The Behavior of HEGY Tests for Quarterly Time Series with Seasonal Mean Shifts*

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Revised: October 4, 2004

Abstract

This paper studies the behavior of the HEGY statistics for quarterly data, for seasonal autoregressive unit roots, when the analyzed time series is deterministic seasonal stationary but exhibits a change in the seasonal pattern. As a by-product, we analyze also the HEGY test for the nonseasonal unit root, the data generation process being trend stationary too. Our results show that when the break magnitudes are finite the HEGY test statistics are not asymptotically biased towards the non-rejection of the seasonal and nonseasonal unit root hypotheses. However, the finite sample power properties may be substancially affected, the behavior of the tests depending on the type of the break. Hence, our results are also useful to understand and to predict this behavior under several circumstances.

Keywords: seasonality; unit roots; structural breaks; HEGY tests.

JEL Classification: C22, C52

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*Previous versions of this paper circulated under the title "The Asymptotic and Finite Sample Behavior of Seasonal Unit Root Tests Under the Presence of a Break in the Deterministic Seasonal Pattern". The first author thanks FCT and Programas Praxis XXI and POCTI (2000/ECO/33778, partially funded by FEDER) for financial support. The second author thanks the Spanish Ministry of Education and Culture for financial support to his stay at Boston University, where part of this paper was written. We are indebted to two anonymous referees and to Philip Hans Franses, Paulo Rodrigues, Andreu Sansó and João Santos Silva for helpful comments on previous versions. We are also grateful to the participants of the 6th CEMAPRE (2000, Lisboa) and the "Seasonality in Economic and Financial Variables" (2000, Faro) Conferences. Obviously, all remaining errors are our own.

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1 Introduction

Since the seminal work of Perron (1989) a new research field emerged. Perron (1989) showed that Dickey and Fuller (1979) (DF) test statistics can often lead to the non rejection of the unit root hypothesis when the analyzed time series is stationary around a segmented deterministic trend. Given this lack of power of DF tests for (trend) stationary time series affected by breaks, new methods for inference have been recently proposed. In this regard we can cite, *inter alia*, the papers by Banerjee *et al.* (1992), Christiano (1992), Perron (1989, 1990, 1997), Perron and Vogelsang (1992), Vogelsang and Perron (1998) and Zivot and Andrews (1992). When more than one break is allowed, one can use the procedures proposed in Lumsdaine and Papell (1997) if the variable exhibits a trend, or Clemente *et al.* (1998) for non-trending series.

A natural extension of conventional unit root testing is the analysis of autoregressive unit roots at seasonal frequencies (for seasonally observed and unadjusted time series). This topic was first analyzed by Hylleberg, Engle, Granger and Yoo (1990) [HEGY] and subsequently the seasonal unit roots or seasonally integrated model has become somewhat popular to model a changing seasonal pattern. Evidence about the presence of unit roots at seasonal frequencies has been reported for some macroeconomic time series [see, e.g., Osborn (1990), Hylleberg *et al.* (1993) and Canova and Hansen (1995)].

However, alternative models are available to model unstable seasonal patterns. Besides the periodic autoregressive (PAR) model [see, e.g., Franses (1996)], the seasonal mean shifts model recently received considerable attention [e.g., Balcombe (1999), Franses *et al.* (1997), Franses and Vogelsang (1998), Hassler and Rodrigues (2004), Lopes (2001), Paap *et al.* (1997) and Smith and Otero (1997)]. In fact, we can think of changing seasonal fluctuations arising from one or more structural breaks in a deterministic seasonal stationary process. Such deterministic seasonal mean shifts may emerge under several circumstances. For instance, one may think of a sudden change in preferences (for holidays, for example) or in production and/or storing technologies, or in institutional arrangements of the economy (e.g., the school calendar). Also, sometimes statistical agencies change the procedures used to measure economic variables.

In recent papers by Franses and Vogelsang (1995, 1998) and Hassler and Rodrigues (2004) new tests for seasonal unit roots which explicitly allow such deterministic mean shifts are put forward. Following Perron (1990) and Ghysels (1994), Franses and Vogelsang (1995) conjecture that HEGY statistics can be biased towards non-rejection of seasonal unit roots when shifting deterministic seasonals are neglected in the testing strategy. This conjecture is confirmed by the Monte Carlo results obtained by Balcombe (1999) and by Smith and Otero (1997). However, a detailed asymptotic analysis is still missing. Such an analysis can be enlightening about the asymptotic behavior of HEGY tests when deterministic seasonal mean shifts are neglected. Furthermore, it may also shed some light on the small sample power properties of the tests. This paper aims to bridge this gap, addressing both issues. Moreover, though our main interest centres on the seasonal unit root tests, as a by-product we also analyze the asymptotic behavior of the HEGY test for the unit root at the zero (nonseasonal or long-run) frequency, the data generation process (DGP) being trend stationary too.

The outline of the paper is as follows. The next section contains the asymptotic analysis. First we obtain the general results and subsequently we consider some cases of particular interest. Our results show that when the break parameters are finite only one of the test statistics converges to zero, the divergence of the remaining statistics allowing us to conclude that there is no asymptotic bias towards non-rejection of the nonseasonal and seasonal unit root hypotheses. Although this result is not very surprising, its importance derives from the similarity of HEGY and DF tests and the claim in Perron (1989) that the latter are inconsistent against certain breaking trend alternatives with finite breaks. However, according to the type of the break, some or all test statistics may require larger sample sizes to maintain their power, and the asymptotic analysis is insightful in this respect too. Section 3 presents the results of some Monte Carlo simulation experiments which aim to ascertain the adherence of these results to small samples. While Appendix A contains some illustrating proofs, in Appendix B we present the critical values used for the simulation study.

The following conventions are used through the paper: a) T denotes the sample size; b) the symbol '$\rightarrow$' denotes convergence in probability and c) frequently, the probability limits of the properly scaled test statistics will be referred simply as the "asymptotic values" or "limit values" for those statistics.

2 Asymptotic Analysis

In this section we derive the probability limits for the seasonal and nonseasonal unit root test statistics when the time series is trend and deterministic seasonal stationary but exhibits a break in its seasonal pattern.

2.1 Assumptions

As the HEGY procedure depends on the frequency of the data, we confine the analysis to the most popular case, i.e., the quarterly data case. Though our main interest centres on the seasonal unit root tests, in order to study also the behavior of the test for the nonseasonal unit root and following Smith and Otero (1997), we assume that the time series is generated according to the model

$$y_t = \sum_{i=1}^{4} \alpha_i D_{it} + \beta\, t + \sum_{i=1}^{4} \delta_i D_{it}[I_{t>\tau}] + u_t, \quad t = -3, -2, -1, 0, 1, 2, ..., T, \qquad (1)$$

where y_t typically denotes a log transformed variable, D_{it} $(i = 1, 2, 3, 4)$ represent the usual seasonal dummy variables and $[I_{t>\tau}]$ is an indicator function (taking the value 1 if $t > \tau$ and 0 otherwise). For the sake of simplicity, $\{u_t\}$ is assumed to be an iid$(0, \sigma_u^2)$ process. We assume that there is at most one shift in each season and denote the time of the break as $\tau = \lambda_s T$, $0 < \lambda_s < 1$. Thus, while until time τ the seasonal cycle is given by the α_i parameters, after the break that same role is performed by the $\alpha_i + \delta_i$ quantities. To simplify the notation but without loss of generality we consider that τ corresponds to a fourth quarter observation.

Contrarily to a postulate which is sometimes used in the literature on conventional unit root testing (see, *e.g.* Perron (1989, p. 1372)), in this paper we confine our attention to the case where the δ_i parameters are finite. Given the presence of the deterministic trend in the DGP, this assumption implies that, in spite of the (seasonal) shift, the relative importance of the seasonal variation decreases as T grows [1], the behavior of the series being asymptotically dominated by the trend. However, this is a feature which is necessarily contained in the assumed DGP, whether there is a break in the seasonal pattern or not.

In other words, as in Perron (*op. cit.*), we could have considered the δ_i as a linear function of τ, e.g., $\delta_i = \gamma_i \tau (i = 1, 2, 3, 4)$, in which case they would tend to infinity with T. We consider that this would not be a reasonable assumption for two essential reasons. First, as we wish to study the behavior of the tests when only some of the seasonal intercepts change, that assumption would imply the values of the series for those seasons to diverge from the remaining (i. e., from those corresponding to $\delta_i = 0$) as the sample size grows, a quite unrealistic situation. Second and more generally, such an assumption would be reasonable only in the case when the pre-break DGP would be given by

$$y_t = \sum_{i=1}^{4} \alpha_i D_{it} + \sum_{i=1}^{4} \beta_i t + u_t,$$

a model which allows increasing (deterministic) seasonal variation, which is rarely considered in empirical work [2], and for which the most popular HEGY regression equation (see below) is not adequate (as it does not provide similar test statistics; see Smith and Taylor (1998)). As one of our main purposes is the study of the implications of a break in the seasonal pattern only, the amplitude of the seasonal variation being kept constant through time in each sub-period, we consider more appropriate that the δ_i do not vary with T. Clearly, this assumption amounts to a serious limitation in what concerns testing for the nonseasonal unit root only. However, it is rarely the

[1] But, given the break, it obviously may alter for the detrended series.

[2] However, see Franses (1998, pp. 115-7) for an example on a previously logarithmized series. This diverging seasonal trends model may be seen also as the alternative hypothesis in the HEGY tests performed in the example provided by Smith and Taylor (1998), but in this case the logarithmic transformation was not used on the original time series.

case that empirical research for seasonally observed time series relies exclusively on the HEGY procedure for such a purpose. In that case, if there is some suspicion of a break, the procedures mentioned in the second paragraph of section 1 should be given priority [3].

2.2 The HEGY procedure: a brief review

Given the characteristics of the variable which is the centre of the study — containing deterministic trend and seasonality — but neglecting the seasonal shifts, the HEGY statistics for testing for autoregressive unit roots are obtained from the OLS estimation of the equation

$$y_{4t} = \sum_{i=1}^{4} \mu_i D_{it} + \gamma\, t + \pi_1\, y_{1,t-1} + \pi_2\, y_{2,t-1} + \pi_3\, y_{3,t-2} + \pi_4\, y_{3,t-1} + \epsilon_t, \quad t = 1, 2, ..., T \quad (2)$$

where $y_{1t} = (1 + L + L^2 + L^3)y_t,\ y_{2t} = (-1 + L - L^2 + L^3)y_t,\ y_{3t} = (-1 + L^2)y_t$ and $y_{4t} = \Delta_4 y_t = y_t - y_{t-4}$, L denoting the usual lag operator, and where $\{\epsilon_t\}$ is assumed to be a white noise process. Since model (1) incorporates the (deterministic) seasonal mean shifts, it is contained neither in the null nor in the alternative hypotheses for which the HEGY statistics have been designed. HEGY showed that when $\pi_1 = 0$ the series contains the (nonseasonal or zero frequency) root 1, when $\pi_2 = 0$ the (semi-annual) root -1 is present, the presence of the (annual) roots $\pm i$ ($i = \sqrt{-1}$) implying $\pi_3 = \pi_4 = 0$ (the stationary alternatives being $\pi_1 < 0$, $\pi_2 < 0$ and $\pi_3 < 0$ and/or $\pi_4 \neq 0$).

Thus, inference on the presence of seasonal unit roots may be carried out through the t-ratios associated to the last three π_i coefficients: t_{π_2}, t_{π_3} and t_{π_4}. On the other hand, evidence on the presence (absence) of a nonseasonal unit root is given by t_{π_1}. However, the analysis of stochastic seasonal non-stationarity becomes simpler if, instead of testing three separate hypotheses, we test some joint null hypotheses. To that end, one can use the F-statistics F_{34}, which tests $H_0 : \pi_3 = \pi_4 = 0$, and F_{234}, associated to $H_0 : \pi_2 = \pi_3 = \pi_4 = 0$. Finally, one can also test whether all the π_i parameters are zero [i.e., whether the $\Delta_4 = (1 - L^4)$ filter is appropriate] using F_{1234}. The first statistic allows testing for the presence of the two complex conjugate roots and, following the argumentation presented in HEGY, it is usually preferred to testing each of the separate single null hypothesis [see also Burridge and Taylor (2001) for an additional argument]. The F_{234} and F_{1234} statistics were proposed by Ghysels *et al.* (1994). The limiting null distributions of the t and F statistics have been derived by

[3] As mentioned in Franses (1996, p. 73), the HEGY procedure may lack power when testing for the conventional unit root. On the other hand, regarding those procedures, and as in the no-break case, we may conjecture that, for seasonally observed time series, the adequate regression equations must contain sufficient lag augmentation; see Ghysels *et al.* (1993).

HEGY and Ghysels *et al.* (1994) [4], where some finite sample critical values are also presented.

2.3 General results

To study the asymptotic behavior of the previous set of statistics when the DGP is given by equation (1), we begin by noting that an asymptotically equivalent way to obtain them is based on the OLS estimation of the regression

$$z_{ht} = \pi_1 z_{1t} + \pi_2 z_{2t} + \pi_3 z_{3t} + \pi_4 z_{4t} + \zeta_t, \quad t = 1, 2, ..., T \tag{3}$$

where z_{ht}, z_{1t}, z_{2t}, z_{3t} and z_{4t} are the residuals of the projections of y_{4t}, $y_{1,t-1}$, $y_{2,t-1}$, $y_{3,t-2}$, $y_{3,t-1}$ over the space defined by $\{D_{1t}, D_{2t}, D_{3t}, D_{4t}, t\}$, respectively. That is, the z_{lt} variables ($l = h, 1, 2, 3, 4$) denote the detrended and seasonally demeaned transformed series of equation (2). Then, first we define some sample moments and evaluate their probability limits. These are reported in the following lemma.

Lemma 1 *Assume that y_t is generated by model (1), where $\{u_t\}$ is a sequence of iid $(0, \sigma_u^2)$ innovations, $\delta_i < \infty$, $i = 1, 2, 3, 4$ and $\tau = \lambda_s T$ $(0 < \lambda_s < 1)$. Denote with z_{ht}, z_{1t}, z_{2t}, z_{3t} and z_{4t}, respectively, the residuals of the projections of y_{4t}, $y_{1,t-1}$, $y_{2,t-1}$, $y_{3,t-2}$ and $y_{3,t-1}$ over the space spanned by $\{D_{1t}, D_{2t}, D_{3t}, D_{4t}, t\}$. Then, as $T \to \infty$:*

a) $T^{-1} \sum_{t=1}^{T} z_{it} z_{ht} \to -\sigma_u^2, \; i = 1, 2, 3,$

b) $T^{-1} \sum_{t=1}^{T} z_{4t} z_{ht} \to 0,$

c) $T^{-1} \sum_{t=1}^{T} z_{it} z_{jt} \to 0, \; \forall i \neq j, \; i, j = 1, 2, 3, 4,$

d) $T^{-1} \sum_{t=1}^{T} z_{ht}^2 \to 2\, \sigma_u^2,$

e) $T^{-1} \sum_{t=1}^{T} z_{1t}^2 \to e_2,$

f) $T^{-1} \sum_{t=1}^{T} z_{2t}^2 \to f_2,$

g) $T^{-1} \sum_{t=1}^{T} z_{3t}^2 \to g_2,$

[4] See also Engle *et al.* (1993), Smith and Taylor (1998) and Ghysels *et al.* (2001).

h) $T^{-1} \sum_{t=1}^{T} z_{4\,t}^2 \;\rightarrow\; h_2,$

where

$$e_2 \;=\; (1 \;-\; 3\,B_1)\; B_1 \left(\sum_{i=1}^{4} \delta_i \right)^2 \;+\; 4\,\sigma_u^2, \tag{4}$$

$$f_2 \;=\; B_1 \left(\sum_{i=1}^{4} (-1\,)^i\, \delta_i \right)^2 \;+\; 4\,\sigma_u^2 \tag{5}$$

and

$$g_2 \;=\; h_2 \;=\; [B_1 \left(\sum_{i=1}^{4} \delta_i^2 \;-\; 2 \sum_{i=1}^{2} \delta_i\, \delta_{i+2} \right) \;+\; 4\,\sigma_u^2]/\,2, \tag{6}$$

with $B_1 = (1 - \lambda_s)\lambda_s$.

Proof: see Appendix A.

Based on these results, the required probability limits are provided in the following theorem.

Theorem 1 *Assume that* y_t *is generated by model (1), where* $\{u_t\}$ *is a sequence of iid (0,* σ_u^2*) innovations,* $\delta_i < \infty$*,* $i = 1, 2, 3, 4$ *and* $\tau = \lambda_s T$ $(0 < \lambda_s < 1)$*. Then, when model (2) is estimated using OLS, as* $T \rightarrow \infty$*:*

a) $T^{-1/2}\, t_{\pi_1} \rightarrow -\sqrt{\frac{f_2\, g_2\, \sigma_u^2}{M}},$

b) $T^{-1/2}\, t_{\pi_2} \rightarrow -\sqrt{\frac{e_2\, g_2\, \sigma_u^2}{M}},$

c) $T^{-1/2}\, t_{\pi_3} \rightarrow -\sqrt{\frac{e_2\, f_2\, \sigma_u^2}{M}},$

d) $t_{\pi_4} \rightarrow 0\,,$

e) $T^{-1}\, F_{34} \rightarrow \frac{e_2\, f_2\, \sigma_u^2}{2\,M},$

f) $T^{-1}\, F_{234} \rightarrow \frac{e_2 (f_2 + g_2)\, \sigma_u^2}{3\,M},$

g) $T^{-1}\, F_{1234} \rightarrow \frac{[e_2 (f_2 + g_2) \,+\, f_2\, g_2] \sigma_u^2}{4\,M},$

where e_2*,* f_2 *and* g_2 *are given in (4), (5) and (6), respectively, and*

$$M = 2\,e_2\,f_2\,g_2 - [e_2(f_2 + g_2) + f_2\,g_2]\,\sigma_u^2.$$

Proof: see Appendix A.

Notice that due to the asymptotic orthogonality of the stochastic regressors in equation (2), the results obey the conditions $F_{34}-(1/2)\sum_{i=3}^{4} t_{\pi_i}^2 \to 0$, $F_{234}-(1/3)\sum_{i=2}^{4} t_{\pi_i}^2 \to 0$, and $F_{1234} - (1/4)\sum_{i=1}^{4} t_{\pi_i}^2 \to 0$. Hence, this context of asymptotic independence implies that the divergence of only one of the t-ratios is sufficient to assure that a F-statistic whose joint null includes the single hypothesis which is being tested by that t-ratio also diverges. Second, these results may be used to establish the probability limits of the (scaled) test statistics when there is no structural change and the most popular HEGY regression is run on a time series whose DGP is given by equation (1) (with $\delta_i = 0,\ i = 1, 2, 3, 4$).

However, the most striking implications that can be drawn from this theorem refer to the consequences of neglecting the seasonal mean shifts. The most important one is that only t_{π_4} goes to zero (the true value of π_4), that is, this is the only statistic whose asymptotic behavior is misleading. The remaining test statistics diverge: the t-ratios diverge towards $-\infty$ at rate $T^{1/2}$ and the F-statistics towards $+\infty$ at a faster rate (T). As evidence on the presence of the complex roots requires also the additional non-rejection of the single null $H_0 : \pi_3 = 0$ or the non-rejection of the joint null $H_0 : \pi_3 = \pi_4 = 0$, the exception for t_{π_4} is irrelevant. Therefore, asymptotically the HEGY tests are not adversely affected by the presence of neglected finite seasonal mean shifts.

Nevertheless, as the limit values depend on the magnitudes of the break parameters, the possibility that large values for these parameters are liable to push the test statistics towards zero in finite samples cannot be discarded. This discussion is motivated by the results of Perron (1989) and Montañés and Reyes (1999), where the behavior of DF tests under the presence of changes in a trend stationary process is considered. When the break affects only the intercept of the trend function the test statistics diverge. However, Montañés and Reyes note that the presence of the break parameter in the denominator of such limit values makes the statistics approach zero when that parameter grows. Thus, in spite of the inherent divergence of DF statistics, in small samples the probability of rejecting the (false) unit root null hypothesis may become very low when a large break occurs. Likewise, we may conjecture that a similar kind of effect can adversely affect the performance of HEGY tests.

2.4 Some particular cases

Following the previous line of discussion, we now try to ascertain the importance of the magnitudes of the break parameters. However, as the complexity of the previous expressions handicaps this kind of analysis, we will subsequently study some particular cases. Besides containing special relevance, these cases allow us to simplify the analysis. Obviously, other cases may also be analyzed using Theorem 1.

The results for the particular cases considered are presented in the following corollaries. Their proofs are direct, simply imposing the assumed parameter restrictions on the expressions of the previous theorem. Let us begin by the one where there is a break that affects the four seasonal intercepts exactly the same way, which amounts to a simple change in the intercept of the trend function, commonly referred to as a level shift (e.g. a *crash*).

Corollary 1 *Assume that $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta$ in model (1). Then, the asymptotic values of the HEGY test statistics are the following:*

a) $T^{-1/2}\, t_{\pi_1} \;\rightarrow\; -\sqrt{\frac{\sigma_u^2}{20\,(1-3\,B_1)\;B_1\,\delta^2 + 4\,\sigma_u^2}}$;

b) $T^{-1/2}\, t_{\pi_2} \;\rightarrow\; -\sqrt{\frac{4\,(1-3\,B_1)\;B_1\,\delta^2 + \sigma_u^2}{20\,(1-3\,B_1)\;B_1\,\delta^2 + 4\,\sigma_u^2}}$;

c) $T^{-1/2}\, t_{\pi_3} \;\rightarrow\; -\sqrt{\frac{4\,(1-3\,B_1)\;B_1\,\delta^2 + \sigma_u^2}{10\,(1-3\,B_1)\;B_1\,\delta^2 + 2\,\sigma_u^2}}$;

d) $T^{-1}\, F_{34} \;\rightarrow\; \frac{4\;(1-3\,B_1)\;B_1\,\delta^2 + \sigma_u^2}{4\left[5\;(1-3\,B_1)\;B_1\,\delta^2 + \sigma_u^2\right]}$;

e) $T^{-1}\, F_{234} \;\rightarrow\; \frac{4\;(1-3\,B_1)\;B_1\,\delta^2 + \sigma_u^2}{4\left[5\;(1-3\,B_1)\;B_1\,\delta^2 + \sigma_u^2\right]}$;

f) $T^{-1}\, F_{1234} \;\rightarrow\; \frac{3\,(1-3\,B_1)\;B_1\,\delta^2 + \sigma_u^2}{4\left[5\;(1-3\,B_1)\;B_1\,\delta^2 + \sigma_u^2\right]}$.

Strictly speaking, the case of this first corollary is not one of a changing seasonal pattern, as the seasonal variation remains constant but, instead, one of a level shift only [as in case A of Perron (1989)]. That is, the seasonal cycle simply moves to a different level. Therefore, the result that the asymptotic value for t_{π_1} is the only one that can be substantially reduced by the size of the break is not unexpected. Hence, a serious loss of power for testing the presence of the nonseasonal unit root can be expected in small samples. That is, although t_{π_1} is not asymptotically biased towards non-rejection, in finite samples there is a tension between two opposite forces: the bias may indeed be large, according to the magnitude of the break and the sample size.

As the seasonal cycle for the adequately detrended time series is unaltered, the remaining test statistics are much less influenced by the break magnitude. In fact, as this parameter appears in both terms of the fractions raised to the same power, no significant changes for the power performance are expected even in small samples.

The second of the particular cases is that where the break affects only one of the seasons. As in the previous case, a change in the level of the series results. However, there is now an additional effect: the seasonal pattern also changes.

Corollary 2 *Assume that* $\delta_i = \delta$ *for a particular* i, $\delta_j = 0$ $(i \neq j,\ i, j = 1, 2, 3, 4)$ *in model (1). Then, the asymptotic values of the HEGY statistics are the following:*

a) $T^{-1/2}\, t_{\pi_1} \rightarrow -\sqrt{\dfrac{\left(B_1\, \delta^2 + 4\sigma_u^2\right)\sigma_u^2}{2\,B_1^2\,(1-3\,B_1)\,\delta^4 + 3\,B_1\,(4-5\,B_1)\,\delta^2\,\sigma_u^2 + 16\,\sigma_u^4}}$;

b) $T^{-1/2}\, t_{\pi_2} \rightarrow -\sqrt{\dfrac{\left[(1-3\,B_1)\,B_1\, \delta^2 + 4\sigma_u^2\right]\sigma_u^2}{2\,B_1^2\,(1-3\,B_1)\,\delta^4 + 3\,B_1\,(4-5\,B_1)\,\delta^2\,\sigma_u^2 + 16\,\sigma_u^4}}$;

c) $T^{-1/2}\, t_{\pi_3} \rightarrow -\sqrt{\dfrac{2\left[(1-3\,B_1)\,B_1\, \delta^2 + 4\sigma_u^2\right]\sigma_u^2}{2\,B_1^2\,(1-3\,B_1)\,\delta^4 + 3\,B_1\,(4-5\,B_1)\,\delta^2\,\sigma_u^2 + 16\,\sigma_u^4}}$;

d) $T^{-1}\, F_{34} \rightarrow \dfrac{\left[(1-3\,B_1)\,B_1\, \delta^2 + 4\sigma_u^2\right]\sigma_u^2}{2\,B_1^2\,(1-3\,B_1)\,\delta^4 + 3\,B_1\,(4-5\,B_1)\,\delta^2\,\sigma_u^2 + 16\,\sigma_u^4}$;

e) $T^{-1}\, F_{234} \rightarrow \dfrac{\left[(1-3\,B_1)\,B_1\, \delta^2 + 4\sigma_u^2\right]\sigma_u^2}{2\,B_1^2\,(1-3\,B_1)\,\delta^4 + 3\,B_1\,(4-5\,B_1)\,\delta^2\,\sigma_u^2 + 16\,\sigma_u^4}$;

f) $T^{-1}\, F_{1234} \rightarrow \dfrac{\left[(4-9\,B_1)\,B_1\, \delta^2 + 16\,\sigma_u^2\right]\sigma_u^2}{4\left[2\,B_1^2\,(1-3\,B_1)\,\delta^4 + 3\,B_1\,(4-5\,B_1)\,\delta^2\,\sigma_u^2 + 16\,\sigma_u^4\right]}$.

Contrarily to the previous case, now it can be observed that, as in all the limit expressions the break parameter appears in the denominator raised to a higher power than in the numerator, all of them will tend to approach zero as the break magnitude grows. Hence, in small samples one should expect a deterioration of the power performance for all the HEGY statistics. That is, these results allow us to understand the Monte Carlo outcomes reported in Smith and Otero (1997). They are useful to explain why a serious loss of power can be expected whenever the break affects only one of the quarters and the magnitude of the break is large. Furthermore, they are also useful to understand the reason why the larger the available sample size is the larger the break magnitude must be for observing a certain loss in power.

The third case under consideration is the one where there is a change in the seasonal pattern but not in the level of the variable, i.e., the sum of all seasonal mean shifts is zero. This is the most interesting case for the seasonal fluctuations analyst. In practical terms, this is probably the most important case too, as seasonal unit root tests are often performed over time series observed for a rather limited number of years, the level of the series remaining about the same. Given this importance, we split the study for this case by imposing two different sets of restrictions; in both cases, only two of the seasons are affected, i.e., there is a *crash* in one of the seasons that is balanced by a *boom* in another. The first assumes that the break affects two consecutive quarters in a year, e.g., "spring becoming summer". The second considers that there is an intermediate observation between the two affected quarters.

Corollary 3 *Assume that* $\delta_i = -\delta_{i+1} = \delta$ *for a particular* i *(*$i = 1, 2, 3$*) and* $\delta_j = 0$ *(*$j \neq i$, $j \neq i+1$, $j = 1, 2, 3, 4$*) in model (1). Then, the asymptotic values of the HEGY statistics are the following:*

a) $T^{-1/2}\, t_{\pi_1} \;\rightarrow\; -\sqrt{\dfrac{\left(B_1\,\delta^2+\sigma_u^2\right)\left(B_1\,\delta^2+2\,\sigma_u^2\right)}{7\,B_1^2\,\delta^4+16\,B_1\,\delta^2\,\sigma_u^2+8\,\sigma_u^4}}$;

b) $T^{-1/2}\, t_{\pi_2} \;\rightarrow\; -\sqrt{\dfrac{\left(B_1\,\delta^2+2\,\sigma_u^2\right)\sigma_u^2}{7\,B_1^2\,\delta^4+16\,B_1\,\delta^2\,\sigma_u^2+8\,\sigma_u^4}}$;

c) $T^{-1/2}\, t_{\pi_3} \;\rightarrow\; -\sqrt{\dfrac{4\left(B_1\,\delta^2+\sigma_u^2\right)\sigma_u^2}{7\,B_1^2\,\delta^4+16\,B_1\,\delta^2\,\sigma_u^2+8\,\sigma_u^4}}$;

d) $T^{-1}\, F_{34} \;\rightarrow\; \dfrac{2\left(B_1\,\delta^2+\sigma_u^2\right)\sigma_u^2}{7\,B_1^2\,\delta^4+16\,B_1\,\delta^2\,\sigma_u^2+8\,\sigma_u^4}$;

e) $T^{-1}\, F_{234} \;\rightarrow\; \dfrac{\left(5\,B_1\,\delta^2+6\,\sigma_u^2\right)\sigma_u^2}{3\left[7\,B_1^2\,\delta^4+16\,B_1\,\delta^2\,\sigma_u^2+8\,\sigma_u^4\right]}$;

f) $T^{-1}\, F_{1234} \;\rightarrow\; \dfrac{B_1^2\,\delta^4+8\,B_1\,\delta^2\,\sigma_u^2+8\,\sigma_u^4}{4\left[7\,B_1^2\,\delta^4+16\,B_1\,\delta^2\,\sigma_u^2+8\,\sigma_u^4\right]}$.

An expected result now clearly emerges: only the behavior of the statistics designed to detect the presence of the seasonal unit roots is now adversely affected by the size of the break. Furthermore, it is also easy to observe that this magnitude is liable to exert a relatively larger influence on the asymptotic value of t_{π_2} than in the one of t_{π_3}. That is, neglecting the help that may be provided by the rejection of $H_0 : \pi_4 = 0$ — on which one cannot rely even asymptotically using t_{π_4} —, in small samples one can expect to find misleading evidence on the presence of a unit root at the bi-annual frequency more often than at the annual frequencies. This effect can be understood intuitively, as this case is the closest to that of a big shock affecting a certain observation and changing the cyclical movements of the series with a periodicity of two, which is precisely the effect one has in mind when the model $(1 + L)y_t = \epsilon_t$ (where the root -1 is present) is considered [Engle *et al.* (1993, p. 277)].

Clearly, the asymptotic value of t_{π_1} depends on δ too. However, since this parameter now appears in both terms of the fraction raised to the same power, no drastic power decreases are predicted.

Finally, the following corollary analyzes the case where the structural change affects two quarters one period apart.

Corollary 4 *Assume that* $\delta_i = -\delta_{i+2} = \delta$ *for a particular* i *(*$i = 1, 2$*) and* $\delta_j = 0$ *(*$j \neq i, j \neq i+2$, $j = 1, 2, 3, 4$*) in model (1). Then, the asymptotic values of the HEGY statistics are the following:*

a) $T^{-1/2}\, t_{\pi_1} \;\rightarrow\; -\sqrt{\frac{B_1\,\delta^2+\sigma_u^2}{6\,B_1\,\delta^2+4\,\sigma_u^2}}$;

b) $T^{-1/2}\, t_{\pi_2} \;\rightarrow\; -\sqrt{\frac{B_1\,\delta^2+\sigma_u^2}{6\,B_1\,\delta^2+4\,\sigma_u^2}}$;

c) $T^{-1/2}\, t_{\pi_3} \;\rightarrow\; -\sqrt{\frac{\sigma_u^2}{3\,B_1\,\delta^2+2\,\sigma_u^2}}$;

d) $T^{-1}\, F_{34} \;\rightarrow\; \frac{\sigma_u^2}{6\,B_1\,\delta^2+4\,\sigma_u^2}$;

e) $T^{-1}\, F_{234} \;\rightarrow\; \frac{B_1\,\delta^2+3\,\sigma_u^2}{6\,\left(3\,B_1\,\delta^2+2\,\sigma_u^2\right)}$;

f) $T^{-1}\, F_{1234} \;\rightarrow\; \frac{B_1\,\delta^2+2\,\sigma_u^2}{4\,\left(3\,B_1\,\delta^2+2\,\sigma_u^4\right)}$.

For the same reason of the previous case, the power behavior of the test for the nonseasonal unit root should depend much less on δ than in the first two cases. However, the most remarkable result is that t_{π_2} now asymptotically behaves as t_{π_1}: in spite of the change in the seasonal pattern, its limit value is not much influenced by the size of the break. This is in sharp contrast with the asymptotic values for t_{π_3} and F_{34}, both approaching zero as δ grows. In other words, the seasonal unit root test statistics no longer exhibit an homogeneous behavior in δ for all the frequencies: although all of them (besides t_{π_4}) diverge when T grows, in small samples and when δ is large, one can now expect to find frequent misleading evidence for the presence of the annual roots, but a similar deficiency should not occur when testing for the semi-annual unit root.

Nevertheless, this discrepancy cannot be considered as unexpected. Actually, the effect of a break affecting only the seasonal pattern and changing the seasonal means for two quarters one period apart is similar to that of a big shock affecting only the seasonal (cyclical) fluctuations of the series with a periodicity of four, which is precisely the kind of effect that the model $(1+L^2)y_t = \epsilon_t$ (where the annual frequency roots $\pm i$ are present) allows to capture. Clearly, under the conditions of the previous corollary the annual seasonal cycle was also indirectly disturbed, which is the reason why the behavior of all the seasonal unit root tests became strongly affected by the size of the break. As is easy to understand, under the assumptions stated in Corollary 4 there is no such (reverse) indirect effect, the annual cycle being the only one that is really affected by the structural change. Finally, a feature that this case shares with the previous one is that, *ceteris paribus*, all the absolute limit values are minimized when $\lambda_s = 0.5$.

3 Small Sample Results

In this section we present the results of some simple Monte Carlo experiments, carried out to ascertain to what extent the asymptotic results are useful to explain and to predict the behavior of the HEGY statistics in small samples. To keep the volume of tables inside reasonable limits, we report only a selection of the simulation results, though briefly mentioning some unreported ones. Additionally, unless stated explicitly otherwise, the power estimates: a) refer to 5% level tests (whose critical values are reported in Appendix B); b) consider $u_t \sim iid\, N(0,1)$, that is, though the results show that the asymptotic values depend also on σ_u^2, to simplify the analysis this parameter will be taken as fixed; c) are based on $10,000$ replications using TSP 4.3; d) refer to the case when $\lambda_s = 0.5$, i. e., the break occurs in the middle of the sample; e) consider a pre-break deterministic seasonal pattern given by $-\alpha_1 = \alpha_2 = -\alpha_3 = \alpha_4 = 1$ and a deterministic trend parameter $\beta = 1$; notice, however, that the results are invariant to the values of these parameters. Since the F_{34} statistic provides a simpler and more powerful procedure for testing the restrictions implied by the presence of the complex roots, the behavior of t_{π_3} and t_{π_4} will not be analyzed; further, these are rarely used in empirical work.

The DGP for the experiments is given by equation (1), its particularizations being provided by the cases studied in the corollaries. We set the values for the single break parameter at $\delta = 0$ (no-break), 1, 3 and 5 [5]. Given that $\sigma_u = 1$, a unity break parameter represents a small break, which is hardly detectable through graphical means. The case $\delta = 3$ is used to represent a moderately large break, one that may be detected through the plot of the detrended series. Finally, $\delta = 5$ represents a large break, which may be discernible even in the plot of the original (non-detrended) series.

Previous to the presentation of the power analysis, we have also considered an additional Monte Carlo study to verify the usefulness of the asymptotic results to predict the values of the statistics in small samples. Towards that end we have adopted the same parameter constellation but a wider range of sample sizes, $T = 24, 32, 48, 64, 96, 120, 200, 400, 600, 800$. Using $5,000$ replications, the mean values of the scaled test statistics were compared with their asymptotic values. Figures 1 and 2 show these deviations for two of the worst case scenarios: those of t_1 and t_2 for the cases of corollaries 1 and 2, respectively.

Figures 1 and 2 about here

The emerging picture is quite clear: the asymptotic results provide a very good approximation to the behavior in small samples. The case depicted in figure 2 seems remarkable: for example, even when $\delta = 3$ and $T = 96$ only the deviation is a mere

[5] To confirm the symetrical behavior of the tests as a function of δ we considered also negative break magnitudes. Although we will not formally report the corresponding results, these being available from the authors on request, it must be mentioned that, as a general result, the prediction of a symmetrical behavior in δ is quite accurate in small samples. The same can be said about relaxing the assumption that τ corresponds to a fourth quarter observation.

0.01 . [6]

For comparison purposes, the power estimates for the no-break case, i.e., $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta = 0$, are presented in Table 1, the remarkable power properties of the seasonal unit root tests for sample sizes as small as $T = 48$ clearly emerging. However, it should be noted that we are considering a very simple DGP, one that, besides the trend, contains only purely deterministic seasonality, i.e., a DGP whose only purpose is to act as a standard against which the power losses resulting from the structural change will be judged.

Table 1 about here

3.1 The case of a level shift

The results of Table 2 reflect the asymptotic behavior reported in Corollary 1, in the sense that only t_{π_1} is clearly affected by the break. In particular, the performance of t_{π_1} depends on the combination of the values of δ and T: on the one hand, the larger the break magnitude, the smaller the power of this statistic; on the other hand, the larger the sample size, the more powerful t_{π_1} is. Further, while t_{π_1} is virtually powerless for the case of a big break even for samples as large as $T = 160$, when $\delta = 5$ but $T = 400$ the estimated power attains 0.930. Contrasting with this behavior, the minor changes implied by the break on the limit values of the seasonal unit root test statistics manifest in small samples through negligible changes in the rejection frequencies.

Table 2 about here

The outcomes of table 2 do not represent, however, the worst case scenario for the performance of the HEGY tests, particularly in what concerns t_{π_1}. Actually, as is easily observed in the results of Corollary 1, the minimum absolute asymptotic values are not attained when $\lambda_s = 0.5$. For t_{π_1} this occurs for $\lambda_s \approx 0.2$ and 0.8, and this behavior holds for small samples too, as some additional simulations show [7]. Quite on the contrary, for all the other test statistics such a displacement of the time of the break produces only marginal power reductions (and only when $T = 48$).

Finally, based on the results we obtained for the cases when $\lambda_s = 0.25$ and $\lambda_s = 0.75$, the power estimates for small samples do not fully reflect the predicted symmetrical behavior in λ_s (around 0.5) but do not strongly contradict it either.

[6] Further, as expected, the approximation performs even better for the cases of the almost unaffected statistics.

[7] For instance when $\delta = 3$ and $\lambda_s = 0.25$, the nonseasonal unit root hypothesis will be rejected only in about 0.7%, 3.6% and 29.3% of the occasions for samples sized as $T = 48, 96$ and 160, respectively. Furthermore, for large breaks, this effect still remains highly significant for samples as large as $T = 400$: when $\delta = 5$ the power estimates are 0.930 and 0.286 when $\lambda_s = 0.5$ and $\lambda_s = 0.25$, respectively.

3.2 A change in the level and in the seasonal pattern

This case is simply illustrated with the situation considered in Corollary 2, where only one of the seasonal means changes. As can be seen readily from those results, the asymptotic values for t_{π_2} and t_{π_3} are more adversely affected than the one for t_{π_1} [8] — this effect becoming clearer as δ grows — but, for a given break, they are not as much affected as this last one in the previous case. Moreover, for a given δ, the limit value for t_{π_1} is not so influenced as in the previous case. All these effects are easily understandable: now the seasonal pattern also changes, thereby affecting the behavior of the seasonal unit root tests, and the effect of a shift in only one of the quarters on the level of the series is now weaker. Obviously, this is also reflected in the probability limits of the scaled F-statistics, now lower than in the precedent case (for the same δ). All these features can be observed in small samples too, as the comparison between the results in table 3 and those of tables 1 and 2 documents.

Table 3 about here

However, while for the tests where at least one seasonal unit root is involved the absolute asymptotic values are minimized when $\lambda_s = 0.5$, for t_{π_1}, and as in the previous case, the same is not generally true. This is reflected in small samples too. For instance, for a large break ($\delta = 5$) located at $\lambda_s = 0.25$ and for $T = 96$, the power losses for the t_{π_2} and F_{34} statistics are not so dramatic as those reported in table 3 as the power estimates are 0.595 and 0.907, respectively. For this same example, the loss in power incurred when testing only the presence of the root 1 is significantly higher than the one which is reported in table 3 (0.559). However, some unreported simulations show that the effect of a break location differing from $\lambda_s = 0.5$ on the behavior of this test is now much less noticeable than in the previous case.

Finally, additional simulations carried out for the cases when $\lambda_s = 0.25$ and 0.75 show that the symmetrical asymptotic behavior in λ_s is now a relatively poor approximation for small samples, particularly when the break is large (and even for samples as large as $T = 160$). However, the performance of the HEGY tests is mixed, the only possible conclusion being that the test for the nonseasonal unit root seems less severely affected when the break occurs in the second half of the sample.

3.3 A change in the seasonal pattern only

The results presented in table 4 confirm the main predictions for small samples based on Corollary 3 [9]: a) the seasonal unit root tests, and particularly the test of the semi-annual root, are now those which are clearly affected by the size of the break ; b) this

[8] For instance, concerning t_{π_2} and t_{π_1}, the limit value for the second will be lower than the one for the first whenever $-3B_1^2\delta^2 < 0$, i.e., whenever there is a break.

[9] In general, the Monte Carlo results for the case $\delta_1 = -\delta_4 = \delta$ and $\delta_2 = \delta_3 = 0$ are very close to those presented in table 4. Significant discrepancies are found only when $T = 48$.

magnitude also plays a role in the behavior of t_{π_1}, but it is now much less important than in the previous cases, the power recovery being almost completed for samples with size $T = 96$ even for a large break ($\delta = 5$).

Table 4 about here

Moreover, we can also observe that the case of a break affecting two consecutive quarters is that one where the power reduction incurred by F_{234} and F_{1234} is the most serious (see also table 5). Unreported additional results also show that while the prediction of the power function for all tests being minimized when $\lambda_s = 0.5$ is observed in small samples, that of a symmetrical behavior in λ_s is not, particularly for large values of δ. For this case, when the break is moderate or large ($\delta = 3, 5$), the power loss experienced by the statistics involving seasonal unit roots is generally higher when the shift occurs in the second half of the sample.

A behavior similar to this may be observed for F_{34}, F_{234} and F_{1234} when the break affects two quarters one period apart (see table 5). However, once again this is the only exception concerning the accuracy of the predictions based on the results of Corollary 4. In particular: a) the performance of t_{π_1} and t_{π_2} is now seen to be very close to the no-break case, their power exhibiting a significant decrease only when $T = 48$; b) contrasting with this, the small sample power performance of F_{34} strongly depends on δ, the growth of this magnitude requiring larger samples that allow the test to recover its power. Also, as expected, the performance of F_{234} and F_{1234} is now less severely affected than in the previous case. Finally, unreported results for the cases when $\lambda_s = 0.25$ and 0.75 also seem to confirm that the misleading evidence produced by all the tests, and particularly when using the F_{34} statistic, is more likely to occur when $\lambda_s = 0.5$.

Table 5 about here

3.4 Relaxing two assumptions

Additional simulation experiments [10] allow us to state that two important assumptions concerning DGP (1) can be relaxed without affecting qualitatively our results. These weaker and more empirically relevant conditions refer to: a) allowing that the innovations process, $\{u_t\}$, is serially correlated according to some stationary and invertible ARMA model; b) relaxing the trend stationarity assumption. In both cases the simulation results are similar to those presented, i.e., the statistics for the seasonal unit root tests still diverge for a finite break and the different types of breaks produce the effects previously observed. Further, this also holds for the non-seasonal unit root test when only a) is assumed.

[10] Though we do not present the results, they are available from the authors on request.

To accomplish a), i.e., to introduce some dynamics in (1), besides the deterministic component we have considered an autoregressive (stationary) polynomial in L for y_t, allowing for non-zero roots at all the frequencies [11]. In other words, we extended (1) to $\alpha(L)y_t = m_t + \epsilon_t$, where m_t denotes the deterministic elements which are present in (1), $\epsilon_t \sim nid(0,1)$ and, for instance, $\alpha(L) = (1-0.5L)^2(1+0.5L)^2(1+0.5^2L^2)$. In this particular case, to prevent the tests from being contaminated by residual autocorrelation, the auxiliary regression (2) was augmented with the first two lags of y_{4t}. Concerning b), instead of (1) we assumed the DGP as being given by $\Delta y_t = \sum_{i=1}^{4} \alpha_i D_{it} + \sum_{i=1}^{4} \delta_i D_{it}[I_{t>\tau}] + u_t$, thereby introducing a stochastic trend at the long-run frequency.

4 Conclusions

The most important conclusion that can be drawn from this paper is that when the seasonal shifts are finite there is not an asymptotic bias towards the (incorrect) non-rejection of seasonal unit root hypotheses when the HEGY procedure is used on a seasonal deterministic time series affected by a structural break. Given that our data generating mechanism is contained neither in the null nor in the alternative hypotheses for which the tests have been designed, this property of the HEGY statistics seems remarkable.

To gain some insight into the small sample behavior of the tests, we further particularized the asymptotic analysis for four basic cases which we consider as specially relevant in empirical research. This proved to be very useful, the different specifications of the break producing rather distinctive features in the limit expressions. As the Monte Carlo experiments confirmed, the asymptotic analysis proved useful in several distinct respects:

a) to understand and to predict the power performance of the HEGY tests in small samples, in many different break cases;

b) to obtain the asymptotic values of the HEGY statistics for data generated by model (1) even when there is no break;

c) to explain clearly the power performance of the HEGY tests reported in the previous literature and to provide an explanation for new cases too;

d) to clarify the effects of the different types of breaks on the power performance of the various test statistics;

[11] However, it should be mentioned that in this case the asymptotic orthogonality conditions observed for (1) no longer hold, further complicating the calculations of the probability limits.

e) to provide a very good approximation to the numeric values of the statistics in samples of moderate size.

Besides this confirmation evidence, the Monte Carlo simulations showed the usefulness of the asymptotic analysis for predicting the small sample behavior of the HEGY tests in other respects too, the only exception being the non symmetrical behavior of the power functions in λ_s for most of the cases. Moreover, the importance of those results for understanding and predicting empirical evidence on seasonal unit roots is strengthened by the fact that, though they were derived for a trend stationary process with independent innovations, they are still valid when both these assumptions are relaxed.

A Appendix A: Proofs

A.1 Lemma 1

As the presentation of the full proof of this lemma would be rather tedious, we consider more appropriate to provide only detailed proofs of some of the reported probability limits. The remaining limit values can be derived following similar arguments and are available on request.

Consider again equation (3):

$$z_{ht} = \pi_1 z_{1t} + \pi_2 z_{2t} + \pi_3 z_{3t} + \pi_4 z_{4t} + \zeta_t, \tag{A.1}$$

where z_{ht}, z_{1t}, z_{2t}, z_{3t} and z_{4t} are, respectively, the residuals of the projections of y_{4t}, $y_{1,t-1}$, $y_{2,t-1}$, $y_{3,t-2}$, $y_{3,t-1}$ over the space defined by $\{D_{1t}, D_{2t}, D_{3t}, D_{4t}, t\}$. Then, this implies, for example, that the vector containing the observations of z_{ht} can be defined as

$$z_h = h - Wb_h \tag{A.2}$$

where h denotes the vector containing the observations of y_{4t}, W is a $(T \times 5)$ matrix containing the observations of the seasonal dummies ($D_{it},\ i = 1, 2, 3, 4$) and of the deterministic trend term, and the vector of OLS estimates, b_h, is given by

$$b_h = (W'W)^{-1}W'h. \tag{A.3}$$

Notice that the matrix $W'W$ can be written as

$$W'W = \begin{bmatrix} \frac{T}{4} & 0 & 0 & 0 & \frac{T^2-2T}{8} \\ 0 & \frac{T}{4} & 0 & 0 & \frac{T^2}{8} \\ 0 & 0 & \frac{T}{4} & 0 & \frac{T^2+2T}{8} \\ 0 & 0 & 0 & \frac{T}{4} & \frac{T^2+4T}{8} \\ \frac{T^2-2T}{8} & \frac{T^2}{8} & \frac{T^2+2T}{8} & \frac{T^2+4T}{8} & \frac{T(T+1)(2T+1)}{6} \end{bmatrix}. \tag{A.4}$$

On the other hand, considering that $\sum_{t=1}^{T} D_{it}h_t = \beta T + o(T), \ i = 1,2,3,4,$ and $\sum_{t=1}^{T} th_t = 2\,\beta T^2 + \left(2\,\beta + \lambda_s \sum_{i=1}^{4} \delta_i\right) T + o_p(T)$, it is not difficult to show that

$$diag(T^{-1}, T^{-1}, T^{-1}, T^{-1}, T^{-2})W'h = \begin{bmatrix} T^{-1}\sum D_{1t}h_t \\ T^{-1}\sum D_{2t}h_t \\ T^{-1}\sum D_{3t}h_t \\ T^{-1}\sum D_{4t}h_t \\ T^{-2}\sum t\,h_t \end{bmatrix} \to \begin{bmatrix} \beta \\ \beta \\ \beta \\ \beta \\ 2\,\beta \end{bmatrix} \tag{A.5}$$

Thus, using (A.4) and (A.5) in (A.3), we can obtain the probability limit for b_h:

$$b_h \to (4\beta, 4\beta, 4\beta, 4\beta, 0)'. \tag{A.6}$$

Then, the asymptotic convergence of $\sum z_{ht}^2$ can be easily evaluated simply considering that

$$\sum z_{ht}^2 = z_h' z_h = (h - Wb_h)'(h - Wb_h) = h'h - h'Wb_h, \tag{A.7}$$

and taking into account that

$$h'h = \sum h_t^2 = 16\beta^2 T + 2\sum u_t^2 + o_p(T). \tag{A.8}$$

Using (A.5), (A.6) and (A.8) in (A.7), it is straightforward to show that

$$T^{-1}\sum_{t=1}^{T} z_{ht}^2 \to 2\ \sigma_u^2, \tag{A.9}$$

as is reported in d) of Lemma 1.

To prove the remaining results reported in Lemma 1, it is now necessary to establish the asymptotic behavior of the sample moments of the z_i variables. To evaluate them, let us begin by denoting $y_{1,t-1},\ y_{2,t-1},\ y_{3,t-2},\ y_{3,t-1}$ simply as $x_{1t},\ x_{2t},\ x_{3t}$ and x_{4t}, respectively (and notice also that when we drop the time subscript we are referring to the corresponding vector of observations). Then, we can define z_i as

$$z_i = x_i - Wb_{x_i},\ i = 1,2,3,4, \tag{A.10}$$

where $b_{x_i} = (W'W)^{-1}W'x_i$ $(i = 1,2,3,4)$.

Some calculations allow us to show that

$$b_{x_1} \to (a_1, a_1, a_1, a_1, 4\beta_1)', \tag{A.11}$$

$$b_{x_2} \to (a_{21}, a_{22}, a_{23}, a_{24}, 0)', \tag{A.12}$$

$$b_{x_3} \to (c_{10}, c_{20}, c_{11}, c_{21}, 0)', \tag{A.13}$$

$$b_{x_4} \to (c_{20}, c_{11}, c_{21}, c_{11}, 0)', \tag{A.14}$$

where:

$$a_1 = \sum_{i=1}^{4} \alpha_i + \left(1 - 4\lambda_s + 3\lambda_s^2\right) \sum_{i=1}^{4} \delta_i - 10\beta, \tag{A.15}$$

$$a_{2j} = (-1)^{j+1} \left[\sum_{i=1}^{4} (-1)^{i+1} \alpha_i + (1 - \lambda_s) \sum_{i=1}^{4} (-1)^{i+1} \delta_i \right] - 2\beta, \; j = 1, 2, 3, 4, \tag{A.16}$$

$$c_{ij} = (-1)^j \left[\alpha_i - \alpha_{i+2} + (1 - \lambda_s) \left(\delta_i - \delta_{i+2} \right) \right] - 2\beta. \tag{A.17}$$

For example, to obtain (A.12), we should consider that

$$\begin{aligned} diag(T^{-1}, T^{-1}, T^{-1}, T^{-1}, T^{-2}) W' x_2 \quad &= \quad \begin{bmatrix} T^{-1} \sum D_{1t} x_{2t} \\ T^{-1} \sum D_{2t} x_{2t} \\ T^{-1} \sum D_{3t} x_{2t} \\ T^{-1} \sum D_{4t} x_{2t} \\ T^{-2} \sum t \, x_{2t} \end{bmatrix} \\ &\rightarrow \quad \begin{bmatrix} \frac{-2\beta - \sum_{i=1}^{4} (-1)^i [\alpha_i + (1-\lambda_s)\, \delta_i]}{4} \\ \frac{-2\beta + \sum_{i=1}^{4} (-1)^i [\alpha_i + (1-\lambda_s)\, \delta_i]}{4} \\ \frac{-2\beta - \sum_{i=1}^{4} (-1)^i [\alpha_i + (1-\lambda_s)\, \delta_i]}{4} \\ \frac{-2\beta + \sum_{i=1}^{4} (-1)^i [\alpha_i + (1-\lambda_s)\, \delta_i]}{4} \\ -\beta \end{bmatrix}. \end{aligned} \tag{A.18}$$

Combining this result with (A.4) the probability limit for b_{x_2} that is reported in (A.12) can be obtained.

Now, to determine for example the probability limit of the cross sample moment between z_2 and z_h, we begin by defining the magnitude

$$\sum z_{2t} \, z_{ht} = z_h' z_2 = (h - W b_h)'(x_2 - W b_{x_2}) = h' x_2 - h' W b_{x_2} \tag{A.19}$$

Thus, taking into account that

$$\sum x_{2t} h_t = h' x_2 = -8\beta^2 T - \sum u_t^2 + o_p(T) \tag{A.20}$$

and using (A.20), (A.5) and (A.12) in (A.19), it can easily be shown that

$$T^{-1} \sum z_{2t} \, z_{ht} = -\sigma_u^2. \tag{A.21}$$

Using a similar technique, it is also possible to prove the result reported in f). However, the calculations are now even more cumbersome and tedious.

A.2 Theorem 1

Let us define the OLS vector of estimators of (4) as $\hat{\pi}' = [\hat{\pi}_1\ \hat{\pi}_2\ \hat{\pi}_3\ \hat{\pi}_4] = (Z'Z)^{-1}Z'z_h$, where $Z = \{z_{1t}, z_{2t}, z_{3t}, z_{4t}\}$. Using the results of Lemma 1, the convergence in probability of the OLS estimator is given by

$$\hat{\pi} = \left(T^{-1}\,ZZ\right)^{-1}\left(T^{-1}\,Z'z_h\right) \rightarrow [Diag\,(e_2,\, f_2,\, g_2,\, h_2)]^{-1}\begin{bmatrix} -\sigma_u^2 \\ -\sigma_u^2 \\ -\sigma_u^2 \\ 0 \end{bmatrix}.$$

Thus, this is a case of asymptotically orthogonal regressors and the calculation of the probability limits for these estimators is very easy:

$$\hat{\pi}_1 \rightarrow -\,\sigma_u^2/e_2,$$

$$\hat{\pi}_2 \rightarrow -\,\sigma_u^2/f_2,$$

$$\hat{\pi}_3 \rightarrow -\,\sigma_u^2/g_2,$$

$$\hat{\pi}_4 \rightarrow 0.$$

Similarly, the probability limit for the estimator of the variance of the innovations is given by

$$\hat{\sigma}_u^2 \rightarrow \frac{2\,e_2\,f_2\,g_2 - [e_2\,(f_2 + g_2) + f_2\,g_2]\,\sigma_u^2}{e_2\,f_2\,g_2}\,\sigma_u^2.$$

Using all these results (including those above for the elements of the $(Z'Z)^{-1}$ matrix), it is straightforward to derive the asymptotic values for the t-ratios.

On the other hand, to derive the probability limits for the (pseudo) F-statistics, we should additionally take into account that, for instance, the statistic for testing the joint null hypothesis $H_0 : \pi_2 = \pi_3 = \pi_4 = 0$ can be defined as $F_{234} = (SSR_{234} - SSR)/3\,\hat{\sigma}^2$, where SSR is the sum of the squared residuals of equation (4) and SSR_{234} is the sum of the squared residuals of the regression

$$z_{4t} = \pi_1 z_{1t} + \eta_t.$$

Moreover, the limit value for this variable is given by

$$T^{-1}\,SSR_{234} \rightarrow \sigma_u^2\,(2\,e_2 - \sigma_u^2)/e_2.$$

Similarly, the statistic for testing $H_0 : \pi_3 = \pi_4 = 0$ can be expressed as $F_{34} = (SSR_{34} - SSR)/2\,\hat{\sigma}^2$, where SSR_{34} is the sum of the squared residuals of the regression

$$z_{4t} = \pi_1 z_{1t} + \pi_2 z_{2t} + \varepsilon_t,$$

whose probability limit is given by

$$T^{-1}\,SSR_{34} \to \frac{2\,e_2\,f_2 \,-\, (\,e_2 \,+\, f_2)\;\sigma_u^2}{e_2\,f_2}\;\sigma_u^2.$$

Finally, the statistic for testing $H_0 : \pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$ can be written as $F_{1234} = \hat{\pi}' Z' z_h / 4\hat{\sigma}^2$. Then, some tedious algebra allows us to show that

$$T^{-1}\,\hat{\pi}'\,Z'\,z_h' \to \frac{e_2\;(f_2 \,+\, g_2) \,+\, f_2\,g_2}{e_2\,f_2\,g_2}\;\sigma_u^2.$$

B Appendix B: Critical values

In the following table we present the 5% critical values used for the Monte Carlo simulations when the auxiliary test regression is (2), using a DGP given by $\Delta_4 y_t = \epsilon_t$, $\epsilon_t \sim iid\,N(0,1)$, and based on 40 000 replications.

Table B about here

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Table 1. Rejection frequencies of the HEGY tests when there is no break ($\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta = 0$)

T	48	96	160
t_{π_1}	0.685	0.997	1.000
t_{π_2}	0.869	1.000	1.000
F_{34}	0.976	1.000	1.000
F_{234}	0.992	1.000	1.000
F_{1234}	0.991	1.000	1.000

Table 2. Rejection frequencies of the HEGY tests for the case of a level shift ($\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta = 1, 3, 5$)

δ	1			3			5		
T	48	96	160	48	96	160	48	96	160
t_{π_1}	0.505	0.963	1.000	0.054	0.206	0.693	0.002	0.002	0.015
t_{π_2}	0.855	1.000	1.000	0.853	1.000	1.000	0.888	1.000	1.000
F_{34}	0.976	1.000	1.000	0.989	1.000	1.000	0.999	1.000	1.000
F_{234}	0.991	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000
F_{1234}	0.985	1.000	1.000	0.984	1.000	1.000	0.999	1.000	1.000

Table 3. Rejection frequencies of the HEGY tests for the case of a change in the level and in the seasonal pattern ($\delta_1 = \delta = 1,\ 3,\ 5; \delta_2 = \delta_3 = \delta_4 = 0$)

δ	1			3			5		
T	48	96	160	48	96	160	48	96	160
t_{π_1}	0.635	0.995	1.000	0.360	0.932	1.000	0.158	0.696	0.993
t_{π_2}	0.810	0.999	1.000	0.335	0.912	1.000	0.032	0.286	0.896
F_{34}	0.950	1.000	1.000	0.528	0.993	1.000	0.065	0.595	0.995
F_{234}	0.976	1.000	1.000	0.624	0.999	1.000	0.079	0.749	1.000
F_{1234}	0.977	1.000	1.000	0.682	1.000	1.000	0.153	0.917	1.000

Table 4. Rejection frequencies of the HEGY tests for the case of a change in the seasonal pattern only, the break affecting two consecutive quarters ($\delta_1 = -\delta_2 = \delta = 1, 3, 5; \delta_3 = \delta_4 = 0$)

δ	1			3			5		
T	48	96	160	48	96	160	48	96	160
t_{π_1}	0.595	0.993	1.000	0.363	0.967	1.000	0.409	0.969	1.000
t_{π_2}	0.646	0.992	1.000	0.008	0.111	0.687	0.000	0.000	0.000
F_{34}	0.905	1.000	1.000	0.152	0.854	1.000	0.001	0.041	0.626
F_{234}	0.935	1.000	1.000	0.103	0.842	1.000	0.000	0.014	0.516
F_{1234}	0.947	1.000	1.000	0.313	0.989	1.000	0.093	0.803	1.000

Table 5. Rejection frequencies of the HEGY tests for the case of a change in the seasonal pattern only, the break affecting two quarters one period apart ($\delta_1 = -\delta_3 = \delta = 1, 3, 5; \delta_2 = \delta_4 = 0$)

δ	1			3			5		
T	48	96	160	48	96	160	48	96	160
t_{π_1}	0.605	0.995	1.000	0.457	0.980	1.000	0.521	0.987	1.000
t_{π_2}	0.827	1.000	1.000	0.752	1.000	1.000	0.815	1.000	1.000
F_{34}	0.841	0.999	1.000	0.022	0.374	0.979	0.000	0.000	0.018
F_{234}	0.944	1.000	1.000	0.403	0.993	1.000	0.208	0.940	1.000
F_{1234}	0.951	1.000	1.000	0.589	1.000	1.000	0.477	0.998	1.000

Table B. 5% critical values for the HEGY tests

T	48	96	160	400
t_{π_1}	−3.35	−3.38	−3.40	−3.39
t_{π_2}	−2.79	−2.82	−2.84	−2.84
F_{34}	6.55	6.61	6.58	6.58
F_{234}	6.13	6.02	5.94	5.87
F_{1234}	6.72	6.47	6.34	6.20

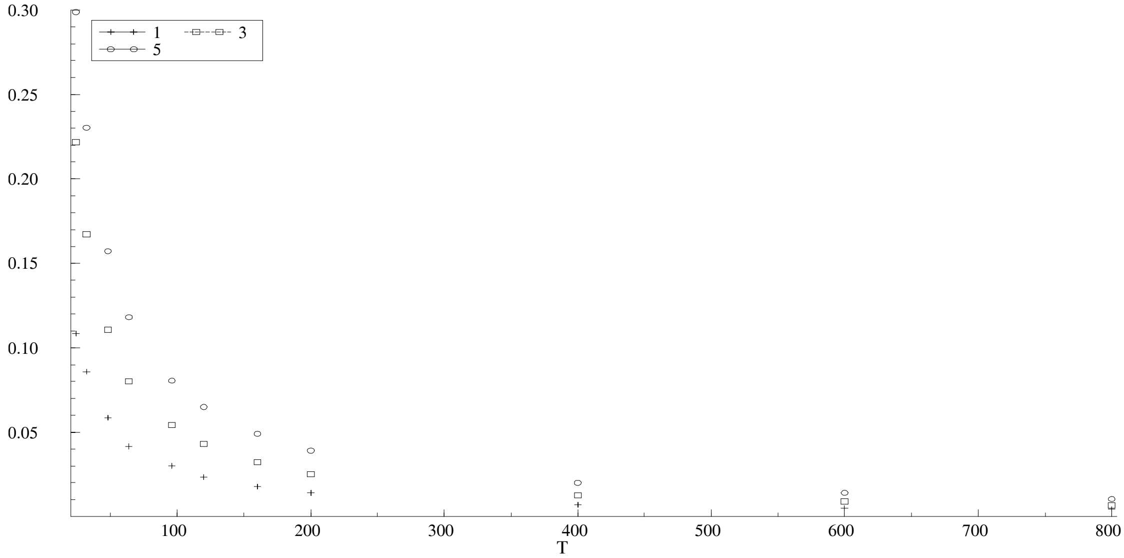

Figure 1: Deviations between the averaged scaled t_1 statistics and its asymptotic value for the case of corollary 1, as a function of δ $(=\ 1, 3, 5)$ and T $(=$ $24, 32, 48, 64, 96, 120, 160, 200, 400, 800)$, based on $5,000$ replications.

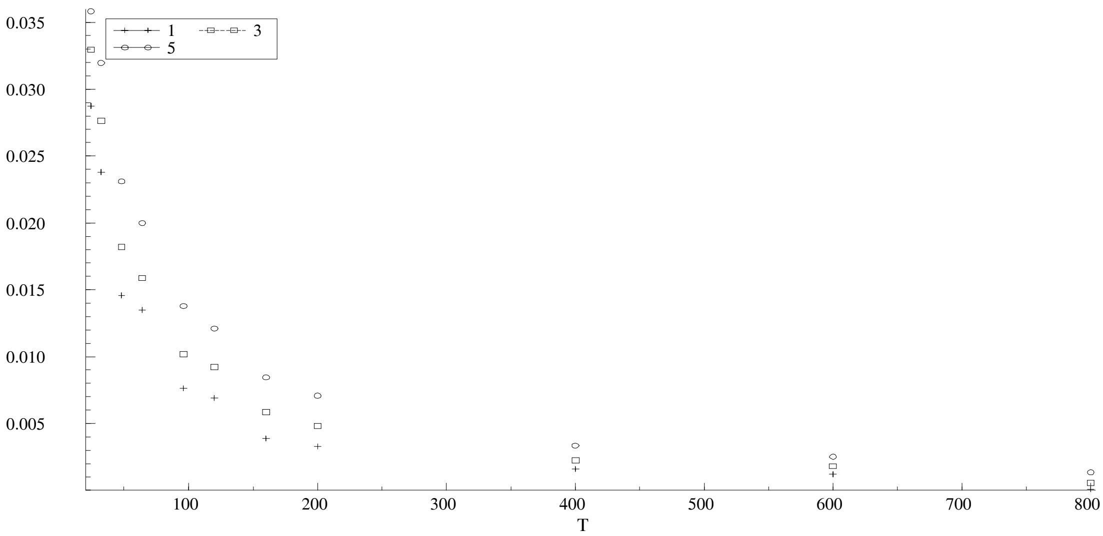

Figure 2: Deviations between the averaged scaled t_2 statistics and its asymptotic value for the case of corollary 2, as a function of δ $(=\ 1, 2, 3)$ and T $(=$ $24, 32, 48, 64, 96, 120, 200, 400, 600, 800)$, based on $5,000$ replications.