The program to run the test can be downloaded for free at
http://home.cerge-ei.cz/kocenda/software.htm

# AN ALTERNATIVE TO THE BDS TEST: INTEGRATION ACROSS THE CORRELATION INTEGRAL

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ABSTRACT

This paper extends and generalizes the BDS test presented by Brock, Dechert, Scheinkman, and LeBaron (1996). In doing so it aims to remove the limitation of having to arbitrarily select a proximity parameter by integrating across the correlation integral. The Monte Carlo simulation is used to tabulate critical values of the alternative statistic. Previously published empirical studies are replicated as well as power tests executed in order to evaluate the relative performance of the suggested alternative to the BDS test. The results are favorable for the suggested alternative.

## 1. Introduction

Applications of deterministic nonlinear dynamics and chaos theory to the analysis of stochastic economic time series are common in contemporary macroeconomics and finance. The pioneering volume on the complexity of the economy, edited by Anderson, Arrow and Pines (1988), includes a paper by Brock (1988) that is closely related to this topic.

A non-parametric method of testing for nonlinear patterns in time series, devised by Brock, Dechert and Scheinkman (1987) and developed in Brock, Dechert, Scheinkman and LeBaron (1996) is known as the BDS test. The null hypothesis is that data in a time series is independently and identically distributed (iid). The test is unique in its ability to detect nonlinearities independent of linear dependencies in the data.

This paper suggests an alternative test that aims to offer some improvement over the BDS test. Both methods are based on the theoretical concept of the correlation integral described by Grassberger and Procaccia (1983). In order to conduct the BDS test, certain parameters must be chosen arbitrarily, ex ante, with limited guidance from statistical theory. It is, therefore, likely that inappropriate values will be chosen. The proposed alternative aims to remove the arbitrary selection of the proximity parameter $\varepsilon$ through integrating across correlation integral.

The paper continues with Section 2 that provides a brief theoretical background. Section 3 describes the new test and further generates critical values using Monte Carlo technique. Section 4 presents power tests and puts forth an empirical comparison of the alternative test statistic with the BDS test by replicating three previously published studies. Section 5 briefly concludes.

## 2. Theoretical Background

Chaotic systems of low dimensionality can generate seemingly random numbers that may give an impression of white noise, thereby hiding their true nature. Under presumed randomness, a nonlinear pattern can hide without being detected. Exchange rates, stock market returns and other macroeconomic variables of generally high frequency may originate from low dimensional-chaos. Detection of nonlinear *hidden patterns* in such time series provides important information about their behavior and improves forecasting ability over short time periods.

The analysis of chaotic systems often starts with computing a correlation dimension. This is because of easy computation and the availability of sampling theory. The aforementioned BDS test is based on such a technique and was designed to detect hidden patterns in stochastic time series. This test is a non-parametric test of the null hypothesis that the data are independently and identically distributed (iid) against an unspecified alternative. The procedure has power against both deterministic and stochastic systems. The ability of this test to deal with stochastic time series makes its application in modern macroeconomics and financial economics very appealing.

The test rests upon the concept of the correlation integral, developed by Grassberger and Procaccia (1983), to distinguish between chaotic deterministic systems and stochastic systems. The definition of this integral is simple: Let $\{x_t\}$ be a scalar time series generated randomly according to a density function *f*. Form *m*-dimensional vectors, called *m*-histories, $x_t^m = (x_t, x_{t+1}, \ldots, x_{t+m-1})$ .

The correlation integral at embedding dimension $m$ is computed as

$$C_{m,T}(\varepsilon) = 2\sum_{t=1}^{T_{m-1}} \sum_{s=t+1}^{T_m} I_\varepsilon(x_t^m, x_s^m) / (T_m(T_m - 1)), \tag{1}$$

where $T_m = T - m + 1$, and $I_\varepsilon(x_t^m, x_s^m)$ is an indicator function of the event

$$\left\| x_t^m - x_s^m \right\| = \max_{i=0,1,\ldots m-1} \left| x_{t+i} - x_{s+i} \right| < \varepsilon . \tag{2}$$

Thus, the correlation integral measures the fraction of pairs that lie within the tolerance distance $\varepsilon$ for the particular embedding dimension $m$.

The BDS statistic, for the time series of length $T$ is then defined as

$$\mathrm{BDS}_{m,T}(\varepsilon) = T^{\frac{1}{2}} \left[ C_{m,T}(\varepsilon) - C_{1,T}(\varepsilon)^m \right] / \sigma_{m,T}(\varepsilon) , \tag{3}$$

where $T$ is the sample size, $\varepsilon$ is an arbitrarily chosen proximity parameter, and $\sigma_{m,T}(\varepsilon)$ is the standard sample deviation of the statistic's numerator that varies with dimension $m$.[1] By using pairs of $m$-histories that too often cluster together within a specific distance $\varepsilon$, the BDS test is able to reveal *hidden patterns* that should not occur in truly randomly distributed data. While the BDS statistic is easy to compute, it suffers from an obvious drawback—the values of two parameters, $m$ and $\varepsilon$ must be determined *ex ante*.[2]

Heretofore, the BDS statistic, when used for testing, has often been evaluated for only few values of the proximity parameter.[3] This was brought about, in part, by the Monte Carlo studies of Hsieh and LeBaron (1988) who tested the asymptotic normality of the statistic for only three values of the parameter, and tabulated the corresponding critical values. It is worthwhile noting that originally an important reason to develop the BDS-test was that point estimates of the correlation dimension were very unstable across $\varepsilon$-values. The alternative test suggests to consider an OLS-estimate of the correlation dimension over a range of $\varepsilon$-values, and is thus closer in spirit to the original correlation dimension than the BDS-test.[4]

This paper proposes an alternative testing method that eliminates the arbitrariness in the choice of the proximity parameter, leaving unresolved only the question regarding the choice of embedding dimension.

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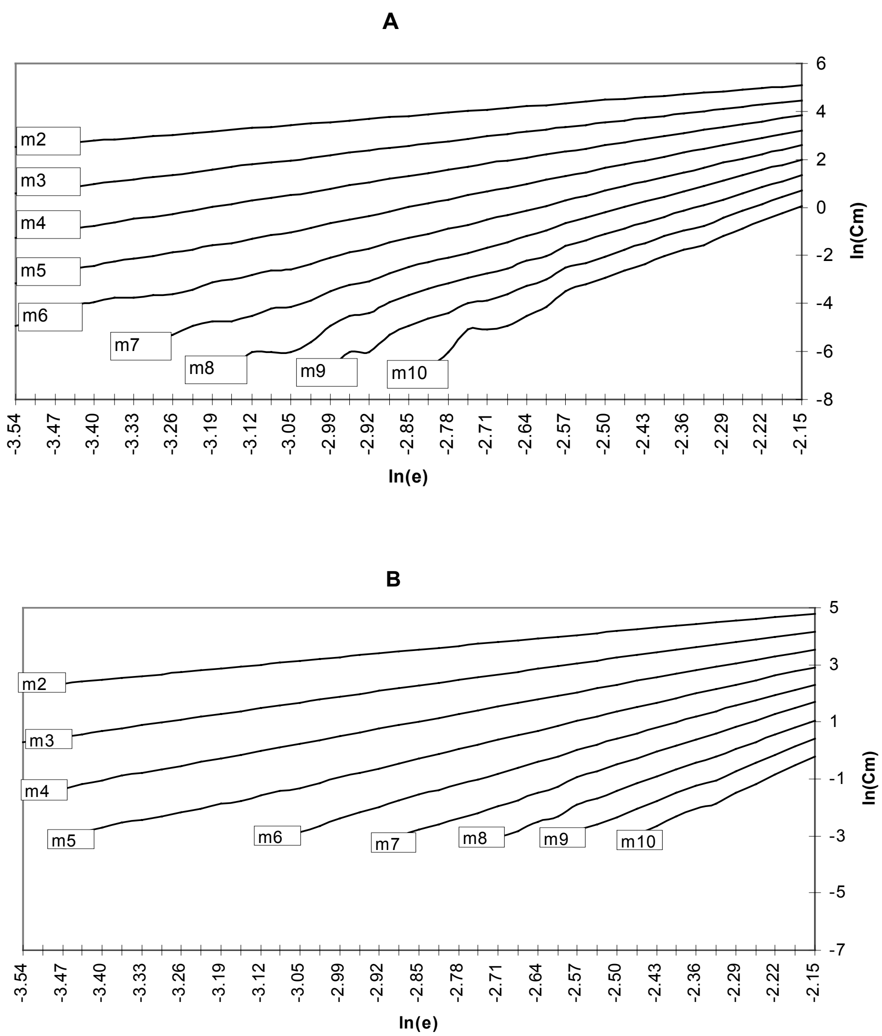
[1] Even though in econometric literature sample statistics are traditionally marked by Greek letters with "hats," or by upper case Latin letters, we preferred here to adhere to the established notation introduced by Brock, Dechert, Scheinkman, and LeBaron (1996).
[2] Limited guidance can be found for example in Dechert (1994), Brock, Dechert, Scheinkman and LeBaron (1996), and de Lima (1992).
[3] For several recent applications see Hsieh (1993), Olmeda and Perez (1995), Cecen and Erkal (1996), Kočenda (1996), Serletis and Gogas (1997), and Chwee (1998) among others . For other sources see section 4 of the paper.
[4] This conscious description was brought up by one of the referees.

### 3. An Alternative Test

The alternative test is based on the correlation integral described by equations (1) and (2). It is, however, constructed in a manner that radically differs from the BDS statistic (3). It uses a number of tolerance distances chosen from a specific range for each particular embedding dimension.

FIG. I

Trajectories of the plotted *ln($C_m$($\varepsilon$))* against *ln($\varepsilon$)* at various embedding dimensions *m*
(A) Unconstrained and (B) Constrained Case




The log of the correlation integral, $\ln(C_m(\varepsilon))$, is plotted against the log of the proximity parameter, $\ln(\varepsilon)$, for a particular embedding dimension *m*. Because numerous tolerance distances $\varepsilon$ are used, such a plot yields a map of trajectories as illustrated in Figure IA, where pure white noise was used as input data to form such trajectories. These trajectories are downward sloping and become steeper as dimension *m* becomes larger. At higher levels of *m*, the absence of pairs lying within the tolerance distance results in increased variance, and the far sections of the trajectories become highly erratic. If a larger number of matched pairs had been included, the variance would asymptotically decrease and the erratic portion of the trajectories would straighten. In order to preserve the sections with a constant slope, value of correlation integral $C_m(\varepsilon)$ is constrained so that it maximizes the power of the test, or implicitly, minimizes error of the second kind. The map of trajectories then looks like Figure IB.[5]

To summarize, an alternative test of the iid hypothesis is developed by calculating the slope of the log of the correlation integral versus the log of the proximity parameter over a broad range of values of the proximity parameter. The slope coefficients, $\beta_m$, can be estimated as

$$\beta_m = \frac{\sum_{\varepsilon}\left(\ln(\varepsilon) - \overline{\ln(\varepsilon)}\right)\cdot\left(\ln(C_m(\varepsilon)) - \overline{\ln(C_m(\varepsilon))}\right)}{\sum_{\varepsilon}\left(\ln(\varepsilon) - \overline{\ln(\varepsilon)}\right)^2}, \tag{4}$$

where $\ln(\varepsilon)$ is the log of proximity parameter (tolerance distance), $\ln(C_m(\varepsilon))$ is the correlation integral value, *m* is the embedding dimension, and the variables with a bar denote the mean of their counterparts without a bar.[6] Since a range of different tolerance distances $\varepsilon$ is used the slope coefficients $\beta_m$ do not depend on an arbitrary choice of $\varepsilon$. The same is true for the choice of dimension *m*. Again, a range of dimensions *m* is used which gives enough variety to capture a more complex dimensional structure without eliminating unexplored opportunities.

One theoretical feature of the slope coefficients $\beta_m$ is that under the null hypothesis that the data are iid, these slopes should equal the respective embedding dimension *m* at which the statistic is calculated (i.e. $\beta_m = m$).[7] However, slope coefficient *estimates* $\beta_m$ are smaller than respective embedding dimension *m*, i.e. $\beta_m \leq m$. To show this we assumed in (1) that under the null

[5] The plot of slopes serves only for a purpose of illustration. Such a representation was used in a conditional variance analysis of exchange rates in Kočenda (1996). We agree with one of the referees that the i*dea of measuring this slope* is shown in the article by Brock (1986) where it is stated that "Natural scientists construct Grassberger-Procaccia dimension plots of $lnC_m(\varepsilon)$ against $ln(\varepsilon)$ and attempt to measure the slopes $\hat{\alpha}_m$ of these G-P plots for each embedding dimension *m*." This idea also occurs in Brock (1988) which is cited in the first section of the paper. However, to our knowledge no one has ever used the estimated coefficients to base a statistical test on them.

[6] As $\beta_m$ is, in fact, an OLS estimate of the slope coefficient, by econometric tradition it should be labeled as $\hat{\beta}_m$. For the sake of notational simplicity, we decided to omit the hat.

[7] See Hsieh (1991).

hypothesis the series $x_t$ is randomly generated according to density function $f$ . Then for small $\varepsilon$

$$\frac{P(|x_t - x_s| < \varepsilon)}{\varepsilon} = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} f(x_t) \int_{(x_t-\varepsilon)}^{(x_t+\varepsilon)} f(x_s)\, \mathrm{d}x_s\ \mathrm{d}x_t \cong 2 \int f(x_t)^2\ \mathrm{d}x_t \equiv k\ . \tag{5}$$

By the virtue of equation (1) for sufficiently small $\varepsilon$ it holds that

$$E(C_{m,T}(\varepsilon)) = 2 \sum_{t=1}^{T_m-1} \sum_{s=t+1}^{T_m} P\left( \max_{i=0,\ldots,m-1} |x_{t+i} - x_{s+i}| < \varepsilon \right) \Big/ (T_m (T_m - 1)) \cong 2(\varepsilon k)^m . \tag{6}$$

It follows that

$$\lim_{\substack{T\to\infty \\ \varepsilon\to 0}} \frac{\ln E(C_{m,T}(\varepsilon))}{\ln(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{m \ln(\varepsilon k)}{\ln(\varepsilon)} = m\ . \tag{7}$$

Besides

$$\lim_{\varepsilon\to 0} \frac{\ln P(|x_t - x_s| < \varepsilon)}{\ln(\varepsilon)} = 1. \tag{8}$$

By Jensen's inequality

$$E(\ln(C_{m,T}(\varepsilon))) \le \ln(E(C_{m,T}(\varepsilon))). \tag{9}$$

Combining previous results, namely (7), (8), and (9), it follows that for *large* $T$ and *small* $\varepsilon$

$$E(\ln(C_{m,T}(\varepsilon))) \le \ln(E(C_{m,T}(\varepsilon))) \cong \ln P(|x_t - x_s| < \varepsilon) \cong m \ln(k) + m \ln(\varepsilon)\ .^{8} \tag{10}$$

As in the regression

$$\ln(C_{m,T}(\varepsilon)) = \alpha + \beta \ln(\varepsilon) + e\ , \tag{11}$$

the left hand variable has a negative bias from

$$m \ln(k) + m \ln(\varepsilon)\ . \tag{12}$$

Therefore, the smaller $\varepsilon$, the smaller the bias, so that the estimated coefficient satisfies

$$E[\beta] \le m. \tag{13}$$

This completes the explanation of why the slope coefficient estimates $\beta_m$ are smaller than the respective embedding dimension $m$.

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[8] Equality holds in linear cases.

If the data are identically and independently distributed, then the slope coefficients $\beta_m$ must stay within certain confidence intervals. Therefore, in order to derive the statistical properties of this test, a Monte Carlo study with 10,000 replications of the distribution of these slopes under the null hypothesis is performed.[9] In order to obtain the "whitest" white noise observations, a compound random number generator was employed. It is based on the idea of Collings (1987) and constructed from 17 generators described by Fishman and Moore (1982). This method was chosen for two reasons. First, a compound random number generator effectively eliminates repetitiveness in the data caused by the limitations of computer hardware. Secondly, other methods such as obtaining hypothetically white noise residuals by estimating a generating process (i.e. AR, ARCH, GARCH, etc.) may possess some unaccounted for structural form which would bias the critical values in a Monte Carlo simulation.[10]

The simulations generated groups of iid samples containing 500, 1000, and 2500 observations distributed normally with a zero mean and unit variance. Each sample was exposed to the computational procedure of the correlation integral allowing for nine embedding dimensions $m$(2-10) and 41 tolerance distances $\varepsilon$ ranging over the interval $\langle 0.25\sigma, 1.0\sigma \rangle$ in equal increments. Then, slope coefficient estimates of $\beta_m$ were calculated according to equation (4).

To obtain the most accurate slope coefficient estimates of $\beta_m$ of the constant slope portions of the trajectories, a cut-off point was set to eliminate the erratic portion of the trajectories at the highest embedding dimensions, $m$(7-10).[11] The cut-off point represents the number of matches that maximizes the power of the test or, implicitly, minimizes error of the second kind. By simulation it was found that such a number lies in the interval between 40 to 50. To be on the safe side, the value of the correlation integral was constrained to be 50.[12] Such a cut-off point does not affect the analysis for lower embedding dimensions $m$, but considerably reduces the increasing variance as embedding dimension $m$ grows larger and tolerance distance $\varepsilon$ becomes smaller.

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[9] Monte Carlo simulations are used instead of offering a distribution theory because the test is non-parametric.

[10] The issues of how the asymptotic distribution of the test statistics might be affected by the estimation process is discussed by de Lima (1998).

[11] The main problem is that as $m$ increases, fewer and fewer non-overlapping $m$-histories will be available. This means that for samples of moderate size only a low-dimensional chaos will be characterized. Deviation of critical values is thus greater for small data size. This is similar also in case of the BDS.

[12] The "cut-off" value for $C_m(\varepsilon)$ must be chosen before slope coefficient estimates are computed. $C_m(\varepsilon) = 50$ resulted from simulations that were compared with various trajectories (see Figure I) resulting from the analysis conducted on different time series.

Finally, quantiles for the slope coefficient estimates $\beta_m$ at different dimensional levels were tabulated.[13] Table I presents the quantiles to allow a hypothesis testing at levels of 1, 2, 5, and 10 percents for a time series of 500 observations. Tables II and III present the quantiles for a time series of the length 1000 and 2500 observations, respectively. Let $L_\alpha$ and $U_\alpha$ be lower and upper bounds of the (100 - $\alpha$) percentage confidence interval. If $(x < L_\alpha) \vee (x > U_\alpha)$, then the null hypothesis of iid can be rejected at the $\alpha$ percent confidence level.

TABLE I

Quantiles of the Slope Coefficients $\beta_m$
for a Sample Size of 500 Observations

| Quantile | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.5% | 1.833 | 2.721 | 3.552 | 4.262 | 4.872 | 5.381 | 5.755 | 5.810 | 3.643 |
| 1.0% | 1.839 | 2.731 | 3.570 | 4.293 | 4.934 | 5.510 | 5.881 | 6.020 | 4.561 |
| 2.5% | 1.848 | 2.751 | 3.603 | 4.341 | 5.020 | 5.614 | 6.055 | 6.343 | 5.493 |
| 5.0% | 1.858 | 2.768 | 3.632 | 4.395 | 5.096 | 5.719 | 6.224 | 6.573 | 6.163 |
| | | | | | | | | | |
| 95.0% | 1.934 | 2.928 | 3.955 | 4.907 | 5.876 | 6.915 | 8.007 | 9.418 | 11.120 |
| 97.5% | 1.940 | 2.944 | 3.983 | 4.954 | 5.955 | 7.043 | 8.215 | 9.903 | 12.009 |
| 99.0% | 1.946 | 2.958 | 4.015 | 5.010 | 6.042 | 7.179 | 8.514 | 10.465 | 13.131 |
| 99.5% | 1.950 | 2.974 | 4.035 | 5.056 | 6.129 | 7.296 | 8.750 | 10.990 | 14.238 |

"m" denotes an embedding dimension. Based on 10,000 replications.

TABLE II

Quantiles of the Slope Coefficients $\beta_m$
for a Sample Size of 1,000 Observations

| Quantile | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.5% | 1.859 | 2.773 | 3.671 | 4.465 | 5.212 | 5.801 | 6.348 | 6.670 | 6.461 |
| 1.0% | 1.865 | 2.787 | 3.687 | 4.481 | 5.247 | 5.844 | 6.444 | 6.779 | 6.908 |
| 2.5% | 1.874 | 2.802 | 3.702 | 4.533 | 5.295 | 5.958 | 6.592 | 6.998 | 7.266 |
| 5.0% | 1.882 | 2.815 | 3.720 | 4.567 | 5.345 | 6.051 | 6.686 | 7.230 | 7.547 |
| | | | | | | | | | |
| 95.0% | 1.935 | 2.914 | 3.931 | 4.927 | 5.885 | 6.868 | 7.883 | 8.986 | 10.335 |
| 97.5% | 1.940 | 2.923 | 3.959 | 4.970 | 5.951 | 6.950 | 8.027 | 9.162 | 10.761 |
| 99.0% | 1.945 | 2.938 | 3.981 | 5.005 | 6.027 | 7.034 | 8.313 | 9.465 | 11.363 |
| 99.5% | 1.947 | 2.947 | 4.017 | 5.045 | 6.076 | 7.108 | 8.448 | 9.642 | 11.979 |

"m" denotes an embedding dimension. Based on 10,000 replications.

[13] The "slope test" does not *simultaneously* test that (theoretically) $\beta_1 = 1$ and $\beta_2 = 2$ ... and $\beta_m = m$ and so it too has a problem when some of the slopes are in the right range, and some are not.

TABLE III

Quantiles of the Slope Coefficients $\beta_m$
for a Sample Size of 2,500 Observations

| Quantile | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.5% | 0.980 | 2.783 | 3.707 | 4.597 | 5.425 | 6.177 | 6.883 | 7.568 | 8.122 |
| 1.0% | 1.103 | 2.789 | 3.718 | 4.616 | 5.447 | 6.211 | 6.927 | 7.635 | 8.199 |
| 2.5% | 1.278 | 2.802 | 3.731 | 4.642 | 5.479 | 6.263 | 7.015 | 7.718 | 8.322 |
| 5.0% | 1.499 | 2.811 | 3.743 | 4.658 | 5.518 | 6.298 | 7.067 | 7.777 | 8.433 |
| | | | | | | | | | |
| 95.0% | 1.920 | 2.884 | 3.855 | 4.858 | 5.822 | 6.757 | 7.688 | 8.618 | 9.572 |
| 97.5% | 1.923 | 2.888 | 3.864 | 4.886 | 5.857 | 6.795 | 7.743 | 8.702 | 9.692 |
| 99.0% | 1.927 | 2.894 | 3.876 | 4.908 | 5.893 | 6.863 | 7.805 | 8.781 | 9.865 |
| 99.5% | 1.928 | 2.897 | 3.886 | 4.931 | 5.923 | 6.891 | 7.855 | 8.842 | 10.006 |

"m" denotes an embedding dimension. Based on 10,000 replications.

## 4. Power Tests and Empirical Comparison

### 4.1 Power Tests

We applied both the BDS and our alternative to the artificially generated nonlinear data resulting from the processes described below. This eliminates the problem of removing linear structure by taking residuals from a fitted linear model. We performed a power test for both tests to judge their performance at 5% significance level (and thus fixed probability of the „first-type" error). When the test is applied to the nonlinear series, the relative number of acceptances of null hypothesis at the given significance level corresponds to the probability that the test is subject to the „second-type" error—accepting null hypothesis when it is not true. The test that has smaller probability of the „second-type" error (probability of the „first-type" error being fixed) is regarded as having the greater power.

The first model used is the nonlinear moving average (NLMA) in the following form:

$$x_t = .5\varepsilon_{t-1}\varepsilon_{t-2} + \varepsilon_t \,. \tag{14}$$

The $\varepsilon_t$ terms are iid normal. The second model is the ARCH model of Engle (1982), which can be represented in the following form:

$$x_t \sim N(0, h_t)\,,\; h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i x_{t-i}^2 \tag{15}$$

where in this case $q = 1$, $\alpha_0 = 1$, and $\alpha_1 = 0.5$.[14]

Table IV shows the power of the test against specific models for lengths of 500, 1000, and

[14] Both processes (models and values of parameters) correspond (due to replication exactness) to those used

2500 observations. The numbers represent the frequency of rejection at the 5% confidence level. Derivation of critical values is described in the previous section. The power of the test against specified models is comparable to the power of the BDS statistic shown in Hsieh and LeBaron (1988) and Brock, Dechert, Scheinkman, and LeBaron (1996). However, due to the characteristics of the correlation integral, the power generally declines at the highest levels of embedding dimension.

TABLE IV

Power Test of Slope Coefficients $\beta_m$
Null of iid Rejected at 5% Level

| Process | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| NLMA, 500 obs. | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 0.99 | 0.50 |
| NLMA, 1000 obs | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| NLMA, 2500 obs. | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| | | | | | | | | | |
| ARCH, 500 obs. | 0.99 | 0.97 | 0.84 | 0.68 | 0.65 | 0.56 | 0.42 | 0.19 | 0.01 |
| ARCH, 1000 obs. | 1.00 | 0.99 | 0.96 | 0.87 | 0.83 | 0.73 | 0.64 | 0.38 | 0.17 |
| ARCH, 2500 obs. | 0.39 | 0.71 | 0.73 | 0.72 | 0.58 | 0.51 | 0.50 | 0.49 | 0.35 |

"m" denotes an embedding dimension.

Series that exhibit zero autocorrelation structure, as the above models do, are evidently atypical. The following examples of empirical comparisons suggest the usefulness and added value of the proposed testing method.

### 4.2 Empirical Comparison

High frequency financial data fully reflect the stylized fact of changing variance over time. Numerous financial time series were studied and found to contain linear as well as non-linear dependencies. An appropriate model that would account for conditional heteroskedasticity in such time series should be able to remove possible nonlinear patterns in the data. Standardized (fitted) or corrected residuals from such a model are ideal for the BDS test, as well as for the suggested alternative method, because they should be independent under the null model. Thus, the test is not only of non-linearity but also of correct specification.

Three empirical studies were replicated in order to yield comparisons between the two tests. For clarity, notation throughout this section is identical to that used in the original studies. Results show that the suggested alternative is able to detect remaining non-linear dependencies in standardized (fitted) residuals where the BDS test does not.

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in Brock, Dechert, Scheinkman, and LeBaron (1996).

### 4.2.1 Analysis of ARCH corrected weekly exchange rates

Kugler and Lenz (1990) analyzed non-linear dependence of weekly exchange rate changes for four currencies against the US dollar from 1979 to 1989 (575 observations, the rate of change of the log exchange rate $x_t = \Delta \log S_t$). The data were corrected to account for the present ARCH process by transformation into the ARCH corrected rate of changes in the form

$$\Delta \log S_t^h = \Delta \log S_t \Bigg/ \left( \hat{\alpha}_0 + \sum_{\tau=1}^{6} \hat{\alpha}_\tau \Delta \log S_{t-\tau}^2 \right)^{0.5} \tag{16}$$

where $\alpha$-coefficients were obtained by OLS regression of $(\Delta \log S_t)^2$ on constant and six lagged variables. Such ARCH corrected rates of changes were subjected to the BDS test using embedding dimensions $N$ = 2,3,4, and 5, and tolerance distances $\varepsilon$ = 0.5, 0.75, 1.0, and 1.5 of the standard deviation of the sample. Kugler and Lenz (1990) found that the described correction successfully removed nonlinearity from the Swiss Franc and Deutsche Mark. However, the BDS test did not allow rejection of the null hypothesis for the French Franc (specifically at levels of $N$ = 4 and 5) and Japanese Yen (specifically at levels of N = 3,4, and 5).

We have replicated the original study with the same results and applied the corrected rates to the alternative test. The results are presented in Table V. As in the original study, the null hypothesis is rejected for the French Franc and Japanese Yen. Contrary to the original analysis, however, the alternative test finds remaining non-linear dependency in the residuals of the Deutsche Mark. The Swiss Franc is the only currency where the null of iid cannot be rejected.

TABLE V

Slope Coefficients $\beta_m$ of
ARCH(6) Corrected Changes in Exchange Rates

| Currency | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| DMUS | 1.850[d] | 2.736[c] | 3.593[c] | 4.328[c] | 5.126 | 5.900 | 6.480 | 6.474[d] | 7.255 |
| FFUS | 1.865 | 2.806 | 3.652 | 4.259[a] | 4.913[b] | 5.701[d] | 6.401 | 6.720 | 7.626 |
| SFUS | 1.876 | 2.811 | 3.807 | 4.809 | 5.748 | 6.533[c] | 7.529 | 7.583 | 8.527 |
| YNUS | 1.818[a] | 2.705[a] | 3.583[c] | 4.481 | 4.990[c] | 5.479[b] | 5.987[c] | 6.783 | 7.566 |

"m" denotes an embedding dimension. Superscripts denote significance at levels of :
(a) 1%, (b) 2%, (c) 5%, and (d) 10%.

### 4.2.2 Analysis of daily exchange rates

Brock, Hsieh, and LeBaron (1991, p.130) analyzed the daily closing bids for the five major currencies in U.S. Dollars: Swiss Franc (SF), Canadian Dollar (CD), Deutsche Mark (DM), British Pound (BP), and Japanese Yen (JY) during the period from January 2, 1974 to December 30, 1983 (2,510 observations).[15]

Rates of change are calculated by taking the first logarithmic differences between successive trading days. The data were prefiltered by an AR process with daily dummies to remove linear dependency. In order to capture variance-nonlinearity, a GARCH model was estimated. The specification of the model resulted in the mean equation:

$$r_t = \beta_0 + \sum_{i=1}^{j} \beta_i r_{i-1} + \beta_M D_{M,t} + \beta_T D_{T,t} + \beta_W D_{W,t} + \beta_R D_{R,t} + \beta_H D_H + u_t \tag{17}$$

where, $u_t \left| \Omega_{t-1} \sim D(0, h_t), \right.$ and variance equation

$$h_t = \phi_0 + \psi u_{t-1}^2 + \phi h_{t-1} + \phi_M D_{M,t} + \phi_T D_{T,t} + \phi_W D_{W,t} + \phi_R D_{R,t} + \phi_H D_H \tag{18}$$

where $r_t$ is the rate of change of the nominal exchange rate at time *t*, $D_{M,t}$, $D_{T,t}$, $D_{W,t}$, and $D_{R,t}$, are dummy variables for Monday, Tuesday, Wednesday, and Thursday; and $D_H$ is the number of holidays between two successive trading days excluding week-ends. Daily dummies were included to capture the daily effects of fluctuations that are known to materialize in correlation at financial markets and thus might affect the analysis. The order of the AR process was determined to be *j* = 6, 5, 6, and 0 respectively for SF, CD, DM, and BP.

After estimation, the overall fit of the model is assessed by performing diagnostic tests on standardized residuals $z_t = u_t / h_t^{\frac{1}{2}}$, where $u_t$ is the residual of the mean equation (17), and $h_t$ is the estimated conditional variance from equation (18). The BDS test finds no evidence of nonlinearity in standardized residuals of SF, some nonlinearity (at dimensions 8, 9, and 10) for the DM, and strong nonlinearity for CD and BP.

The findings of Brock, Hsieh, and LeBaron (1991, pp. 140 and 155) were replicated.[16] The standardized residuals were then subjected to the alternative test. The slope coefficients derived from this test are presented in the Table VI. DM and BP show the presence of nonlinearity at the 1% significance level no matter what embedding dimension is considered. CD and SF show some presence of nonlinearity at various significance levels depending on embedding dimension *m*. The

[15] Japanese Yen was dropped from the replication because of data inconsistency.

[16] There is a descriptive error in Brock, Hsieh, and LeBaron (1991) on this subject. On p. 140 it is claimed that the BDS test finds no evidence of nonlinearity in CD. However, on p. 155 the table shows that statistics for CD are significant at the 1% level, thus, supporting evidence of nonlinearity in standardized residuals of this currency.

alternative test confirmed the presence of nonlinearity in DM, CD, and BP and, contrary to original study, detected remaining nonlinearity in the supposedly independent residuals of SF.

TABLE VI

Slope Coefficients $\beta_m$ of
Standardized Residuals from AR(p)-GARCH(1,1) Model

| Currency | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| CD | 1.875 | 2.808[d] | 3.718[b] | 4.595[a] | 5.491[d] | 6.405 | 7.193 | 8.016 | 8.568 |
| SF | 1.861 | 2.784[b] | 3.707[a] | 4.631[c] | 5.488[d] | 6.205[b] | 6.835[a] | 7.308[a] | 8.171[b] |
| DM | 1.855 | 2.774[a] | 3.702[a] | 4.557[a] | 5.289[a] | 5.918[a] | 6.383[a] | 6.756[a] | 7.140[a] |
| BP | 0.998[b] | 2.475[a] | 3.140[a] | 3.661[a] | 4.010[a] | 4.328[a] | 4.629[a] | 4.935[a] | 4.952[a] |

"m" denotes an embedding dimension. Superscripts denote significance at levels of :
(a) 1%, (b) 2%, (c) 5%, and (d) 10%.

### 4.2.3 Analysis of weekly exchange rates

Kugler and Lenz (1993) analyzed the non-linear dependence of exchange rate changes for ten currencies against the US dollar. They used weekly end of period data of the Australian dollar (ADUS), Canadian dollar (CDUS), Belgian Franc (BFUS), French Franc (FFUS), Deutsche Mark (DMUS), Dutch Guilder (HFUS), Italian Lira (LTUS), Spanish Peseta (PTUS), Swiss Franc (SFUS), and Japanese Yen (YNUS). The sample period is from 1979 to 1989 (575 observations, the rate of change of the log exchange rate $x_t = \Delta \log S_t$). The LM test performed on the rates of change clearly indicates the presence of ARCH process, and the BDS test decisively rejects the null of iid. In order to check whether the detected dependence can be attributed solely to an ARCH process, the authors estimated the following GARCH-M model

$$\Delta \log S_t = \beta_0 + \sum_{\tau=1}^{3} \beta_\tau \Delta \log S_{t-\tau} + \beta_4 \sqrt{h_t} + \eta_t \tag{19}$$

$$h_t = \alpha_0 + \alpha_1 \eta_{t-1}^2 + \alpha_2 h_{t-1} \qquad \eta_t = \varepsilon_t \sqrt{h_t}\ .$$

In equation (19) linear dependencies of the AR type are allowed for, as the estimated pure GARCH-M model showed signs of residual autocorrelation for some currencies. For all currencies the GARCH coefficients $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are highly significant from zero. Thus, ARCH effects are important for all currencies. Finally, the fitted residuals $\hat{\varepsilon}_t = \eta_t / \sqrt{h_t}$ were subjected to the BDS test (tolerance distance of one standard deviation and embedding dimensions $N$ = 2, 3, 4, and 5 were used). Results revealed no indication of dependence in the fitted residuals of any currency.

We have replicated the study with the same results. The fitted residuals $\hat{\varepsilon}_t$ were then subjected to the alternative test. The results, which are presented in Table VII, confirmed the original findings of independence for only 5 of the 10 currencies (CDUS, FBUS, FFUS, HFUS, and SFUS). Contrary to the original analysis the alternative test detected remaining non-linear dependencies in the fitted residuals for the rest of the supposedly independent currencies (ADUS, DMUS, LTUS, PTUS, and YNUS).

TABLE VII

Slope Coefficients $\beta_m$ of
Fitted Residuals from AR(3)-GARCH(1,1)-M Model

| Currency | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| ADUS | 1.751[a] | 2.630[a] | 3.498[a] | 4.385[d] | 5.001[c] | 5.726 | 6.462 | 6.848 | 7.470 |
| CDUS | 1.843[c] | 2.772 | 3.725 | 4.614 | 5.270 | 5.799 | 6.472 | 6.947 | 7.214 |
| DMUS | 1.860 | 2.739[c] | 3.557[b] | 4.368[d] | 5.188 | 5.896 | 6.606 | 7.272 | 8.247 |
| FBUS | 1.839[b] | 2.783 | 3.764 | 4.534 | 5.304 | 5.751 | 6.416 | 7.263 | 7.988 |
| FFUS | 1.874 | 2.825 | 3.643 | 4.335[c] | 5.124 | 5.992 | 6.833 | 7.258 | 8.278 |
| HFUS | 1.876 | 2.826 | 3.682 | 4.615 | 5.417 | 6.099 | 6.691 | 7.258 | 7.267 |
| LTUS | 1.860 | 2.772 | 3.563[b] | 4.242[a] | 4.912[b] | 5.441[b] | 5.755[a] | 6.191[c] | 7.325 |
| PTUS | 1.769[a] | 2.524[a] | 3.064[a] | 3.323[c] | 3.498[a] | 3.901[a] | 4.102[a] | 4.451[a] | 4.845[a] |
| SFUS | 1.877 | 2.832 | 3.850 | 4.703 | 5.471 | 6.216 | 7.239 | 8.124 | 9.107 |
| YNUS | 1.844[c] | 2.741[c] | 3.709 | 4.535 | 5.314 | 5.700 | 6.307[d] | 7.414 | 8.170 |

"m" denotes an embedding dimension. Superscripts denote significance at levels of :
(a) 1%, (b) 2%, (c) 5%, and (d) 10%.

**5. Conclusion**

This paper has presented a new method of testing for iid. The method originates in chaos theory and is based on the concept of the correlation integral. The test is suggested as an alternative to the widely used nonparametric BDS test. The paper extends and generalizes the BDS test presented by Brock, Dechert, Scheinkman, and LeBaron (1996). In doing so it aims to remove a limitation of arbitrary selection of a proximity parameter $\varepsilon$ through integrating across correlation integral. The alternative statistic is developed by calculating the slope of the log of the correlation integral versus the log of the proximity parameter over a broad range of values of the proximity parameter for different embedding dimensions. Monte Carlo simulations are used to tabulate critical values of the slope coefficients $\beta_m$ at different significance levels.

The power of the new method is tested against artificial nonlinear data. In addition, three previously published empirical studies (that used the BDS test) are replicated in order to evaluate the relative performance of the suggested alternative to the BDS test. The proposed test is applied to standardized (corrected) residuals from different models and finds nonlinear dependencies in cases where the published results using the BDS test did not find them.

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