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Were Cobb and Douglas Prejudiced? A Critical Re-analysis of their 1928 Production Model Identification

Abstract

In 1928 Cobb and Douglas (C&D) presented a system analysis which established the first empirically identified production model, which forms the foundation for Solow's growth theory and research into productivity growth factors, such as "technological progress " and "human capital development ". C&D claimed that their production model ("function") showed neutral economies of scale, *i.e.*, constant returns to scale, with a labor production elasticity of 3/4 and a capital production elasticity of 1/4. A simple CLS analysis shows that C&D's data were incorrectly identified by an (n,q)=(3,1) linear model. C&Ds claim that their neutral "constant returns of scale " was the inevitable scientific conclusion of their analysis was also incorrect, since that conclusion is strictly determined by their subjectively chosen projection direction. In fact, the data shows that with their model and identification technology constant, increasing and diminishing returns to scale are all three compatible with the uncertain data. Their (n,q) = (3,1) model was never identified with an acceptable level of scientific accuracy, with a maximum coefficient value variation of 212%). In contrast, a simple two-equation (n,q) = (3,2) system model can be accurately identified from C&Ds data set, with an acceptable level of accuracy, with a maximum coefficient value variation of 7.4%).

Acknowledgement 1 *With thanks to my academic mentor Prof. Em. Rudolf E. Kalman, who in 1985 showed me the "via luminis."*

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1 Introduction

In 1928 Cobb and Douglas (C&D) presented a system analysis of some macro-economic data they had collected, which established the first empirically identified production model (Cobb & Douglas, 1928). C&D called their resulting empirical equation a production *function*, although they did not establish that fact. Their single equation model was after all only a statistical *relation* between three variables: output, labor and capital.

The importance of C&D's path-breaking economic reseach is undeniable and well-recognized. Their economic production model formed the foundation for Robert Solow's neo-classical growth theory (Solow, 1956, 1970), in which he used an aggregate production function. His growth theory, which became part of the "mainstream" of economics of the post-war period launched the Cambridge Capital Controversy, which pitted Solow and Samuelson against Joan Robinson and the Cambridge Keynesians, but earned him a Nobel Memorial Prize in 1987.

Solow's growth theory and his co-authored constant elasticity of substitution (CES) production function (Arrow *et al.*, 1961) also formed the basis for the follow-up research into productivity growth factors, such as "technological progress" and "human capital development," which led next to similar production modeling and later to massive efforts led by the Wold Bank in the 1960s and 1970s, to invest in lage-scale physical and social infrastucture and education projects in developing countries in Latin-Ameica, Africa and Asia. But was such a massive theoretical superstucture and following policy development justified by the rather shaky "scientific" foundation? This paper re-examines the empirical evidence of C&D's 1928 data set with *Complete* Least Squares (CLS) analysis, and comes to the conclusion that the answer to this question has to be: "No!"

This paper is organized as follows. First, we formulate the correct *and* the incorrect claims of Cobb and Douglas (1928) (C&D). In the following section we provide the evidence by computing the logarithms of the original data set of C&D and by completely re-analyzing the resulting information matrix. This enables us to identify the specific scientific prejudices of C&D's empirical economic analysis. In the third section we demonstrate that constant, increasing, and diminishing returns to scale are all three compatible with the very uncertain data, depending on the chosen projection directions. and that the choice of the constant retuns of scale was a subjective choice. We also find out why C&D concluded their labor and capital elasticities to be 3/4 and 1/4, respectively. Next, we accurately identify the correct $(n, q) = (3, 2)$ simultaneous equation model from the CLS projections, both algebaically and geometically. This re-analysis is strictly algebraic-geometric and not based on any statistical (limit or probability) theory. The only statististics we use are the data covaiance matrix and some measurements of coefficient (range) accuracy. Finally, we summarize our conclusions and discuss some of the consequences of this re-analysis.

2 Cobb and Douglas (1928) Correct and Incorrect Claims

Cobb and Douglas (C&D) claimed that their production model showed neutral economies of scale, *i.e.*, constant returns to scale, with a production elasticity of labor of about 3/4 and a production elasticity of capital of about 1/4 (Cobb and Douglas, 1928, p.). A simple Complete Least Squares (CLS) analysis (Kalman, 1980, 1982a & b; Los, 1989a &b, 1991, 1999, 2001) shows that Cobb and Douglas were incorrect in asserting that their logarithmically transformed data were to be described by an $(n, q) = (3, 1)$ linear model, *i.e.*, by an $n = 3$ variable model consisting of only $q = 1$ linear relation, since it is to be described by an $(n, q) = (3, 2)$ linear model, *i.e.*, by a 3-variable model consisting of $q = 2$ independent linear relations.

Moreover C&D were incorrect in claiming that their neutral "constant returns of scale " was the only inevitable scientific conclusion from their set of data, given their own $(n, q) = (3, 1)$ model. Their numerical conclusion. regarding the production elasticities of the labor and capital inputs, was strictly dependent on their single equation model and their subjectively chosen projection direction. In fact, the data show that given their single equation model constant, increasing and diminishing returns to scale are all three compatible with the uncertain data, depending on which projection direction is chosen and that linear combinations these elasticities are much too uncertain for a definite scientific identification.

Indeed, scores of subsequent empirical researchers have found their own $(n, q) = (3, 1)$ production model "regression "coefficients to be highly "unstable," because of the model misspecification, in addition to the possible non - stationarity and non - homogeneity of their own data sets.

But more impotantly, we will see that C&D's own data tells us that their model was misspecified, since the covariance structure of thei noisy data set algebraically and geometically dictates that it must be identified by a simultaneous two-equation $(n, q) = (3, 2)$ model

3 Cobb and Douglas' Information Matrix Reanalyzed

We compute in an EXCEL spreadsheet first the logarithmic transformations of Cobb-Douglas' original data set of $T = 24$ observations on $n = 3$ variables in Table 1. The data are taken from the original data set published in the C&D (1928) paper, from the Capital Table II on their page.145, from the Labor Table III on their page 148, and from the Production Output Table IV on their page 149.

[FIGURE 1 ABOUT HERE]

After this simple logarithmic transformation, we compute the two $(24x3)$ matrices of deviations from the means of these transformed data in Table 2. We check that the means of these deviations from the means equal, indeed, zero:

[FIGURE 2 ABOUT HERE]

Next, we first transpose the (24×3) matrix of deviations from the means of the logarithmic data $\mathbf{x}$. We show only the first seven of the 24 columns of the (3×24) matrices $\mathbf{x}\prime$:

[FIGURE 3 ABOUT HERE]

Then the data covariance matrices $\mathbf{\Sigma} = \mathbf{x}'\mathbf{x}$ are easily computed by applying the matrix function MMULT in EXCEL to matrix-multiply the transposed (3×24) matrix $\mathbf{x}\prime$ with the

(24×3) matrix $\mathbf{x}$:[1]

[FIGURE 4 ABOUT HERE]

Similarly, we compute the information matrix $\mathbf{\Sigma}^{-1}$ by using the EXCEL matrix function MINVERSE:

[FIGURE 5 ABOUT HERE]

Since each row of the information matrix $\mathbf{\Sigma}^{-1}$ represents a $3-$variable single equation regression, or orthogonal projection – for the first row the projection of variable x_1 on $x_2[= \ln(Labor)]$ and $x_3[= \ln(Capital]$, for the second row the projection of x_2 on x_1 and x_3, and, for the third row the projection of x_3 on x_1 and x_2 – we can now unambiguously conclude that one of the scientific prejudices of Cobb and Douglas (C&D) empirical economic analysis consists of publishing only the first regression Gassmanian projection coefficients. C&D's scientific report was incomplete, since they did not report the slope coefficients of the second and third (= so-called "reverse") regression projections.

To focus the mind a bit in the conventional (logarithmic) linear system way: the linear system investigated by C&D has the form:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \tag{1}$$

which C&D normalized on x_1 to make it equivalent to

$$\begin{aligned} x_1 + (a_2/a_1)x2 + (a_3/a_1)x3 &= \\ x1 - b.x2 - c.x3 &= 0 \end{aligned} \tag{2}$$

where the coefficients $b = -(a_2/a_1)$ and $c = -(a_3/a_1)$, or, what is the same as in their paper:

$$x_1 = bx_2 + cx_3 \tag{3}$$

[1] We do not need to normalize on either the number of obsevations $T = 24$, or on the degrees of freedom $T - 1 = 23$, since these nomalizations cancel out later in the projection analysis. This shows that the number of obsevations in a data set has nothing to do with the quality of the projection analysis of the covariance structure of the data, as statisticians erroneously claim. It may have something to do, however, with the stability or invariance of the computed covaiance matix $\mathbf{\Sigma}$.

All three $(n, q) = (3, 1)$ LS projection results are summarized in Fig. 6, from which we can easily assess the Grassmanian coefficient uncertainty. The uncertainty of these projection coeffcients is maximally 93.0% from the mean b coefficient value and 211.9% from the mean c coefficient for this presumed $(n, q) = (3, 1)$ C&D model. This effectively implies that the C&D presumed $(n, q) = (3, 1)$ model was never scientifically identified.[2]

[FIGURE 6 ABOUT HERE]

Let's now look at the same issue, not from a pure mathematical or a scientific accuracy point of view, but from C&D's economic point of view. C&D's empirical result, based on their own logarithmic data, is in the first row of their normalized information matrix. Their so-called "elasticity of labor" coefficient $b = -(a_2/a_1) = 0.80728$ and their so-called "elasticity of labor" coefficient $c = -(a_3/a_1) = 0.23305$. These two elasticity coefficients add up to 1.04033, which is almost equal to 100%, suggesting that industrial production shows *constant* returns to scale. CD rounded this empirical result off to $b + c = 0.75 + 0.25 = 1.00$. Since in their pure competition production theory these elasticities are equivalent to the allocation shares of labor and capital in total revenues, this allocation of revenues of $\frac{3}{4}$ for labor and $\frac{1}{4}$ for capital became almost a statement of absolute religion for the labor unions in the 1930s and later years.

The second set of $(3, 1)$ LS projected Grassmanian coeffcients, on the second row of their infomation matix, from the projection of x_2 on x_1 and x_3, shows that the "elasticity of capital" $b = -(a_2/a_1) = 1.35478$ and the "elasticity of capital" $c = -(a_3/a_1) = 0.01498$. These coefficient add up to $b+c = 1.36976$, suggesting that industrial production shows *increasing* returns to scale.

The third "regresion" result, from the orthogonal projection of x_3 on x_1 and x_2, shows that the "elasticity of capital" is only $b = -(a_2/a_1) = 0.05189$ and the "elasticity of labor" $c = -(a_3/a_1) = 0.59673$. These coefficients add up to $b+c = 0.64862$, suggesting that industrial production shows *decreasing* returns to scale.

[2] Like C&D, I have normalized these other projections also on the first variable x_1 to make a rational comparison possible with C&D's 1928 esults, but altenative normalizations are, of course, all allowed and are immateial for the esults of the re-analysis, since they produce the same conclusion. These alternative projections within the infomation are indicated by arrows in Fig. 6.

These very different conclusions of constant, increasing, respectively, decreasing returns to scale of industrial production, have major implications for the revenue allocations of labor and capital in the competitive theory of production, which are very different from the still widely accepted Cobb and Douglas' 1928 conclusions, which were adopted by Solow c.s.

These three projection results, based on the same data covariance matrix, suggest that all three conclusions of constant, increasing and decreasing returns to scale of the production technology are compatible with the C&D empirical data, depending on which orthogonal projection direction is adopted and given their $(n, q) = (3, 1)$ single equation model !

But the infomation in their data set was just too different from that model specification. A quite different way of looking at the problem is presented in Fig. 7, where I've plotted one isocontour (size = 125) of C&Ds covariance information, based on their own infomation matix.

[FIGURE 7 ABOUT HERE]

C&D fitted one single equation plane through this information ellipsoid, repesented by the first row of Grassmanian coefficients in the information matrix, as visualized in Fig.8.

[FIGURE 8 ABOUT HERE]

However, the two other rows of the information matrix provide the Grassmanian coefficients for two more planes, and thus there are actually three planes possible, given the C&D 1928 data, as visualized in Fig. 9.

[FIGURE 9 ABOUT HERE]

If C&D were correct with the postulation of their plane-like logarithmic-linear production relationship, these three planes should have (almost) coincided. But they clearly do not coincide, as can be seen by viewing along the max-axis of the information ellipsoid Fig 10, Fig. 10 is the same as Fig. 9, but tilted forward so that we can look along the max-axis into the information ellipsoid. Since we look now at the crossing planes from their sides, they appear as (almost) crossing lines. They cut each closely along the max-axis, but do not coincide.

[FIGURE 10 ABOUT HERE]

Moreover, the elongated ellipsoidal shape of the iso-contour information ellipsoid, together with the fact that the cross-lines of the three planes (almost) coincide with the max-axis of the information ellipsoid, make clear that the more acceptable production structure is a system of two independent equations relating production and labor, and production and capital, espectively (implying another relationship between labor and capital), as follows.

4 Complete (3,2) System Identification

The coefficients of the production system are found from two $(n, q) = (3, 2)$ CLS projections (by selecting three times $q = 2$ rows at a time from the 3×3 information matrix), as follows.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + 0.x_3 &= 0 \\
a_{21}x_1 + 0.x_2 + a_{23}x_3 &= 0
\end{aligned} \tag{4}
$$

Fig. 11 provides all three $(n, q) = (3, 2)$ LS projected Grassmanian coeffients:

[FIGURE 11 ABOUT HERE]

This gives us the simultaneous equation *system*

$$
\begin{aligned}
x_1 &= dx_2 \\
x_1 &= ex_3
\end{aligned} \tag{5}
$$

For example, from the first $(3, 2)$ LS projection, we obtain

$$
\begin{aligned}
d &= -a_{12}/a_{11} = -1/-0.71819 = 1.39238 \text{ and} \\
e &= -a_{23}/a_{21} = -1/-1.80312 = 0.55459
\end{aligned}
$$

Notice that this $(3, 2)$ system model implies also that $x_2 = (0.55459/1.39238)x3 = 0.398\,3x_3$, *i.e.*, the "labor elasticity of capital" $= 39.83\%$, or that the "capital elasticity of labor" $= 1/0.3983 = 251.06\%$, etc., which are concepts not earlier discussed in the economics literature.

But in addition there are two alternative $(3,2)$ LS projections possible. Because of the finite data uncertainty, the three possible orthogonal LS projections produce three very narrow sets of Grassmanian coefficients values.[3] Thus, we find very limited ranges of possible values for the "production elasticity of labor" d and the "production elasticity of capital" e, in the following bracketed $[.,.,]$ sets for the d and e coefficients:

$$\begin{aligned} d &= -a_{12}/a_{11} = [1.39238, 1.38833, 1.29134] \text{ and} \\ e &= -a_{23}/a_{21} = [0.55459, 0.61990, 0.62172], \text{ respectively} \end{aligned} \tag{6}$$

The limited coefficient variation is less than 4.9% from the mean for d and less than 7.4% from the mean for e. This means that this $(n,q) = (3,2)$ model is precisely and acceptably identified by CLS projection, despite the low data quality, and about 28 times more accurately than the C&D $(n,q) = (3,1)$ model.

This algebraic and geometric demonstration of a much more precise model identification from noisy data vacuates the whole subsequent theoretical debate in the economic literature about the instability of the elasticity results of C&D. No definite conclusion about the returns to scale and thus about the allocation of total revenue over the two production factors labor and capital should have been drawn by Cobb and Douglas of by Solow c.s. on the basis of their $(n,q) = (3,1)$ plane model and their published data, since their model was never identified with any acceptable level of accuracy. Moreove, their presentation of the possible coefficient results was highly selective, prejudiced and biased.

In contrast, by reanalyzing their data set with CLS projections we can scientifically identify a simple $(n,q) = (3,2)$ model with two simultaneous independent equations (together repesenting a line model), with high and acceptable levels of scientific accuracy from C&D's own noisy data set.

[3] In this case, a good first appoximation would have been provided by the coefficients of the principal components, but then there would have been no expression of the magnitude of the epistemic uncertainty in the data set, as expressed by these backeted sets of identified $(n,q) = (3,2)$ Grassmanian coefficients.

5 Conclusion

Cobb and Douglas' (1928) conclusions are unscientific, because their structural covariance analysis is *incomplete* and, thus, prejudiced, and, consequently, misleading. It provides an incomplete and biased picture of the systematic variation observable in their own empirical data covariance matrix. Their conclusion that production shows "cosntant retuns to scale" is only one of three possible conclusions obtained from their own data with their own $(n, q) = (3, 1)$ model specification and using their own analytic technology of least squares projection. The other two conclusions - increasing and deceasing returns to scale" - are also consistent with the same data and their single equatio model, using altenative LS projections in different diections than selected by Cobb and Douglas. But then their own single equation $(n, q) = (3, 1)$ model can never be identified with an acceptable level of scientific accuracy (C&Ds computed "output elasticity of capital actually by 212% from its mean), strongly suggesting that their model was misspecified.

Consequently, the subsequent superstructual growth theory, which erroneously adopts the constant retuns to scale conclusion as "proved," is built on a quick-sand empirical foundation, and all subsequent policy conclusions, which ae based on similar misspecified data-inconsistent models are similarly scientifically not well-founded and misleading.

But what is more, the covariance structure of the Cobb and Douglas (1928) data set can now be reanalyzed by Complete Least Squaes (CLS) projections and both algebraically and geometrically shown to identified from their data by a simple $(n, q) = (3, 2)$ three-variable-two-equation "line" model or *system*, instead of the single one-equation "plane" *relationship* ("function") they use. The CLS projected coefficient values vary less than 7.4% in this $(3, 2)$ model. This is a rather high and acceptable level of model identification accuracy, given C&Ds low quality, noisy data set.[4]

These logically inescapable conclusions are important for two reasons. Because of the technological advance of our analytic technology, all economic production reseach results have become

[4] We don't blame C&D for the low quality of their data set. It was one of the first and it was collected thanks to their heroic efforts to bring some scientific efforts into the economic policy-making of their time.

now questionable and the empirical reseach has to be redone in a complete and unprejeudiced fashion. All available data sets need to be re-analyzed using a complete and not a prejudiced and biased covariance analysis. Also, the subsequent growth and development theories and their implied policy conclusions have to be critically reviewed from this complete data-analytic point of view. The billions of tax dollars spent on the macro-economic development projects are too valuable and important to be wasted on prejudiced models and their resulting misguided policy conclusions.

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7 Figures

Table 1		Original Cobb & Douglas data			Logarithm of Original Data		
Year	**No#**	**output**	**labor**	**capital**	**output**	**labor**	**capital**
1899	1	100	100	100	4.6052	4.6052	4.6052
1900	2	101	105	107	4.6151	4.6540	4.6728
1901	3	112	110	114	4.7185	4.7005	4.7362
1902	4	122	118	122	4.8040	4.7707	4.8040
1903	5	124	123	131	4.8203	4.8122	4.8752
1904	6	122	116	138	4.8040	4.7536	4.9273
1905	7	143	125	149	4.9628	4.8283	5.0039
1906	8	152	133	163	5.0239	4.8903	5.0938
1907	9	151	138	176	5.0173	4.9273	5.1705
1908	10	126	121	185	4.8363	4.7958	5.2204
1909	11	155	140	198	5.0434	4.9416	5.2883
1910	12	159	144	208	5.0689	4.9698	5.3375
1911	13	153	145	216	5.0304	4.9767	5.3753
1912	14	177	152	226	5.1761	5.0239	5.4205
1913	15	184	154	236	5.2149	5.0370	5.4638
1914	16	169	149	244	5.1299	5.0039	5.4972
1915	17	189	154	266	5.2417	5.0370	5.5835
1916	18	225	182	298	5.4161	5.2040	5.6971
1917	19	227	196	335	5.4250	5.2781	5.8141
1918	20	223	200	366	5.4072	5.2983	5.9026
1919	21	218	193	387	5.3845	5.2627	5.9584
1920	22	231	193	407	5.4424	5.2627	6.0088
1921	23	179	147	417	5.1874	4.9904	6.0331
1922	24	240	161	431	5.4806	5.0814	6.0661
	Means	165.9	145.8	234.2	5.1	5.0	5.4

Figure 1: Cobb and Douglas' (1928) original set of three data series and their logarithmic transformation. Source: Charles W. Cobb and Paul H. Douglas, Theory of Production, American Economic Review, Vol. 18, No. 1 (1928) (Supplement), pp. 145, 148 and 149 (Their Tables II, III and IV, respectively).

Table 2

Year	No#	lnx1	lnx2	lnx3
1899	1	-0.4722	-0.3576	-0.7513
1900	2	-0.4622	-0.3088	-0.6837
1901	3	-0.3588	-0.2622	-0.6203
1902	4	-0.2733	-0.1920	-0.5525
1903	5	-0.2571	-0.1505	-0.4813
1904	6	-0.2733	-0.2091	-0.4292
1905	7	-0.1145	-0.1344	-0.3525
1906	8	-0.0535	-0.0724	-0.2627
1907	9	-0.0601	-0.0355	-0.1860
1908	10	-0.2411	-0.1669	-0.1361
1909	11	-0.0339	-0.0211	-0.0682
1910	12	-0.0084	0.0071	-0.0189
1911	13	-0.0469	0.0140	0.0188
1912	14	0.0988	0.0612	0.0641
1913	15	0.1376	0.0742	0.1073
1914	16	0.0526	0.0412	0.1407
1915	17	0.1644	0.0742	0.2270
1916	18	0.3388	0.2413	0.3406
1917	19	0.3476	0.3154	0.4576
1918	20	0.3298	0.3356	0.5461
1919	21	0.3072	0.3000	0.6019
1920	22	0.3651	0.3000	0.6523
1921	23	0.1101	0.0277	0.6766
1922	24	0.4033	0.1187	0.7096
	Means	0.0	0.0	0.0

Figure 2: Computation of deviations from the means of the logarithmically transformed data in Fig. 1, with a means = 0 check.

-0.472166	-0.462215	-0.358837	-0.273315	-0.257054	-0.273315	-0.114491
-0.357553	-0.308763	-0.262243	-0.192038	-0.150539	-0.209133	-0.134409
-0.751314	-0.683655	-0.620285	-0.552463	-0.481286	-0.42923	-0.352537

Figure 3: Part (first seven columns) of the transpose of the (24 x 3) deviations from the means matrix in Fig. 2.

Covariance Matrix
Logarithmic data

0.069467	0.050036	0.112061
0.050036	0.038747	0.080481
0.112061	0.080481	0.202059

Figure 4: The (3 x 3) covariance matrix of the logarithmically transformed Cobb and Douglas (1928) data of Fig. 2 (not divided by the number of variables T or by the degrees of freedom $T-1$).

Information Matrix
Logarithmic data

338.1156	-272.9534	-78.79902
-272.9534	369.7927	4.088999
-78.79902	4.088999	47.02191

Figure 5: The (3 x 3) information matix Σ^{-1} (= inverse of the covaiance matrix $\mathbf{\Sigma}$ in Fig. 4).

Normalized Information Matrix
Logarithmic data

1	**-0.807278**	**-0.233053**
1	-1.354783	-0.014981
1	-0.051891	-0.596732

CD's empirical result (in bold face)
set to -3/4=-0.75 and -1/4=-0.25, respectively,
Alternative projections

Figure 6: The iinformation matrix normalized on the first variable x_1, for easy comparison with the Cobb and Douglas (1928) results. These $(n, q) = (3, 1)$ Least Squares projection coefficients are directly obtained from the information matrix $\mathbf{\Sigma}^{-1}$ by multiplying each row by the inverse of the cell values of the first column (= normalized on variable x_1= $\ln(output)$. Notice the wide (but limited) coefficient value variation of this model.

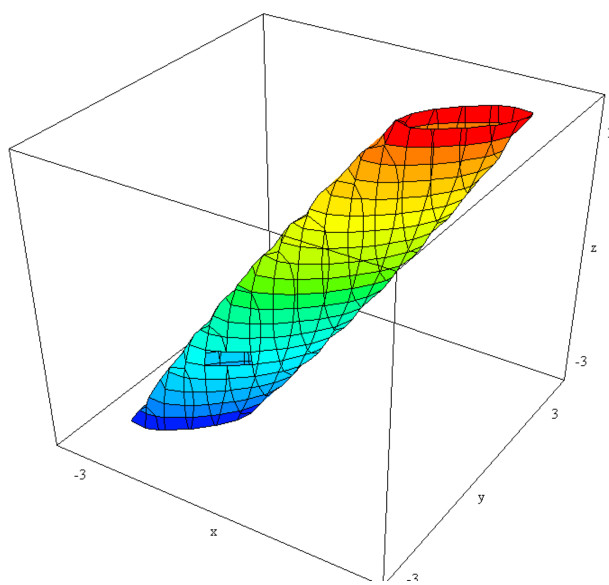

Figure 7: Information ellipsoid $x\Sigma^{-1}x = 125$, based on the 3×3 covariance matrix Σ of the logarithmic transformation of the data of Cobb and Douglas (1928). Notice its elongated and relatively flat shape, as can be seen at both its tips.

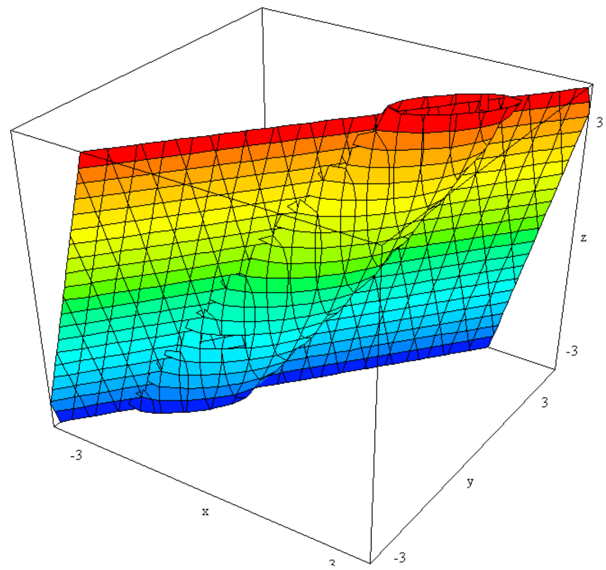

Figure 8: Information ellipsoid $x\Sigma^{-1}x = 125$, based on the 3×3 covariance matrix $\mathbf{\Sigma}$ of the logarithmic transformation of the data of Cobb and Douglas (1928), together with the plane fitted by Cobb and Douglas, *i.e.*, with the Grassmanian coefficients of the first row of the information matrix Σ^{-1}.

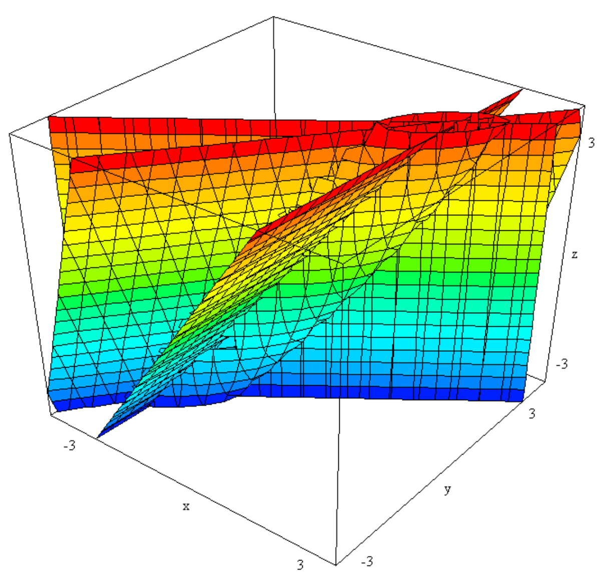

Figure 9: Information ellipsoid $x\Sigma^{-1}x = 125$, based on the 3×3 covariance matrix $\mathbf{\Sigma}$ of the logarithmic transformation of the data of Cobb and Douglas (1928), together with all three possible planes, *i.e.*, with the Grassmanian coefficients of each of the three rows of the information matrix Σ^{-1}.

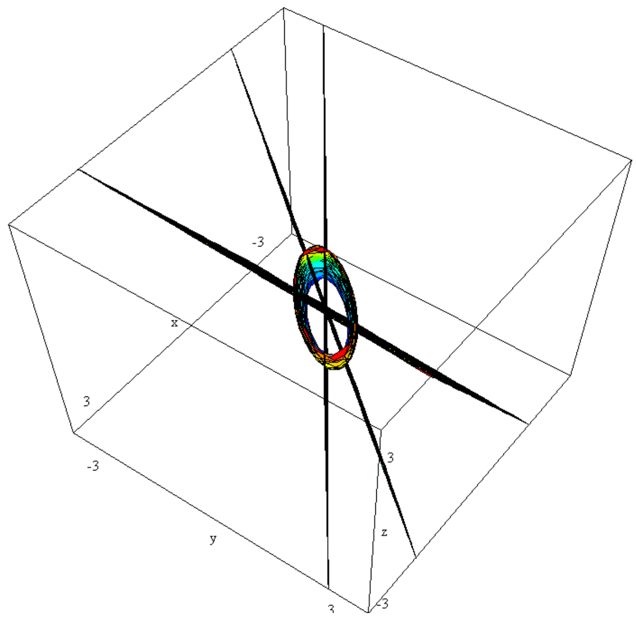

Figure 10: Information ellipsoid $x\Sigma^{-1}x = 125$, based on the 3×3 covariance matrix $\mathbf{\Sigma}$ of the logarithmic transformation of the data of Cobb and Douglas (1928), together with all three possible planes, *i.e.*, with the Grassmanian coefficients of each of the three rows of the information matrix Σ^{-1}. This is the same configuration as in the preceding figure, except that now we look along the max-axis into the information ellipsoid to clearly visualize the crossing of the three planes along the max-axis of the infomation ellipsoid.

-0.71819	1.00000	0.00000
-1.80312	0.00000	1.00000

-0.72029	1.00000	0.00000
-1.61316	0.00000	1.00000

-0.77439	1.00000	0.00000
-1.60845	0.00000	1.00000

Figure 11: This is the set of three possible orthogonal LS projected Grassmanian coefficients, by selecting rows 1 and 2, rows 2 and 3, and rows 1 and 3 from the infomation matrix Σ^{-1} and normalizing on the variables x_2 and x_3.