Powerful Trend Function Tests That are Robust to Strong Serial Correlation with an Application to the Prebish Singer Hypothesis*

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Abstract

In this paper we propose tests for hypothesis regarding the parameters of a the deterministic trend function of a univariate time series. The tests do not require knowledge of the form of serial correlation in the data and they are robust to strong serial correlation. The data can contain a unit root and the tests still have the correct size asymptotically. The tests we analyze are standard heteroskedasticity autocorrelation (HAC) robust tests based on nonparametric kernel variance estimators. We analyze these tests using the small-b asymptotic framework recently proposed by Kiefer and Vogelsang (2002). This analysis allows us to analyze the power properties of the tests with regards to bandwidth and kernel choices. Our analysis shows that among popular kernels, there are specific kernel and bandwidth choices that deliver tests with maximal power within a specific class of tests. We apply the recommended tests to the logarithm of a net barter terms of trade series and we find that this series has a statistically significant negative slope. This finding is consistent with the well known Prebisch-Singer hypothesis. Because our tests are robust to strong serial correlation or a unit root in the data, our results in support of the Prebisch-Singer hypothesis are relatively strong.

Keywords: HAC Estimator, Fixed-b Asymptotics, Power Envelope, Unit Root, Nearly Integrated, Partial Sum, Deterministic Trend, Linear Trend.

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1 Introduction

In this paper we propose tests of linear hypotheses on the parameters in a univariate deterministic trend model. The tests are designed to have optimal power when the errors are stationary and to be size-robust to strong serial correlation in the errors including the case of a unit root in the errors. Robustness to serial correlation is obtained using well known nonparametric heteroskedasticity autocorrelation (HAC) robust standard errors. Using the newly developed fixed bandwidth, i.e. fixed-b, asymptotics of Kiefer and Vogelsang (2002) we show that standard HAC robust trend tests have asymptotic distributions free of serial correlation nuisance parameters regardless of the bandwidth or kernel used to compute the HAC robust standard errors. This asymptotic pivotal result holds for stationary errors as well as unit root errors although the limiting distributions are different in the case of unit root errors compared to the case of stationary errors. This difference in limiting distributions explains the usual over-rejection problem of HAC robust tests. Because the tests are asymptotically pivotal, we are able to control the over-rejection problem by implementing the scaling correction factor proposed by Vogelsang (1998). Therefore, the tests we propose have well behaved size even when the errors have strong serial correlation.

For the special case of the simple linear trend model, we use a local asymptotic power analysis to guide the choice of kernel and bandwidth. Confining attention to tests with correct size, we consider a class of well known and popular kernels and we compute asymptotic power envelopes that represent maximal power across the kernels and bandwidths. We then show that tests based on the Daniell kernel have power that effectively attains the power envelope. If we let $b = M/T$ where T is the sample size and M is the truncation lag or bandwidth used in the HAC estimator, then $b = 0.02$ delivers a test with power nearly identical to the stationary power envelope when the errors are stationary. When the errors have a unit root, $b = 0.16$ delivers a test with power nearly identical to the unit root power envelope. The fact that we make concrete and specific recommendations for kernel and bandwidth choices should appeal to practitioners.

We use the newly developed tests to investigate the well known Prebisch (1950) and Singer (1950) hypothesis that postulates that over time the net barter terms of trade should be declining between countries that primarily export commodities and countries that primarily export manufactures. This empirical conjecture has received considerable attention in the international economics literature. See Ardeni and Wright (1992), Cuddington and Urzua (1989), Grilli and Yang (1988), Lutz (1992), Powell (1991), Sapsford (1985), Spraos (1980) and Trivedi (1995) among others. The empirical results in this literature have been mixed. Many authors have interpreted evidence in support of the Prebisch-Singer hypothesis with caution because of the potential over-rejection problem caused by strong serial correlation/unit root in the errors. In fact, many authors have focused on, and in our opinion been distracted by, the question as to whether or not the innovations have

a unit root or are stationary. Because a time series can have a decreasing deterministic trend whether the innovations are stationary or have a unit root, the unit root issue is simply a nuisance parameter in the context of the Prebisch-Singer hypothesis. The advantage of our tests is that they allow a direct test on the slope coefficient of the linear trend that is robust to the unit root question. When applied to the net barter terms of trade series used by Lutz (1992)[1] we find strong and consistent evidence to support the Prebisch-Singer hypothesis. Our results are not subject to the usual "over-rejection problem" critique because of the robust properties of the tests. Further tests indicate that the trend function of this series is stable over time. Our results confirm what many authors have been saying for over 20 years: Prebisch and Singer were right!

The rest of the paper is organized as follows. In Section 2 the trend function model is described in detail, the required assumptions are stated, and some of the basic asymptotic results are presented. Section 3 describes the scaling procedure that is used to control the over-rejection problem caused by strong serial correlation. In Section 4 we derive and discuss the asymptotic results under the new fixed-b asymptotics. In Section 4 the test statistics are defined and the asymptotic distribution theory is developed. In Section 5 we examine the asymptotic properties of the test statistics in the simple linear trend model. We compute asymptotic power envelopes and determine kernels and bandwidths that deliver tests with power close to the envelopes. In Section 6 the results of some finite sample simulation experiments are reported. The empirical results on the Prebisch-Singer hypothesis are given in Section 7. Section 8 concludes and proofs of important results are collected in the appendix.

2 The Model Setup

We are interested in the following model of a time series with deterministic trends:

$$y_t = f(t)' \beta + u_t, \quad t = 1, ..., T, \tag{1}$$

where $f(t)$ denotes a $(k \times 1)$ vector of trend functions, β is a $(k \times 1)$ vector of parameters, and " $'$ " denotes the transpose, when used in the context of a vector. This type of model is used frequently in macroeconomics and finance to determine the composition of individual data series, like GDP. When performing tests on β, for example to determine whether a given trend should be included, the presence of serial correlation and heteroskedasticity in the errors must be taken into account. In this paper, we will concern ourselves with the situation where the exact error structure is not of interest. In that case, there is no need to model the error structure explicitly, as tests on the coefficients on the trends can be tested without doing so. Testing hypotheses on the coefficients of the trends without modelling the error structure is virtually always done by using HAC estimators

[1] Lutz (1992) extended the well known Grilli and Yang (1988) series to 1995.

to estimate the asymptotic variance of the parameter estimates, and that follow that approach in this paper.

Throughout the paper we assume that u_t is a scalar, mean zero time series. The time series process $\{u_t\}$ may be stationary or have a unit root, and may exhibit serial correlation or conditional heteroskedasticity. For the purpose of studying the impact of these various error specifications on the testing procedures, we make the following flexible assumptions about u_t.

Assumption 1

$$u_t = \alpha u_{t-1} + \varepsilon_t, \quad t = 2, 3, ..., T, \quad u_1 = \varepsilon_1,$$

$$\varepsilon_t = d(L) e_t, \quad d(L) = \sum_{i=0}^{\infty} d_i L^i, \quad \sum_{i=0}^{\infty} i |d_i| < \infty, \quad d(1)^2 = d(1)^2 > 0,$$

where $\{e_t\}$ is a martingale difference sequence with $E\left(e_t^2 | e_{t-1}, e_{t-2}, ...\right) = 1$ and $\sup_t E\left(e_t^4\right) < \infty$. Under this specification, the errors are stationary when $|\alpha| < 1$. Alternatively, the errors can be modeled as local to a unit root by letting $\alpha = \left(1 - \frac{\overline{\alpha}}{T}\right)$ where $\overline{\alpha} = 0$ corresponds to a pure unit root process.

Under Assumption 1 the following functional central limit theorems follow from well known results (see Chan and Wei (1988), Phillips (1987) and Phillips and Solo (1992)):

$$\begin{aligned} T^{-1/2} \sum_{t=1}^{[rT]} u_t &\Rightarrow \sigma w(r) \text{ if } |\alpha| < 1 \\ T^{-1/2} u_{[rT]} &\Rightarrow d(1) V_{\overline{\alpha}}(r) \text{ if } \alpha = 1 - \frac{\overline{\alpha}}{T}, \end{aligned}$$

where $\sigma^2 = d(1)^2/(1-\alpha)^2$, $w(r)$ is a standard Wiener process, $V_{\overline{\alpha}}(r) = \int_0^r \exp\left(-\overline{\alpha}(r-s)\right) dw(s)$ and $\Rightarrow$ denotes weak convergence.

At times it will be useful to stack the equations in (1) and rewrite them as

$$y = \mathbf{f}(T) \beta + u. \tag{2}$$

Here $\mathbf{f}(T)$ is the $(T \times k)$ stacked vector of trend functions. The following assumptions on the trend are sufficient to obtain the main results of the paper:

Assumption 2 $f(t)$ includes a constant, there exists a $(k \times k)$ diagonal matrix τ_T and a vector of functions F, such that $\tau_T f(t) = F\left(\frac{t}{T}\right) + o(1)$, $\int_0^1 F_i(r) dr < \infty$, $i = 1, ..., k$, and $\det\left[\int_0^1 F(r) F(r)' dr\right] > 0$.

Assumption 2 is essentially the same assumption used by Vogelsang (1998) and is fairly standard. We include the additional assumption that an intercept is included in the model.

Model (1) is estimated using Ordinary Least Squares (OLS) and $\widehat{\beta} = (\mathbf{f}(T)'\mathbf{f}(T))^{-1}\mathbf{f}(T)'y$ denotes the OLS estimate of β, while $\widehat{u} = y - \mathbf{f}(T)\widehat{\beta}$ denotes the OLS residuals. The limiting distribution of $\widehat{\beta}$ is well known for both stationary and unit root errors:

$$\begin{aligned} T^{1/2}\tau_T^{-1}\left(\widehat{\beta}-\beta\right) &\Rightarrow \sigma\left(\int_0^1 F(r)F(r)'dr\right)^{-1}\int_0^1 F(r)\,dw(r) \quad \text{if} \quad |\alpha|<1, \\ T^{-1/2}\tau_T^{-1}\left(\widehat{\beta}-\beta\right) &\Rightarrow d(1)\left(\int_0^1 F(r)F(r)'dr\right)^{-1}\int_0^1 F(r)\,V_{\overline{\alpha}}(r)\,dr \quad \text{if} \quad \alpha = 1-\frac{\overline{\alpha}}{T}. \end{aligned}$$

Notice that when the errors are stationary the only unknown nuisance parameter in the limiting distribution is σ and when the errors are integrated the only unknown nuisance parameters are $d(1)$ and $\overline{\alpha}$. We focus on the result for stationary errors and consider standard HAC robust tests designed for that case. The tests will be asymptotically pivotal when the errors are stationary and only depend on $\overline{\alpha}$ when the errors are integrated. We deal with the dependence on $\overline{\alpha}$ by using the scaling factor approach proposed by Vogelsang (1998).

To construct the usual HAC robust t or Wald tests, an estimator of σ^2 is often used.[2] We consider the case where σ^2 is estimated nonparametrically using the OLS residuals, $\widehat{u}_t$:

$$\widehat{\sigma}^2 = \widehat{\gamma}_0 + 2\sum_{j=1}^{T-1} k(j/M)\,\widehat{\gamma}_j, \tag{3}$$

where $\widehat{\gamma}_j = T^{-1}\sum_{t=j+1}^{T}\widehat{u}_t\widehat{u}_{t-j}$ and $k(x)$ is a kernel function satisfying $k(x) = k(-x)$, $k(0) = 1$, $|k(x)| \leq 1$, $k(x)$ continuous at $x = 0$ and $\int_0^1 k^2(x)\,dx < \infty$. M is called the bandwidth or the truncation lag. For $\widehat{\sigma}^2$ to be consistent, it is necessary to downweight or eliminate the sample autocovariances for high values of j. Specifically, it is necessary that $M \to \infty$ and $M/T \to 0$ as $T \to \infty$. Most commonly used kernel functions have the property that $k(x) = 0$ for $|x| > 1$, effectively eliminating the sample autocovariances for all values of j greater than M, inspiring the name truncation lag.

We are interested in testing hypotheses of the form $H_0 : R\beta = q$. Typically, R is simply a matrix selecting single entries of β, and q is a vector of zeros, but we maintain the hypothesis in its general form. As a rule, the test statistics used to test this type of hypothesis on the trend function

[2]The fact that a single nuisance parameter appears in the limiting distribution of the OLS estimates occurs because the regressors are deterministic. In a regression model with random regressors, the asymptotic variance of the OLS estimates depends on a zero-frequency spectral density matrix with rank equal to the number of regression parameters. In that case, the HAC robust standard errors are computed using a vector of time series comprised of products of the regressors and OLS residuals.

are either t or Wald statistics of the form:

$$t = \frac{R\widehat{\beta} - r}{\sqrt{\widehat{\sigma}^2 R \left(\mathbf{f}\left(T\right)' \mathbf{f}\left(T\right)\right)^{-1} R'}}$$

$$W_T = \left(R\widehat{\beta} - r\right)' \left[\widehat{\sigma}^2 R \left(\mathbf{f}\left(T\right)' \mathbf{f}\left(T\right)\right)^{-1} R'\right]^{-1} \left(R\widehat{\beta} - r\right).$$

If the errors are stationary and $\widehat{\sigma}^2$ is a consistent estimator, then the t test has a standard normal limiting distribution and W has a limiting chi-square distribution. Unfortunately, when there is strong serial correlation in the errors, these standard asymptotic approximations are often inaccurate and the tests suffer from severe over-rejection problems (see Vogelsang (1998, Table I)). In addition, the finite sample behavior of the tests are sensitive to the choice of bandwidth and kernel, yet the standard asymptotics is the same regardless of the kernel or bandwidth. We address both of these issues. We control the over-rejection problem using a scaling factor proposed by Vogelsang (1998). We address the bandwidth and kernel problem by deriving the limiting distributions of the scaled tests under the fixed-b asymptotic framework proposed by Kiefer and Vogelsang (2002).

3 Scaled Statistics

We now describe the scaling procedure proposed by Vogelsang (1998) and introduce a new variant of the approach. The basic idea is to multiplicatively scale the t and W tests by a factor that converges to one when the errors are stationary but converges to a nuisance parameter free random variable when the errors have a unit root. We consider two scaling factors based on two unit root tests. Let J denote the unit root test proposed by Park (1990) and Park and Choi (1988). Consider the regression

$$y_t = f\left(t\right)'\beta + \sum_{i=j}^{9} \alpha_i t^i + u_t, \tag{4}$$

where t^{j-1} is the highest order polynomial of t included in $f(t)$. Then the J statistic is defined as

$$J = \frac{SSR_{(1)} - SSR_{(4)}}{SSR_{(4)}},$$

where $SSR_{(4)}$ is the sum of squared residuals obtained from the estimation of (4) by OLS, and $SSR_{(1)}$ be the sum of squared residuals from the OLS estimation of (1). The second unit root test is the test proposed by Breitung (2002) defined as

$$BG = \frac{\sum_{t=1}^{T} \widehat{S}_t^2}{SSR_{(1)}},$$

where $\widehat{S}_t = \sum_{j=1}^{t} \widehat{u}_j$ are the partial sums of the OLS residuals from Model (1). Both the J and BG statistics share the property that they are asymptotically invariant to nuisance parameters when the errors have a unit root and they converge to zero when the errors have a unit root.

Let UR generically denote either J or BG and let c denote a constant. The scaling factor

$$\exp(-cUR),$$

converges to a well defined nuisance parameter free random variable when the errors have a unit root but converges to one when the errors are stationary. Using the scaling factor we now redefine the t and W statistics as

$$t = \left(\frac{R\widehat{\beta} - q}{\sqrt{\widehat{\sigma}^2 R \left(\mathbf{f}(T)' \mathbf{f}(T) \right)^{-1} R'}} \right) \exp(-cUR),$$

$$W_T = \left(\left(R\widehat{\beta} - q \right)' \left[\widehat{\sigma}^2 R \left(\mathbf{f}(T)' \mathbf{f}(T) \right)^{-1} R' \right]^{-1} \left(R\widehat{\beta} - q \right) \right) \exp(-cUR). \tag{5}$$

The limiting distributions of t and W are unaffected by the scaling when the errors are stationary. When the errors have a unit root, the scaling factor affects the limiting distribution. For a given percentage point, it will possible to choose the constant c so that the asymptotic critical values of t and W are the same for stationary errors and unit root errors for a specific value of $\overline{\alpha}$. We follow Vogelsang (1998) and compute c for the case of $\overline{\alpha} = 0$. Asymptotically the scaling factors solve the over-rejection problem caused by strong serial correlation in the errors.

The versions of the statistics given by (5) will be used for the remainder of the paper. Note that the value of c used in practice depends on the significance level of the test and depends on which unit root statistic is chosen for the scaling factor. A detailed discussion of the choice of c is given below.

4 Limiting Distributions Under Fixed-b Asymptotics

In this section we provide the limiting null distributions of t and W as defined in (5) under the assumption that $M = bT$ where $b \in (0, 1]$. This asymptotic nesting for the bandwidth was proposed by Kiefer and Vogelsang (2002) and results were obtained for stationary models estimated by generalized method of moments. The results in Kiefer and Vogelsang (2002) do not apply to parameters associated with deterministic trends. Therefore, the results given here are new.

Before we proceed, some additional notation and definitions are required. As is well known, estimators of coefficients on different trends will often converge at different rates. Specifically, the coefficients entering the constraint which converge the slowest will dominate the asymptotic distribution. In order to formalize this, let μ_i be the largest non-positive power of time, t, in the

nonzero elements in the i'th row of $R\tau_T$. Then define the $d \times d$ diagonal matrix A in such a way that $A_{ii} = T^{\mu_i}$, and let $R^* = \lim_{T\to\infty} A^{-1} R\tau_T$. Under small-$b$ asymptotics, the limiting distributions depend on the type of kernel used in computing $\widehat{\sigma}^2$. The following definition describes the types of kernel we analyze.

Definition 1 *A kernel is labelled Type 1 if $k(x)$ is twice continuously differentiable everywhere and as a Type 2 kernel if $k(x)$ is continuous, $k(x) = 0$ for $|x| \geq 1$ and $k(x)$ is twice continuously differentiable everywhere except at $|x| = 1$.*

In addition to kernels which fall in these two categories, we consider the Bartlett kernel (which is neither Type 1 or 2) separately.

The limiting distributions are expressed in terms of the following functions and random variables.

Definition 2

$$N^F = \begin{cases} \int_0^1 F(s)\, dw(s), & \text{if } |\alpha| < 1 \\ \int_0^1 F(s)\, V_{\overline{\alpha}}(s)\, ds, & \text{if } \alpha = 1 - \frac{\overline{\alpha}}{T} \end{cases}$$

$$H(r) = \begin{cases} w(r) & \text{if } |\alpha| < 1 \\ \int_0^r V_{\overline{\alpha}}(s) ds & \text{if } \alpha = 1 - \frac{\overline{\alpha}}{T} \end{cases}$$

$$Q^F(r) = H(r) - \int_0^r F(s)'\, ds \left(\int_0^1 F(s) F(s)'\, ds \right)^{-1} N^F$$

$$k^*(x) = k\left(\frac{x}{b}\right),$$

$k_-^{*\prime}$ is the first derivative of k^* from below

$$\Phi^F(b,k) = \begin{cases} \int_0^1 \int_0^1 -k^{*\prime\prime}(r-s)\, Q^F(r)\, Q^F(s)'\, dr ds & \text{if } k(x) \text{ is Type 1} \\ \int\int_{|r-s|<b} -k^{*\prime\prime}(r-s)\, Q^F(r)\, Q^F(s)\, dr ds & \\ \quad + 2k_-^{*\prime}(b) \int_0^{1-b} Q^F(r+b)\, Q^F(r)\, dr & \text{if } k(x) \text{ is Type 2} \\ \frac{2}{b}\int_0^1 Q^F(r)^2\, dr - \frac{2}{b}\int_0^{1-b} Q^F(r+b)\, Q^F(r)\, dr & \text{if } k(x) \text{ is Bartlett} \end{cases}$$

In the case of $I(1)$ errors, the limiting distributions of the tests depend on the limiting distributions of the unit root tests. Let $\widehat{V}_{\overline{\alpha}}(r)$ denote the residuals from the projection of $V_{\overline{\alpha}}(r)$ onto the space spanned by $F(r)$, and let $V^*_{\overline{\alpha}}(r)$ denote the residuals from the projection of $V_{\overline{\alpha}}(r)$ onto the space spanned by $\left(F(r)', r^j, r^{j+1}, ..., r^9\right)'$. The following lemma follows directly from Park (1990), Park and Choi (1988) and Breitung (2002).

Lemma 1 *Suppose Assumptions 1 and 2 hold. If* $|\alpha| < 1$*, then as* $T \to \infty$*,* $J \Rightarrow 0, BG \Rightarrow 0$*. If* $\alpha = 1 - \frac{\overline{\alpha}}{T}$*, then as* $T \to \infty$*,*

$$\begin{aligned} J &\Rightarrow \frac{\int_0^1 \widehat{V}_{\overline{\alpha}}(r)^2\,dr - \int_0^1 V_{\overline{\alpha}}^*(r)^2\,dr}{\int_0^1 V_{\overline{\alpha}}^*(r)^2\,dr} \\ BG &\Rightarrow \frac{\int_0^1 Q^F(r)^2\,dr}{\int_0^1 \widehat{V}_{\overline{\alpha}}(r)^2\,dr}. \end{aligned}$$

We generically denote these limiting distributions by UR_∞ in what follows.

We can now state the main theorem.

Theorem 1 *Let* $M = bT$*,* $b \in (0,1]$*. Then under Assumptions 1 and 2 as* $T \to \infty$

a)

$$\begin{aligned} \widehat{\sigma}^2 &\Rightarrow \sigma^2 \Phi^F(b,k) \quad \text{if } |\alpha| < 1, \\ T^{-2}\widehat{\sigma}^2 &\Rightarrow d(1)^2 \Phi^F(b,k) \quad \text{if } \alpha = 1 - \frac{\overline{\alpha}}{T}, \end{aligned}$$

b)

$$\begin{aligned} W_T \Rightarrow &\left(R^* \left(\int_0^1 F(s) F(s)' ds \right)^{-1} N^F \right)' \left[\Phi^F(b,k) R^* \left(\int_0^1 F(s) F(s)' ds \right)^{-1} R^{*\prime} \right]^{-1} \\ &\times \left(R^* \left(\int_0^1 F(s) F(s)' ds \right)^{-1} N^F \right) \exp(-cUR_\infty), \end{aligned}$$

c)

$$t \Rightarrow \left(\frac{R^* \left(\int_0^1 F(s) F(s)' ds \right)^{-1} N^F}{\sqrt{\Phi^F(b,k) R^* \left(\int_0^1 F(s) F(s)' ds \right)^{-1} R^{*\prime}}} \right) \exp(-cUR_\infty).$$

Theorem 1 demonstrates that pivotal test statistics are obtained under fixed-b asymptotics regardless of kernel or bandwidth, although the limiting distributions of the test statistics depend upon the choice of kernel and bandwidth. The limiting distributions are clearly different when the errors are stationary compared to when the errors have a unit root. Given the kernel, bandwidth, scaling factor and percentage point, c can be chosen so that the critical values are the same for both stationary errors and unit root errors ($\overline{\alpha} = 0$). The critical values corresponding to the asymptotic distributions in Theorem 1 along with the values of c are simple to compute numerically. A power analysis in the next section indicates specific kernels and bandwidth values that lead to tests with pseudo-optimal power properties in a model with a simple linear trend. Critical values and details of their computation are given for the recommended tests in the simple linear trend model following a discussion power.

5 Asymptotic Power in the Simple Linear Trend Model and Optimal Kernel/Bandwidths

In this section extensive analysis of local asymptotic power of the simple linear trend model is provided. We focus on tests of the slope parameter and we derive limiting distributions under a local alternative. This allows us to compute local asymptotic power for a wide range of kernels and bandwidths. We base the choice of kernel and bandwidth on how they affect power.

The simple linear trend model is given by

$$y_t = \beta_1 + \beta_2 t + u_t,\ t = 1, ..., T. \tag{6}$$

The null hypothesis under consideration is $H_0 : \beta_2 \leq \beta_0$. The alternative is given by

$$H_A : \beta_2 = \beta_0 + \delta g(T),$$

where $g(T) = T^{-3/2}$ if $|\alpha| < 1$ and $g(T) = T^{-1/2}$ if $\alpha = 1 - \frac{\overline{\alpha}}{T}$. The t statistic for this test is given by

$$t = \left(\frac{T^{3/2}\left(\widehat{\beta}_2 - \beta_0\right)}{\sqrt{\widehat{\sigma}^2 \left(T^{-3}\sum_{t=1}^{T}(t-\overline{t})^2\right)^{-1}}} \right) \exp\left(-cUR\right). \tag{7}$$

The limiting null distribution of t follows from Theorem 1. Note that $\widehat{\sigma}^2$, J and BG are exactly invariant to the true value of β_2 and are hence exactly invariant to the value of δ. Therefore, only $\widehat{\beta}_2 - \beta_0$ depends on the local alternative. The following theorem gives the limiting distribution of t under the local the alternative.

Theorem 2 *Let $M = bT$, $b \in (0, 1]$. Suppose Assumptions 1 and 2 hold. Let t be given by (7) and let $F(r) = (1, r)'$ and $R^* = (0, 1)'$. Then under the local alternative, H_A, as $T \to \infty$*

$$t \Rightarrow \left(\frac{\nu + R^* \left(\int_0^1 F(s) F(s)' ds\right)^{-1} N^F}{\sqrt{\Phi^F(b, k) R^* \left(\int_0^1 F(s) F(s)' ds\right)^{-1} R^{*\prime}}} \right) \exp(-cUR_\infty),$$

where $\nu = \delta/\sigma$ if $|\alpha| < 1$ and $\nu = \delta/d(1)$ if $\alpha = 1 - \overline{\alpha}/T$.

Using the results of this theorem, it is easy to simulate asymptotic power of the t statistic for different choices of kernels and bandwidths. The first step is to simulate asymptotic critical values under the null hypothesis. This was done using 50,000 replications. For each replication, we approximated the Wiener processes implicit in the limiting distributions using normalized partial sums of 1,000 iid N(0,1) random deviates. We focused on five well known kernels: Bartlett, Parzen,

Bohman, Daniell and Quadratic Spectral (QS). Formulas for the kernels are given in an appendix. We considered the grid of bandwidths given by $b = 0.02,\ 0.04, ..., 1$. Given a percentage point, for a given bandwidth and kernel we computed values of c such that the asymptotic critical values are the same for $|\alpha| < 1$ and $\alpha = 1$. These values of c are different for the J and BG scaling factors. Given the values of c and the critical values, the second step is to compute rejection probabilities for a grid of values of δ using simulation methods thus producing asymptotic power curves.

To guide the choice of kernel and bandwidth, we computed power envelopes and then searched for specific kernels and bandwidths that deliver power close to the envelopes. When the errors are stationary, the scaling factor does not play a role asymptotically. For each value of δ, the point on the power envelope is the maximal power across the five kernels and across the grid of b's. For the case of unit root errors, we consider $\overline{\alpha} = 0, 10, 20$. For each value of $\overline{\alpha}$ we computed power envelopes in the same way as was done for the stationary case except that power depends on whether J or BG is used as the scaling factor. For unit root errors and a given value of $\overline{\alpha}$, the power envelope is the maximal power between the J and BG scaling factors and across the five kernels and the grid of b's.

In Figure 1 we plot asymptotic power for the case of stationary errors. We plot the power envelope and power of the tests using the five kernels each with $b = 0.02$. We see that, regardless of the kernel, power virtually equals the power envelope. Therefore, using $b = 0.02$ delivers essentially optimal tests when the errors are stationary.

Next we consider the asymptotic power of the tests when the errors are $I(1)$. Figures 2 and 3 display the power when $\overline{\alpha} = 0$. These two figures plot the overall power envelope as well as envelopes for the individual kernels. Figure 2 gives power when the J scaling factor is used and Figure 3 gives power when the BG scaling factor is used. The figures show that the Daniell kernel with the BG scaling factor attains the power envelope. We label this test Dan-BG. Figure 4 plots the power of the power envelope and Dan-BG using various values of b. This figure shows that the Dan-BG test with $b = 0.16$ essentially attains the $\overline{\alpha} = 0$ power envelope and is, for all practical purposes, optimal.

When the errors have a pure unit root, we recommend that the Dan-BG test with $b = 0.16$ be used in practice. When the errors are $I(0)$ we recommend that the Daniell kernel be used with the J scaling factor and bandwidth $b = 0.02$. The kernel does not matter when $b = 0.02$ and the errors are $I(0)$ and we recommend the Daniell kernel for the sake of convenience. Because the choice of scaling factor does not matter asymptotically when the errors are $I(0)$ we use the J scaling factor when $b = 0.02$ because it delivers higher power than the BG scaling factor when the errors are nearly integrated, $\overline{\alpha} = 10, 20$ (see below)[3].

[3] These values of $\overline{\alpha}$ correspond to $AR(1)$ processes in a sample of size 100 with $\alpha = 0.9, 0.8$ which are empirically relevant for economic data.

Figures 5-8 compare power of the recommended tests and the t-PS-J test of Vogelsang (1998).[4] Figure 5 plots power for stationary errors. In this case, the Daniell kernel with $b = 0.02$ is optimal. Using $b = 0.16$ results in some power loss but delivers a tests with power still slightly higher than t-PS-J. Figure 6 plots power for the case of pure unit root errors ($\overline{\alpha} = 0$). Here the Dan-BG test with $b = 0.16$ is optimal whereas using $b = 0.02$ results in a loss of power. As $\overline{\alpha}$ increases, the rankings of the tests begin to switch and the ranking depends on the value of the alternative, δ. With $\overline{\alpha} = 10$ Dan-J with $b = 0.02$ attains the power envelope for small values of δ. For large values of δ Dan-BG with $b = 0.16$ attains the power envelope. Dan-BG with $b = 0.02$ is dominated by the other tests and is the reason we do not recommend that the BG scaling factor be used when $b = 0.02$. It is interesting to note that while the t-PS-J test does not attain the power envelope for any value of δ, it is relatively close to the envelope for all values of δ and provides a good compromise to the relative power strengths of Dan-BG with $b = 0.16$ and Dan-J with $b = 0.02$. Figure 8 plots power when $\overline{\alpha} = 20$. Here we see that Dan-J with $b = 0.02$ have power that is, on average, the closest to the power envelope and using the J scaling factor when $b = 0.02$ clearly dominates using the BG scaling factor.

The power results can be summarized as follows. The two recommended tests, Dan-BG with $b = 0.16$ and Dan-J with $b = 0.02$ have complementary power with the former test optimal for unit root errors and the latter optimal for stationary errors. The t-PS-J test can also be recommended given its good average power for errors with strong serial correlation. All three tests are configured to have robust size for both stationary and unit root errors.

One small but important practical note is needed for Dan-J with $b = 0.02$. In very small samples, $b = 0.02$ will generate a bandwidth, M, that is less than 1. Therefore, in practice, we recommend using $M = \max(0.02T, 2)$ for the Dan-J test.

6 Asymptotic Critical Values For the Recommended Tests in the Simple Linear Trend Model

The limiting distributions of the recommended test statistics in the simple linear trend model, which are given by Theorem 1, are non-standard. Because they are functions of Brownian motions, critical values can easily be computed using simulations. We provide right tail critical values for the t statistics, the left tail critical value, as usual, follow by symmetry around zero. The critical values can be found in Table 1. The critical values were calculated via simulation methods using 50,000 replications. Normalized partial sums of 1000 $i.i.d.$ $N(0,1)$ random deviates were used to approximate the standard Brownian Motions in the distributions found in Theorem 1. Below each critical value, we provide the values of c required for the J or BG scaling factors. Although

[4] The t-PS-J test is calculated using the regression $z_t = \beta_1 t + \beta_2 \left[\frac{1}{2}\left(t^2 + t\right)\right] + S_t$,where $z_t = \sum_{i=1}^{t} y_i$ and $S_t = \sum_{i=1}^{t} u_i$. The t-PS-J statistic is the standard OLS t-statistic divided by $T^{1/2}$ and the J scaling factor is used.

our theoretical power results only apply to tests regarding the slope parameter, we also provide asymptotic critical values for t-tests on the intercept in the simple linear trend model for the convenience of practitioners.

7 Finite Sample Evidence

In this section, we discuss some finite sample simulations designed to assess the accuracy of the asymptotic approximations and compare the finite sample performance of the recommended tests.

For the finite sample simulations, we again use the model given in (6). We test the hypothesis that $\beta_2 \leq 0$ against $\beta_2 > 0$ at the 5% significance level. The errors are generated according to $u_t = \alpha u_{t-1} + e_t + \phi e_{t-1}$, where e_t is $i.i.d.$ $N(0,1)$. Simulations are reported for $\alpha = 0.0$, 0.3, 0.5, 0.7, 0.8, 0.9, 0.95, 1.0, and $\phi = -0.8$, -0.4, -0.0, 0.4, 0.8 and for sample sizes 50, 100, and 200. $5,000$ replications were used in all cases. Table 2 provides empirical rejection probabilities of the t-tests. It is clear that unless a large negative MA-term and a unit root are simultaneously present, all of these tests have empirical rejection probabilities either close to 0.05 or below. Therefore, the J and BG scaling factors work well in practice. This contrasts with standard HAC robust tests were it is well known that strong serial correlation causes over-rejections that can be severe. See Vogelsang (1998, Table I). The reason the tests over-reject when there is a unit root and a large negative MA component is because the J and BG statistics are oversized as unit root tests. In other words, J and BG tend to be too small in finite samples and they do not scale down the t-statistics enough to control the over-rejection problem. Note that the test that tends to over-reject the least is Dan-BG. This is a fortunate given that Dan-BG is the test with the highest power when the errors have a unit root.

We also report some finite sample power results to show that power in practice is qualitatively similar to that implied by the local asymptotic analysis. Figures 9-16 plot power for $\alpha = 0.0$, 0.8, 0.9, 1.0, $\phi = 0$, for $T = 50$ and 100. The results show that the asymptotic patterns are also reflected in the finite sample results. Dan-BG with $b = 0.16$ performs best when serial correlation is high, while Dan-J with $b = 0.02$ performs best when serial correlation is low. In addition the Daniell t-statistics perform better than the t-PS-J test, sometimes providing substantial power gains. The highest power gains over t-PS-J test are obtained when serial correlation is very high, and the gain increases as T increases from 50 to 100.

8 Evidence on the Prebisch-Singer Hypothesis

In this section we provide empirical evidence on the Prebisch-Singer hypothesis. We analyze the logarithm of the net barter terms of trade series constructed by Grilli and Yang (1988) and extended by Lutz (1992). See Grilli and Yang (1988) and Lutz (1992) for details on the construction of this

time series. The data is annual from 1900-1995. The net barter terms of trade is the ratio of a non-fuel primary commodities price index to a manufacturing price index. The Prebisch-Singer hypothesis asserts that the net barter terms of trade should be falling over time. We plot the data in Figure 17 and it is clear from the plot that the logarithm of net barter terms of trade has been decreasing over time. Is this decrease systematic? If we take regression (6) as a reasonable model of the statistical time series behavior of the logarithm of the net barter terms of trade, then the Prebisch-Singer hypothesis asserts that the trend slope coefficient is negative. If we take as the null hypothesis that the Prebisch-Singer hypothesis does not hold against the alternative that the Prebisch-Singer hypothesis holds, then we can parameterize the hypothesis as $H_0 : \beta_2 \leq 0$, $H_1 : \beta_2 > 0$.

Note that the Prebisch-Singer hypothesis is an empirical notion about the long run behavior of a time series; namely that the time series is steadily decreasing over time. It is important to keep in mind that this notion has nothing to do with the correlation in the data. More specifically, the Prebisch-Singer hypothesis has nothing to do with whether the error term is stationary or has a unit root. In our opinion, the empirical literature on the Prebisch-Singer hypothesis has become distracted by the unit root issue. This is not surprising given the technical difficulties the presence of a unit root brings with it. The advantage of the tests proposed in this paper is that they allow a direct and very simple test of the Prebisch-Singer hypothesis that does not depend on whether or not a unit root is in the errors.

Using the logarithm of the net barter terms of trade series, we estimated regression (6) by OLS and obtained $\widehat{\beta}_2 = -0.0645$. We computed the Dan-$J(b = 0.02)$ and Dan-$BG(b = 0.16)$ t-statistics. Recall that the value of c used for the scaling factors depends on the significance level of the tests and we provide results for significance levels 5%, 4% and 3%. We also report results using the t-PS-J test. The results are given in Table 3. All three tests indicate that the null hypothesis that the Prebisch-Singer hypothesis does not hold can be rejected at the 5% level. The null can be rejected at the 4% level using the Dan-$J(b = 0.02)$ statistic. These rejections are robust because the tests do not suffer from over-rejection problems even if the errors have a unit root. Our results suggest that there is relatively strong evidence that Prebisch-Singer hypothesis holds implying that Prebisch and Singer were right[5].

[5] As an additional robustness check, we applied the partial sum trend function structural change tests proposed by Vogelsang (1999). We computed variants of the Vogelsang (1999) tests designed to jointly detect a shift in intercept and/or slope in the deterministic trend function. The break date was treated as unknown. The tests also use the J scaling factor to control the over-rejection problem caused by strong serial correlation. We computed the mean, mean-exponential and supremum statistics using 1% trimming (see Vogelsang (1999) for details). The results were: mean=0.084, mean-exponential=0.0103 and supremum=0.0948. The 5% asymptotic critical values for these tests when using the J scaling factor are 2.0917, 1.3325 and 5.1651 respectively. Therefore, the null hypothesis that the trend function is stable over time cannot be rejected.

9 Conclusion

In this paper we have proposed tests for hypotheses regarding the parameters of a the deterministic trend function of a univariate time series. The tests do not require knowledge of the form of serial correlation in the data and they are robust to strong serial correlation. The data can even contain a unit root and the tests still have the correct size asymptotically. The tests we analyze are standard HAC robust tests based on nonparametric variance estimators. We extend the small-b asymptotic framework for HAC robust tests recently proposed by Kiefer and Vogelsang (2002). This allows us to analyze the power properties of the tests with regards to bandwidth and kernel choices. Our analysis shows that among popular kernels, there are specific kernel and bandwidth choices that deliver tests with maximal power within a specific class of tests that have the correct asymptotic size whether the errors are stationary or have a unit root. We achieve this size robustness using the J scaling factor proposed by Vogelsang (1998) and a new scaling factor based on the unit root test of Breitung (2002).

For inference regarding the slope parameter in the simple linear trend model, our analysis suggests that three specific tests should be used in practice. When the errors are stationary, the Daniell kernel with bandwidth equal to 0.02 times the sample size provides a test with optimal power. We recommend that the J scaling factor be used with this test. When the errors have a unit root, the Daniell kernel with bandwidth equal to 0.16 times the sample size provides a test with optimal power when the Breitung scaling factor is used. Although not optimal, the partial sum test of Vogelsang (1998) is also recommended because it provides a viable compromise between the two Daniell kernel tests when serial correlation is strong. In this case neither Daniell tests are optimal and the partial sum test can have average power closer to the power envelope.

We applied the three recommended tests to the logarithm of a net barter terms of trade series and the tests suggest that this series has a statistically significant negative slope. This finding is consistent with the well known Prebisch-Singer hypothesis. Because our tests are robust to strong serial correlation or a unit root in the data, our results in support of the Prebisch-Singer hypothesis are relatively strong.

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Appendix

In this appendix we give the proof of Theorem 1. Theorem 2 follows easily from Theorem 1 using simple algebra and details are omitted.

Proof of Theorem 1.

Proof of part a):

Following Kiefer and Vogelsang (2002), we define

$$k^*(x) = k\left(\frac{x}{b}\right),$$

and

$$\Delta^2\kappa_{ij} = \left\{k\left(\frac{i-j}{[bT]}\right) - k\left(\frac{i-j-1}{[bT]}\right)\right\} - \left\{k\left(\frac{i-j+1}{[bT]}\right) - k\left(\frac{i-j}{[bT]}\right)\right\},$$

and use this expression to rewrite $\widehat{\sigma}^2$ as

$$\widehat{\sigma}^2 = -T^{-1}\sum_{l=1}^{T-1} T^{-1}\sum_{i=1}^{T-1} T^2\Delta^2\kappa_{il}\left(T^{-1/2}\widehat{S}_i\right)\left(T^{-1/2}\widehat{S}_l\right). \tag{8}$$

For (8) to be valid it must be the case that the residuals sum to zero. So, for the asymptotic results to hold, a constant must be included in the model. The following lemma provides the distribution of $T^{-1/2}\widehat{S}_t$.

Lemma 2 $T^{-1/2}\widehat{S}_{[rT]} \Rightarrow \sigma Q^F(r)$.

Proof of Lemma 2: Simple matrix manipulations yield:

$$T^{-1/2}\widehat{S}_{[rT]} = T^{-1/2}\sum_{t=1}^{[rT]} u_t - \left(T^{-1}\sum_{t=1}^{[rT]} f(t)'\tau_T\right) T^{1/2}\tau_T^{-1}\left(\widehat{\beta}-\beta\right). \tag{9}$$

where

$$T^{1/2}\tau_T^{-1}\left(\widehat{\beta}-\beta\right) = \left(T^{-1}\tau_T\mathbf{f}(T)'\mathbf{f}(T)\tau_T\right)^{-1}\left(T^{-1/2}\tau_T\mathbf{f}(T)'u\right). \tag{10}$$

Clearly the terms consisting only of trend functions will have limiting distributions which do not depend on whether or not u_t is stationary. It is well know that these terms have the following limits:

$$T^{-1}\tau_T\mathbf{f}(T)'\mathbf{f}(T)\tau_T \Rightarrow \int_0^1 F(s)F(s)'ds, \text{ and} \tag{11}$$

$$T^{-1}\sum_{t=1}^{[rT]} f(t)'\tau_T \Rightarrow \int_0^r F(s)'ds. \tag{12}$$

The last term in (10) and the first term in (9) depend on u_t and therefore their limiting distributions will depend on whether or not u_t is stationary. Again using standard results, those asymptotic distributions are:

$$\begin{aligned}
T^{-1/2}\tau_T \mathbf{f}(T)' u &\Rightarrow \sigma \int_0^1 F(s)\, dw(s) \quad \text{if} \quad |\alpha| < 1, \\
T^{-3/2}\tau_T \mathbf{f}(T)' u &\Rightarrow d(1) \int_0^1 F(s) V_{\overline{\alpha}}(s)\, ds \quad \text{if} \quad \alpha = 1 - \frac{\overline{\alpha}}{T}, \\
T^{-1/2} \sum_{t=1}^{[rT]} u_t &\Rightarrow \sigma w(s) \quad \text{if} \quad |\alpha| < 1, \\
T^{-3/2} \sum_{t=1}^{[rT]} u_t &\Rightarrow d(1) \int_0^r V_{\overline{\alpha}}(s)\, ds \quad \text{if} \quad \alpha = 1 - \frac{\overline{\alpha}}{T}.
\end{aligned}$$

Using these limits the asymptotic distribution of $\widehat{S}_{[rT]}$ is as follows.

$$\begin{aligned}
T^{-1/2}\widehat{S}_{[rT]} &\Rightarrow \sigma \left(w(r) - \int_0^r F(s)' ds \left(\int_0^1 F(s) F(s)' ds \right)^{-1} \int_0^1 F(s)\, dw(s) \right) \\
&= \sigma Q^F(r) \quad \text{if} \quad |\alpha| < 1, \\
T^{-3/2}\widehat{S}_{[rT]} &\Rightarrow d(1) \left(\int_0^r V_{\overline{\alpha}}(s)\, ds - \int_0^r F(s)' ds \left(\int_0^1 F(s) F(s)' ds \right)^{-1} \int_0^1 F(s) V_{\overline{\alpha}}(s)\, ds \right) \\
&= d(1) Q^F(r) \quad \text{if} \quad \alpha = 1 - \frac{\overline{\alpha}}{T}.
\end{aligned}$$

The rest of the proof is split into three cases, corresponding to Type 1, Type 2 and the Bartlett kernels.

Case 1: $k(x)$ is a Type 1 kernel. By definition of the second derivative, $T^2\Delta^2\kappa_{il} \to k''$, and using Lemma (2) it follows easily for the case when $|\alpha| < 1$ that

$$\begin{aligned}
\widehat{\sigma}^2 &= T^{-1} \sum_{l=1}^{T-1} T^{-1} \sum_{i=1}^{T-1} -T^2\Delta^2\kappa_{il} T^{-1/2}\widehat{S}_i T^{-1/2}\widehat{S}_l \\
&\Rightarrow \sigma^2 \int_0^1 \int_0^1 -k''(r-s)\, Q^F(r)\, Q^F(s)\, dr ds.
\end{aligned}$$

When $\alpha = 1 - \frac{\overline{\alpha}}{T}$ we have

$$\begin{aligned}
T^{-2}\widehat{\sigma}^2 &= T^{-1} \sum_{l=1}^{T-1} T^{-1} \sum_{i=1}^{T-1} -T^2\Delta^2\kappa_{il} T^{-3/2}\widehat{S}_i T^{-3/2}\widehat{S}_l \\
&\Rightarrow d(1)^2 \int_0^1 \int_0^1 -k''(r-s)\, Q^F(r)\, Q^F(s)\, dr ds.
\end{aligned}$$

Case 2: $k(x)$ is a Type 2 kernel. Following Kiefer and Vogelsang (2002), we use simple algebra and the definition of $\Delta^2\kappa_{ij}$, to establish that when $|i-j| > [bT]$, $\Delta^2\kappa_{ij} = 0$, and when $|i-j| = [bT]$, $\Delta^2\kappa_{ij} = -k\left(\frac{[bT]-1}{[bT]}\right)$. Also recall that when $|i-j| < [bT]$ $k(x)$ is twice continuously differentiable. First consider the case when $|\alpha| < 1$. We split up the expression of $\widehat{\sigma}^2$ as follows:

$$\begin{aligned}
\widehat{\sigma}^2 &= T^{-1}\sum_{l=1}^{T-1} T^{-1}\sum_{i=1}^{T-1} -T^2\Delta^2\kappa_{il}T^{-1/2}\widehat{S}_i T^{-1/2}\widehat{S}_l \\
&= T^{-1}\sum_{l=1}^{T-1} T^{-1}\sum_{i=1}^{T-1} -1_{\{|i-j|<[bT]\}}T^2\Delta^2\kappa_{il}T^{-1/2}\widehat{S}_i T^{-1/2}\widehat{S}_l \\
&\quad + 2T^{-2}\sum_{l=1}^{T-[bT]-1} T^2 k\left(\frac{[bT]-1}{[bT]}\right)T^{-1/2}\widehat{S}_i T^{-1/2}\widehat{S}_{i+[bT]} \\
&= T^{-1}\sum_{l=1}^{T-1} T^{-1}\sum_{i=1}^{T-1} -1_{\{|i-j|<[bT]\}}T^2\Delta^2\kappa_{il}T^{-1/2}\widehat{S}_i T^{-1/2}\widehat{S}_l \\
&\quad + 2k\left(1-\frac{1}{[bT]}\right)\sum_{l=1}^{T-[bT]-1} T^{-1/2}\widehat{S}_i T^{-1/2}\widehat{S}_{i+[bT]} \\
&\Rightarrow \sigma^2\left(\int\int_{|r-s|<b} -k^{*\prime\prime}(r-s)\,Q^F(r)\,Q^F(s)\,drds + 2\sigma^2 k_-^{*\prime}(b)\int_0^{1-b} Q^F(r+b)\,Q^F(r)\,dr\right),
\end{aligned}$$

where the asymptotic distribution follows directly from Lemma (2) and Kiefer and Vogelsang (2002). The result when $\alpha = 1 - \frac{\overline{\alpha}}{T}$ follows analogously for $T^{-2}\widehat{\sigma}^2$ where $\widehat{S}_i$ is normalized by $T^{-3/2}$ instead of $T^{-1/2}$.

Case 3: $k(x)$ is the Bartlett Kernel. Here again using simple algebra following Kiefer and Vogelsang (2002), it can be verified that when $|i-j| = 0$, $\Delta^2\kappa_{ij} = \frac{2}{[bT]}$, and when $|i-j| = [bT]$, $\Delta^2\kappa_{ij} = -\frac{1}{[bT]}$. Using these expressions and Lemma (2) in (8), we obtain the following limiting distribution when $|\alpha| < 1$:

$$\begin{aligned}
\widehat{\sigma}^2 &= T^{-1}\sum_{l=1}^{T-1} T^{-1}\sum_{i=1}^{T-1} T^2\Delta^2\kappa_{il}T^{-1/2}\widehat{S}_i T^{-1/2}\widehat{S}_l \\
&= \frac{2}{[bT]}\sum_{i=1}^{T-1}\left(T^{-1/2}\widehat{S}_i\right)^2 - \frac{2}{bT}\sum_{i=1}^{T-[bT]-1} T^{-1/2}\widehat{S}_i T^{-1/2}\widehat{S}_{i+[bT]} \\
&\Rightarrow \sigma^2\left(\frac{2}{b}\int_0^1 Q^F(r)^2\,dr - \frac{2}{b}\int_0^{1-b} Q^F(r+b)\,Q^F(r)\,dr\right),
\end{aligned}$$

The result when $\alpha = 1 - \frac{\overline{\alpha}}{T}$ follows analogously for $T^{-2}\widehat{\sigma}^2$ where $\widehat{S}_i$ is normalized by $T^{-3/2}$ instead of $T^{-1/2}$. Comparing the distributions from Cases 1-3 with the definition of $\Phi^F(b,k)$ completes the proof of **a)**.

Proof of part b): First note that W_T can be written as

$$\begin{aligned} W_T &= \left(R\widehat{\beta}-q\right)'\left[\widehat{\sigma}^2 R\left(\mathbf{f}\left(T\right)'\mathbf{f}\left(T\right)\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-q\right)\exp\left(-cUR\right). \\ &= T\left(R\widehat{\beta}-q\right)'\left[\widehat{\sigma}^2 R\tau_T\left(\frac{1}{T}\tau_T\mathbf{f}\left(T\right)'\mathbf{f}\left(T\right)\tau_T\right)^{-1}\tau_T R'\right]^{-1}\left(R\widehat{\beta}-q\right)\exp\left(-cUR\right) \\ &= \left[\left(A^{-1}R\tau_T\right)\tau_T^{-1}T^{1/2}\left(\widehat{\beta}-\beta\right)\right]'\left[\widehat{\sigma}^2\left(A^{-1}R\tau_T\right)\left(\frac{1}{T}\tau_T\mathbf{f}\left(T\right)'\mathbf{f}\left(T\right)\tau_T\right)^{-1}\left(\tau_T R'A^{-1}\right)\right]^{-1} \\ &\qquad\times\left[\left(A^{-1}R\tau_T\right)\tau_T^{-1}T^{1/2}\left(\widehat{\beta}-\beta\right)\right]\exp\left(-cUR\right), \end{aligned}$$

By definition $A^{-1}R\tau_T \to R^*$. Furthermore we established the asymptotic distribution of $\widehat{\sigma}^2$ in **a)**. It therefore directly follows that when $|\alpha|<1$

$$\begin{aligned} W_T \;\Rightarrow\; & \left[\sigma R^*\left(\int_0^1 F\left(s\right)F\left(s\right)'ds\right)^{-1}N^F\right]'\left[\sigma^2\Phi^F\left(b,k\right)R^*\left(\int_0^1 F\left(s\right)F\left(s\right)'ds\right)^{-1}\left(R^*\right)'\right]^{-1} \\ &\qquad\times\left(\sigma R^*\left(\int_0^1 F\left(s\right)F\left(s\right)'ds\right)^{-1}N^F\right)\exp(-cUR_\infty) \\ = \;& \left[R^*\left(\int_0^1 F\left(s\right)F\left(s\right)'ds\right)^{-1}N^F\right]'\left[\Phi^F\left(b,k\right)R^*\left(\int_0^1 F\left(s\right)F\left(s\right)'ds\right)^{-1}\left(R^*\right)'\right]^{-1} \\ &\qquad\times\left(R^*\left(\int_0^1 F\left(s\right)F\left(s\right)'ds\right)^{-1}N^F\right)\exp(-cUR_\infty). \end{aligned}$$

When $\alpha = 1-\frac{\overline{\alpha}}{T}$ the desired result follows by normalizing $\left(\widehat{\beta}-\beta\right)$ by $T^{-1/2}$ and normalizing $\widehat{\sigma}^2$ by T^{-2}. Part c) of the theorem follows directly from part b). ■

A List of Kernels

The kernels we use:

$$\begin{aligned}
\text{Bartlett} \quad k(x) &= \begin{cases} 1-|x| \text{ for } |x| \leq 1 \\ 0 \text{ otherwise} \end{cases} \\
\text{Parzen (a)} \quad k(x) &= \begin{cases} 1-6x^2+6|x|^3 \text{ for } |x| \leq \frac{1}{2} \\ 2(1-|x|)^3 \text{ for } \frac{1}{2} \leq |x| \leq 1 \\ 0 \text{ otherwise} \end{cases} \\
\text{Quadratic Spectral (QS)} \quad k(x) &= \frac{25}{12\pi^2 x^2}\left(\frac{\sin(6\pi x/5)}{6\pi x/5} - \cos(6\pi x/5)\right) \\
\text{Daniell} \quad k(x) &= \frac{\sin(\pi x)}{\pi x} \\
\text{Bohman} \quad k(x) &= \begin{cases} (1-x)\cos(\pi x) + \sin(\pi x)/\pi \text{ for } |x| \leq 1 \\ 0 \text{ otherwise} \end{cases}
\end{aligned}$$

The second derivatives of the kernels we use are:

$$\begin{aligned}
\text{Parzen (a)} \quad k''(x) &= \begin{cases} -12+36|x| \text{ for } |x| \leq \frac{1}{2} \\ 12(1-|x|) \text{ for } \frac{1}{2} \leq |x| \leq 1 \end{cases} \\
\text{QS} \quad k''(x) &= \begin{cases} -\frac{36\pi^2}{125} \text{ for } x = 0 \\ \frac{125}{72\pi^3 x^5}\left[\left(12 - \frac{36\pi^2 x^2}{5}\right)\sin(6\pi x/5) + \right. \\ \left.\left(\frac{216\pi^3 x^3}{125} - \frac{72\pi x}{5}\right)\cos(6\pi x/5)\right] \text{ otherwise} \end{cases} \\
\text{Daniell} \quad k''(x) &= \begin{cases} -\frac{1}{3}\pi^2 \text{ for } x = 0 \\ \frac{2(\sin(\pi x) - \pi x\cos(\pi x))}{\pi x^3} - \frac{\pi\sin(\pi x)}{x} \text{ otherwise} \end{cases} \\
\text{Bohman} \quad k''(x) &= \pi\sin(\pi x) - \pi^2(1-x)\cos(\pi x)
\end{aligned}$$

Table 1: Asymptotic Critical Values of t-tests in the Simple Linear Trend Model $y_t = \beta_1 + \beta_2 t + u_t$.

Dan-J, $b = 0.02$			Dan-BG, $b = 0.16$			t-PS-J		
	β_1	β_2		β_1	β_2		β_1	β_2
90%	1.337	1.329	90%	1.773	1.752	90%	1.857	1.321
(c)	(.5791)	(.9648)	(c)	(53.84)	(148.8)	(c)	(.1748)	(.5039)
95%	1.726	1.710	95%	2.405	2.391	95%	2.451	1.737
(c)	(.7315)	(1.322)	(c)	(57.42)	(191.1)	(c)	(.2275)	(.7266)
97.5%	2.064	2.052	97.5%	3.065	3.035	97.5%	3.021	2.134
(c)	(.8975)	(1.795)	(c)	(60.55)	(241.5)	(c)	(.2915)	(1.017)
99%	2.436	2.462	99%	3.964	3.995	99%	3.699	2.599
(c)	(1.167)	(2.466)	(c)	(66.21)	(317.3)	(c)	(.3877)	(1.479)

TABLE 2: Empirical Null Rejection Probabilities in the Simple Trend Model
5% Nominal Level, 5,000 Replications

$y_t = \beta_1 + \beta_2 t + u_t$, $u_t = \alpha u_{t-1} + e_t + \phi e_{t-1}$, e_t, $i.i.d.N(0,1)$, $u_0 = 0$, $H_0 : \beta_2 \leq 0$, $H_A : \beta_2 > 0$.

		$T = 50$			$T = 100$			$T = 200$		
ϕ	α	Dan-J	Dan-BG	t-PS-J	Dan-J	Dan-BG	t-PS-J	Dan-J	Dan-BG	t-PS-J
-0.8	0.00	0.000	0.022	0.000	0.000	0.041	0.000	0.001	0.046	0.002
	0.50	0.001	0.018	0.004	0.001	0.036	0.009	0.003	0.047	0.019
	0.70	0.005	0.025	0.016	0.010	0.042	0.030	0.018	0.045	0.037
	0.80	0.020	0.034	0.034	0.041	0.049	0.052	0.049	0.046	0.053
	0.90	0.063	0.056	0.074	0.128	0.063	0.093	0.129	0.050	0.081
	0.95	0.106	0.075	0.099	0.196	0.082	0.128	0.193	0.057	0.108
	1.00	0.190	0.126	0.165	0.297	0.152	0.213	0.314	0.133	0.208
-0.4	0.00	0.003	0.039	0.020	0.014	0.045	0.033	0.036	0.045	0.039
	0.50	0.026	0.025	0.031	0.045	0.037	0.044	0.047	0.040	0.047
	0.70	0.043	0.023	0.033	0.071	0.031	0.049	0.064	0.034	0.049
	0.80	0.046	0.023	0.034	0.077	0.028	0.047	0.075	0.029	0.047
	0.90	0.054	0.026	0.042	0.071	0.022	0.042	0.076	0.023	0.042
	0.95	0.067	0.034	0.049	0.070	0.024	0.043	0.058	0.018	0.034
	1.00	0.113	0.078	0.100	0.107	0.069	0.092	0.081	0.058	0.072
0.0	0.00	0.020	0.030	0.033	0.034	0.040	0.043	0.043	0.042	0.045
	0.50	0.035	0.018	0.026	0.050	0.029	0.041	0.046	0.034	0.045
	0.70	0.028	0.016	0.020	0.052	0.024	0.035	0.052	0.026	0.041
	0.80	0.025	0.014	0.018	0.045	0.019	0.031	0.055	0.022	0.038
	0.90	0.019	0.013	0.017	0.036	0.016	0.023	0.045	0.016	0.027
	0.95	0.023	0.021	0.022	0.030	0.016	0.021	0.033	0.013	0.022
	1.00	0.045	0.051	0.045	0.052	0.049	0.052	0.048	0.048	0.048
0.4	0.00	0.032	0.022	0.028	0.036	0.035	0.041	0.040	0.038	0.045
	0.50	0.030	0.015	0.020	0.043	0.027	0.036	0.042	0.030	0.043
	0.70	0.020	0.012	0.015	0.041	0.020	0.031	0.047	0.024	0.038
	0.80	0.015	0.011	0.013	0.036	0.017	0.027	0.048	0.020	0.033
	0.90	0.013	0.012	0.011	0.026	0.014	0.020	0.038	0.014	0.025
	0.95	0.014	0.017	0.014	0.024	0.014	0.017	0.029	0.012	0.019
	1.00	0.028	0.042	0.030	0.040	0.046	0.042	0.042	0.045	0.043
0.8	0.00	0.034	0.021	0.026	0.033	0.033	0.040	0.038	0.036	0.045
	0.50	0.027	0.015	0.018	0.041	0.026	0.034	0.041	0.029	0.041
	0.70	0.018	0.011	0.014	0.039	0.020	0.029	0.046	0.024	0.037
	0.80	0.015	0.009	0.012	0.034	0.017	0.026	0.047	0.020	0.033
	0.90	0.012	0.011	0.009	0.024	0.013	0.019	0.037	0.014	0.024
	0.95	0.013	0.015	0.013	0.022	0.013	0.017	0.028	0.012	0.018
	1.00	0.026	0.041	0.028	0.037	0.045	0.040	0.041	0.044	0.042

Table 3: Empirical Results for the Logarithm of Net Barter Terms of Trade
Annual Data, 1900-1995

	Dan-J ($b = 0.02$)	Dan-BG ($b = 0.16$)	t-PS-J
Slope Estimate	-0.0645	-0.0645	-0.0598
t-stat (5% c)	-2.490	-2.427	-1.818
(5% cv)	(-1.710)	(-2.391)	(-1.737)
t-stat (4% c)	-2.185	-2.279	-1.686
(4% cv)	(-1.824)	(-2.588)	(-1.867)
t-stat (3% c)	-1.830	-2.119	-1.498
(3% cv)	(-1.960)	(-2.860)	(-2.029)

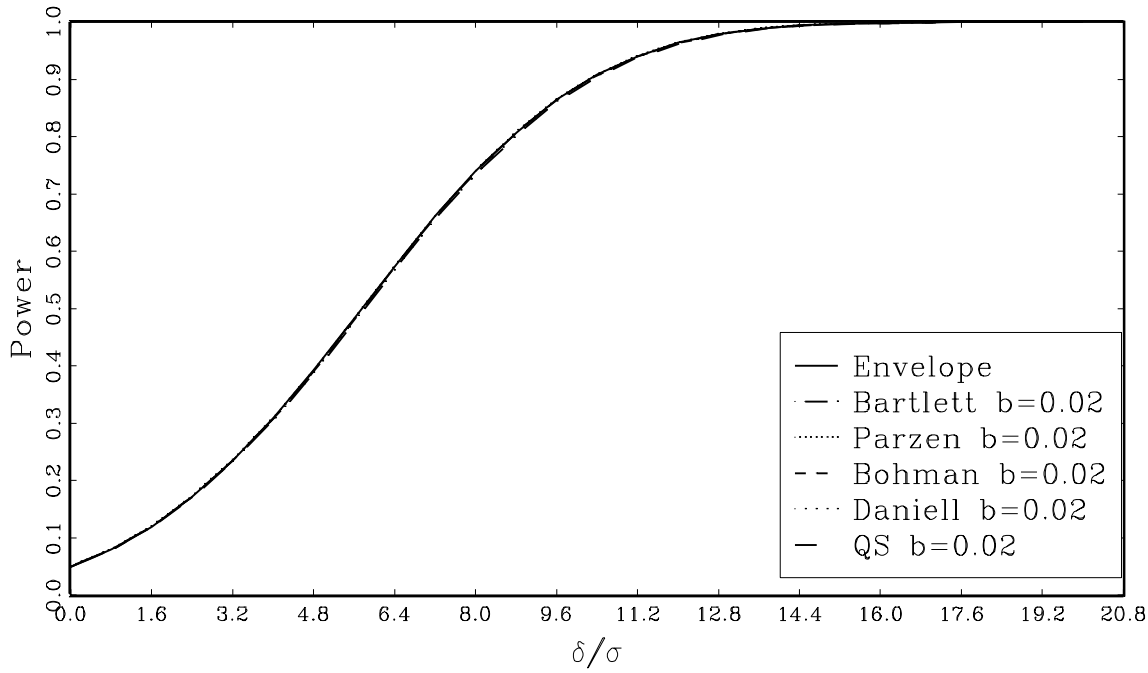

Figure 1: Asymptotic Power Stationary Errors

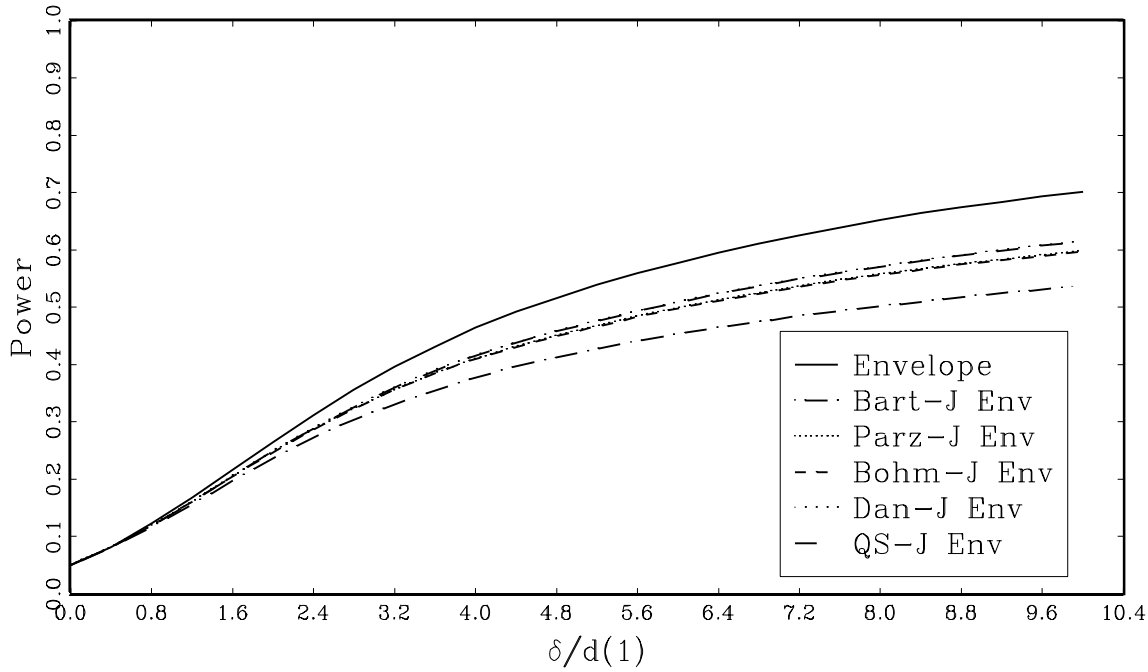

Figure 2: Asymptotic Power I(1) Errors, abar=0

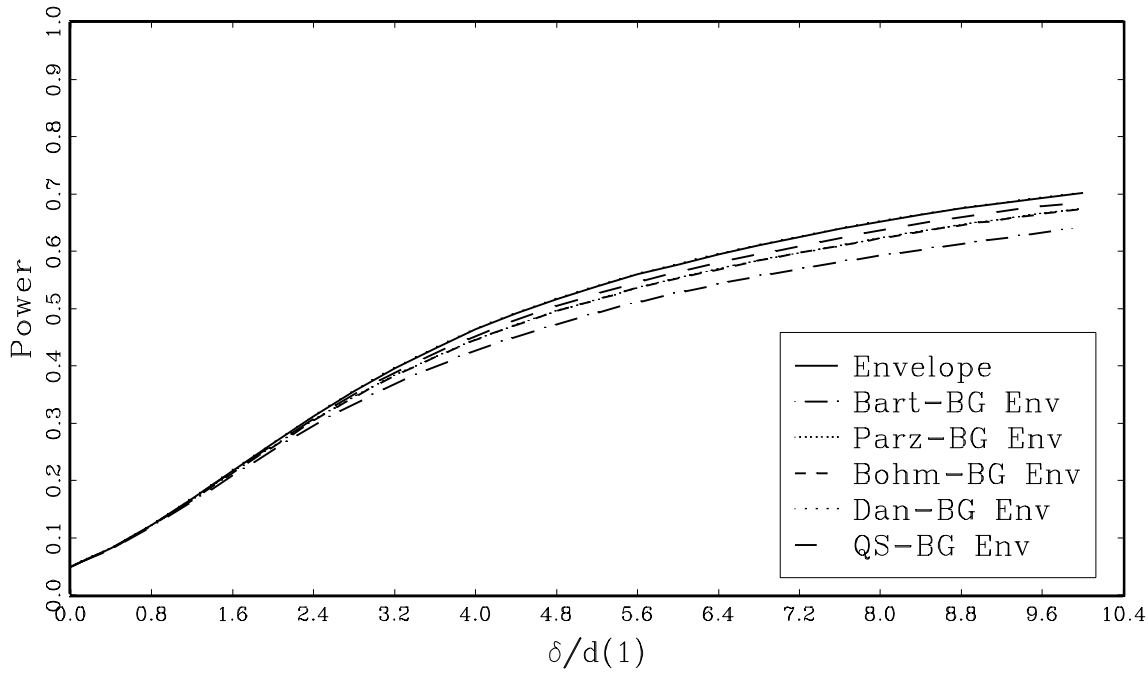

Figure 3: Asymptotic Power I(1) Errors, abar=0

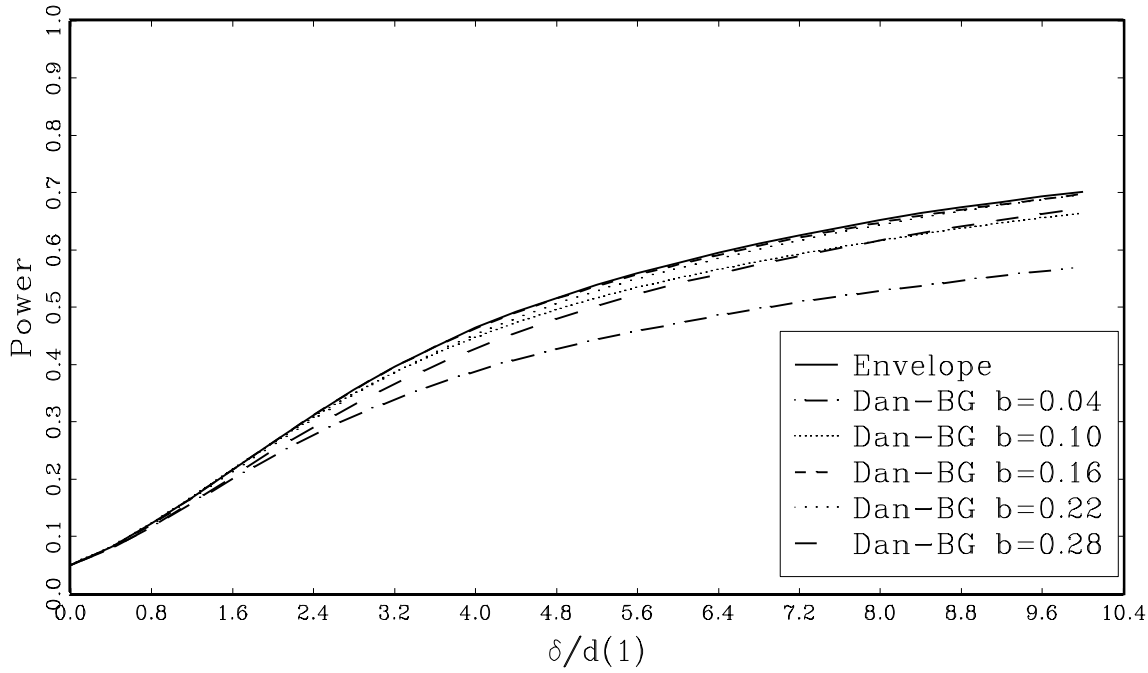

Figure 4: Asymptotic Power I(1) Errors, abar=0

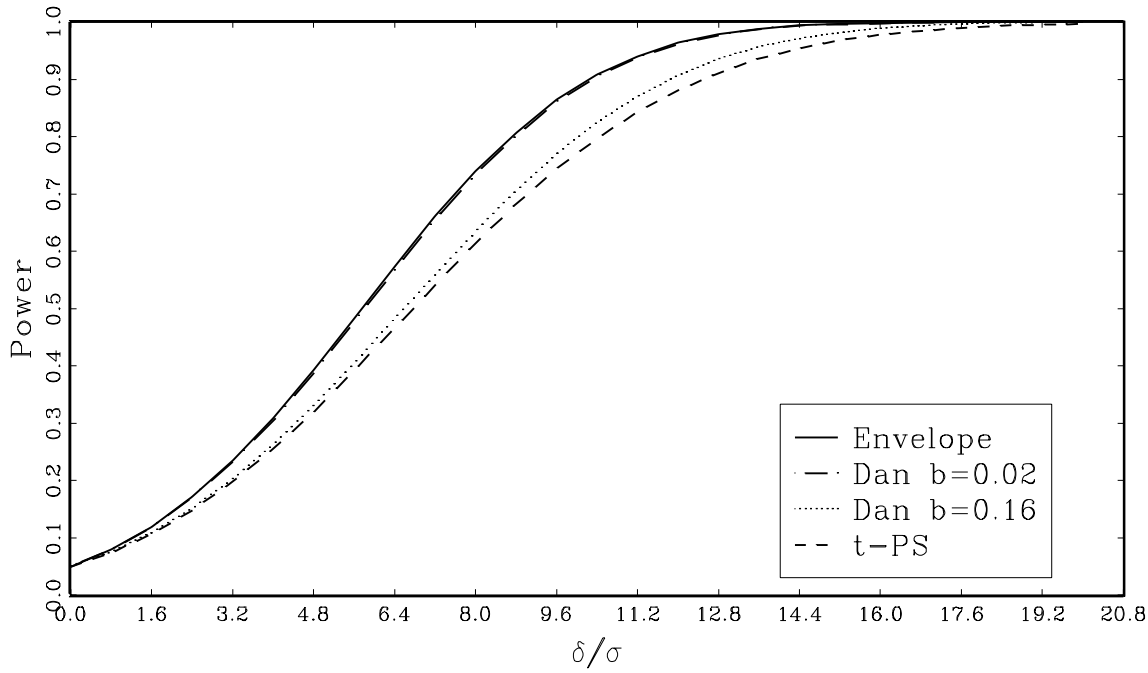

Figure 5: Asymptotic Power Stationary Errors

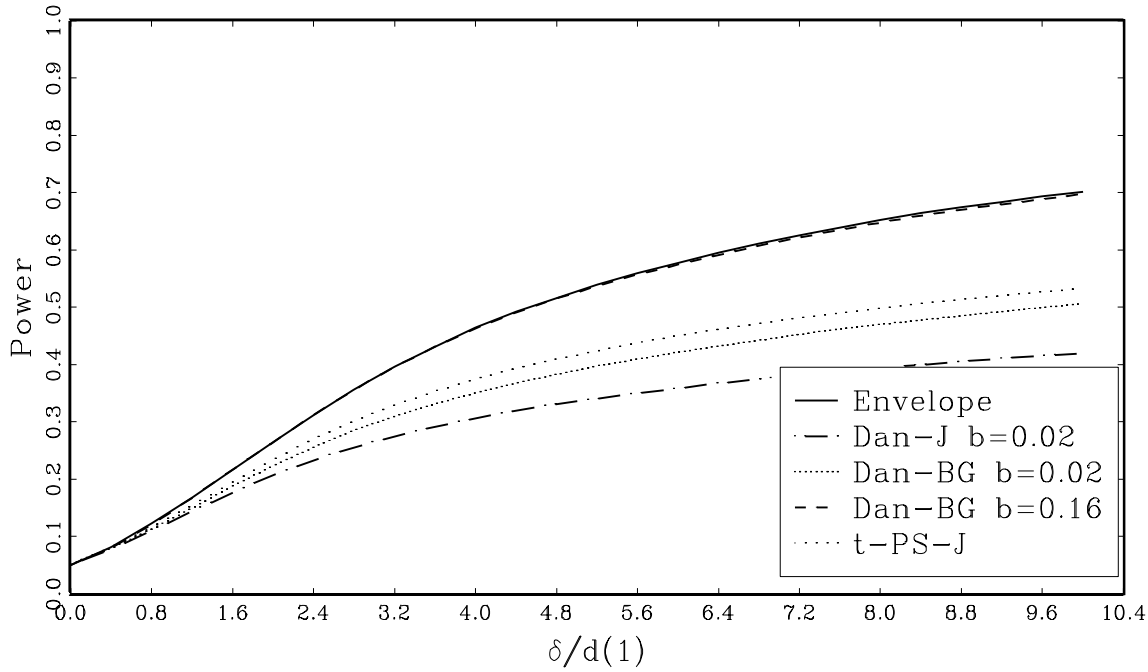

Figure 6: Asymptotic Power I(1) Errors, abar=0

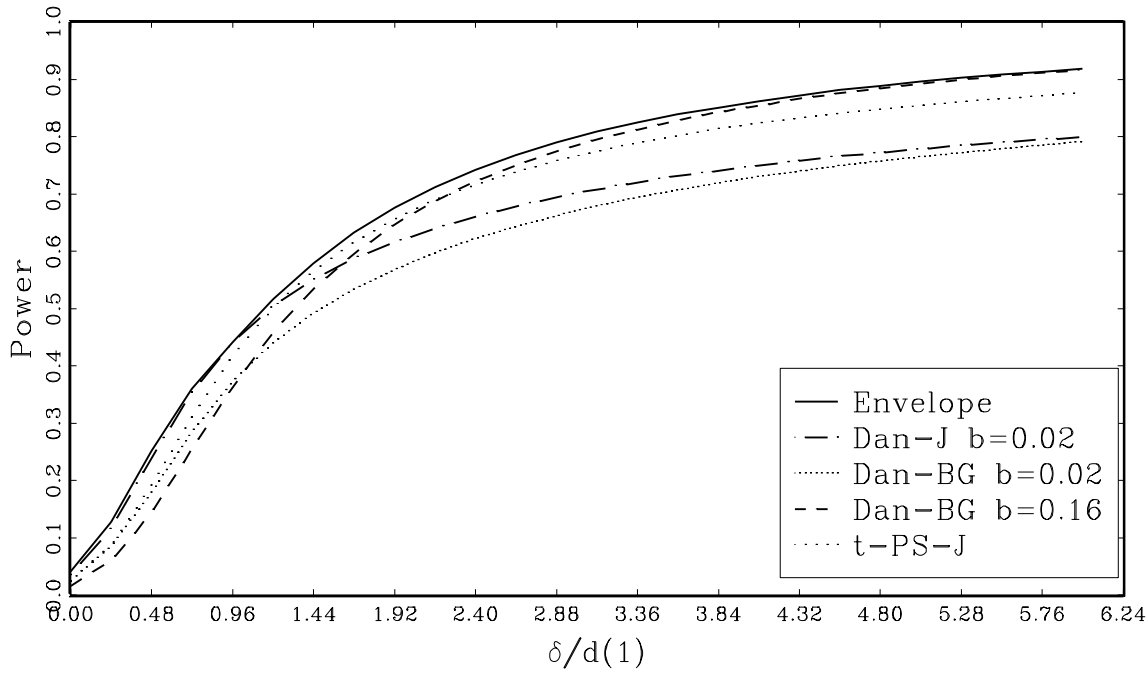

Figure 7: Asymptotic Power I(1) Errors, abar=10

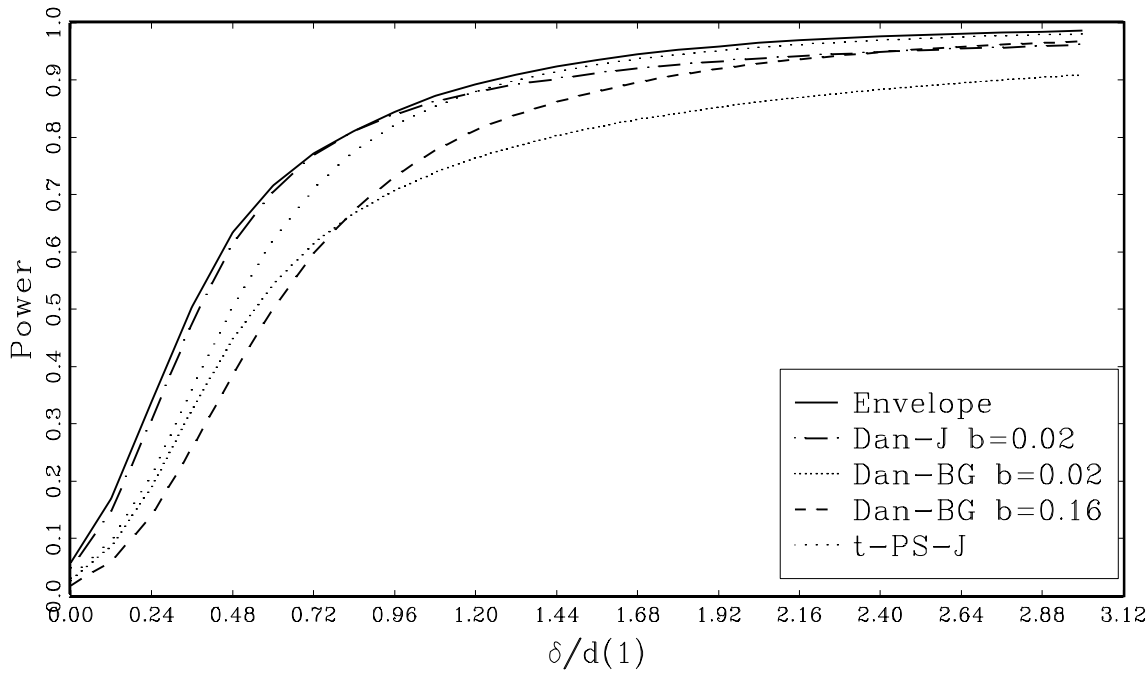

Figure 8: Asymptotic Power I(1) Errors, abar=20

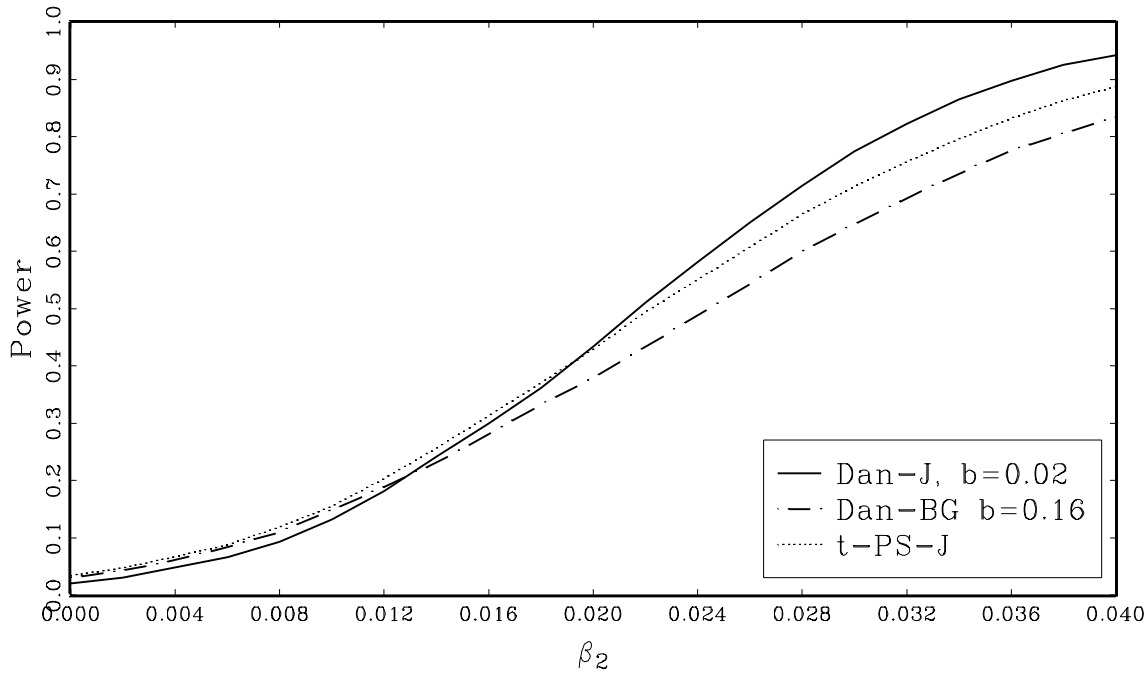

Figure 9: Finite Sample Power, AR(1) Errors, α=0, T=50.

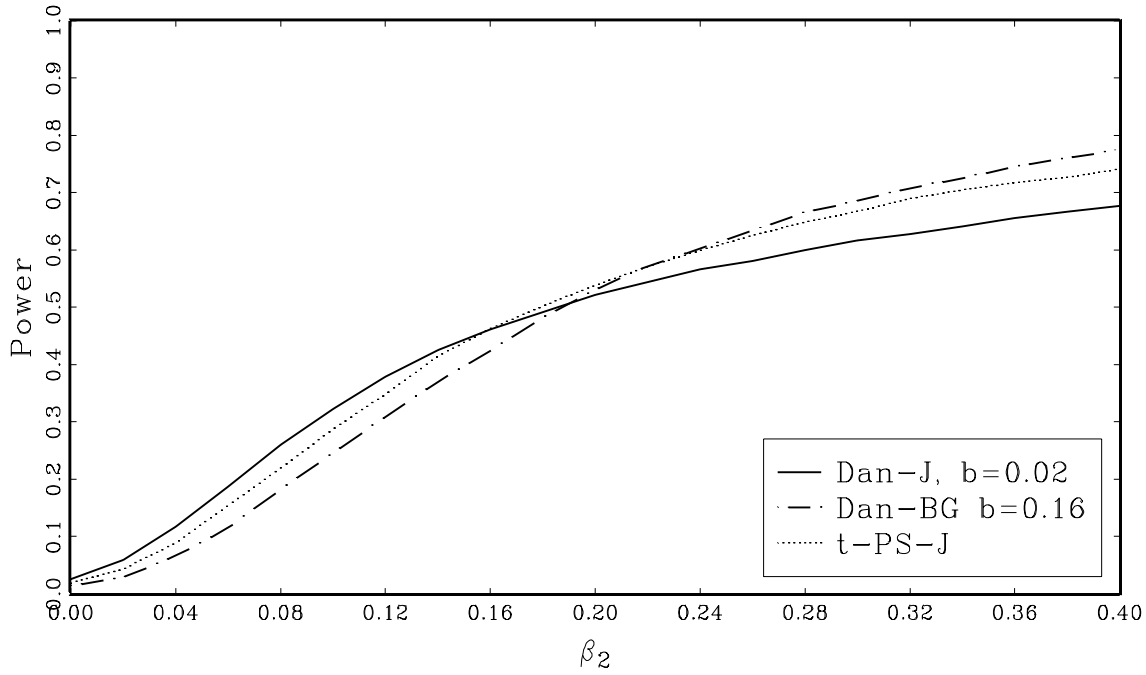

Figure 10: Finite Sample Power, AR(1) Errors, α=0.8, T=50.

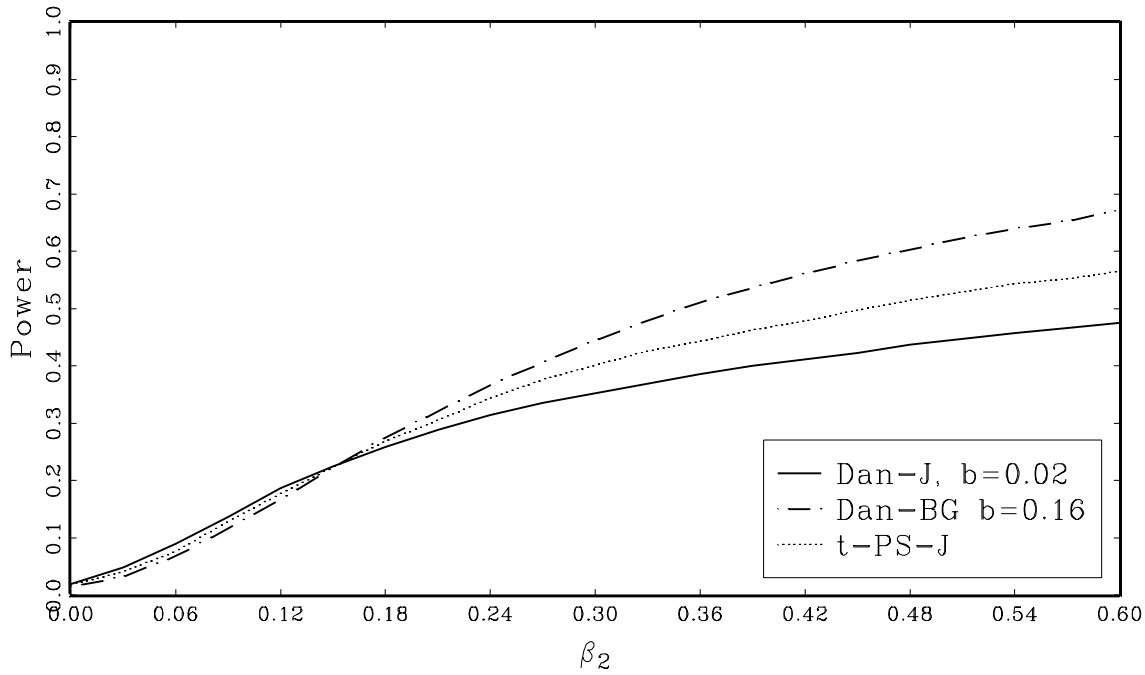

Figure 11: Finite Sample Power, AR(1) Errors, α=0.9, T=50.

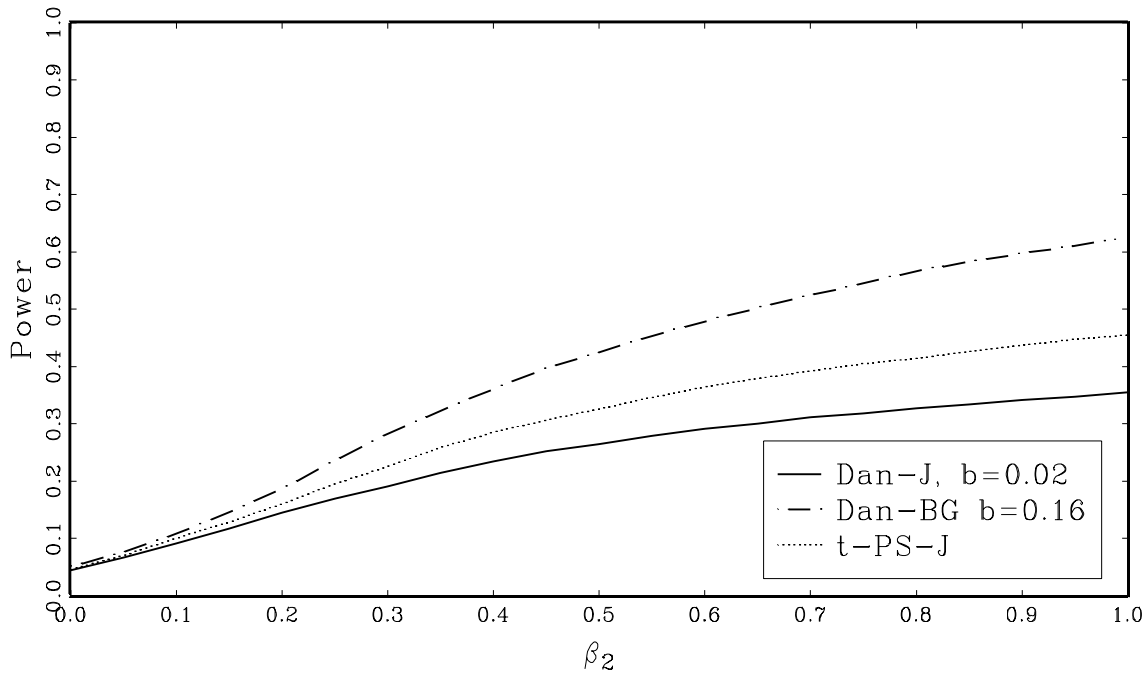

Figure 12: Finite Sample Power, AR(1) Errors, α=1.0, T=50.

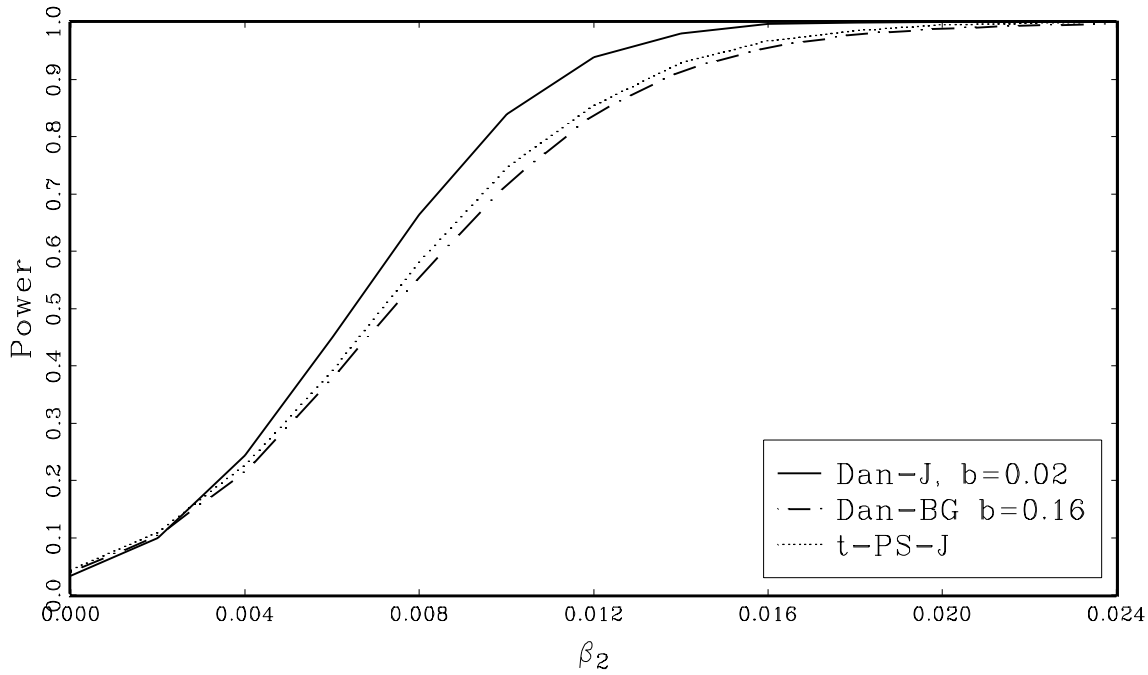

Figure 13: Finite Sample Power, AR(1) Errors, α=0, T=100.

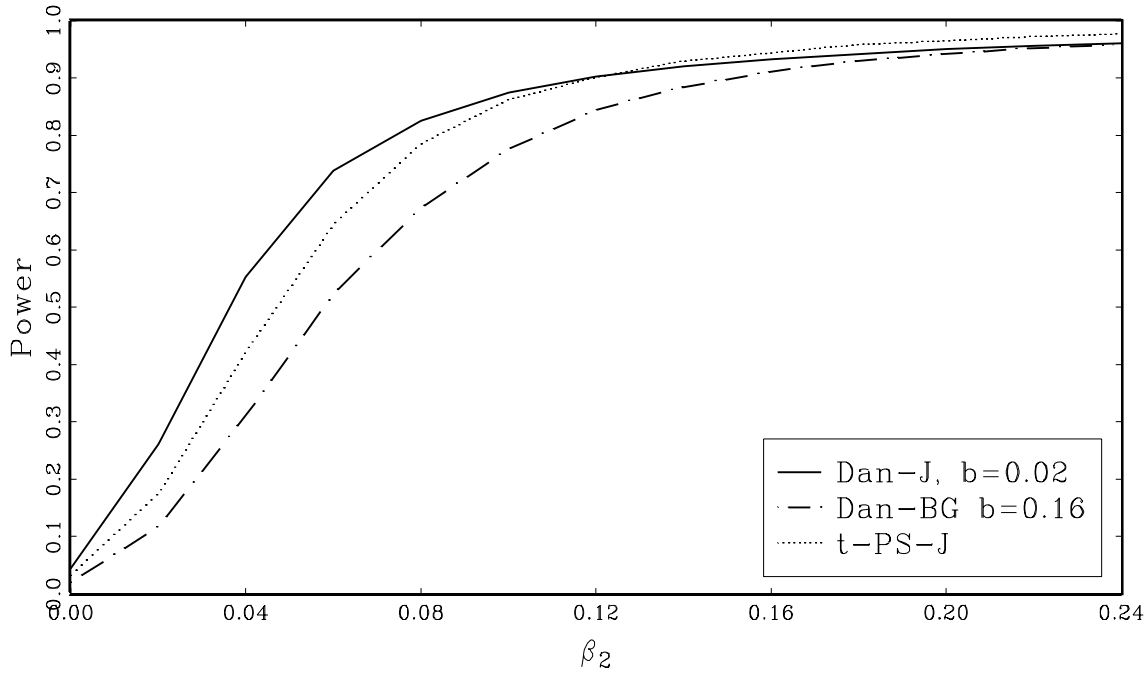

Figure 14: Finite Sample Power, AR(1) Errors, α=0.8, T=100.

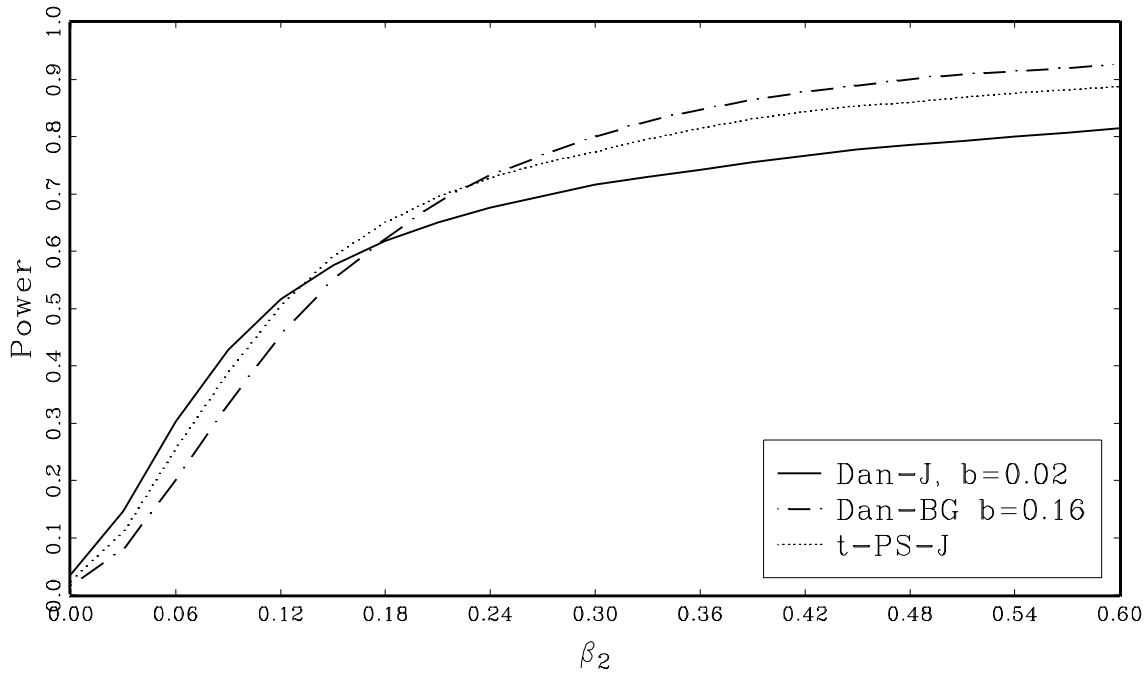

Figure 15: Finite Sample Power, AR(1) Errors, α=0.9, T=100.

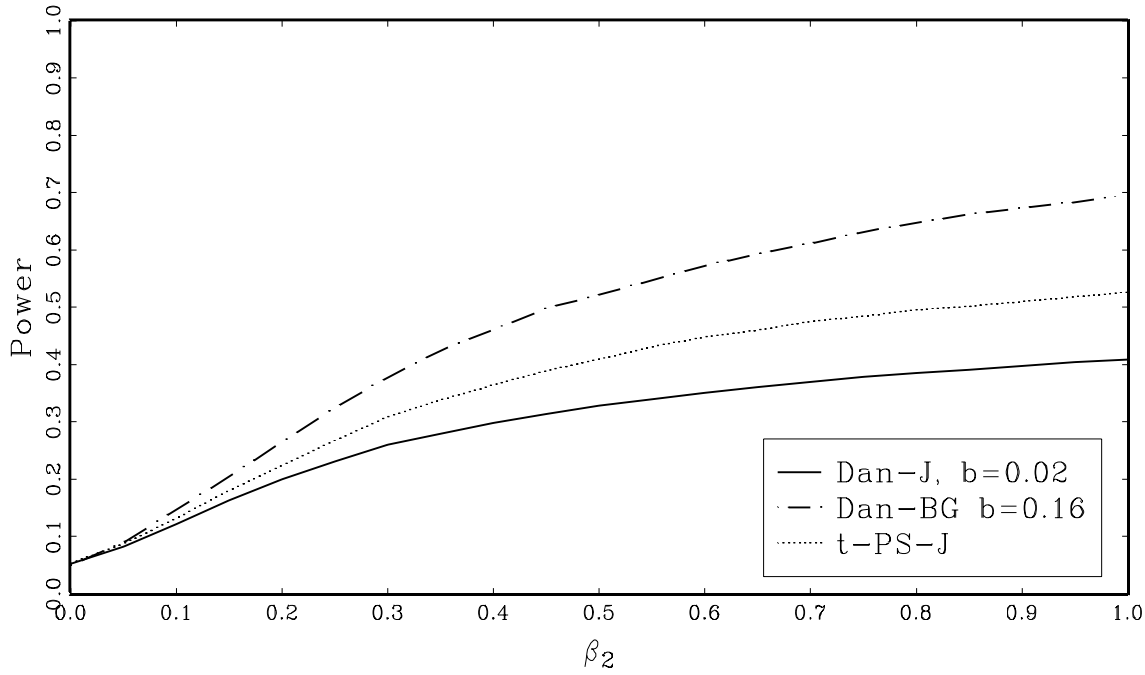

Figure 16: Finite Sample Power, AR(1) Errors, α=1.0, T=100.

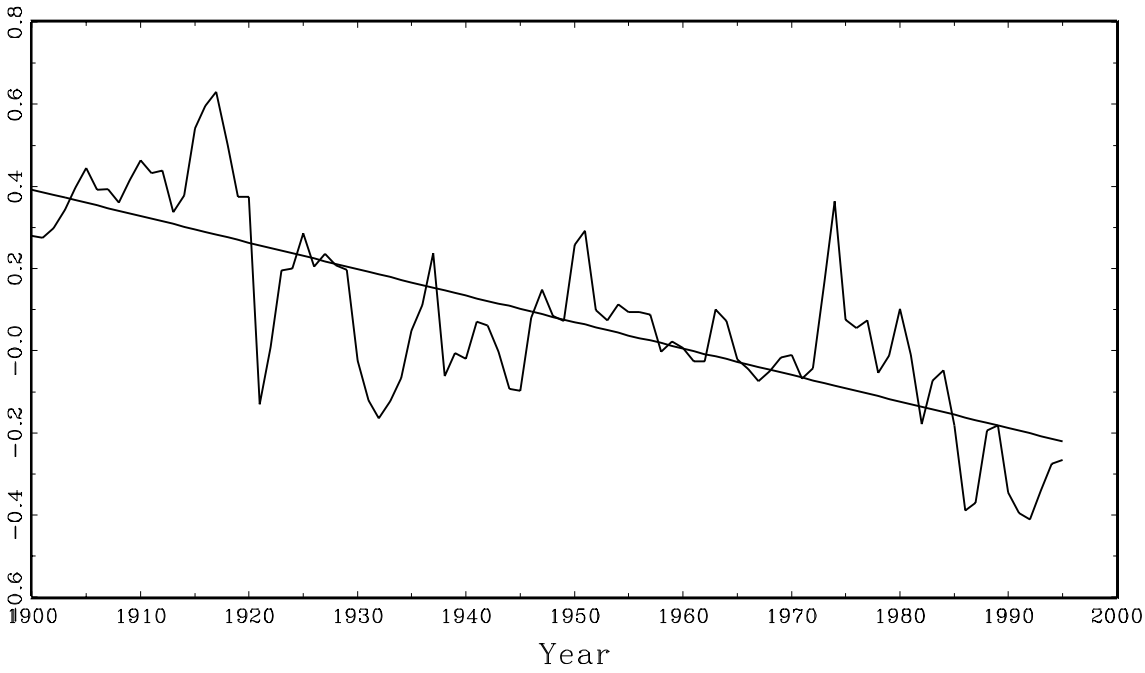

Figure 17: Logarithm of Net Barter Terms of Trade and Fitted Trend