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RESEARCH REPORT

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Conditional Distribution of the Limit Order Book Given the History of the Best Quote Process

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Abstract

Recently, models of limit order markets, particularly those of the continuous double auction, are subject to an intense research. Due to their complexity, the models are regarded as analytically intractable. In the present paper, nonetheless, a closed form result is derived: the conditional distribution of the limit order book given the history of the best quote process.

Keywords: limit order markets, continuous double auction, limit order book, conditional distribution, spatial immigration-death process, market microstructure

AMS classification: 91B26
JEL classification: C51,G10

1 Introduction

Gone are the days when it was widely believed that the prices reach their equilibrium under all circumstances - it is clear now that the behavior of the prices depends strongly on the (micro)structure of the market [O'Hara, 1995].

In the present paper, the markets with the following rules are studied:

1. At any time, each agent may place a *buy (limit) order*, containing a maximal price and an order size, or a *sell (limit) order*, containing a minimal price and an order size. For simplicity, we assume that the order size is always unit.[1]

2. If a newly arrived limit order matches with the best (i.e. most advantageous) waiting limit order of the opposite type (let us call it *best counterpart*) then a trade is made for the limit price of the best counterpart (if there is more then one counterpart with the best limit price then the oldest one, i.e. the one with the earliest placement date, is executed).

3. If a newly arrived limit order finds no counterpart then it remains waiting until it is executed or it is canceled by its submitter.

The type of trading, described here, is usually called *continuous double auction* (CDA),[2] the list of all the currently waiting buy orders is called *buy limit order*

[1] Cf. Smith et al. [2003] for a partial justification of the assumption of the unit order size.

[2] Usually, also so called market orders, i.e. those without the limit price condition, are considered in the literature [Smith et al., 2003]. We do not handle them separately because they can be naturally modeled by means of the limit orders with the minimal/maximal possible limit price.

book, the highest limit price from the buy limit order book is called *(best) bid*, the list of all the currently valid sell orders is called *sell limit order book* and the lowest limit price from the sell limit order book is called *(best) ask*.

In reality, many markets possess the structure described above: some financial markets, first of all, but also various marketplaces, real estate markets,[3] trading made by means of the advertising in newspapers etc.

In the present paper, the complete randomness of the agents' actions is assumed. In particular, the times of the arrivals of the limit orders are assumed to follow a Poisson process, their limit prices are regarded as i.i.d. random variables independent on the arrival times and the lifetimes of the limit orders are supposed to be exponentially distributed and independent both on the arrival times and on the limit prices.

For a greater formal elegance and shorter notation, both the flow of the buy orders and the flow of the sell orders are regarded as time-spatial point processes. Though the point processes are increasingly used in modern finance [e.g. Engle and Russel, 1998], many readers may still be unfamiliar with this notion; therefore, the basic definitions and the properties of the point processes applied in the present work are summarized in a separate section of the paper so that no additional study of this area is needed to understand our results. For more information on the point processes, we refer the reader to Kallenberg [2002] or Daley and Vere-Jones [2003].

The model introduced by the present paper is a generalization of the one of Smith et al. [2003];[4] as opposed to the cited model, we allow a non-uniform density of the limit prices, our model comprises also the case of a continuous price space (the lack of ticks) and, moreover, the time is not discretized as in Smith et al. [2003].

The main achievement of the present paper is a description of the conditional distribution of the buy limit order book given the history of the best bid (the case of the sell limit order book is completely symmetric). The usage of this result may be wide, mainly when the limit order book is only partially published: having the best bid history, which is usually available, the rest of the limit order book may be estimated.

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[3] Here, however, there could be a problem with the heterogeneity of the commodity.

[4] To be precise: there are minor differences between the present model and the one of Smith et al. [2003]. At first, Smith et al. [2003] consider neither buy orders with a higher limit price then the best ask nor the sell orders with a lower limit price then the best bid. Second, while a finite intensity of the order flow is postulated by us, Smith et al. [2003] assume an infinite number of ticks, each with a separate Poisson flow of the orders, which leads to an infinite expected number of the orders per a finite time. The first difference is not crucial: the main result of the present paper would remain valid even if our model was modified not to include the mentioned orders. Either the second difference is minor: the limit orders "far from the bid/ask" play a little role in the area "near the bid/ask" which is of the main interest.

The paper is organized as follows: In Section 2, the mathematical model of the continuous double auction is introduced, in Section 3, the spatial immigration-death process is defined and several its properties are mentioned, in Section 4, the main result and its corollaries are stated, the paper is concluded by Section 5. In the Appendix, a formal definition of the CDA and some longer proofs may be found.

2 The Model of CDA

Denote $\tau^1_{b\star} \leq \tau^2_{b\star} \leq \dots$ the times of the arrivals of the buy orders. Further, denote x^i the limit price of the i-th buy order and $\tau^i_{b\dagger}$ the time at which the i-th buy order will be canceled if it is not executed until $\tau^i_{b\dagger}$ (let us call the time *cancelation time*). Analogously, denote $\tau^1_{s\star} \leq \tau^2_{s\star} \leq \dots$ the arrival times of the sell orders, y^j the limit price of the j-th sell order and $\tau^j_{s\dagger}$ the cancelation time of the j-th sell order. Assume that no pair of the events (i.e. the arrivals of the orders and their cancelations) happen at the same time.

If follows from the informal description of the CDA, given in the Introduction, that the buy order may find itself in four possible states:

> The buy order is *prenatal* if it has not yet arrived.
>
> The buy order is *waiting* if it has arrived but it did not find a counterpart yet.
>
> The buy order is *executed* if either it has found a matching counterpart immediately at the time of its arrival or it has been "found" by a newly arrived sell order after some time of waiting.
>
> The buy order is *canceled* if, after some time of waiting, it has been canceled by its submitter.

Clearly, the i-th buy order is in the state *prenatal* at the time 0 and it may change its state only at the times $\tau^i_{b\star}, \tau^i_{b\dagger}$, $\tau^1_{s\star}, \tau^2_{s\star}, \tau^3_{s\star}, \dots$. The following list specifies what may happen at those times:

At the time $\tau^i_{b\star}$: The order becomes either *executed* (if it matches the counterpart) or *waiting* (otherwise).

At the time $\tau^i_{b\dagger}$: If the order is still *waiting*, it becomes *canceled*, otherwise it remains in its current state (i.e. *executed*).

At a time $\tau^j_{s\star}$: The buy order becomes *executed* if it is currently the best buy order (i.e. the oldest of the waiting buy orders with the limit price equating to best bid) and if the limit price of the newly arrived sell order is no greater than the best bid. Otherwise, it remains in its current state (i.e. *prenatal*, *waiting*, *executed* or *canceled*).

Completely symmetric rules hold for the sell orders.

The limit order books are represented by counting measures[5] on $\mathbb{R}$. In particular, we define the *buy limit order book* as

$$B_t(Z) \triangleq |\{i : x^i \in Z, X_t^i = \mathit{waiting}\}|, \qquad Z \in \mathbb{B}(\mathbb{R}),$$

where X_t^i denotes the state of the i-th buy order at the time t, and the *sell limit order book* as

$$S_t(Z) \triangleq |\{j : y^j \in Z, Y_t^j = \mathit{waiting}\}|, \qquad Z \in \mathbb{B}(\mathbb{R}),$$

where Y_t^j denotes the state of the j-th sell order at the time t (the symbol $|\bullet|$ denotes the number of elements of the set, $\mathbb{B}(\Xi)$ denotes the Borel σ-algebra of a metric space Ξ).

Finally, we define the *(best) bid* as

$$b_t = \max\{p \in \mathbb{R} : B_t[p, \infty) > 0\}$$

and the *(best) ask* as

$$a_t = \min\{p \in \mathbb{R} : S_t(-\infty, p] > 0\}$$

(it is understood that $\max\emptyset = -\infty$ and $\min\emptyset = \infty$). The precise mathematical definition of the CDA is given in Appendix A.

3 Spatial Immigration-Death Process

First let us recall the definition of the Poisson point process.

Definition 1 *Let ρ be a σ-finite measure on a measurable space $(Z, \mathcal{Z})$. A random counting measure $\mathcal{N}$ on $(Z, \mathcal{Z})$ is a* Poisson point process (p.p.p.) with the intensity ρ *if*

(i) $\mathcal{N}(A) \sim \text{Poisson}(\rho(A))$ *for each $A \in \mathcal{Z}$ such that $\rho(A) < \infty$,*

(ii) for any disjoint $A^1, A^2, \ldots, A^i \in \mathcal{Z}$ the random variables $\mathcal{N}(A^1)$, $\mathcal{N}(A^2)$, $\ldots$, $\mathcal{N}(A^i)$ are independent.

[5] i.e. the measures whose values on the measurable sets are nonnegative integers

To illustrate the notion of the Poisson point process, let us show a simple example:

Example. Let $Z^1, Z^2, \ldots, Z^N$ be mutually independent standard Poisson processes with the intensities $m^1, m^2, \ldots, m^N$. Then the random measure $\mathcal{N}$ on $(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{B}(\mathbb{R}_0^+ \times \mathbb{R})\}$ defined by

$$\mathcal{N}((t^1, t^2] \times [s^1, s^2]) \triangleq \sum_{i \in \{1,2,\ldots,N\}, z^1 \leq i \leq z^2} (Z^i_{t^2} - Z^i_{t^1})$$

is a p.p.p. with the intensity

$$\lambda \otimes \left(\sum_{i=1}^{N} \delta_i \right)$$

where λ is the Lesbegue measure and δ_i is the Dirac measure concentrated in the point i.

Some p.p.p.'s may be represented as follows:

Lemma 1 *Let η be a finite measure on $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$, let $e^1, e^2, \ldots$ be i.i.d. random variables having the distribution $\eta/|\eta|$ and let $\Delta\tau^1_\star, \Delta\tau^2_\star, \ldots$ be i.i.d. random variables having the exponential distribution with mean $1/|\eta|$ such that e^i is independent of $\Delta\tau^j_\star$ for each $i, j \in \mathbb{N}$. Denote $\tau^i_\star \triangleq \sum_{j=1}^{i} \Delta\tau^i_\star$. Then the random measure*

$$\mathcal{N} \triangleq \sum_{i \in \mathbb{N}} \delta_{(\tau^i_\star, e^i)}$$

is a Poisson point process on $\mathbb{R}_0^+ \times \mathbb{R}$ with the intensity $\lambda \otimes \eta$.

Proof. See Kushner and Dupuis [2001], p. 30. □

Since Definition 1 determines the distribution of the p.p.p. uniquely [Kallenberg, 2002, Lemma 12.1.], we can assume that any p.p.p. with an intensity $\lambda \otimes \eta$, where η is some finite measure, is represented the way shown in Lemma 1.

Let us proceed with the definition of the spatial immigration and death process now:

Definition 2 *Let $c > 0$ be a real constant and let η be a finite measure on $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$. The signed random measure $\mathcal{I}$ is a* spatial immigration-death process (s.i.d.p.) with the intensity η and with the death rate c *if*

$$\mathcal{I} \stackrel{d}{=} \mathcal{N} - \mathcal{D}^c_{\mathcal{N}}, \qquad \mathcal{D}^c_{\mathcal{N}} = \sum_{i=1}^{\infty} \delta_{\xi^i + (\Delta\tau^i_\dagger, 0)},$$

where $\mathcal{N}$ is a Poisson point process with the intensity $\lambda \otimes \eta$, where $\Delta\tau_\dagger^i \sim \mathrm{Exp}(c)$, $i \in \mathbb{N}$, are mutually independent random variables independent of $\mathcal{N}$ and where $\xi^1, \xi^2, \ldots$ denote the atoms of $\mathcal{N}$ (the symbol $\stackrel{d}{=}$ means the identity of the distributions, the symbol $\mathrm{Exp}(m)$ denotes the exponential distribution with mean $1/m$).

Clearly, the immigration-death process may be represented by the collection $(e^i, \tau_\star^i, \tau_\dagger^i)_{i\in\mathbb{N}}$ where $\tau_\dagger^i \stackrel{\triangle}{=} \tau_\star^i + \Delta\tau_\dagger^i$ for each $i \in \mathbb{N}$. Moreover, it may be easily shown that $\tau_\star^1, \tau_\dagger^1, \tau_\star^2, \tau_\dagger^2, \ldots$ mutually differ almost sure and that, if η is absolutely continuous with respect to the Lesbegue measure, then also $e^1, e^2, \ldots$ mutually differ almost sure.

In the following Lemma, two useful properties of the s.i.d.p. are listed:

Lemma 2 *Let $\mathcal{I} = \mathcal{N} - \mathcal{D}^c_{\mathcal{N}}$ be an immigration-death process with the intensity η and with the death rate c. Put*

$$\mathcal{I}_M \stackrel{\triangle}{=} \mathcal{N}|_M - (\mathcal{D}^c_{\mathcal{N}|_M})|_M$$

for each $M \in \mathbb{B}(\mathbb{R}_0^+ \times \mathbb{R})$ (the symbol $\rho|_A$ denotes the restriction of the measure ρ to the subspace A). The following two statements hold true:

(i) If $M = (t, \tau] \times B$ for some $B \in \mathbb{B}(\mathbb{R})$ then

$$\mathcal{L}(|\mathcal{I}_M| = k) = \mathrm{Poisson}\left(\frac{\eta(B)}{c}[1 - \exp\{-c(\tau - t)\}]\right).$$

(ii) If $M, N \in \mathbb{B}(\mathbb{R}_0^+ \times \mathbb{R})$ are disjoint then $\mathcal{I}_M$ and $\mathcal{I}_N$ are independent.

Proof. See Appendix B. $\square$

4 CDA with Complete Randomness

Assume that the flow of the buy orders is ruled by a spatial immigration-death process $\mathcal{B}$ with an intensity ν, $|\nu| < \infty$, and with a positive death rate c: each atom with the positive weight standing for an arrival of a buy order, each atom with the negative weight denoting a cancelation of an order (naturally, the first coordinate of the atoms denotes the time and their second coordinate stands for the limit price).

Remark 1 *The flow of the buy orders, described above, may be equivalently defined by assuming that $\Delta\tau_{b\star}^i \stackrel{\triangle}{=} (\tau_{b\star}^i - \tau_{b\star}^{i-1}) \sim \mathrm{Exp}(|\nu|)$, that $x^i \sim \nu/|\nu|$, $\Delta\tau_{b\dagger}^i \stackrel{\triangle}{=} (\tau_{b\dagger}^i - \tau_{b\star}^i) \sim \mathrm{Exp}(c)$, $i \in \mathbb{N}$, and that $(\Delta\tau_{b\star}^i)_{i=1}^\infty$, $(\Delta\tau_{b\dagger}^i)_{i=1}^\infty$ and $(x^i)_{i=1}^\infty$ are mutually independent.*

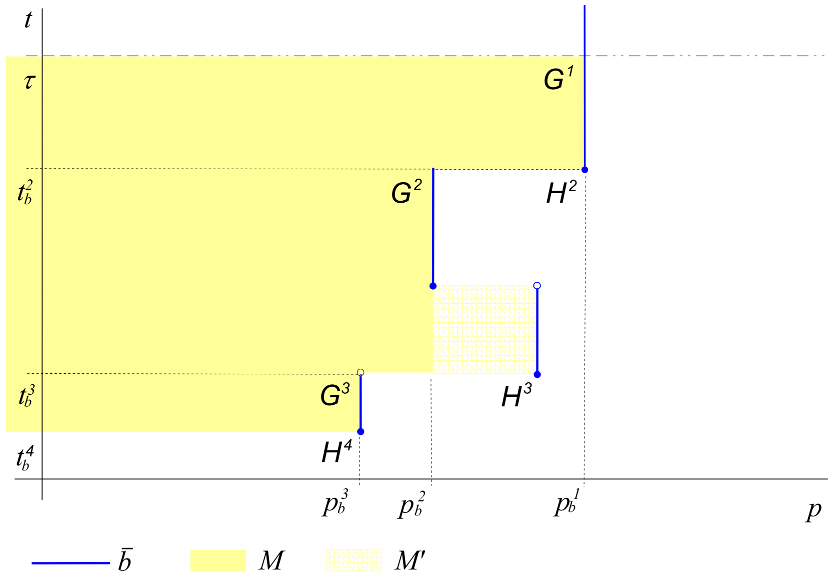

Figure 1: Illustration of the preview

Further, assume that the flow of the sell orders is ruled by a s.i.d.p. $\mathcal{S}$ with an intensity μ, $|\mu| < \infty$, and with a positive death rate d. An equivalent definition of the sell orders' flow, analogous to the one of the buy orders' flow, may be formulated.

Finally, assume that the processes $\mathcal{B}$ and $\mathcal{S}$ are independent.

As it was already mentioned, the main outcome of our paper is a description of the distribution of the buy limit order book at the time $\tau > 0$ given the history of the best bid process until τ. Since the exact specification of our result is a bit complicated, we start with a less formal "preview":

Suppose that the trajectory of the process $b_t, t \in [0, \tau]$, realized itself as $\bar{b}$ and that we want to determine the distribution of $|B_\tau|$, i.e. of the total number of the buy orders waiting at the time τ.

Our situation is illustrated by Figure 1. In addition to the objects depicted there, we may imagine the arrivals and the cancelations of the orders as points on the plane. It is clear from the definition of b_t that no arrival of a waiting buy order may appear right from $\bar{b}$ and that no waiting buy order whose arrival lies in M' may "live" at the time τ. Moreover, since only a single event may happen at a single time and since each upward jump of $\bar{b}$ mean an arrival of a new buy order with the limit price equal to the value the trajectory jumped to, no arrival of a buy order may lie either on the open segments $[G^2, H^2]$, $[G^3, H^3]$ or on the open ray started from H^4.

Hence, if a buy order is waiting at the time τ, then

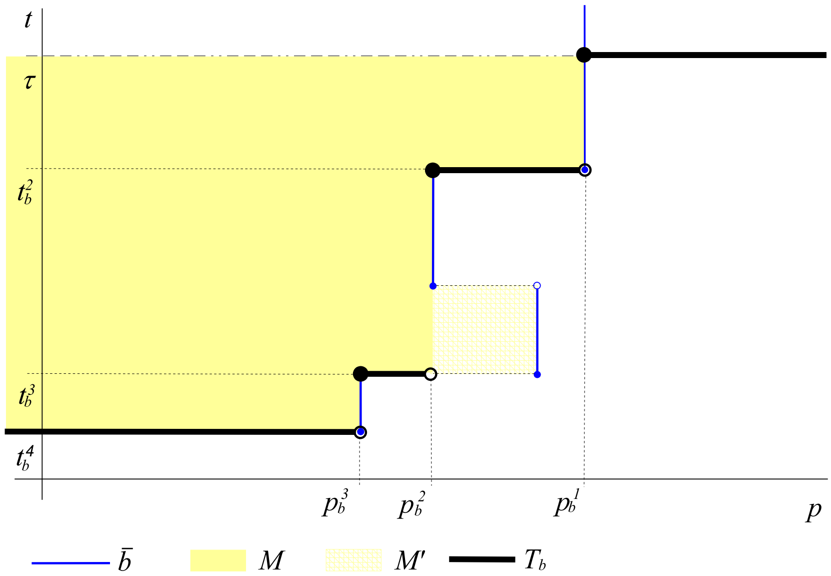

Figure 2: Illustration of the Theorem 1

1. either its arrival lies in the set M or its limit price is $p^i_{\bar{b}}$ for some i and it was waiting at the time $t^i_{\bar{b}}$ (it is understood that M contains the adjacent parts of the trajectory $\bar{b}$ but it does not contain the horizontal segments at its "bottom"),

2. the order has not been canceled before τ,

i.e.

$$|B_\tau| = |\mathcal{B}_M| + h^1 + h^2 + h^3$$

where h^i is the number of the buy orders with the limit price p^i_β waiting both at τ and at t^i_β.

Denote γ^i the number of the buy orders with the limit price p^i_β waiting at the time $t^i_{\bar{b}}$. If $\bar{b}$, γ^1, γ^2 and γ^3 were non-random, our work would be finished because the distribution of $|\mathcal{B}_M|$ is known (Lemma 2) and the distribution of h^i is easy to be determined; however, since $\bar{b}$ and γ^i are random, we have to examine the conditional distribution of $|\mathcal{B}_M|$ and h^i given $\bar{b}$ and γ^i instead of the unconditional one. Luckily, the conditional distribution appears to be the same as if $\bar{b}$ and γ^i were constant (see Theorem 1 and its proof).

Now let us tell the same in a precise way:

Theorem 1 *Let $n^1, n^2, \dots \in \mathbb{N}$. Denote $\mathcal{J}_\tau$ the space of all the right continuous piecewise constant mappings from $[0, \tau]$ into $\mathbb{R} \cup \{-\infty\}$ and let $\beta \in \mathcal{J}_\tau$.*

Denote

$$T_\beta(p) = \begin{cases} \min\{y \in [0,\tau] : p < \beta(z) \text{ for each } z \in [y,\tau]\} & p < \beta(\tau) \\ \tau & p \geq \beta(\tau), \end{cases}$$

denote I_β *the number of the jumps of* T_β, *denote*

$$p_\beta^1 > p_\beta^2 > \cdots > p_\beta^{I_\beta}$$

the points at which T_β *jumps (note that* $\beta(\tau) = p_\beta^1$*) and put* $p_\beta^i = -\infty$ *for each* $i > I_\beta$. *Finally, put*

$$t_\beta^i = \begin{cases} T(p_\beta^i) & \text{for each } i = 1,2,\dots,I_\beta \\ T(-\infty+) & \text{for } i = I_\beta + 1. \\ 0 & \text{for each } i > I_\beta + 1 \end{cases}$$

(see Figure 2 for an illustration). Finally, put $\gamma_\beta^i = B_{t_\beta^i}\{p_\beta^i\}$ *(it is understood that* $B_{t_\beta^i}\{-\infty\} = 0$*). Then, for each* $u \in \mathbb{R}$,

$$\mathcal{L}(B_\tau(-\infty,u)|\bar{b} = \beta, \gamma_{\bar{b}}^1 = n^1, \gamma_{\bar{b}}^2 = n^2, \dots) = \mathcal{L}\left(U + \sum_{1 \leq i \leq I_\beta, p_\beta^i < u} V^i\right) \qquad (1)$$

where

$$U \sim \text{Poisson}\left(\sum_{i=1}^{I_\beta} \frac{\nu[p_\beta^{i+1}, p_\beta^i \wedge u)}{c}[1 - \exp\{-c(\tau - t_\beta^{i+1})\}]\right)$$

and

$$V^i \sim \text{Binomial}(n^i, \exp\{-(\tau - t_\beta^i)c\})$$

such that $U, V^1, V^2, \dots, V^{I_\beta}$ *are mutually independent.*

Proof. See Appendix C. □

Corollary 1.1 *Let* $-\infty < u < v < \infty$. *Then the random variable* $B_\tau(-\infty,u)$ *is conditionally independent of* $B_\tau(u,v)$ *given* $(\bar{b}, \gamma_{\bar{b}}^1, \gamma_{\bar{b}}^2, \dots)$.

Proof. The Corollary may be proved analogously to the proof of Theorem 1. □

Corollary 1.2 *Let ν be absolutely continuous with respect to the Lesbegue measure. Then, for each $u \in \mathbb{R}$,*

$$\mathcal{L}(B_\tau(-\infty,u)|\bar{b}=\beta) = \mathcal{L}\left(U + \sum_{1\le i\le I_\beta, p^i_\beta<u} V^i\right)$$

where

$$U \sim \text{Poisson}\left(\sum_{i=1}^{I_\beta} \frac{\nu[p^{i+1}_\beta, p^i_\beta \wedge u)}{c}[1-\exp\{-c(\tau - t^{i+1}_\beta)\}]\right),$$

$$V^i \sim \text{Alternative}(\exp\{-(\tau - t^i_\beta)c\})$$

and $U, V^1, V^2, \dots, V^{I_\beta}$ are mutually independent.

Proof. Let $1 \le i \le I_\beta$. Since $x^1, x^2, \dots$ mutually differ almost sure, it has to be

$$\gamma^i_\beta \le 1 \tag{2}$$

almost sure. On the other hand, since

$$B_{t^i_\beta -}\{p^i_\beta\} \ge 1$$

from the definitions of t^i_β and p^i_β, since, at the time t^i_β, a new waiting buy order necessarily arrived and since only a single event may happen at a single time, nothing could have happened to the buy order with the limit price p^i_β at the time t^i_β hence

$$\gamma^i_\beta = B_{t^i_\beta}\{p^i_\beta\} \ge 1. \tag{3}$$

By the combination of (2) and (3) we finally get

$$[\bar{b}=\beta] \Longrightarrow [(\gamma^1_{\bar{b}}, \gamma^2_{\bar{b}}, \dots) = e_\beta], \qquad e_\beta = (\underbrace{1,1,\dots,1}_{I_\beta \text{ times}}, 0, 0, \dots)$$

so that

$$\mathcal{L}(B_\tau(-\infty,u)|\bar{b}=\beta) = \mathcal{L}(B_\tau(-\infty,u)|\bar{b}=\beta, (\gamma^1_{\bar{b}}, \gamma^2_{\bar{b}}, \dots) = e_\beta).$$

The rest may be got from Theorem 1.

□

Remark 2 *The assertion, concerning the distribution of the sell limit order book given the best ask history, is completely symmetric to Theorem 1.*

Remark 3 *Analogously to Theorem 1, the conditional distribution of the buy limit order book given the history of the i-th best waiting buy order may be derived.*

Remark 4 *Denote* $D_\tau(p) \stackrel{\triangle}{=} B_\tau[p,\infty)$ *the "aggregate" buy limit order book and assume, for simplicity, that* η *is absolutely continuous with respect to the Lesbegue measure. Then, by Theorem 1,*

$$\mathbb{E}(D_\tau(p)|\bar{b} = \beta) = u_\beta(p) + v_\beta(p)$$

where

$$u_\beta(p) = \sum_{i=1}^{I_\beta} \frac{\nu[p \vee p_\beta^{i+1}, p_\beta^i)}{c}[1 - \exp\{-c(\tau - t_\beta^{i+1})\}]$$

$$v_\beta(p) = \sum_{1 \le i \le I_\beta, p \le p_\beta^i} \exp\{-(\tau - t_\beta^i)c\}.$$

Since $t_\beta^1 > t_\beta^1 > \cdots > t_\beta^{I_\beta+1}$, *it is clear that* $u_\beta(p)$ *is convex while* $u_\beta(p)$ *is "concave" (i.e. the increments of its jumps decrease as i increases). Hence, it seems that, in thin markets, where the influence of the old buy orders is low, the limit order book should be convex. On the other hand, with the increasing intensity of the orders' flow, the aggregate buy limit order book should tend to be more and more "concave".*

5 Conclusion

In the paper, a formula of the conditional distribution of the limit order book given the best quote history has been derived. Using this result, some calculations, computed by the simulation so far, may be done directly (e.g. estimation of the price impact). Moreover, our findings may be compared with empirical data to confirm or falsify the assumptions of the model. In addition, it seems to be possible to use our results to estimate the parameters of the model.

It is clear that many of the assumptions of the model are restrictive: for instance, the Poisson flow of the orders implies that the activity of the agents does not depend on the development of the price. However, there is a large potential in future generalizations of the model.

A Definition of the Model of CDA

Let

$$X_i \in \{prenatal, waiting, executed, canceled\}, \qquad i = 1, 2, \ldots,$$

and

$$Y_j \in \{\mathit{prenatal}, \mathit{waiting}, \mathit{executed}, \mathit{canceled}\}, \qquad j = 1, 2, \dots,$$

bc processes on $\mathbb{R}_0^+$ defined as follows:

$$\boxed{\begin{array}{l} X_t^i = \begin{cases} \mathit{prenatal} & \text{if } t = 0 \\ \mathit{waiting} & \text{if } t = \tau_{b\star}^i \text{ and } x^i < a_{t-} \\ \mathit{executed} & \left[\begin{array}{l} \text{if } t = \tau_{b\star}^i \text{ and } x^i \geq a_{t-} \\ \text{or if } \left[\begin{array}{l} t = \tau_{s\star}^j \text{ for some } j \in \mathbb{N} \\ X_{t-}^i = \mathit{waiting} \\ x^i = b_{t-} \\ y^j \leq x^i \\ \{k < i : x^k = x^i, X_{t-}^k = \mathit{waiting}\} = \emptyset \end{array}\right. \end{array}\right. \\ \mathit{canceled} & \text{if } t = \tau_{b\dagger}^i \text{ and } X_{t-}^i = \mathit{waiting} \end{cases} \\ X_\bullet \text{ is constant right-continuous otherwise} \end{array}} \tag{4}$$

for each $i \in \mathbb{N}$ and

$$\boxed{\begin{array}{l} Y_t^j = \begin{cases} \mathit{prenatal} & \text{if } t = 0 \\ \mathit{waiting} & \text{if } t = \tau_{s\star}^j \text{ and } y^j > b_{t-} \\ \mathit{executed} & \left[\begin{array}{l} \text{if } t = \tau_{s\star}^j \text{ and } y^j \leq b_{t-} \\ \text{or if } \left[\begin{array}{l} t = \tau_{b\star}^i \text{ for some } i \in \mathbb{N} \\ Y_{t-}^j = \mathit{waiting} \\ y^j = a_{t-} \\ x^i \geq y^j \\ \{k < j : y^k = y^j, Y_{t-}^k = \mathit{waiting}\} = \emptyset \end{array}\right. \end{array}\right. \\ \mathit{canceled} & \text{if } t = \tau_{s\dagger}^j \text{ and } Y_{t-}^j = \mathit{waiting} \end{cases} \\ Y_\bullet \text{ is constant right-continuous otherwise} \end{array}} \tag{5}$$

for each $j \in \mathbb{N}$ ($\tau_{b\star}^i, \tau_{b\dagger}^i, \tau_{s\star}^j, \tau_{s\dagger}^j$, b_t and a_t are defined in Section 2).

B Proof of Lemma 2

(i) Let $\mathcal{N}$ be a p.p.p. with the intensity $\lambda \otimes \eta$. Clearly, $\mathcal{N}|_{(t,\infty)\times B}$ is a p.p.p. with the intensity $(\lambda \otimes \eta)|_B$ so that, by Lemma 1, it may be represented by $(e^i, \Delta\tau_\star^i)_{i\in\mathbb{N}}$ where $\Delta\tau_\star^i \sim \mathrm{Exp}(\eta(B))$ and $e^i \sim \frac{\eta|_B}{|\eta|_B|}$ such that $e^1, \Delta\tau_\star^1, e^2, \Delta\tau_\star^2, \dots$ are independent.

Further, denote $D_s \overset{\triangle}{=} \mathcal{I}_{(t,t+s]\times B}$. By a procedure, usual in the queueing theory (i.e. making the limits of the transition probabilities) we can get that D_s is a continuous time Markov chain with the intensity matrix

$$\begin{pmatrix} -\eta(B) & \eta(B) & 0 & 0 & \dots \\ c & -\eta(B)-c & \eta(B) & 0 & \dots \\ 0 & 2c & -\eta(B)-2c & \eta(B) & \dots \\ 0 & 0 & 3c & -\eta(B)-3c & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

whose distribution is known to be Poisson with mean

$$\frac{\eta(B)}{c}(1-\exp\{-cs\})$$

given that the process started from zero [Osaki, 1992, par. 9.3.3.]. Since $|\mathcal{I}_M| = D_{\tau-t}$, the part (i) is proved.

(ii) Let $\tau_\dagger^1, \dot{\tau_\dagger}^1, \bar{\tau_\dagger}^1, \tau_\dagger^2, \dot{\tau_\dagger}^2, \bar{\tau_\dagger}^2, \dots$ be a sequence of i.i.d. variables independent of $\mathcal{N}$ with the distribution $\mathrm{Exp}(c)$. Put

$$\Delta^M = \sum_{i=1}^{\infty} \mathbf{1}_M(\xi^i)\delta_{\xi^i+(\dot{\tau_\dagger}^i,0)}, \qquad \Delta^N = \sum_{i=1}^{\infty} \mathbf{1}_N(\xi^i)\delta_{\xi^i+(\bar{\tau_\dagger}^i,0)},$$

and

$$\Delta = \sum_{i=1}^{\infty} \mathbf{1}_{(\mathbb{R}_0^+\times\mathbb{R})\setminus(M\cup N)}(\xi^i)\delta_{\xi^i+(\tau_\dagger^i,0)},$$

where ξ^i are the atoms of $\mathcal{N}$ and $\mathbf{1}$ is the indicator function. Clearly,

$$\mathcal{I} \overset{d}{=} \mathcal{N} - (\Delta + \Delta^M + \Delta^N),$$

which implies, together with the definition of $\mathcal{I}_M$ and $\mathcal{I}_N$, that

$$(\mathcal{I}_M, \mathcal{I}_N) \overset{d}{=} \left(\mathcal{N}|_M - (\Delta^M)|_M, \mathcal{N}|_N - (\Delta^N)|_N\right).$$

Since, by the definition of the p.p.p., $\mathcal{N}|_M$ is independent of $\mathcal{N}|_N$ and since, by the definition of $(\tau_\dagger^1, \dot{\tau_\dagger}^1, \bar{\tau_\dagger}^1)_{i=1}^\infty$, Δ^M is independent of $(\mathcal{N}|_N, \Delta^N)$ and Δ^N is independent of $(\mathcal{N}|_M, \Delta^M)$, the independence of $\mathcal{I}_M$ and $\mathcal{I}_N$ is proved.

C Proof of Theorem 1

During the proof, we shall use the calculus of conditional probabilities. We refer the reader to monographs Hoffmann-Jørgensen [1994], chp. 6, or Kallenberg [2002], chp. 6, for rules of handling of conditional probabilities.

Clearly, if $p \in \mathbb{R}$, then

$$B_\tau\{p\} = |\{i : x^i = p, X^i_\tau = waiting\}| = m_p + n_p \tag{6}$$

where m_p is the number of the buy orders with the limit price p having risen during $(T_{\bar{b}}(p), \tau]$ and not being canceled or executed at τ and n_p is the number of the buy orders with the limit price p waiting at the time $T_{\bar{b}}(p)$ and not being canceled or executed at τ.

Since, by (4),

$$\boxed{\text{A buy order with the limit price } p \text{ is executed at the time } t.} \Longrightarrow \bar{b}_t \leq p.$$

and since $\bar{b}_t > p$ for each $t > T_{\bar{b}}(p)$, it has to be

$$t > T_{\bar{b}}(p) \Longrightarrow \boxed{\text{No buy order with the limit price } p \text{ was executed at the time } t}$$

hence m_p equals to the number of the buy orders having risen during $(T_{\bar{b}}(p), \tau]$ and not being canceled at τ and n_p equals to the number of the buy orders waiting at $T_{\bar{b}}(p)$ and not being canceled at τ, i.e.

$$m_p = |\mathcal{B}_{(T_{\bar{b}}(p),\tau]\times\{p\}}|,$$

and

$$n_p = W(p, T_{\bar{b}}(p)), \qquad W(p,t) = \left|\{i \in \mathbb{N} : x^i = p, X^i_t = waiting, \tau^i_{b\dagger} > \tau\}\right|.$$

Moreover, since $B_{T_{\bar{b}}(p)}\{p\} = 0$ for each $p^{i+1}_{\bar{b}} < p < p^i_{\bar{b}}$, $i = 0, 1, \ldots I_{\bar{b}}$[6] (we take $p^0_\beta = \infty$ and $t^0_\beta = \tau$), it has to be

$$W(p, T_{\bar{b}}(p)) = 0 \tag{7}$$

for each $p \notin \{p^1_{\bar{b}}, p^2_{\bar{b}}, \ldots\}$.

By summing (6) over all $p < u$ while applying (7), we are getting

$$B_\tau(-\infty, u) = |\mathcal{B}_{M_{\bar{b},u}}| + \sum_{1 \leq i \leq I_{\bar{b}}, p^i_{\bar{b}} < u} w^i_{\bar{b}}, \tag{8}$$

where

$$M_{\beta,u} = \{(t,p) \in \mathbb{R}^2 : T_\beta(p) < t \leq \tau\} \cap (\mathbb{R} \times (-\infty, u)).$$

and

$$w^i_{\bar{b}} = W(p^i_{\bar{b}}, t^i_{\bar{b}}).$$

[6] If $i = 0$ then the fact is trivial. If $i > 0$ then, from the definition, $T_{\bar{b}}(p) = T_{\bar{b}}(t^{i+1}_{\bar{b}})$. Since b_t jumped from $p^{i+1}_{\bar{b}}$ to $p^i_{\bar{b}}$ at the time $t^{i+1}_{\bar{b}}$ and since only a single event may happen at a single time, it follows that the limit order book B had no waiting orders with the limit price p at the time $t^{i+1}_{\bar{b}}$.

Further, denote

$$\Gamma_\beta \triangleq (\gamma_\beta^{)}, \gamma_\beta^2, \dots)$$

and

$$\upsilon \triangleq (n^1, n^2, \dots).$$

From (8) it follows that, to prove the Theorem, it suffices to verify the following three statements:

(a) $\mathcal{L}(|\mathcal{B}_{M_{\bar{b},u}}||\bar{b} = \beta, \Gamma_{\bar{b}} = \upsilon) = \mathcal{L}(U)$

(b) $\mathcal{L}(w_{\bar{b}}^i|\bar{b} = \beta, \Gamma_{\bar{b}} = \upsilon) = \mathcal{L}(V^i)$

(c) $|\mathcal{B}_{M_{\bar{b},u}}|, w_{\bar{b}}^1, w_{\bar{b}}^2, \dots$ are conditionally independent given $\bar{b} = \beta$ and $\Gamma_{\bar{b}} = \upsilon$.

We start with (a). Denote $F = [0, \tau] \times \mathbb{R}$ and let $e \in \mathcal{J}_\tau$. Clearly

$$(\bar{b}, \Gamma_{\bar{b}}) = G_e(\mathcal{B}_F, \mathcal{S}_F)$$

for some mapping G. Further, denote

$$C_e = \{f \in \mathcal{J}_\tau : f \geq e\}.$$

It follows from definitions (4) and (5) that neither the points risen in $M_{e,u}$ nor the anti-points of the points risen in $F \setminus M_{e,u}$ that fell outside $F \setminus M_{e,u}$ change $\bar{b}$ or $\Gamma_{\bar{b}}$ given that $\bar{b} \in C_e$. In other words, it means that G is constant both in $\mathcal{B}|_{M_{e,u}}$ and in $[(\mathcal{B}|_{F \setminus M_{e,u}})^-]|_{M_{e,u}}$ on the set $[\bar{b} \in C_e]$, i.e. there exists G_e such that

$$(\bar{b}, \Gamma_{\bar{b}}) = G_e(\mathcal{B}_{F \setminus M_{e,u}}, \mathcal{S}_F)$$

on the set $[\bar{b} \in C_e]$.

Since $\mathcal{B}_{M_{e,u}}$ is independent both of $\mathcal{B}_{F \setminus M_{e,u}}$ (by Lemma 2) and of $\mathcal{S}_F$ (by our assumptions), we have

$$\begin{aligned}
&\mathbb{P}(|\mathcal{B}_{M_{e,u}}| = k, \bar{b} \geq f, \Gamma_{\bar{b}} \geq p) \\
&\qquad = \mathbb{P}(|\mathcal{B}_{M_{e,u}}| = k, G_e(\mathcal{B}_{F - M_{e,u}}, \mathcal{S}_F) \geq (f, p)) \\
&\qquad = \mathbb{P}(|\mathcal{B}_{M_{e,u}}| = k)\mathbb{P}(G_e(\mathcal{B}_{F - M_{e,u}}, \mathcal{S}_F) \geq (f, p)) \\
&\qquad = \pi_{e,u}(k)\mathbb{P}(\bar{b} \geq f, \Gamma_{\bar{b}} \geq p),
\end{aligned}$$

where $\pi_{e,u}(k) = \mathbb{P}(|\mathcal{B}_{M_{e,u}}| = k)$ for each $f \in C_e$ and $p \in \mathbb{N}^\infty$ where $\mathbb{N}^\infty \triangleq \mathbb{N} \times \mathbb{N} \times \dots$. Using it, we get

$$\begin{aligned}
&\int_{h \geq f, \zeta \geq p} \mathbb{P}(|\mathcal{B}_{M_{e,u}}| = k|\bar{b} = h, \Gamma = \zeta) dQ(h, \zeta) \\
&\qquad = \mathbb{P}(|\mathcal{B}_{M_{e,u}}| = k, \bar{b} \geq f, \Gamma_{\bar{b}} \geq p) = \pi_{e,u}(k) \int_{h \geq f, \zeta \geq p} dQ(h, \zeta) \\
&\qquad = \int_{h \geq f, \zeta \geq p} \pi_{e,u}(k) dQ(h, \zeta),
\end{aligned}$$

where Q is the distribution of $(\bar{b}, \Gamma_{\bar{b}})$, for each $f \in C_e$ and $p \in \mathbb{N}^\infty$. Hence, it has to be

$$\mathbb{P}(|\mathcal{B}_{M_{e,u}}| = k|\bar{b} = h, \Gamma_{\bar{b}} = \zeta) = \pi_{e,u}(k)$$

almost sure for each $e \in \mathcal{J}_\tau$, $h \in C_e$ and $\zeta \in \mathbb{N}^\infty$ and, particularly,

$$\mathbb{P}(|\mathcal{B}_{M_{\beta,u}}| = k|\bar{b} = \beta, \Gamma_{\bar{b}} = \upsilon) = \pi_{M_{\beta,u}}(k).$$

Finally, since

$$M_{\beta,u} = \bigcup_{i=0}^{I_\tau} (t_\beta^{i+1}, \tau] \times [p_\beta^{i+1}, p_\beta^i \wedge u)$$

and since $t_{\bar{b}}^1 \equiv \tau$, it is easy to get (a) by means of Lemma 2.

(b) Denote

$$J^i = \{j \in \mathbb{N} : x^j = p_\beta^j, X_{t_\beta^i}^j = waiting\}.$$

Clearly,

$$j \in J^i \iff [x^j = p_\beta^j] \wedge [\tau_{b\dagger}^j > t_\beta^i] \wedge [\tau_{b\star}^j \leq t^i] \wedge C_{i,j} \tag{9}$$

where

$$C_{i,j} = [\text{the } j\text{-th buy order was not executed until } t_\beta^i].$$

Denote $J = (J^1, J^2, \dots)$. By using the Complete Probability Theorem, we get

$$\begin{aligned} &\mathbb{P}(w_\beta^i = k|\bar{b} = \beta, \Gamma_{\bar{b}} = \upsilon) = \\ &= \sum_{\pi \subset \mathbb{N}^\infty} \mathbb{P}(w_\beta^i = k|\bar{b} = \beta, \Gamma_{\bar{b}} = \upsilon, J = \pi)\mathbb{P}(J = \pi|\bar{b} = \beta, \Gamma_{\bar{b}} = \upsilon) \\ &= \sum_{\substack{\pi \subset \mathbb{N}^\infty, |\pi^1| = n^1, |\pi^2| = w_p, \dots, \\ \pi^m \cap \pi^n = \emptyset}} \mathbb{P}(w_\beta^i = k|\bar{b} = \beta, J = \pi)\mathbb{P}(J = \pi|\bar{b} = \beta, \Gamma_{\bar{b}} = \upsilon) \end{aligned} \tag{10}$$

where, π^j denotes the j-th component of π (it is because $\Gamma_{\bar{b}} = \upsilon$ is implied by $J = \pi$ given that π fulfils the condition below the sum). Clearly

$$\begin{aligned} &\mathbb{P}(w_\beta^i = k|\bar{b} = \beta, J = \pi) \\ &= \mathbb{P}(\text{exactly } k \text{ values from } \{t_1, \dots, t_{|\pi^i|}\} \text{ is greater then } \tau|\bar{b} = \beta, J = \pi) \end{aligned} \tag{11}$$

where $\tau_{b\dagger}^{\pi^i} \triangleq (\tau_{b\dagger}^j)_{j \in \pi^i}$.

Further, denote $\mathcal{E}^{-\pi^i}$ the vector of all the variables

$$(\tau_{b\star}^1, \Delta\tau_{b\dagger}^1, x^1, \tau_{s\star}^1, \Delta\tau_{s\dagger}^1, y^1, \tau_{b\star}^2, \Delta\tau_{b\dagger}^2, x^2, \tau_{s\star}^2, \Delta\tau_{s\dagger}^2, y^2, \dots).$$

except for $(\tau_{b\star}^{\pi^i}, \Delta\tau_{b\dagger}^{\pi^i})$. Since $(\tau_{b\star}^{\pi^i}, \Delta\tau_{b\dagger}^{\pi^i}, \mathcal{E}^{-\pi^i})$ describes all the randomness of our system, there has to exist a mapping F_{π^i} such that

$$(\bar{b}, J) = F_{\pi^i}(\tau_{b\star}^{\pi^i}, \Delta\tau_{b\dagger}^{\pi^i}, \mathcal{E}^{-\pi^i}).$$

Since, by (9),

$$
\begin{aligned}
[J=\pi]\cap[\bar b=\beta] = [x^j=p^j_\beta] &\cap [j\in J^i \text{ for each } j\in\pi^i, i\in\mathbb{N}] \\
&\cap [j\notin J^i \text{ for each } j\notin\pi^i, i\in\mathbb{N}]\cap[\bar b=\beta] \\
= [x^j=p^j_\beta] &\cap [\Delta\tau^{\pi^i}_{b\dagger}+\tau^{\pi^i}_{b\star}>t^i_\beta \text{ for each } i\in\mathbb{N}] \qquad (12)\\
&\cap [\tau^{\pi^i}_{b\star}\le t^i, i\in\mathbb{N}]\cap[C_{i,j}, j\in\pi^i, i\in\mathbb{N}] \\
&\cap [j\notin J^i \text{ for each } j\notin\pi^i, i\in\mathbb{N}]\cap[\bar b=\beta]
\end{aligned}
$$

and since none of the conditions in (12) depend on $(\Delta\tau^{\pi^i}_{b\dagger})$ given that $\Delta\tau^{\pi^i}_{b\dagger}+\tau^{\pi^i}_{b\star}>t^i_\beta$ holds true, F_{π^i} has to be constant in $(\Delta\tau^{\pi^i}_{b\dagger})$ given $\Delta\tau^{\pi^i}_{b\dagger}+\tau^{\pi^i}_{b\star}>t^i_\beta$ so there has to exist a mapping f_{π^i} such that

$$
[J=\pi]\cap[\bar b=\beta]=[\Delta\tau^{\pi^i}_{b\dagger}+\tau^{\pi^i}_{b\star}>t^i_\beta]\cap[f_{\pi^i}(\tau^{\pi^i}_{b\star},\mathcal{E}^{-\pi^i})=(\beta,\pi)]. \tag{13}
$$

Let $(t,\epsilon)\in f^{-1}_{\pi^i}(\beta,\pi)$ and $s=(s^1,\dots,s^{|\pi^i|})\ge(t^i_\beta,\dots,t^i_\beta)$. Since $t\le t^i_\beta$, by (12), it has to be

$$
s-t\ge t^i_\beta-t\ge 0.
$$

Using this and the independence of $\Delta\tau^{\pi^i}_{b\dagger}$ of $(\tau^{\pi^i}_{b\star},\mathcal{E}^{-\pi^i})$, we get

$$
\begin{aligned}
&\mathbb{P}(\tau^{\pi^i}_{b\dagger}>s|\Delta\tau^{\pi^i}_{b\dagger}+\tau^{\pi^i}_{b\star}>t^i_\beta,\tau^{\pi^i}_{b\star}=t,\mathcal{E}^{-\pi^i}=\epsilon)\\
&\qquad=\mathbb{P}(\Delta\tau^{\pi^i}_{b\dagger}>s-t|\Delta\tau^{\pi^i}_{b\dagger}>t^i_\beta-t,\tau^{\pi^i}_{b\star}=t,\mathcal{E}^{-\pi^i}=\epsilon)\\
&\qquad=\frac{\mathbb{P}(\Delta\tau^{\pi^i}_{b\dagger}>s-t|\tau^{\pi^i}_{b\star}=t,\mathcal{E}^{-\pi^i}=\epsilon)}{\mathbb{P}(\Delta\tau^{\pi^i}_{b\dagger}>t^i_\beta-t|\tau^{\pi^i}_{b\star}=t,\mathcal{E}^{-\pi^i}=\epsilon)}\\
&\qquad=\frac{\mathbb{P}(\Delta\tau^{\pi^i}_{b\dagger}>s-t)}{\mathbb{P}(\Delta\tau^{\pi^i}_{b\dagger}>t^i_\beta-t)} \qquad\qquad (14)\\
&\qquad=\prod_{p=1}^{|\pi^i|}\exp\{-c(s^p-t^i_\beta)\}.
\end{aligned}
$$

be an easy calculation.

Finally, since the conditional probability (14) is constant on the set $\{(t,\epsilon)\in f^{-1}_{\pi^i}(\beta,\pi)\}$ for each (β,π), it is measurable with respect to $\sigma(\bar b,J)$, so that it may serve as the conditional probability with respect to $(\bar b,J)$, i.e.

$$
\mathbb{P}(\tau^{\pi^i}_{b\dagger}>s|J=\pi,\bar b=\beta)=\prod_{p=1}^{|\pi^i|}\exp\{-c(s^p-t^i_\beta)\}. \tag{15}
$$

which yields, together with (11), that

$$
\mathbb{P}(w^i_\beta=k|J=\pi,\bar b=\beta)=q_{\beta,|\pi^i|,i,k}, \tag{16}
$$

where

$$q_{\beta,n,i,k} = \mathbb{P}(\text{Binomial}(n, \exp\{-c(\tau - t^i_\beta)\}) = k).$$

By imposing (16) into (10), we finally get

$$\mathbb{P}(w^i_\beta = k|\bar{b} = \beta, \Gamma_{\bar{b}} = \upsilon) = q_{\beta,n^i,i,k}$$

which was to prove.

(c) Fix J from part (b) and observe that none of the conditions from (12) changes if we vary $\tau^{\pi^i}_{b\dagger}$ or $\tau^{\pi^j}_{b\dagger}$ given that $\Delta\tau^{\pi^i}_{b\dagger} + \tau^{\pi^i}_{b\star} > t^i_\beta$ and $\Delta\tau^{\pi^j}_{b\dagger} + \tau^{\pi^j}_{b\star} > t^j_\beta$. Hence, it may be shown analogously to the part (b) that

$$\begin{aligned}
&\mathbb{P}(\tau^{\pi^i}_{b\dagger} > s_1, \tau^{\pi^j}_{b\dagger} > s_2|J = \pi, \bar{b} = \beta) \\
&\qquad\qquad = \prod_{p=1}^{|\pi^i|} \exp\{-c(s^p_1 - t^i_\beta)\} \prod_{p=1}^{|\pi^j|} \exp\{-c(s^p_2 - t^j_\beta)\}
\end{aligned}$$

which suffices for the conditional independence of $|w^i_{\bar{b}}|$ and $|w^j_{\bar{b}}|$.

It remains to prove the conditional independence of $|\mathcal{B}_{M_{\beta,u}}|$ and w^i_β. But it is easy. Since $|\mathcal{B}_{M_{\beta,u}}| = \phi_{\pi^i}(\mathcal{E}^{-\pi^i})$ for some function ϕ_{π^i} given that $J = \pi$, we have

$$\begin{aligned}
&\mathbb{P}(\tau^{\pi^i}_{b\dagger} > s||\mathcal{B}_{M_{\beta,u}}| = k, J = \pi, \bar{b} = \beta) \\
&\overset{(9)}{=} \mathbb{P}(\tau^{\pi^i}_{b\dagger} > s|\mathcal{E}^{-\pi^i} \in \phi^{-1}_{\pi^i}(m), \Delta\tau^{\pi^i}_{b\dagger} + \tau^{\pi^i}_{b\star} > t^i_\beta, (\tau^{\pi^i}_{b\star}, \mathcal{E}^{-\pi^i}) \in f^{-1}_{\pi^i}(\pi, \beta)) \\
&= \prod_{p=1}^{|\pi^i|} \exp\{-c(s^p - t^i_\beta)\}
\end{aligned} \tag{17}$$

by the same logic as in (b), hence

$$\mathbb{P}(\tau^{\pi^i}_{b\dagger} > s||\mathcal{B}_{M_{\bar{b},u}}| = k, J = \pi, \bar{b} = \beta) \overset{(15),(17)}{=} \mathbb{P}(\tau^{\pi^i}_{b\dagger} > s|J = \pi, \bar{b} = \beta). \tag{18}$$

which proves the conditional independence of $\tau^{\pi^i}_{b\dagger}$ and $|\mathcal{B}_{M_{\bar{b},u}}|$. Since $w^i_{\bar{b}}$ is a function of $\tau^{\pi^i}_{b\dagger}$ given that $J = \pi$, also the conditional independence of $|\mathcal{B}_{M_{\bar{b},u}}|$ and $w^i_{\bar{b}}$ is proved by (18).

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