estimates We can keep track of revisions, due to the accrual of further observations, by using a *fixed-point smoothing* algorithm. Elaborating results in de Jong (1989), and assuming that the system has reached a steady state, we have, for a fixed t and for $l \geq 0$, the following smoothing recursions:

$$\begin{array}{rclrcl} \tilde{\boldsymbol{\alpha}}_{t|t+l} & = & \tilde{\boldsymbol{\alpha}}_{t|t} + \boldsymbol{P}\boldsymbol{L}'\boldsymbol{r}_{t|t+l}, & \boldsymbol{P}_{t|t+l} & = & \bar{\boldsymbol{P}}_{t|t} - \boldsymbol{P}\boldsymbol{L}'\boldsymbol{N}_{t|t+l}\boldsymbol{L}\boldsymbol{P}, \\ \boldsymbol{r}_{j|t+l} & = & \boldsymbol{L}'\boldsymbol{r}_{j+1|t+l} + \boldsymbol{z}f^{-1}\xi_{j+1}, & \boldsymbol{N}_{j|t+l} & = & \boldsymbol{L}'\boldsymbol{N}_{j+1|t+l}\boldsymbol{L} + \boldsymbol{z}\boldsymbol{z}'f^{-1}, \end{array} \tag{22}$$

$j = t+l, t+l-1, \ldots, t$, where $\bar{\boldsymbol{P}}_{t|t} = \boldsymbol{P} - \boldsymbol{P}\boldsymbol{z}\boldsymbol{z}'\boldsymbol{P}f^{-1}$ and the backwards recursions are initialised $\boldsymbol{r}_{t+l|t+l} = \boldsymbol{0}$, $\boldsymbol{N}_{t+l|t+l} = \boldsymbol{0}$.

Now, as $l \to \infty$ (i.e. assuming a doubly infinite sample), $\boldsymbol{r}_{j|t+l}$ is a backward first order stationary vector autoregression, and $\boldsymbol{N}_{j|t+l}$ is its covariance matrix. The final state estimation error covariance matrix, denoted $\boldsymbol{P}_{t|\infty}$, solves $\boldsymbol{P}_{t|\infty} = \boldsymbol{P} - \boldsymbol{P}\boldsymbol{N}\boldsymbol{P}$, where $\boldsymbol{N}$ is the steady state solution of the backward smoothing equation, $\boldsymbol{N}_{j|t+l} = \boldsymbol{L}'\boldsymbol{N}_{j+1|t+l}\boldsymbol{L} + \boldsymbol{z}\boldsymbol{z}'f^{-1}, j = t+l, \ldots, t$, as $l \to \infty$; a unique stable solution for $\boldsymbol{N}$ exists provided the characteristic roots of $\boldsymbol{L}$ are less than unity in modulus, which is already the condition for a steady state solution. The elements of the solution are obtained from

$$\mathrm{vec}(\boldsymbol{N}) = (\boldsymbol{I} - \boldsymbol{L}' \otimes \boldsymbol{L}')^{-1}\mathrm{vec}(\boldsymbol{z}\boldsymbol{z}'f^{-1}).$$

Hence, $\boldsymbol{P}_{t|\infty}$ contains the final estimation error covariance matrix, and can be written:

$$\boldsymbol{P}_{t|\infty} = \bar{\boldsymbol{P}}_{t|t} - \boldsymbol{P}(\boldsymbol{z}\boldsymbol{z}'f^{-1} - \boldsymbol{N})\boldsymbol{P}.$$

The second term on the right hand side, which is obviously positive semi-definite, measures the total reduction in the estimation uncertainty as we go from the real time to the final estimates.

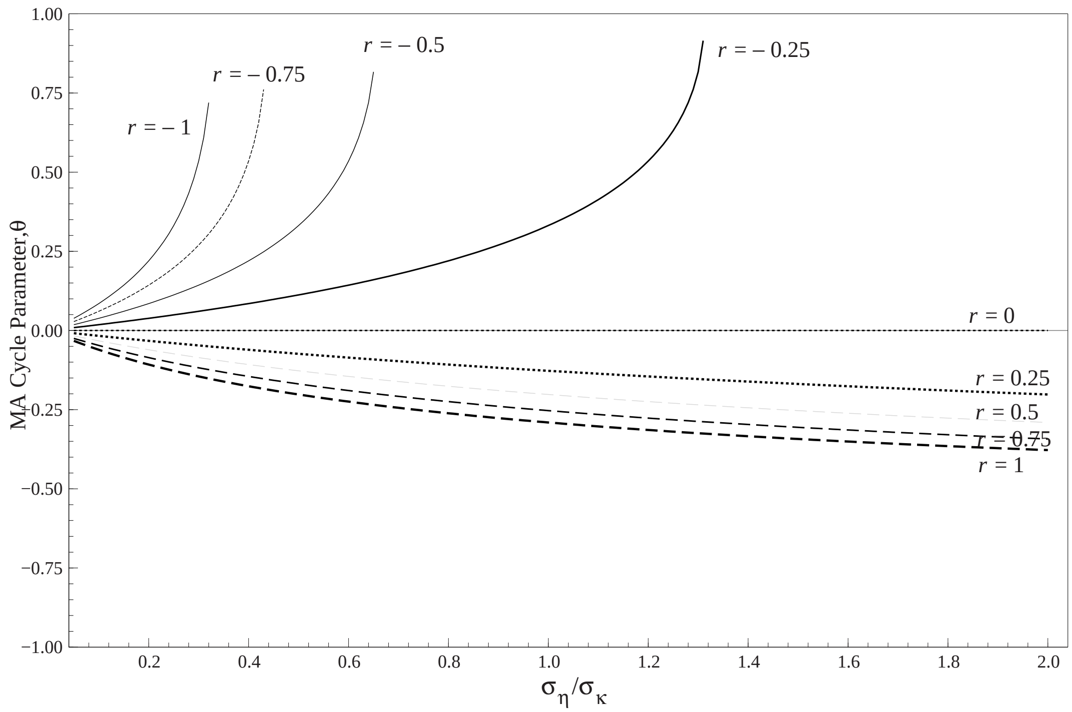

Figure 1: Plot of θ (vertical axis) in UC$(0, \theta)$ as a function of the parameters $\sigma_\eta/\sigma_\kappa$ (horizontal axis) and r of an UC$(r, 0)$ model.

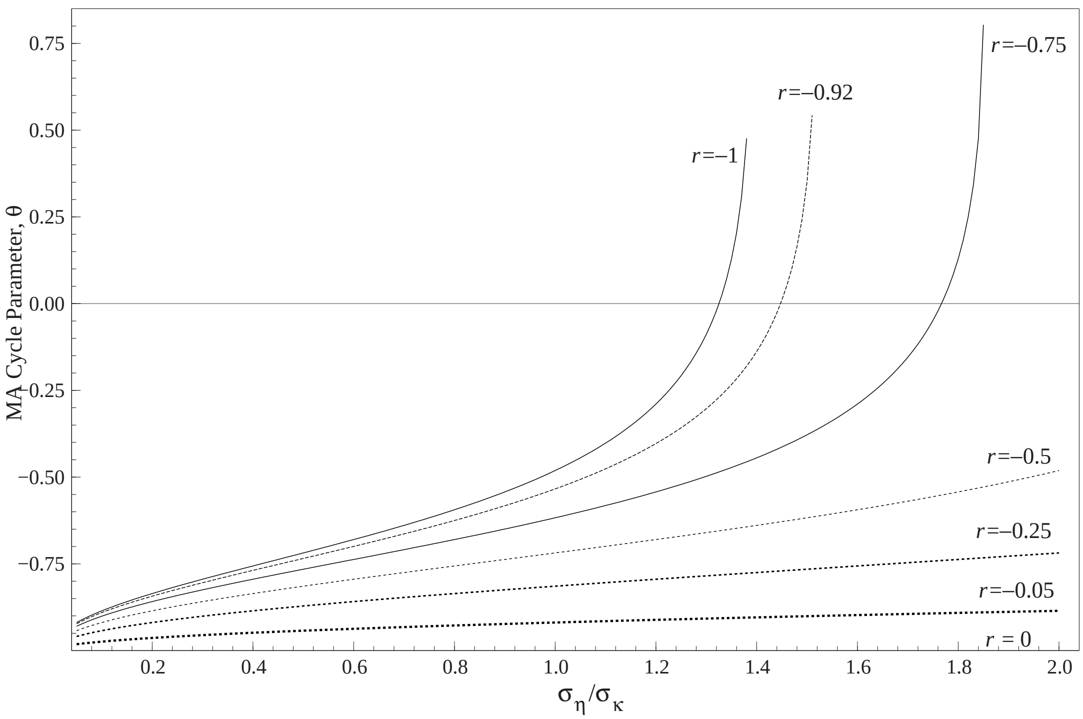

Figure 2: Plot of θ (vertical axis), the MA parameter of the cyclical growth model, as a function of the parameters $\sigma_\eta/\sigma_\kappa$ (horizontal axis) and r of an UC$(r, 0)$ model.

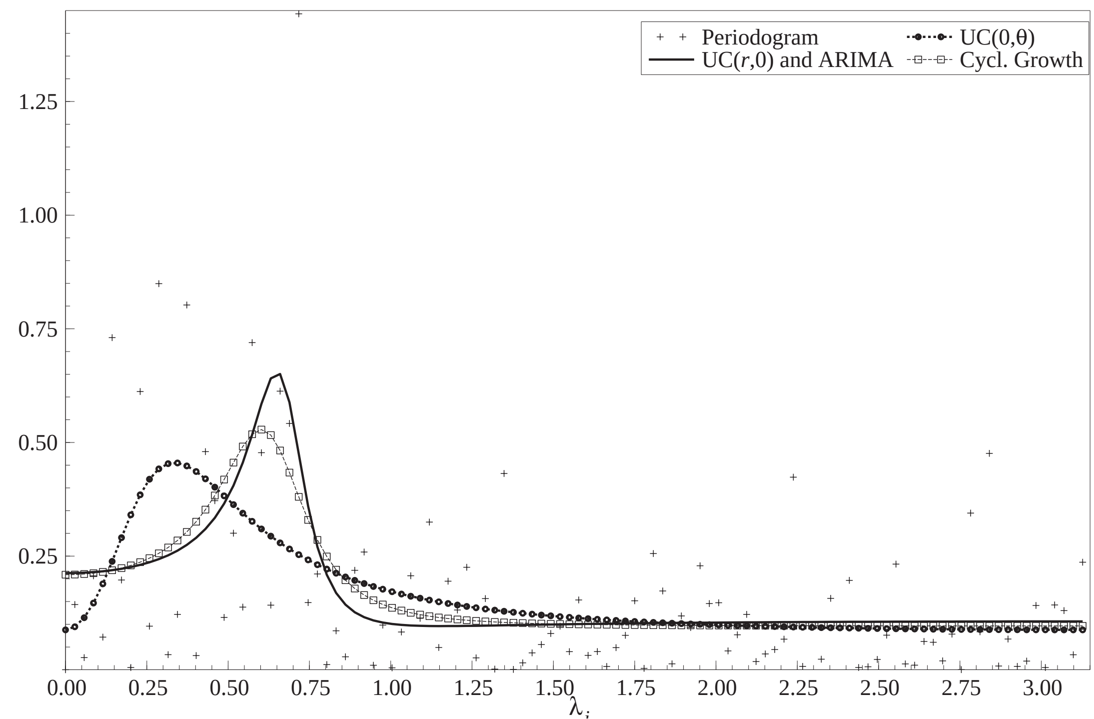

Figure 3: U.S. GDP, 1947.1-2001.4. Periodogram, $I(\lambda_j)$, and parametric spectral densities of Δy_t, $g_m(\lambda_j)/(2\pi)$, estimated by the ARIMA(2,1,2) model, the UC(r,0), UC(0,θ) and cyclical growth model.

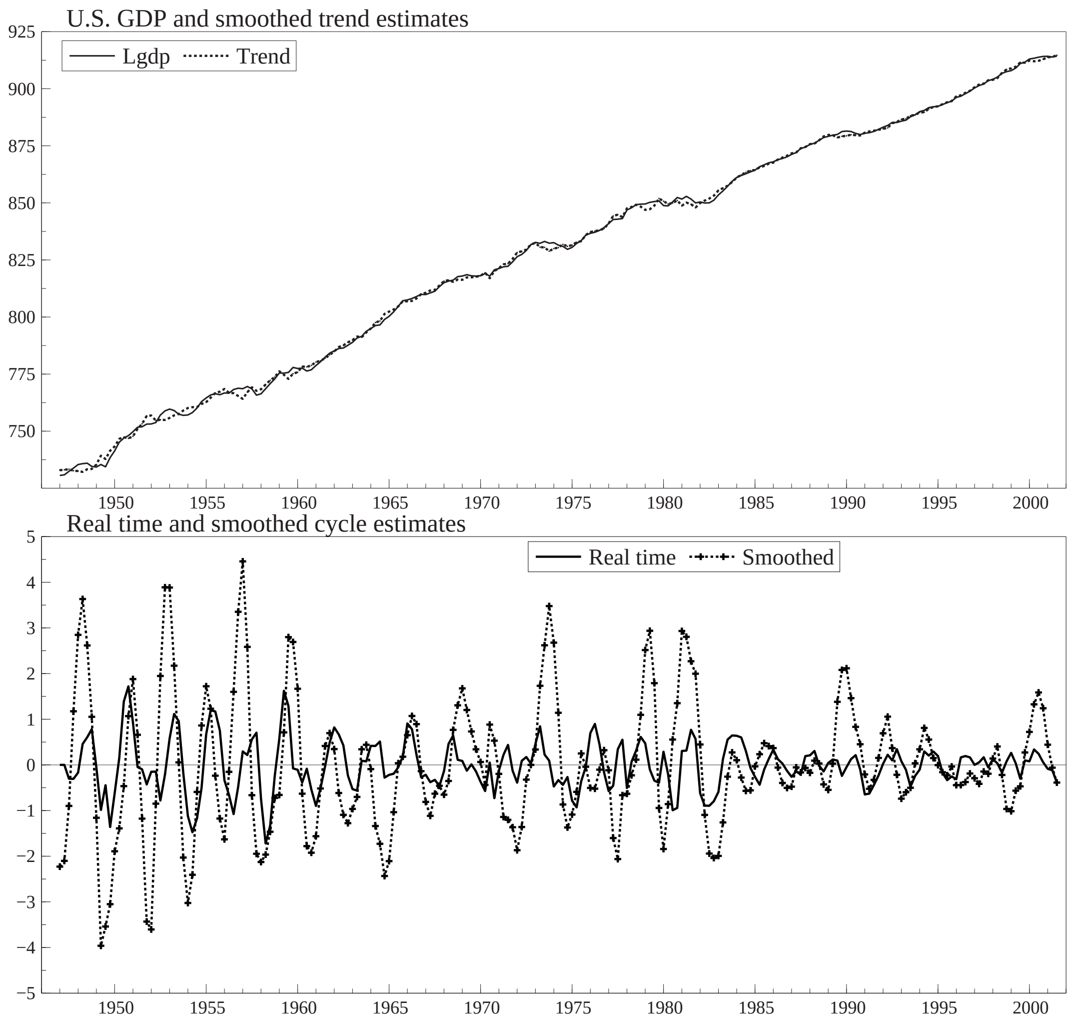

Figure 4: Plot of series and smoothed estimates arising from the model UC(r,0). Series with trend, $\tilde{\mu}_{t|T}$ (first panel); real time ($\tilde{\psi}_{t|t}$) and smoothed ($\tilde{\psi}_{t|T}$) estimates of the cyclical component.

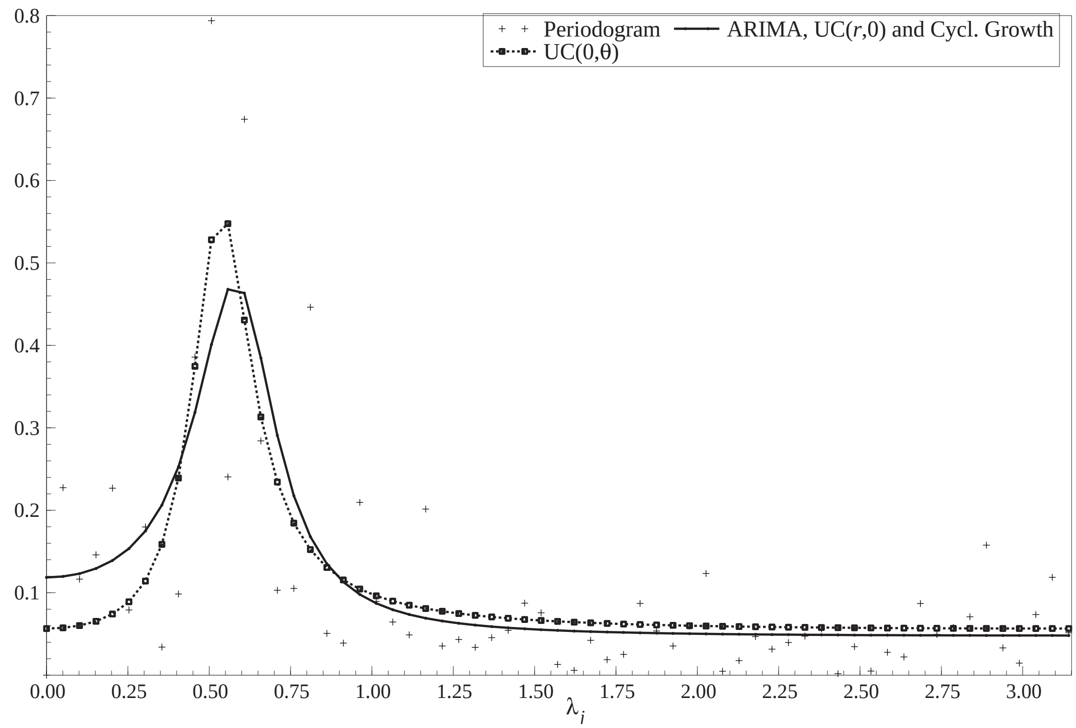

Figure 5: Italy, GDP, 1970.1-2001.2. Periodogram, $I(\lambda_j)$, and parametric spectral densities of Δy_t, $g_m(\lambda_j)/(2\pi)$, estimated by the ARIMA(2,1,2) model, the UC(r,0), UC(0,θ) and cyclical growth model.

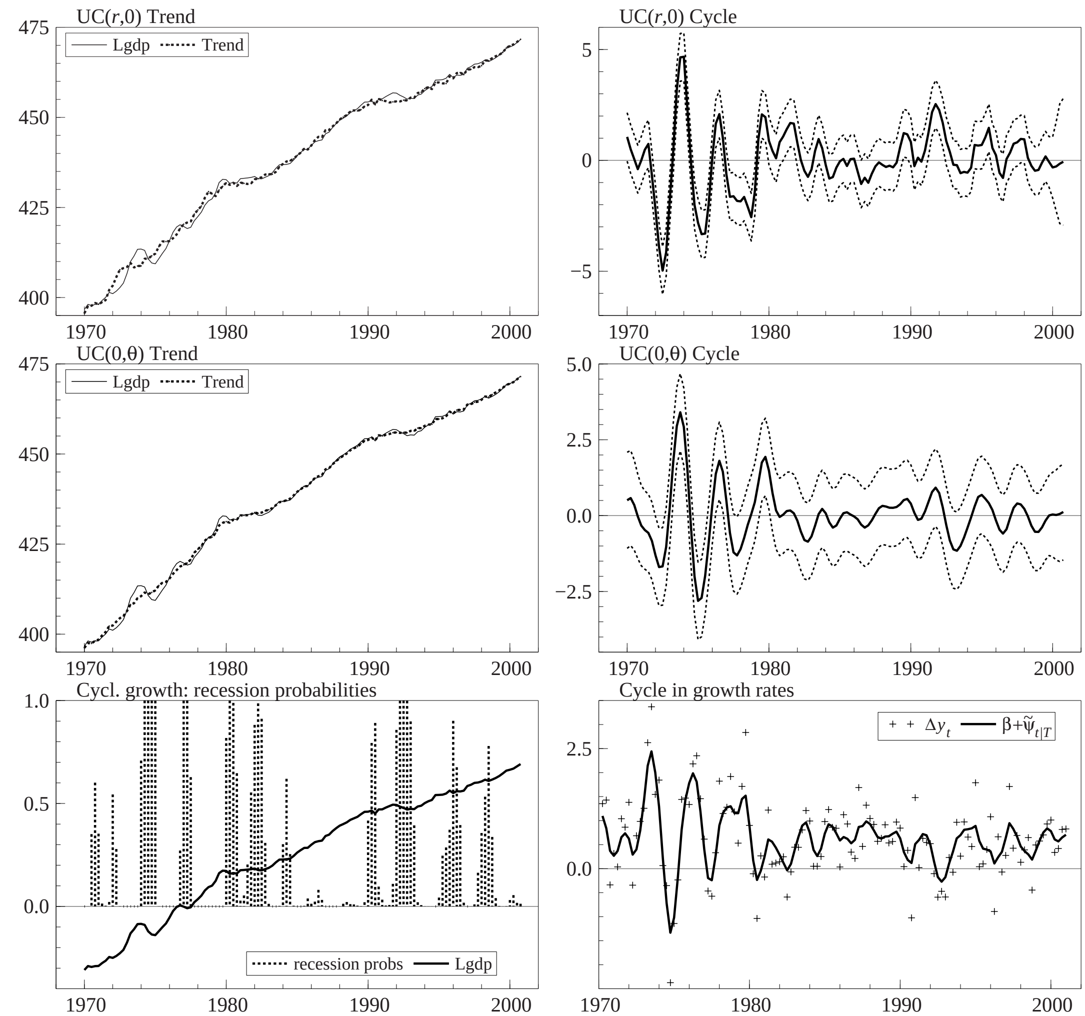

Figure 6: Italian GDP, 1970.1-2001.2. Smoothed estimates of trend and cycle arising from the UC(r,0) model (top), UC(0,θ) (centre). For the cyclical growth model we present Prob($\psi_t + \beta < 0|Y_T$) (bottom left) and the smoothed estimates of underlying growth, $\beta + \tilde{\psi}_{t|T}$.