# Understanding the Brazilian Unemployment Structure: A Mixed Autoregressive Approach

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**Abstract**

The aims of this paper are estimate and forecast the *Non-Accelerating Inflation Rate of Unemployment*, or NAIRU, for Brazilian unemployment time series data. In doing so, we introduce a methodology for estimating mixed additive seasonal autoregressive (MASAR) models, by the Generalized Method of Moments (GMM). Furthermore, in order to cover a lack in econometric literature, an asymptotic theory for the Yule-Walker estimators of autoregressive parameters is developed. The paper provides some insights on estimating MASAR models when one of its component has a possible unit root. The obtained results are consistent to the literature and produce reasonable forecasts for NAIRU.

**Keywords:** Time series, Inflation, NAIRU, Seasonality, Unit Root.
**J.E.L. Classification:** E24, C13, C22, C51 and C53.

## 1 Introduction

The prospect of unemployment is one of the most important topics in Macroeconomics. It involves two crucial issues. First, one has to deal with the determinants of average unemployment over extended periods. Though the relevant question here

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is related to the fact that unemployment corresponds to a legitimate failure of markets to clear, its causes and consequences are not less important. There is a wide range of possible views. At one extreme is the perception that unemployment is largely illusory, or the working out of no great concern frictions in the process of matching up workers and jobs. Alternatively is the view that unemployment is the result of non-Walrasian features of the economy and it fundamentally corresponds to a misuse of resources.

Second issue concerns the cyclical behavior of the labor markets. Mostly, the real wage does not appear to be highly procyclical, an evidence consistent to the view that labor markets follow a Walrasian pattern, either the labor supply is highly elastic or shifts in labor supply play an important role in employment fluctuations. In this paper we will address the first issue.

The natural rate of unemployment is a key concept in modern Macroeconomics. Such a concept originated upon Milton Friedman's 1968 Presidential Address to the American Economic Association. At that time, he argued that there was no long-run trade-off between inflation and unemployment: as the economy adjusts to any average rate of inflation, unemployment returns to its *natural* rate. Higher inflation brings neither benefit in terms of lower average unemployment, nor does lower inflation involve any cost in terms of higher average unemployment. Instead, the microeconomic structure of labor markets, households and individual firms' decisions affecting labor supply and labor demand determine the natural rate of unemployment. If monetary policy cannot affect the natural rate of unemployment, then its appropriate role might be control inflation. In fact, monetary policy would help stabilizing the economy around the natural rate in the short run, fitting to the objective of maintaining low and stable inflation rates.

Another important concept of unemployment rate is the *Non-Accelerating Inflation Rate of Unemployment*, or NAIRU. This is the unemployment rate that matches the objective of maintaining stable inflation rates. According to the standard macroeconomic theory enshrined in most undergraduate textbooks, inflation will tend to rise if the unemployment rate falls below the natural rate. Conversely, when the unemployment rate rises above the natural rate, inflation tends to fall. Thus, the natural rate and the NAIRU are often viewed as two names for the same thing, providing an important benchmark for gauging the state of the business cycle, the outlook for future inflation, and the appropriate stance of monetary policy. Indeed, natural rate of unemployment is not the same thing of NAIRU. While the two are often viewed as synonyms, Estrella e Mishkin (1998) argues on the importance of distinguishing them. The natural rate is the unemployment rate that would be observed when short-run cyclical factors have played themselves out. Because wages and prices adjust sluggishly, the natural rate can be viewed as the unemployment rate when wages have had time for adjusting labor demand and labor supply. It depends on structural factors characterizing the labor market and is generally assumed to change slowly over time. At the circumstance when cyclical factors can take significant time to work themselves out, the natural rate may be less useful for policymakers concerned with the outlook for inflation over a short period of time.

The NAIRU, in Estrella e Mishkin (1998) view, should be interpreted as the unemployment rate consistent with steady inflation in the near term; i.e. over the next 12 months. The level of unemployment consistent with a steady inflation rate over such a time horizon can change significantly. Hence, in the absence of cyclical factors it need not to be the same as that, which is consistent to steady inflation in the short-run, and the short-run NAIRU will fluctuate much more than the natural rate. However, it is possible to look at unemployment without posit reference to a Phillips Curve, admitting that the unemployment rate reverts to its natural rate over medium to long horizons. In either case, the implication is that univariate data on unemployment can be used to extract an estimate of NAIRU as a local average of the series. Moreover, it can be seen that natural rate of unemployment and NAIRU are the same. Note that in a MASAR model, the NAIRU is defined as a sum of two constant averages, an annual and a monthly one.

The purpose of this paper is to estimate Brazilian NAIRU taking an autoregressive time series model into account that is able to separate seasonal and local mean components. The paper is organized as follow. Section 2 contains a review of a simple economic model for unemployment rate determination, based on Neoclassical theory. Next section presents the data used in this paper. Section 4 derives the econometric model and shows the results we have reached upon. Finally, section 5 concludes.

## 2 Economic Foundations of Unemployment

The purpose of this section is to present an economic model of labor market, and derive the employment rate. The neoclassical macroeconomics as the Keynesian one, admits a stable relationship, in short run, between the volume of employment ($n$) and the output ($y$),

$$y = f(n)$$

where the so-called *short run production function*, $f(n)$ is increasing, strictly concave and differentiable, with $f(0) = 0$. In this fashion, the average productivity of labor $f(n)/n$ tends to zero when $n$ tend to infinity. The stability of this function results from the hypothesis that capital stock, technological knowledge and firm's structure are given in the short run. Since these factors are fixed, the output $y$ is related to employment $y$ through the law of decreasing returns.

Additionaly, we suppose the economy is competitive and a firm acts so as to maximize its profits. In this way, given the output price $p$ and the nominal wage $w$, the employment level $n$ is chosen to maximize the difference between revenue and costs:

$$py - wn = pf(n) - wn \tag{1}$$

Since $f(n)$ is supposed to be concave and differentiable, the condition for profit maximization give us the following:

$$\frac{w}{p} = f'(n) \tag{2}$$

which equates the real wage $(w/p)$ to marginal productivity of labor, setting up the labor demand's curve.

Assembling the short run production function and the labor demand's curve, allow us to express output in terms of (p/w), the price/wage relation:

$$y = h(p/w) \tag{3}$$

which is nothing else than aggregate supply.

Suppose now the labor demand's curve is known, $(w/p) = g(n)$. Hence, since $g(n) = f'(n)$ and $f(0) = 0$:

$$f(n) = \int_0^n G(x)dx \tag{4}$$

Finally, taking equation (3) and its inverse function, the following can be written:

$$\frac{p}{w} = h^{-1}(y) \tag{5}$$

Nevertheless:

$$\frac{p}{w} = \frac{1}{f'(n)} = \frac{dn}{dy} \tag{6}$$

and, in view of the fact that $y = 0$ for $n = 0$:

$$n = \int_0^y h^{-1}(z)dz \tag{7}$$

In this equation, $n$ is the level of employment consistent to output $y$. If we label $l$ the number of workers currently employed and $u$ the unemployed ones, the sum

$l + u$ must be equal to $n$. Therefore,the definition of unemployment rate is given by:

$$ur = \frac{u}{n} \tag{8}$$

As we discuss at the beginning of this paper, there are quite a few interpretations for this quantity. The simplest one is to name it as the *natural rate of employment.* If this quantity is suitable for a steady-state inflation rate,the *Non-Accelerate Inflation Rate of Unemployment*, NAIRU is defined.

Although the simplicity, models of labor market have proved useful for explaining the cyclical components of unemployment and provide a reasonable basis for the existence of a short run Phillips Curve (see, for example, Caballero e Bertola (1992), Staiger, Stock e Watson (1996) ). While most of the work on searching models focuses on understanding cyclical variations, they also provide a conceptual framework for modelling NAIRU, which can be viewed as the model's steady-state unemployment rate.

If inflation expectations are unbiased and if a supply shock variable has zero or an absent mean, then the average unemployment rate will equal the NAIRU. Alternatively, we can simple posit the problem without reference to a Phillips Curve where unemployment rate reverts to its natural rate at medium to long horizons. In either case, the implication is that univariate data on unemployment can be used in order to extract an estimate of the NAIRU as a local mean of the series. This methodology is the framework upon which our empirical model is implemented next section.

## 3 Data and Methodology

The time series taken into account in this paper is the Open Unemployment Rate (OUR) corresponding to five Brazilian metropolitan areas - Recife, Salvador, Belo Horizonte, Rio de Janeiro, São Paulo and Porto Alegre - given in a monthly basis, including fifteen years old people and above[1]. The data set covers the period from 1980.01 to 2002.12. We decided to use this series because (i) the quality of the data is a striking fact compared to other related series and (ii) it covers the longest period available, 22 years, a time span that seems appropriated to calculate the natural rate of unemployment. Figure (1) portraits Brazilian Natural Rate of Unemployment time series behavior:

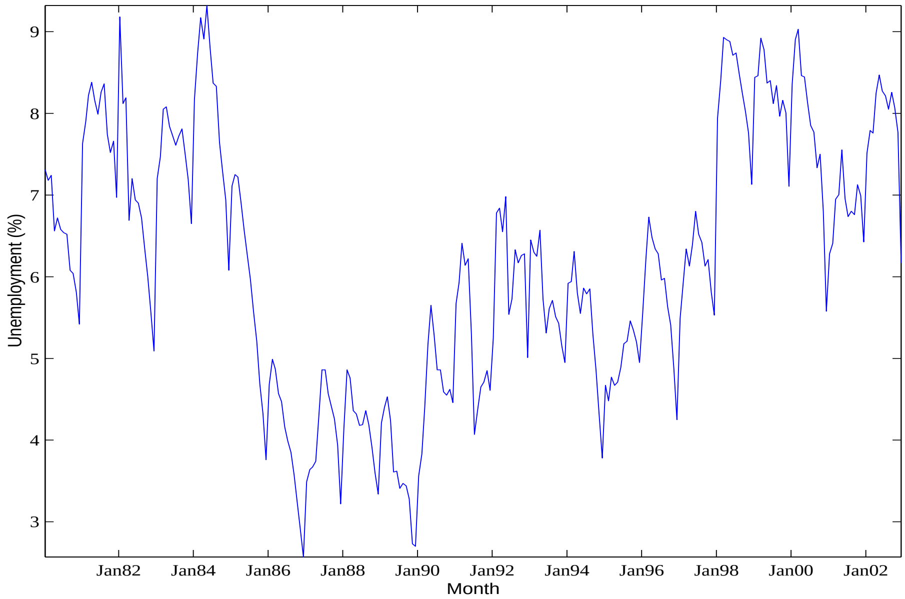


Figure 1: Brazilian Open Unemployment Rate - BOUR.

Concerning methodology, we admit a univariate time series model able to distinguish between deterministic and stochastic components of unemployment rate. The deterministic components are a sum of two averages, one that is annual and another one, a monthly seasonal structure. The first is modelled as an autoregressive process with drift and can be interpreted as the NAIRU. In turn, the relevant parameters of this time series are estimated, which allow us to forecast upon the results.

[1]The data comes from Brazilian Institute of Geography and Statistics - IBGE, (PME).

## 4 The Econometric Model and Results

The aim of this section is to present a methodological sequence of useful procedures for finding accurate estimates for a mixed additive seasonal autoregressive (MASAR) model when a possible unit root is present in one of its components. Our empirical implementation of the univariate approach starts with the autoregressive model:

$$X_{t(r,m)} = \alpha_r + \mu_m + \sigma_m Z_{t(r,m)} \tag{9}$$

for $t(r,m) = S(r-1) + m$ for $r = 1, 2, \ldots, T$ and $m = 1, 2, \ldots, S$, where $T$ is the time series dimension and $S$ the seasonal length in each period. The component $\mu_m$ refers to seasonality, $\alpha_r$ is the annual average series and $Z_{t(r,m)}$ constitutes the stochastic structure of the process. Then, the model's properties can be summarized as follow:

**Assumption 4.1** *Let $X_{t(r,m)}$ be a covariance stationary composite stochastic process where autocovariances are assumed to be absolute. Then:*

1. *$\Psi(L)Z_{t(r,m)} = a_{t(r,m)}$ is an autoregressive process and the lag length $p$ is endogenously choose. Additionally:*

   *(a) $E(Z_{t(r,m)}) = 0$;*

   *(b) $a_{t(r,m)} \sim$ i.i.d $N(0, \sigma_a^2)$;*

2. *The annual average series $\alpha_r$ is a first order autoregressive process,*

$$\alpha_r = \mu_\alpha + \gamma \alpha_{r-1} + \omega_r$$

   *with a possible unit root when $\gamma$ is near unity. The innovations $\omega_r$ are independent and identically distributed as $N(0, \sigma_\omega^2)$.*

*3. Finally, independence between the innovations are assumed, hence,*

$$E(\omega_r a_{t(r,m)}) = 0.$$

The assumptions above are summarized by the following structural model:

$$\begin{aligned} X_{t(r,m)} &= \alpha_r + Y_{t(r,m)} & (10) \\ \alpha_r &= \mu_\alpha + \gamma\alpha_{r-1} & (11) \\ Y_{t(r,m)} &= \mu_m + \sigma_m Z_{t(r,m)} & (12) \\ Z_{t(r,m)} &= \sum_{j=1}^{p} \phi_j Z_{t(r,m)} + a_{t(r,m)}. & (13) \end{aligned}$$

Generalized method of moments is the method used in this paper (GMM) for estimating the parameters of the model. Convergence characteristics for moments estimators was stated by Hansen (1982).

In a first step one has to be careful on estimating the parameters of a MASAR model as stated above, since every component should be checked out for stationarity, specially the additive seasonal ones. Using data described in section 3, we found that equation (11) may have a unit root, hence the vector of parameters $\Theta_1 = (\mu_\alpha, \gamma, \sigma_\omega^2)$ has a different asymptotic distribution of the remaining ones.

In order to deal with inference about the parameters, we first calculate a unit root test, which ought to be robust for small samples, as this series length is only $T = 23$ years. Therefore, we choose the DF-GLS test due to Elliot, Rothenberg e Stock (1996). The basic procedure involves the estimation of an autoregression form as follows:

$$\Delta\alpha_r^{dm} = \gamma^*\alpha_{r-1}^{dm} + \beta_1\Delta\alpha_{r-1}^{dm} + \ldots + \beta_p\Delta\alpha_{r-p}^{dm} + \varpi_r \quad (14)$$

where the superscript $^{dm}$ means that the time series was demeaned, by $\mu_\alpha$, specifying that the alternative hypothesis is the stationarity around a mean. The moment estimator for $\mu_\alpha$ is given by[2]:

$$\widehat{\mu}_\alpha = \frac{\alpha_r - \alpha_1}{r-1} \tag{15}$$

In this case, we can write an estimator for $\gamma^*$, $\widehat{\gamma}^*$, as a function of residual moments estimated $\widehat{\varpi}_r$:

$$\widehat{\gamma}^* = \frac{(1/r)\sum_{r=1}^{r} \widehat{\varpi}_r \widehat{\varpi}_{r-1}}{(1/r)\sum_{r=1}^{r} \widehat{\varpi}_{r-1}^2} \tag{16}$$

and the error's variance as

$$\widehat{\sigma^*}_{\widehat{\alpha}_r}^2 = \frac{1}{r}\frac{\sum_{r=1}^{r}(\alpha_r - \gamma\alpha_{r-1})^2}{\sum_{r=1}^{r}\alpha_{r-1}^2} \tag{17}$$

The t-test statistics for null hypothesis of $\gamma^* = 1$ is calculated by a traditional ordinary least squares approach, however, its limiting distribution is not Gaussian, but a functional of a Brownian motion, $W(s)$, defined as $0.5\left(\int W(s)^2\right)^{-1/2}\left[W(1)^2 - 1\right]$. The difference between this test and the tradicional Dickey e Fuller (1979) unit root test is that the time-series is transformed via a generalized least squares (GLS) regression prior performing the test.

Elliot, Rothenberg e Stock and subsequent studies have shown that this test has significantly higher power than the previous versions of the augmented Dickey-Fuller test, specially for small samples. Lag length for autoregressive estimation was choosen by using Perron e Ng (2001) sequential's *t*-test criterion.

Straightaway, we must now present estimators for the vector $\Theta_2 = (\mu_m, \sigma_m, \phi_j, \sigma_\varepsilon^2)$. Lets first take a look to equation (12). The estimators for vector parameters $(\mu_m, \sigma_m,)$ are:

$$\widehat{\mu}_m = \frac{1}{T}\sum_{r=1}^{T} Y_{t(r,m)} \tag{18}$$

$$\widehat{\sigma}_m^2 = \frac{1}{T}\sum_{r=1}^{T}\left(Y_{t(r,m)} - \widehat{\mu}_m\right)^2 \tag{19}$$

[2]Since the model do not have a mean in the usual sense, this estimator correspond to a recursive estimation of a parameter of a diffusion process.

In order to draw close the MASAR model presented at the beginning of this section, we ought to derive the estimators for the autoregressive parameters $(\phi_j, \sigma_a^2)$ in equation (13). We choose Yule-Walker equations approach, which are a special case of moments estimators. Let the $\rho_k$ be a positive definite autocorrelation function which satisfies the same form of difference equation:

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \ldots + \phi_p \rho_{k-p} \quad \text{for} \quad k > 0$$

Note that if we write:

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} \quad \boldsymbol{\rho}_p = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_p \end{pmatrix} \quad \boldsymbol{P}_p = \begin{pmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \cdots & 1 \end{pmatrix}$$

the solution of the system for the parameters $\boldsymbol{\phi}$, in terms of autocorrelations, may be written in the following way

$$\boldsymbol{\phi} = \boldsymbol{P}_p^{-1} \boldsymbol{\rho}_p \tag{20}$$

Admitting $\rho_0 = 1$, the Yule-Walker estimators are replacing by replacing the theoretical autocorrelations $\rho_k$ through its sample analogues $r_k$, which results in

$$\begin{aligned} \widehat{\boldsymbol{\phi}} &= \boldsymbol{R}_p^{-1} \boldsymbol{\rho}_p && (21) \\ \widehat{\sigma}_z^2 &= \frac{\sigma_\varepsilon^2}{1 - \sum_{l=1}^{p} \rho_l \phi_l} && (22) \end{aligned}$$

where $\boldsymbol{R}_p$ are the matrix of sample correlations.

For the sake of covering a lack in time series literature, we decided to present an accurate proof of asymptotic normality of the Yule-Walker estimators. This proof is obtained in two ways: first, assuming independence, under the hypothesis that the sequence $Z_{t(r,m)}$ is stationary of order 4 and writing it as an infinite moving average, and second by assuming that $Z_{t(r,m)}$ takes the form of a Martingale difference sequence. Sticking with the former approach, we prove the following proposition in appendix A:

**Proposition 4.2** ***Asymptotic Normality of Yule-Walker Estimators.*** *If* $Z_{t(r,m)}$ *is a causal AR(p) process with* $\{a_t\}$ *i.i.d.* $(0, \sigma^2)$*, and* $\hat{\boldsymbol{\phi}}$ *is the Yule-Walker estimator of* $\boldsymbol{\phi}$*, then*

$$n^{1/2}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \Rightarrow N(\mathbf{0}, \sigma^2\Gamma_p^{-1}) \tag{23}$$

*where* $\Gamma_p$ *is the covariance matrix* $[\gamma(i-j)]_{i,j=0}^{p}$*. Moreover:*

$$\hat{\sigma}_z^2 \to \sigma_z^2$$

At this point, before presenting the model's estimation results, we should provide some idea about the econometric set up settled above. As a first point for noticing, it would be acceptable choosing a unit root model directly, including a constant an a time trend to fit the model. However, objections can be raised concerning this procedure, justifying, indeed, the appeal for constructing of a new methodology, named mixed additive seasonal autoregressive (MASAR) modelling.

First, the lower span frequency of annual average time series is desirable for monthly observations when performing unit root tests, as stated in Campbell e Perron (1992) and showed in Ghysels e Perron (1993). Moreover, at least theoretically, our empirical implementation shed light on Brazilian Unemployment Structure, admitting the natural rate, or NAIRU, as a sum of two components. Finally, after controlling for the integrated part, which lies at the annual model's structure, we are able to work on time serie's monthly transient dynamics and perform good forecasts. Let's now show the results obtained.

Setting about the stationary component, i.e., the coefficients $\boldsymbol{\phi}$ of $Z_{t(r,m)}$, whose lag length is endogenously chosen, using the Bayesian Information Criteria (BIC), we found that the transient dynamic of Brazilian unemployment rate is better adjusted by a first order autoregressive model with a coefficient equal to 0.6338, when BIC $= -0.5077$.

We ought to note that in a MASAR model, the NAIRU is defined as a sum of two averages constants, $(\mu_\alpha + \mu_m)$, an annual and a monthly one. The value reached for NAIRU is around 6.2417%. Alternatively, the transient dynamics, or the deviations of the unemployment around its trend, is annually determined by the autoregressive unit root process and monthly determined by the stationary process $Z_{t(r,m)}$.

The DF-GLS unit root test clearly indicates it is not possible to reject the null hypothesis of a unit root for annual averages series, constituting the accelerating component of NAIRU. Using a different model setup, Corseuil, Gonzaga e Issler (1999) found the same results using disaggregate time series for Brazilian regions. In this fashion, the results presented here confirm the thesis that persistency in Brazilian unemployment data is not restrict to less develop regions, but is a phenomenon that affects the economy at all. Our results is summarized in table (4) below:

| DF-GLS Unit Root Test | | |
|---|---|---|
| $\tau - Stat$ | 1% Critical Value | 5% Critical Value |
| -1.515 | -2.66 | -2.544 |
| Estimates for Non-Stationary Components | | |
| $\hat{\gamma}^*$ | $\hat{\mu}_\alpha$ | $\hat{\sigma^*}^2_\omega$ |
| 0.9856 | 0.0627 | 0.0225 |
| Estimates for Stationary Components | | |
| $\hat{\phi}$ | $\hat{\mu}_m$ | $\hat{\sigma}^2_m$ |
| 0.6338 | 6.179 | 2.4871 |

Table 1: MASAR Model Estimations for Brazilian Unemployment Rate.

Those findings raise two implications. First one concerns forecasting. The DF-GLS test assumes the model is defined as a pure AR(1) after demeaning, and, since it has a unit root, we cannot apply difference transformations in order to perform the forecasts. However, as we will show next section, it is possible to re-parametrize it in a way to include the constant term and then performs the forecasts.

An important fact arising upon these findings involves a theoretical issue which resists to be fully explained in light of economic theory. During the last fourty years, the unemployment rate has raised in most developed and underdeveloped countries. The lowest value encountered for one of the components of Brazilian NAIRU was 1987, when the long-run unemployment rate was only 3.98%. In previous periods, evidences are toward high unemployment rates. However, this was specially due to the debt crisis and world depression avowed at the beginning of the eighth's. Analyzing evolution along time, one realizes the trend for this component has been scalating, reaching 8.34% in 1999, though experiencing a slow decrease to 7.88% in 2003.

There exist some hypotheses which attempt to shed light on this trend. Labor economists argue that the Second World War changed the labor force composition, introducing a new net flow of young workers into the labor market and, consequently, rising the unemployment rate[3]. Another issue deals with the evident entrance of women in the labor market, leading to an increase in the number of family members earning wages. Finally, another approach goes toward the fact that generalization of sectorial disjoint rise the unemployment rate increasing the job separation, or, the time lag between loosing and finding a new job. However, these issues are beyond the scope of this paper.

## 5 Forecasting

Along this section minimum mean square error(MSE) are taking into account for evaluating models' forecasting, as estimated in previous section. Our goal is to look upon the behavior of NAIRU, however,since we are dealing with a MASAR model, we must construct the minimum MSE for every model's component. Lets admit a model

[3]Of course, this explanation is based in a Search Type Model which assumes the flow of new jobs opening are slower than the flow of new workers.

denoted by equation (9). Taking conditional expectations at time $t(r,m)$:

$$E\left(X_{t(r,m)_{+1}}|\mathcal{H}_{t(r,m)}\right) = E\left(\alpha_{r_{+1}}|\mathcal{H}_{t(r,m)}\right) + E\left(Y_{t(r,m)_{+1}}|\mathcal{H}_{t(r,m)}\right) \tag{24}$$

where $\mathcal{H}_{t(r,m)}$ is the filtering that represents all information up to time $t(r,m)$.

Counting on the monthly average series component, $\alpha_r$, the conditional expectation up to time $r$ is given by:

$$E\left(\alpha_{r_{+1}}|\mathcal{H}_{t(r,m)}\right) = E\left(\alpha_{r_{+1}}|\alpha_r\right) \tag{25}$$

As expected, the minimum MSE for a forecasting model is

$$\hat{\alpha}_{r+1} = E\left(\alpha_{r_{+1}}|\alpha_r\right) = \mu_\alpha + \alpha_r \tag{26}$$

Since $\alpha_r$ is not directly observed, an recursive equation of the type applies

$$\hat{\alpha}_{r+1} = (r-1)\mu_\alpha + \hat{\alpha}_1 \tag{27}$$

where $\alpha_1$ matches an GMM estimate for $\alpha_1$ and is given by

$$\hat{\alpha}_1 = \frac{1}{S}\sum_{m=1}^{S} X_{t(1,m)} \tag{28}$$

Or alternatively:

$$E\left(Y_{t(r,m)_{+1}}|\mathcal{H}_{t(r,m)}\right) = \mu_m + \sigma_m E\left(Z_{t(r,m)_{+1}}|\mathcal{H}_{t(r,m)}\right) \tag{29}$$

Once an AR(1) model for $Z_{t(r,m)}$ is adjusted in the previous section, so, in this case, the following can be written:

$$E\left(Z_{t(r,m)_{+1}}|\mathcal{H}_{t(r,m)}\right) = \hat{Z}_{t(r,m)_{+1}} = \hat{\phi}_1\hat{Z}_{t(r,m)} \tag{30}$$

Finally, the minimum MSE forecast for the entire MASAR process is obtained, which is given by:

$$\hat{X}_{t(n,m)+1} = (n-1)\hat{\mu}_\alpha + \hat{\alpha}_1 + \hat{\mu}_m + \hat{\sigma}_m \hat{Z}_{t(n,m)+1}. \tag{31}$$

for $m = 1, 2, \ldots, S$.

Figure (2) shows the results for the model's forecast.

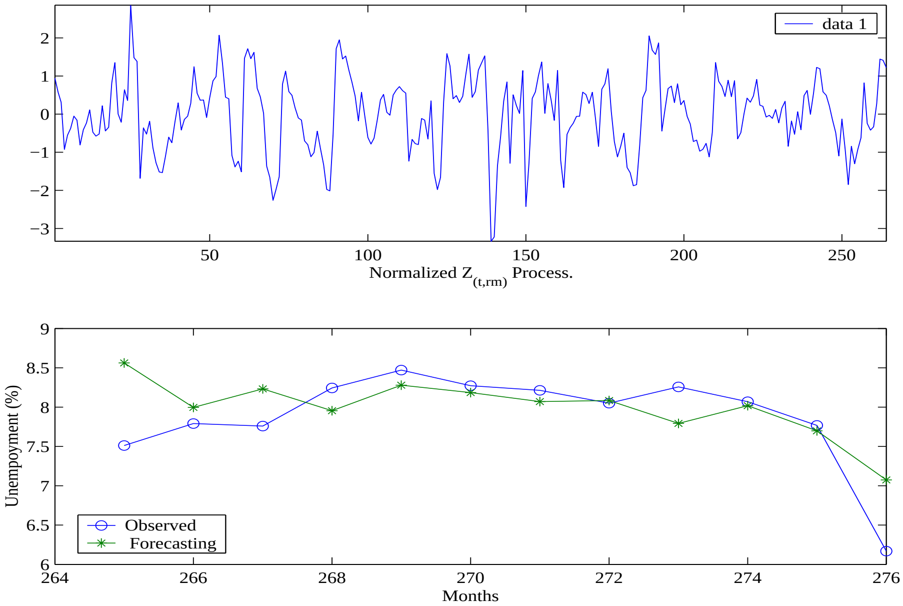


Figure 2: Forecasting for Unemployment Time Series.

Updating the forecasting based on conditional expectations, we obtain future values for a 12 months length. The square root of MSE is 0.4618, and the mean absolute percentage error of the forecasts is 4.45%. Chow's[4] forecast test for the 12 forecasts returns an $F$ value of 0.7217, with a corresponding $P-value$ of 0.99, So the hypothesis of a stable MASAR scheme for this specification is not rejected.

[4]This test is based on Chow's influential 1960 article, Chow (1960).

## 6 Concluding Remarks

For the last fifty years, Macroeconomics has been a matter of concern among academics and policy makers. Searching for a reasonable explanation for stressing its importance, one can raise many of them. However, nothing strikes the researchers most than puzzles or controversies generated by discussions involving macroeconomic issues. Along this paper, we dealt with a controversy that has been lasting for forty years, approximately: the estimation and interpretation of NAIRU.

Once this issue is still an open question, we decided to get into this debate by estimating Brazilian NAIRU taking an autoregressive time series model into account that is able to separate seasonal and local mean components. In doing so, we admit an univariate time series model that is able to distinguish between deterministic and stochastic components of unemployment rate. The deterministic components are a sum of two averages, one that is annual and other: a monthly seasonal structure. The first was modelled as an autoregressive process with drift and can be interpreted as the NAIRU. In this sense, the methodology developed along this paper is new and allow us to bring more information to the discussion.

Upon the results obtained by estimating the econometric model designed in section 4 of this paper, we highlight two concluding remarks. First, a formal proof for the asymptotic normality of the Yule-Walker estimator is provided, which is indeed a puzzling task. Second, the coefficient estimates of the MASAR model are consistent, besides showing separate estimates for stationary and non-stationary components.

From the macroeconomic point of view, the results presented in Table 1 are striking. Once the estimate coefficients for the NAIRU reaches, approximately, 6.25%, three main conclusive features arises from our analysis. First, since the existence of a unit root cannot be rejected, the long run trend of Brazilian unemployment is increasing. Second, the estimate for $\hat{\gamma}^*$ reinforces the non-rejection of a null hypothesis about the existence of a unit root in Brazilian unemployment data, suggesting the upward trend for unemployment is permanent. Finally, concerning the cyclical or transitory component, we interpret it as the transient dynamics of unemployment, given a government intervention through a macroeconomic policy.

Finally, but certainly not last, our results set challenges to the future design of macroeconomic policies. Moreover, there is also room for researching on this issue. Indeed, the unemployment rate long run trend is upward and the degree of success of a macroeconomic policy on job creation is, at least, doubtful. The reasons for our findings is still a matter of further research to be pursued after this paper.

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# A An Asymptotic Theory for the Yule-Walker Equations

Asymptotic normality for Yule-Walker equations can be found in a variety of ways. However, a formal and consistent proof for the AR(p) model is relatively hard to find, since it takes converge properties of infinity series into account, due to its MA($\infty$) (moving average) representation. In this appendix, we aim to proof Proposition (4.2) assuming that $Z_{t(r,m)}$ is a causal and stationary AR(p) process of order 4 that can be expressed in a infinite moving average form. Writing this model in a matrix notation:

$$\boldsymbol{Z} = \boldsymbol{\phi}\boldsymbol{W} + \boldsymbol{E} \tag{32}$$

we obtain the following estimator:

$$\hat{\boldsymbol{\phi}} = (\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'\boldsymbol{Z}$$

onde $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_{t(r,m)})$, $\boldsymbol{E} = (E_1, E_2, \ldots, E_{t(r,m)})$ and $\boldsymbol{W}$ is a matrix given by

$$\boldsymbol{W} = \begin{bmatrix} W_0 & W_{-1} & \cdots & W_{1-p} \\ W_1 & W_0 & \cdots & W_{2-p} \\ \vdots & \vdots & \ddots & \vdots \\ W_{t(r,m)-n-1} & W_{t(r,m)-n-2} & \cdots & W_{t(r,m)-p} \end{bmatrix} \tag{33}$$

Note that, following equation (21):

$$\hat{\boldsymbol{\phi}} = \boldsymbol{R}_p^{-1}\hat{\boldsymbol{\rho}}_p$$

In this fashion, the first step to show asymptotic normality of Yule-Walker estimators is the following:

$$n^{-1}(\boldsymbol{W}'\boldsymbol{W}) \Rightarrow \boldsymbol{R}_p \tag{34}$$

and

$$n^{-1}(\boldsymbol{W}'\boldsymbol{Z}) \Rightarrow \boldsymbol{\rho}_p \tag{35}$$

where the symbol " $\Rightarrow$ " indicates weak convergence (or in probability).

**<u>PROOF</u>**: Let $\boldsymbol{R}_p = \rho\left[(i-j)\right]_{i,j=0}^{p}$ be the covariance matrix of the autoregressive process $\boldsymbol{Z}_t$. Writing the model in an infinity moving average representation,

$$\boldsymbol{W}_t = \sum_{j=0}^{\infty} \Psi_j \boldsymbol{Z}_{t-j}, \qquad \boldsymbol{Z}_{t-j} \sim iid(0,\sigma^2) \tag{36}$$

Then, for proving propositions (34) and (35) it is enough to show that:

$$\boldsymbol{\rho}_p^*(h) \Rightarrow \left(\sum_{j=0}^{\infty} \Psi_j \Psi_{j+h}\right) \sigma^2 = \boldsymbol{\rho}_p \tag{37}$$

Ensuring $\sum_{j=0}^{\infty} \Psi_j < \infty$ and $\sum_{j=0}^{\infty} \Psi_j^2 |j| < \infty$, this is, the process is covariance stationary, we shall prove the result (37) for $h = 0$. The general case is similar. Now:

$$\boldsymbol{\rho}_p^*(0) = n^{-1} \sum_{t=1}^{n} \sum_i \Psi_i^2 \boldsymbol{Z}_{t-1}^2 + Y_n \tag{38}$$

where $Y_n = \sum\sum_{i \neq j} \Psi_i \Psi_j n^{-1} \sum_{t=1}^{n} \boldsymbol{Z}_{t-i} \boldsymbol{Z}_{t-j}$ By the weak law of large number for moving averages, lets show the converge of first term:

$$\begin{aligned} E\left(n^{-1} \sum_{t=1}^{n} \sum_i \Psi_i^2 \boldsymbol{Z}_{t-1}^2\right) &\leq n^{-1} \sum_{t=1}^{n} \sum_i E\left(\Psi_i^2\right) E\left(\boldsymbol{Z}_{t-1}^2\right) \\ &\leq n^{-1} E\left(\boldsymbol{Z}_{t-1}^2\right) \sum_i E\left(\Psi_i^2\right) \\ &\leq n^{-1} \sigma^2 \left(\sum_i \Psi_i^2\right) \end{aligned}$$

Since $\left(\sum_i \Psi_i^2\right)$ directly converges by applying Slutsky Lemma, then:

$$n^{-1} \sum_{t=1}^{n} \sum_i \Psi_i^2 \boldsymbol{Z}_{t-1}^2 \Rightarrow n^{-1} \sigma^2 \left(\sum_i \Psi_i^2\right) \qquad \blacksquare \tag{39}$$

So it suffices to show: $Y_n \Rightarrow 0$. For $i \neq j$, $\{\boldsymbol{Z}_{t-i}\boldsymbol{Z}_{t-j}\} \sim\ iid\ \ WN(0, \sigma^4)$, and hence[5],

$$Var\left(n^{-1}\sum_{t=1}^{n}\boldsymbol{Z}_{t-i}\boldsymbol{Z}_{t-j}\right) = n^{-1}\sigma^4 \to 0.$$

Thus, for an integer $k$, we are able to write the following:

$$Y_{nk} = \sum_{|i|\leq k}\sum_{|j|\leq k, i\neq j}\Psi_i\Psi_j n^{-1}\sum_{t=1}^{n}\boldsymbol{Z}_{t-i}\boldsymbol{Z}_{t-j} \tag{40}$$

which enables us to show that:

$$\lim_{k\to\infty}\limsup_{n\to\infty} E|Y_n - Y_{nk}| \Rightarrow 0 \tag{41}$$

Taking

$$\begin{aligned}
P\left(|Y_n - Y_{nk}| \geq \epsilon\right) &= P\left(\sum\sum_{i\neq j}\Psi_i\Psi_j n^{-1}\sum_{t=1}^{n}\boldsymbol{Z}_{t-i}\boldsymbol{Z}_{t-j} - \sum_{|i|\leq k}\sum_{|j|\leq k, i\neq j}\Psi_i\Psi_j n^{-1}\sum_{t=1}^{n}\boldsymbol{Z}_{t-i}\boldsymbol{Z}_{t-j}\right) \\
&= P\left(\left|\sum_{|i|>k}\sum_{|j|>k}\Psi_i\Psi_j n^{-1}\sum_{t=1}^{n}\boldsymbol{Z}_{t-i}\boldsymbol{Z}_{t-j}\right|\right) \\
&= P\left(1/n\left|\sum_{|i|>k}\sum_{|j|>k}\Psi_i\Psi_j\sum_{t=1}^{n}\boldsymbol{Z}_{t-i}\boldsymbol{Z}_{t-j}\right|\right) \\
P\left(|Y_n - Y_{nk}|\right) &\leq E\left(1/n\epsilon\left|\sum\sum\Psi_i\Psi_j\sum_{t=1}^{n}\boldsymbol{Z}_{t-i}\boldsymbol{Z}_{t-j}\right|\right) \\
&\leq \frac{1}{n\epsilon}E\left(\sum\sum\Psi_i\Psi_j\right)Cov\left(\boldsymbol{Z}_{t-i}\boldsymbol{Z}_{t-j}\right) \Rightarrow 0
\end{aligned}$$

Therefore $Y_n \Rightarrow 0$ which concludes the proof.

Ensuring the results stated above, we go further by demonstrating the asymptotic normality of Yule-Walker estimators. In order to do that a Cholesky factorization

[5] $WN$ denotes a White Noise process.

is necessary. Let $\boldsymbol{U}_t = (W_{t-1}, \ldots, W_{t-p})'\boldsymbol{E}, \;\; t \geq 1$. Using projections we get

$$\begin{aligned}
\boldsymbol{U}_1 &= (W_0, \ldots, W_{-p}) \\
\boldsymbol{U}_2 &= (W_1, \ldots, W_{1-p}) \\
\vdots & \quad \vdots \\
\boldsymbol{U}_n &= (W_{n-1}, \ldots, W_{n-p})
\end{aligned}$$

By the Gram-Schmidt process, we write the sum $\sum_{t=1}^{n} \boldsymbol{W}'\boldsymbol{E}$, which has the following properties:

$$\begin{aligned}
E(\boldsymbol{U}_t) &= 0 \\
Var(\boldsymbol{U}_t\boldsymbol{U}_t') &= \sigma^2 \boldsymbol{R}_p
\end{aligned}$$

We want to show that, if $\hat{\boldsymbol{\phi}}$ is the Yule-Walker estimator of $\boldsymbol{\phi}$, then we prove that:

$$n^{1/2}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \Rightarrow N(\mathbf{0}, \sigma^2 \Gamma_p^{-1}) \tag{42}$$

where $\boldsymbol{R}_p$ is the covariance matrix $[\boldsymbol{\rho}(i-j)]_{i,j=0}^{p}$.

**PROOF**: Observe, initially,

$$\begin{aligned}
n^{1/2}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) &= n^{1/2}\left[\left(\boldsymbol{W}'\boldsymbol{W}\right)^{-1}\boldsymbol{W}'\boldsymbol{Z} - \phi\right] \\
&= n^{1/2}\left[\left(\boldsymbol{W}'\boldsymbol{W}\right)^{-1}\boldsymbol{W}'(\boldsymbol{W}\phi + \boldsymbol{E}) - \phi\right] \\
&= n^{1/2}\left[\left(\boldsymbol{W}'\boldsymbol{W}\right)^{-1}(\boldsymbol{W}'\boldsymbol{W}\phi + \boldsymbol{W}'\boldsymbol{Z}) - \phi\right] \\
&= n\left(\boldsymbol{W}'\boldsymbol{W}\right)\left(n^{-1/2}\boldsymbol{W}'\boldsymbol{E}\right)
\end{aligned}$$

One can observe that working in this fashion the independent random variable $\boldsymbol{U}$, presented above is perfectly useful, to express the term $\boldsymbol{W}'\boldsymbol{E}$. Writing $Z_{t(r,m)}$ in its moving average form,

$$\boldsymbol{W}_t = \sum_{j=0}^{\infty} \Psi_j \boldsymbol{Z}_{t-j}.$$

Let now $m$ be any fixed integer positive and set

$$\boldsymbol{W}_t^{(m)} = \sum_{j=0}^{m} \Psi_j \boldsymbol{Z}_{t-j}$$

and

$$\boldsymbol{U}_t^{(m)} = (W_{t-1}^{(m)}, \ldots, W_{t-p}^{(m)})' \boldsymbol{E}.$$

Now, admitting $\boldsymbol{\lambda}$, a fixed element of $\mathbb{R}^p$, then $\boldsymbol{\lambda} \boldsymbol{U}_t^{(m)}$ is a strictly stationary $(m+p) -$ *dependent* white noise sequence with variance given by $\sigma^2 \boldsymbol{\lambda}' \boldsymbol{R}_p^{(m)} \boldsymbol{\lambda}$. Since we are dealing with $\boldsymbol{U}_t$ that is an independent random variable, we claim the Central Limit Theorem (CLT) to demonstrate the following:

$$n^{-1/2} \sum_{t=1}^{n} \boldsymbol{\lambda}' \boldsymbol{U}_t^{(m)} \Rightarrow N(\mathbf{0}, \boldsymbol{\lambda}' \sigma^2 \boldsymbol{\lambda} \boldsymbol{R}_p^{(m)}) \tag{43}$$

which it suffices to show (42). Writing $\overline{\boldsymbol{U}}_{mn} = n^{-1/2} \sum_{t=1}^{n} \boldsymbol{\lambda}' \boldsymbol{U}_t^{(m)}$, a sum of $(n-m)$ independent random variables with zero mean and variance $\sigma^2 \boldsymbol{\lambda}' \boldsymbol{R}_p \boldsymbol{\lambda}$ is obtained. Then, by Lindeberg-Feller Central Limit Theorem we conclude:

$$\overline{\boldsymbol{U}}_{mn} \sim \; iid \; (\mathbf{0}, \sigma^2 \boldsymbol{\lambda}' \boldsymbol{R}_p \boldsymbol{\lambda}).$$

Given that:

$$\sigma^2 \boldsymbol{R}_p^{(m)} \to \sigma^2 \boldsymbol{R}_p \quad \text{as} \;\; m \to \infty$$

directly the following are obtained:

$$n^{-1/2} \sum_{t=1}^{m} \boldsymbol{\lambda}' \boldsymbol{U}_t^{(m)} \Rightarrow N\left(0, \sigma^2 \boldsymbol{R}_p\right)$$

Moreover:

$$\frac{1}{n}Var\left(\boldsymbol{\lambda}'\sum_{t=1}^{m}\left(\boldsymbol{U}_t^{(m)}-\boldsymbol{U}_t\right)\right)=\boldsymbol{\lambda}'E\left[\left(\boldsymbol{U}_t^{(m)}-\boldsymbol{U}_t\right)\left(\boldsymbol{U}_t^{(m)}-\boldsymbol{U}_t\right)'\right]\boldsymbol{\lambda} \tag{44}$$

As $\boldsymbol{U}_t^{(m)} \to \boldsymbol{U}_t$ the expression above goes to zero as $n \to \infty$. Recall $\boldsymbol{U}$ is written in order to replace exactly $\boldsymbol{W}'\boldsymbol{E}$, then we go back to: $\boldsymbol{W}$,the process of interest.

Using Tchebychev inequality and the fact: $E(\boldsymbol{E}) = 0$, we obtain:

$$\begin{aligned} P\left(\left|\boldsymbol{W}_t^{(m)}-\boldsymbol{W}\right|>\epsilon\right) &= E\left|\boldsymbol{W}_t^{(m)}-\boldsymbol{W}\right|^2 \\ &= \epsilon^{-2}Var\left(\boldsymbol{W}_t^{(m)}-\boldsymbol{W}\right)\to 0 \end{aligned} \tag{45}$$

upon what we conclude $\boldsymbol{W}_t^{(m)} \to \boldsymbol{W}_t$ in mean square when $m \to \infty$.

Writing the characteristic function of $\boldsymbol{W}$ that is Normal by hypothesis and applying the Cramer-Wold device:

$$n^{-1/2}\boldsymbol{W}'\boldsymbol{E} \Rightarrow N\left(0,\sigma^2\boldsymbol{R}_p\right)$$

. Using the results (34) and (35), already proved above, we reach the conclusion:

$$\hat{\boldsymbol{\phi}} \Rightarrow N\left(\boldsymbol{\phi}, n^{-1}\sigma^2\boldsymbol{R}_p^{-1}\right) \qquad \blacksquare \tag{46}$$

which concludes the proof.