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KNOWLEDGE EXCHANGE, MATCHING, AND AGGLOMERATION

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Abstract

Despite wide recognition of their significant role in explaining sustained growth and economic development, uncompensated knowledge spillovers have not yet been fully modeled with a microeconomic foundation. This paper illustrates the exchange of knowledge as well as its consequences for agglomerative activity in a general-equilibrium search-theoretic framework. Agents, possessing differentiated types of knowledge, search for partners to exchange ideas in order to improve production efficacy. When individuals' types of knowledge are too diverse, a match is less likely to generate significant innovations. We demonstrate that the extent of agglomeration has significant implications for the patterns of information flows in economies. By simultaneously determining the patterns of knowledge exchange and the population agglomeration of an economy, we identify additional channels for interaction between agglomerative activity and knowledge exchange. The main implications of the model are a negative correlation between city population and diversity of knowledge exchange and a positive correlation between city population and per capita knowledge or patent output. Contrary to previous work in urban economics and growth theory, it is possible that a decentralized equilibrium is under-populated or over-populated and under-selective or over-selective in knowledge exchange, compared to the social optimum. By allowing for perpetual knowledge accumulation, we find that population agglomeration is generally accompanied by higher growth. The main findings remain qualitatively unchanged even if we allow individual knowledge types to change over time, though the creation of new types of knowledge may result in multiple equilibria.

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I. Introduction

Uncompensated knowledge spillovers have played a central role in explaining sustained growth and economic development. In their pioneering work, Romer (1986) and Lucas (1988) develop models in which the positive external effects of society's aggregate knowledge or human capital stock promote economic growth. The incorporation of this type of positive externality has resulted in abundant research in the areas of growth and development. These insights, however, raise many important but unsettled questions. How do knowledge spillovers occur? What are the consequences of knowledge spillovers for the advancement and concentration of economic activity? Lucas points out that ***interaction*** among economic agents is the key for the development of knowledge: "human capital accumulation is a social activity involving groups of people" (p. 19). Given that interaction serves to promote both knowledge acquisition and creation, various types of economic clusters may emerge as economic organizations to foster the transmission of information. The present paper is devoted to examining these important but largely open issues.

Our paper establishes a microfoundation to explain the patterns and implications of knowledge exchange. Knowledge exchange involves an interpersonal externality: "if one man starts a new idea, it is taken up by others and combined with suggestions of their own; and thus it becomes the source of new ideas." (Marshall 1890, p.352). Kuznets (1962) echoes this view by emphasizing: "creative effort flourishes in a dense intellectual atmosphere, and it is hardly accident that the locus of intellectual progress lies in the larger cities; ... the possibility of more intensive intellectual contact ... afforded by greater numbers may be an important factor in stepping up the rate of additions to new knowledge" (pp.328-9). Jacobs (1969) also stresses that knowledge spillovers are the primary force for agglomeration, such as city formation, firm clustering, and geographical concentration of research activity.[1]

[1]More recently, Rauch (1993) and Saxenian (1994) provide empirical evidence that cities promote the transmission of knowledge. Jaffe et al (1993) show that patents are more likely to cite previous patents from the same area. Audretsch and Feldman (1996) find that even after controlling for the geographical concentration of production, innovative activity clusters more in industries where knowledge spillovers are crucial. Glaeser et al (1992) and Henderson et al (1995) suggest that spillovers occur both within and between industries. Ciccone and Hall (1997) document that locally increasing returns resulting from geographical concentration can explain more than half of labor productivity variation across U.S. states.

These arguments suggest that agglomeration promotes the transmission of knowledge due to lower costs of communication in dense environments and fosters growth. Yet, despite the clearly important role of geography for the propagation of knowledge, spatial considerations have received limited attention in the theoretical literature. We attempt to fill this gap by developing a simple search-theoretic model particularly suitable for analyzing the knowledge transmission mechanism and its interactions with agglomerative activity. We believe the random-matching model to be the most appropriate for studying these issues because it provides an explicit notion of transactions costs (search and entry frictions) and patterns of interaction (knowledge exchange). The latter aspect, in particular, allows us to analyze the relationships between ***endogenous knowledge exchange*** and ***endogenous population agglomeration***.

In our economy, agents, such as individual consumers/workers, firms and patent holders, possess ***horizontally differentiated*** types of knowledge and search for partners to exchange ideas, so as to improve production efficacy. We consider that heterogeneity (in terms of different types of knowledge) plays a role in the transfer of knowledge. When individuals' types of knowledge are too diverse, a match is associated with less knowledge exchange. The same applies when individuals' types are too similar, so little is obtained through collaboration. We first endogenously determine the range of agents with whom an individual will undertake knowledge exchange, hereafter called the ***knowledge spread***, and characterize its determinants in a closed-city model. We later consider the case of a small, open city. This allows us to study the interactions between the determinants of knowledge exchange and the endogenous process of population agglomeration. By characterizing the role of heterogeneity for the flow of ideas, our structure provides insights into questions regarding human capital accumulation, the patterns of information flows, and their interactions with agglomerative activity.

In our first model, we consider the case of a closed-city where the population size is fixed. This exercise is important for two reasons. First, it is valuable in the endogenous growth context as we provide explanations behind the determinants of knowledge exchange. We show that economies with higher search or market frictions will have more diversified patterns of information exchange so that individuals obtain more, but generally less effective, interactions with others. Thus, market frictions play an important role in the patterns of human capital accumulation and an economy's rate of growth. Second, we show that the extent of agglomeration may influence an economy's pattern of information flows and development. Economies with a higher population size will have more selective patterns of knowledge exchange and higher welfare.

In our second model, we simultaneously determine the patterns of knowledge exchange and the spatial agglomeration of the economy. Extending our analysis to study the economy's population size provides additional channels for interaction between agglomerative activity and knowledge exchange. Within the closed-city framework, the level of technology does not affect the range of individuals agents try to meet. Under the endogenous population migration setup, however, a higher level of technology improves production efficacy and further encourages agglomeration. In contrast with conventional urban models with a Lucas-Romer production externality, our paper provides theoretical predictions of the pattern of knowledge exchange and characterizes how an array of matching and knowledge exchange parameters affect population agglomeration and knowledge flows. In response to changes in matching, technology and knowledge exchange parameters, the equilibrium population mass and the equilibrium knowledge spread change in opposite directions, whereas the equilibrium population mass and the equilibrium per capita knowledge output (which may be measured by per capita patents in a local economy) change in the same direction.

Contrary to previous work in spatial agglomeration or endogenous growth, we demonstrate the possibility that a decentralized equilibrium is under-populated or over-populated and under-selective or over-selective in knowledge exchange, compared to the social optimum. This occurs for two reasons. First, as individuals decide to migrate, they do not take into account the effect of their entry on the total population mass. This is a ***congestion externality*** which is common in the urban economics literature. Second, we provide a channel for agglomeration inefficiencies to arise due to distortions associated with knowledge exchange. This results from a ***matching externality*** - since the arrival rate of potential collaborators may depend on the remaining mass of unmatched individuals, agents do not take into account the effect of their choices on the pool of potential collaborators and the probability of matching for other individuals. This establishes an additional reason for knowledge transfer to be socially inefficient. Because of the matching externality, the decentralized equilibrium may be under-selective in knowledge exchange or under-populated relative to the social optimum.

In an array of extensions, we depart from the benchmark model by incorporating perpetual knowledge accumulation, thereby allowing us to study the interactions between population agglomeration and economic growth. In particular, we find that higher growth is accompanied by a larger population. We also consider how knowledge exchange leads to the development of new types of knowledge which further expands the breadth of knowledge of society. In this context, we find that multiple equilibria emerge. Finally, we examine how an individual's base of knowledge may evolve as a result of matching.

Related Literature

There have been a number of papers that have emphasized the role of knowledge spillovers in urban economics. In their pioneering work, Fujita and Ogawa (1982) construct a "locational potential function" in which firms' profits are lower when they are located farther apart. Importantly, they show how such externalities can be responsible for different types of urban configurations. Berliant, Peng and Wang (2002) extend their model to examine urban structures in the presence of uncompensated inter-firm knowledge spillovers which decrease with the distance between firms. However, both regard the mechanism of knowledge spillovers as exogenously given, thereby ruling out any two-way interactions between endogenous patterns of knowledge transmission and population agglomeration. This unexplored issue is the main focus of the present paper. In another related paper, Glaeser (1999) considers the role of cities for the propagation of knowledge. His focus, however, is more on the role of cities in promoting knowledge acquisition by younger, less skilled workers from older and more skilled workers. In contrast, we focus on the potential to learn from individuals with different *types* of knowledge or ideas as a stimulus for the evolution of knowledge and agglomeration.

Our work is also connected with contributions by Helsley and Strange (1990, 2002) which study the role of matching for agglomeration economies. Although Helsley and Strange (1990) demonstrate how agglomeration results from a matching process between firms and heterogeneous workers in a system of cities, our framework emphasizes that knowledge exchange provides the driving force for agglomeration. In addition, our model is explicitly dynamic. We also conduct both positive and normative analyses by characterizing the decentralized equilibrium as well as the social optimum.

Helsley and Strange (2002) study a model of matching between intermediate inputs and entrepreneurs in which firms attempt to take their new "projects" to market in anticipation of receiving ex-post monopoly rents. Thus, the compensation for newly developed ideas provides incentives for innovation. Although both their paper and ours emphasize the role of knowledge for population agglomeration, there are a number of important differences. While they construct a deterministic intermediate input matching model, we develop a two-sided random-matching framework to provide a microfoundation for knowledge exchange. There are three additional issues. In our model, agents know the quality of a match before they make their decision to produce. In their paper, the quality of a match is not known until production occurs. Second, the incentives in the exchange or creation of knowledge are substantially different. In our paper, the surplus from matching and exchanging ideas is ***freely available to both agents***. Consequently, the surplus is a pure externality since there are no prices in our model and the exchange of knowledge is uncompensated. Finally, we consider that production (the matching surplus) is a non-monotone function of the difference in agent characteristics while they assume it is monotonically decreasing in the difference between matching characteristics. (The reader may also refer to their paper for a discussion contrasting their model with ours.) We claim that neither framework is a special case of the other. In particular, there is no obvious mapping between the equilibrium concepts and the predictions of the two models.

II. The Basic Structure of the Economy

This section specifies the economic environment and outlines the mechanisms through which knowledge spillovers occur among agents. We use a continuous-time framework where each infinitely-lived agent has an identical discount rate of $r > 0$. For illustrative purposes, we present in this section a simple benchmark setup of a steady-state economy with knowledge exchange, relegating the discussion of knowledge accumulation and endogenous growth to Section V below.

II.A. Economic Agents

Our goal is to investigate the impact of heterogeneity on the patterns of knowledge accumulation, as well as its interactions with agglomerative activity. We emphasize the 'horizontal' aspects of knowledge rather than its 'vertical' aspects.[2] Each agent is endowed with a specific type of knowledge from the set $\mathcal{K}$ which embodies the set of ***ideas*** or types of ***knowledge*** that society has available. We refer to $\mathcal{K}$ as the "knowledge space" of the economy. The knowledge space may contain any fields of relevance, such as art, biology, history, physics, and economics. As agents in this economy might be regarded as individual workers/consumers, firms, or patent holders, one could interpret $k \in \mathcal{K}$ as an individual's primary field of expertise. In the basic structure, we assume the individual knowledge type does not change over time. An extension that allows for individual knowledge evolution will be given in Section VI below.

We make the following additional assumptions about the economy. First, there is a continuum of agents in the economy with a total population of Lebesgue measure N. Second, we assume that agents' knowledge types are uniformly distributed across the economy's knowledge space. In addition, the knowledge space, $\mathcal{K}$, is a circle of unit circumference. Figure 1 depicts the knowledge space, where each point along the circle indicates a particular knowledge type with two specific knowledge types k and k' highlighted. As we have described, this could represent two different fields such as art and biology. Finally, note that since N is the total population in the economy and knowledge types are uniformly distributed across $\mathcal{K}$, the density of individuals of each knowledge type is also given by N.

II.B. Intellectual Exchange

Agents can meet with others, collaborate and share their knowledge, which enables them to produce more effectively when matched. To begin, consider two individuals k and $k' \in \mathcal{K}$ currently matched and exchanging information with each other. Obviously, heterogeneity among agents plays an important role in the transfer of knowledge. To model the effects of heterogeneity on the knowledge exchange process, we

[2]Jovanovic and Rob (1988) study the diffusion and growth of knowledge in a model where agents exhibit heterogeneity in the "vertical" aspects of knowledge (i.e., of the same type but of different quality). Our approach differs from theirs as we emphasize the "horizontal" aspects of knowledge and allow for interactions between knowledge exchange decisions and agglomerative activities.

consider the following possibility. When individuals are too alike, they cannot accomplish much and little knowledge will be obtained. In contrast, if individuals are too different, they will not have productive exchange. This latter point can be envisioned by contemplating the results of a match between a brain surgeon and an opera singer, as they have little in common to communicate and hence nothing to exchange.

Therefore, it is important to define a distance measure in the knowledge space. Let the knowledge distance between k and $k' \in \boldsymbol{K}$ be measured by the Euclidean metric $d(k,k')$.[3] Under our construction regarding the efficacy of knowledge exchange, it is natural to assert there is an optimal level of idea-diversity among agents denoted by $\bar{\delta}$. Here, we initially assume $\bar{\delta}>0$ and hence knowledge exchange is increasing in d for $d<\bar{\delta}$ but decreasing in d for $d>\bar{\delta}$.[4]

The additional knowledge obtained by an individual k, when collaborating with another individual k', is denoted as $S(k,k')$ and is given by:

$$S(k,k') = q_0 + s_0\,(a_0 - a_1|\bar{\delta} - d(k,k')|) \tag{1}$$

The term q_0 refers to the additional knowledge that an agent obtains from a match independent of the knowledge type of a partner. The parameter a_1 reflects the sensitivity of knowledge exchange to heterogeneity among agents with different types of expertise or ideas. Finally, a_0 reflects the maximum increase in production that results from differences in ideas while s_0 is a positive scaling factor for knowledge exchange. We assume throughout that each parameter in $S(k,k')$ is non-negative.

We illustrate the role of heterogeneity among agents for knowledge exchange in Figure 2. This figure is depicted from the perspective of an individual of knowledge type k, where the horizontal axis represents the set of knowledge types that the individual may meet and the vertical axis gives the flow value of matching with

[3]In general, one may define an individual's knowledge expertise as a set. Such generalization would, however, require the adoption of the Hausdorff metric to measure the knowledge distance between different sets of individual knowledge. For simplicity, the present paper labels agents by a single point representing their expertise, allowing us to adopt the conventional Euclidean metric to measure the distance between two individuals in knowledge space.

[4]See Appendix A where we show that our results are robust to the alternative possibility that knowledge exchange is monotonically more effective when agents are alike (i.e., $\bar{\delta} = 0$).

each type of agent. Figure 2 emphasizes that agents would generate the most new knowledge upon producing with an individual who is $\bar{\delta}$ units away in idea space. For an individual of knowledge type k, Figure 2 shows that the best matches would occur upon meeting with either an individual of knowledge type k - $\bar{\delta}$ or k + $\bar{\delta}$. This setup is just a simple way of attempting to uncover the impact of heterogeneity among agents on the process of knowledge exchange and human capital accumulation in actual economies.[5] Since the additional knowledge obtained through matching depends on the distance between $d(k,k')$ and $\bar{\delta}$, we find it useful to refer to the distance, $|\bar{\delta} - d(k,k')|$, as the ***match-specific knowledge spread***. This match-specific knowledge spread, denoted as $\delta(k,k')$, measures the distance away from an ideal match between a pair of agents k and $k' \in \mathcal{K}$.

II.C. Production and Tastes

By meeting and exchanging ideas with each other, individuals enhance their ability to produce a homogeneous consumption good. With their additional knowledge stock, $S(k,k')$, agents produce flow output, $y(k,k')$ given by:

$$y(k,k') = AS(k,k') \tag{2}$$

where A > 0 is a scaling factor capturing the overall level of technology in the economy. In addition, everyone in the economy has the same preferences over the homogeneous consumption good with flow utility given by:

$$u(y) = y \tag{3}$$

where y is the consumption of output which occurs upon matching and creating new knowledge. There is no disutility of effort. Moreover, flow utility is intertemporally separable. Individuals make choices, as described below, to maximize their expected lifetime utility.

II.D. Meetings

In our economy, each agent enters as unmatched to search for a partner to exchange ideas. Unmatched

[5] Admittedly, our structure has two limitations in order to provide tractability. On the one hand, we do not allow for an individual-specific quality measure which may play a role in affecting the efficacy of knowledge exchange (e.g., a high ability agent may gain little from a low ability agent regardless of their knowledge heterogeneity). On the other, new knowledge obtained from matching does not permanently augment an individual's human capital level. We address this second limitation in section V.B.

agents meet via a random- matching technology. Let U denote the mass of unmatched individuals and let M denote the mass of matched individuals in the economy, where $M = N - U$. In order to illustrate how a dense economic environment fosters more opportunities for interaction, we assume that the flow of meetings is given by a well-defined, standard random- meeting technology. That is, the ***aggregate*** number of meetings per unit of time is given by a function m that has as its first argument the number of unmatched agents who can be in the first position of a meeting, and as its second argument the number of unmatched agents who can be in the second position. If there were two distinct populations, then the two positions or arguments could differ in a feasible allocation. However, since our agents meet symmetrically (in that any agent can meet with any other), at a ***feasible allocation*** the number of eligible agents in each argument of m is the same.

Specifically, we write $m(U, U')$, where m is strictly increasing and concave in each argument and homogeneous of degree $\gamma > 1$ (exhibiting increasing returns to scale) and where m satisfies standard boundary conditions $m(U, 0) = m(0, U') = 0$. We can thus rewrite the random-meeting technology for feasible allocations ($U = U'$) as: $m(U, U) = U^{\gamma} m(1, 1)$. This follows the specification in Diamond (1982), implying that the flow probability for an unmatched agent to locate another is higher in economies with a higher population density of unmatched agents. For feasible allocations, symmetry makes this flow meeting rate resemble the arrival rate of meetings in one-sided search and matching models – thus, we will use flow meeting rate and arrival rate interchangeably. More explicitly, denoting this arrival rate ***per unmatched individual*** by μ, we have $\mu(U) = m(U, U)/U = U^{\gamma-1} m(1, 1)$. (It is the arrival rate for each individual that is important in each agent's optimization problem.) When we come to solving the model analytically, we will for simplicity make the assumption that $\gamma = 2$, under which the arrival rate becomes linear in the mass of unmatched agents: $\mu(U) = \alpha U$, with $\alpha \equiv m(1, 1) > 0$ measuring the arrival intensity.[6] Meeting is costly – there is a stochastic amount of time agents wait to meet others – as long as α is finite.

Empirically, there is evidence suggesting that this random- matching technology in the labor market

[6]Note that we could also allow for decreasing returns to scale in the matching technology if $\gamma \in (0, 1)$. Under this interpretation, there would be congestion in matching. Although allowing for decreasing returns is possible in our framework, it is clearly not appropriate for our study and does not appear to be relevant empirically. (See the discussion that follows in the next paragraph of the text.)

may exhibit constant or increasing returns to scale. For example, Blanchard and Diamond (1989) use national-level U.S. data to estimate the aggregate matching function and find it either of constant or mild increasing returns. Constructing the matching function from disaggregate individual-level U.S. data, Anderson and Burgess (2000) lend support to increasing returns in labor matching functions at the state level.[7] In the context of micro matching for individual knowledge exchange and agglomeration, casual empirics seem to be more favorable toward increasing returns. The basic idea is straightforward: with a larger mass of individuals residing in a given physical area, individuals interact more frequently, which results in a higher arrival rate of potential partners for knowledge exchange. The idea that population density may stimulate knowledge exchange and production is emphasized in Kuznets (1962) and Jacobs (1969), as illustrated in introduction. One could also interpret the arrival rate μ as the inverse of the amount of time it takes for information flows to accrue across sectors in an economy rather than an explicit search-theoretic interpretation, which is still consistent with our random- matching framework.[8]

Further, given the effects of heterogeneity on the efficacy of knowledge exchange in this economy, it is important to distinguish between ***meetings*** and ***matches***. Meetings occur between any two agents with flow probability $\mu(U) = \alpha U$, but only a subset of meetings result in matches. Agents do not want to produce with individuals whose areas of expertise are too alike or too different, since such a match would result in less effective knowledge exchange. This is reflected in the match-specific knowledge spread, $\delta(k,k')$. Note that an agent's optimal match-specific knowledge spread with no transactions cost is $\delta(k,k')=0$. Due to the expected delay between meetings, individuals will accept matches with a positive match-specific knowledge spread. Agents will not accept all matches, however, because individuals cannot meet other potential partners while matched and separation is not instantaneous. Thus, individuals will choose a range of acceptable matches,

[7]For a comprehensive survey of empirical matching functions, the reader is referred to a recent article by Petrongolo and Pissarides (2001).

[8]This is, in fact, the interpretation adopted by Marshall (1985), "so great are the advantages which people following the same skilled trade get from near neighborhood to one another. The mysteries of the trade become no mysteries; but are as it were in the air, and children learn many of them unconsciously" (p.352).

reflecting a trade-off between the quantity and quality of matches.

Throughout the paper, we will focus only on steady-state pure-strategy symmetric Nash equilibria. We hereafter refer to the individual agent's choice of acceptable matches simply as the ***knowledge spread***, δ_k, which is the lifetime utility maximizing match-specific knowledge spread to agent k with any agent $k' \in \mathcal{K}$. As illustrated in Figure 2, the choice of δ_k leads to acceptance by an agent of type k of matches in two intervals, $[k-\bar{\delta}-\delta_k, k-\bar{\delta}+\delta_k]$ and $[k+\bar{\delta}-\delta_k, k+\bar{\delta}+\delta_k]$. The agent's knowledge spread, in turn, affects the frequency of matches. We therefore denote the endogenous ***flow probability of a match*** for an individual agent k as $\beta(\delta_k;U)$. Of course, as we demonstrate below, $\beta(\delta_k;U) < \mu(U)$.

As mentioned above, $\beta(\delta_k;U)$ will depend on the range of types of knowledge that an agent k accepts for intellectual exchange. Specifically, k will select a range of agents with whom to exchange ideas given by:

$$R(k) \equiv [k-\bar{\delta}-\delta_k, k-\max\{0, \bar{\delta}-\delta_k\}] \cup [k+\max\{0, \bar{\delta}-\delta_k\}, k+\bar{\delta}+\delta_k] \tag{4}$$

as depicted in Figure 1. In addition, matches between agents are terminated with an exogenous flow probability η, i.e., the exogenous detachment rate in the economy is given by η.[9] If two individuals meet, but do not match, detachment occurs immediately.

The selection over which agents are accepted for matches is restricted to pure-strategy best responses taking as given the behavior of other agents. It will depend on both the effectiveness of knowledge exchange and primitives of the economic environment such as the ability of individuals to meet in the economy. For example, as it becomes easier for unmatched agents to meet, individuals would be expected to be more selective in the range of agents they will accept for engaging in knowledge exchange.

[9]An exogenous separation rate provides tractability. We could endogenize the separation rate if the productivity of each match is not known ex-ante and agents update their beliefs regarding the productivity of a match over time. Once an agent determines that a match is not sufficiently worthwhile, endogenous separation would occur. This extension is not likely to add more insights into the fundamental issues we study.

II.E. Asset Values

Recall that in any time period, agents will either be matched or unmatched. Each state is associated with a different level of expected lifetime utility because agents' consumption opportunities will vary depending on whether they are currently matched or not. Therefore, let V_{Mt} (*k*, *k'*; *U*) denote the expected lifetime utility for an agent of knowledge type *k* who is currently matched with an agent of knowledge type *k*' in time period *t*. The expected lifetime utility for an unmatched agent of knowledge type *k* is given by V_{Ut} (*k*; *U*) in time period *t*.

We begin by describing the evolution of the expected lifetime utility for an agent of knowledge type *k*.[10] A derivation of the agent's Bellman equation is easiest to see by considering time in discrete units of length Δ. Under this time convention, the expected lifetime utility for an agent who is currently matched in time period *t* is given by:

$$V_{Mt} = \frac{\eta\Delta[y(k,k')\Delta + V_{U(t+\Delta)}] + (1-\eta\Delta)[y(k,k')\Delta + V_{M(t+\Delta)}]}{1 + r\Delta} \tag{5}$$

where under the Poisson assumption for the separation rate, the probability that a breakup will occur at the end of the time interval of length Δ is given by ηΔ. Until the break-up occurs, agents are exchanging information and producing. As of time (t+Δ), the individual will have an expected lifetime utility of $V_{Ut+\Delta}$(*k*; *U*). In contrast, with probability (1- ηΔ), agents remain matched and therefore have an expected discounted lifetime utility of $V_{Mt+\Delta}$(*k*, *k'*; *U*) as of time (t+Δ). Rearranging (5), dividing by Δ, and taking limits as $\Delta\to 0$ yields:

$$rV_{Mt} = y(k,k') + \eta[V_{U,t} - V_{M,t}] + \dot{V}_{Mt} \tag{6}$$

We study the steady-state of the economy. In this setting, the values of all variables are assumed to be constant over time. In particular, the expected lifetime utility of a matched agent is independent of time and, from (6), the Bellman equation for an agent of knowledge type *k* who is currently matched with an agent of type *k'* is:

$$rV_M(k,k';U) = y(k,k') + \eta[V_U(k;U) - V_M(k,k';U)] \tag{7}$$

[10]See Diamond and Fudenberg (1989) for the construction of analogous evolution equations in their search models.

This implies that the flow value of matches is the sum of the flow output produced based on new knowledge obtained and the expected capital loss associated with the change of state from a matched to an unmatched agent. Specifically, as a consequence of the use of steady state, note that the value of matching is independent of time, which rules out cyclical behavior.

Analogously, we can express the corresponding Bellman equation for unmatched agents of type k. It is more complicated than that for matched agents, however, because one must specify the general matching rule in all possible cases concerning the probability of a match between any pair of agents. Specifically, denote by $f(k,k',\delta_k,\delta_{k'})$ the probability that a match between k and k' occurs, given their choices of $\{\delta_k,\delta_{k'}\}$. Then, the general matching rule is given by,

(i) $f(k,k',\delta_k,\delta_{k'}) = 1$ if $k \in R(k')$ and $k' \in R(k)$;

(ii) $f(k,k',\delta_k,\delta_{k'}) = f_0 \in (0,1)$ if $k \in R(k')$ and $k' \notin R(k)$ or $k \notin R(k')$ and $k' \in R(k)$;

(iii) $f(k,k',\delta_k,\delta_{k'}) = 0$ if $k \notin R(k')$ and $k' \notin R(k)$,

where we select equilibria at $f_0 = 0$ that are approximated by equilibria where $f_0 > 0$ but where $f_0 \to 0$. These are the equilibria of interest, since the matching rule at $f_0 = 0$ makes the most sense (if one of a pair does not want to match, then there is no match), but among these, we want robust equilibria. For the remainder of the paper, we assume $f_0 = 0$ but select the robust equilibria.

Given the set of knowledge types, the range that an individual of knowledge type k selects for matching, $R(k) = [k-\bar{\delta}-\delta_k, k-\max\{0,\bar{\delta}-\delta_k\}] \cup [k+\max\{0,\bar{\delta}-\delta_k\}, k+\bar{\delta}+\delta_k]$, and the general matching rule, $f(k,k',\delta_k,\delta_{k'})$, the Bellman equation for an unmatched agent of type k is:

$$rV_U(k;U,\{\delta_{k'}\}_{k'\in K,k'\neq k}) = \mu(U)\max_{\delta_k}\int_{R(k)} f(k,k',\delta_k,\delta_{k'})[V_M(k,k';U) - V_U(k;U)]\,dk' \tag{8}$$

For the ease of the notation, we will refer to agent k's ***best response*** of δ_k (the argmax of the optimization problem given in equation (9)) as $\hat{\delta}_k$. It is important to note that the selection strategy will be chosen by recognizing the tradeoff between a higher number of matches (a ***contact rate effect***) and more effective matches (a ***knowledge efficacy effect***) because knowledge exchange would not be effective when an agent is matched with agents who are too similar or too different.

II.F. Symmetry

In any time period, agents will either be matched or unmatched. Without imposing additional *a priori* heterogeneity on the knowledge space, we assume that the distribution of unmatched (and thus matched) agents is uniform. Constant population in the steady-state therefore requires that the inflows and outflows of population in each category are equal in each time period.

For each type of agent, there is a given mass of unmatched individuals U_k. Also, let M_k denote the pool of matched agents of type k. From the pool of unmatched agents, the flow probability that an agent will find a suitable match is given by $\beta(\delta_k; U)$. Suppressing arguments for notational convenience, this implies that an outflow of $\beta(\delta_k; U)U_k$ agents of type k from the unmatched pool will become matched within the period. Similarly, given the exogenous detachment rate η, an inflow of ηM_k agents of type k will enter the unmatched pool. In computing the flow probability of a match under symmetry, $\beta(\delta;U)$, we may therefore write:[11]

$$\beta(\delta;U) = \mu(U)\left(\frac{\int_{R(k)} U\, dk'}{U}\right) \tag{9}$$

Note that the term $\int_{R(k)} U\, dk'$ reflects the total mass of individuals that agent k selects as potential matches. This is divided by the total mass of unmatched agents to obtain the proportion of unmatched agents in the economy that agent k selects to try to engage in intellectual exchange. The flow probability of a match is thus given by the flow probability of a meeting multiplied by the proportion of unmatched agents selected for knowledge exchange.

Throughout, we will assume that:

Assumption 1. (Knowledge Diversity) *The optimal level of idea-diversity satisfies:* $\bar{\delta} \geq 2(q_0 + s_0 a_0)/(s_0 a_1)$.

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[11] In focusing on steady-state equilibrium allocations in our economy, the total mass of unmatched agents, U, will be constant over time and the flow probability of matching will be the same in each period. Thus, there is no incentive for individuals to choose a different knowledge spread in each time period.

As we demonstrate below, Assumption 1 provides conditions in which agents will not choose a knowledge spread larger than $\bar{\delta}$. From (4) and Assumption 1, we obtain: $\int_{R(k)} dk' = 4\delta$. As a result, (9) implies that $\beta(\delta; U) = 4\mu(U)\delta$.

In order to study an unmatched agent's best response, it is useful to define his payoff assuming that he can choose unilaterally whether or not a match occurs. This gives us a particular optimal knowledge spread $\hat{\delta}$. We will then show in Section III that $\hat{\delta}$ is in fact a best response given our matching rule (as $f_0 \to 0$).

Lemma 1: (Unmatched Value) *An agent's unmatched value, given that he can choose his partners unilaterally, can be written as a function of* $\hat{\delta}$ *and U:* $V_U(\hat{\delta}; U)$, *where* $\hat{\delta}$ = ***argmax***$_\delta$ $V_U(\delta; U)$ and

$$V_U(\delta; U) = \begin{cases} \dfrac{4\mu(U)\delta}{r+\eta+4\mu(U)\delta}\dfrac{A}{r}\left(q_0 + s_0 a_0 - \dfrac{1}{2}s_0 a_1 \delta\right), & \text{if } \delta < \bar{\delta} \\ \dfrac{2\mu(U)(\delta+\bar{\delta})}{r+\eta+2\mu(U)(\delta+\bar{\delta})}\dfrac{A}{r}\left(q_0 + s_0 a_0 - \dfrac{1}{2}s_0 a_1 \dfrac{\bar{\delta}^2+\delta^2}{\bar{\delta}+\delta}\right), & \text{if } \delta \geq \bar{\delta} \end{cases} \tag{10}$$

Proof. All proofs are in Appendix B.

Note that by Lemma 1, an agent's unmatched value function has a kink at $\delta = \bar{\delta}$. Under Assumption 1, however, this kink will occur $V_U(\delta; U)$ is negative. Therefore, the value of $\bar{\delta}$ does not affect the agent's choice of her knowledge spread. This considerably simplifies the analysis. Appendix A presents some additional technical details for the case where Assumption 1 does not necessarily hold.

II.G. Steady-State Populations

Under symmetry, steady-state equilibrium requires the following equalities in order for the populations of matched and unmatched agents to remain constant over time:

$$\beta(\delta; U)\, U = \eta M = \eta (N - U) \tag{11}$$

where the reference to agents of type k is removed in the interest of a symmetric equilibrium. From (11) and the result that $\beta(\delta; U) = 4\mu(U)\delta$ we find

$$U = \frac{\eta}{\eta + 4\mu(U)\delta} N. \tag{12}$$

Individuals regard their own selection of the knowledge spread as having no influence on the steady-state population of unmatched agents.

III. Steady-State Equilibrium

In this section, we focus on determining the ***steady-state pure-strategy symmetric Nash equilibrium*** in the context of an environment where the population mass is exogenously given. This allows us to highlight the influence of the extent of agglomeration on the knowledge exchange process, as well as to obtain insights concerning the knowledge spread.

Definition 1: (Steady-State Equilibrium) *A **non-degenerate**, **symmetric**, **steady-state equilibrium** (SSE) is a tuple* $\left\{ \{R(k)\}_{k \in \mathbf{K}}, \hat{\delta}, U \right\}$ *satisfying the following conditions*:

(E-1) *agents maximize their expected lifetime utilities through their choice of the knowledge spread, that is, the best response* $\hat{\delta}_k$ *is such that:* $\hat{\delta}_k \equiv \underset{\delta_k}{\arg\max}\; V_U(\delta_k; U)$;

(E-2) *equilibrium range of agents for k to exchange ideas:* (4);

(E-3) *steady-state population:* (12);

(E-4) *symmetry:* $\hat{\delta}_k = \hat{\delta}$, $\forall k \in \boldsymbol{K}$;

(E-5) *there is interaction among agents* (*the steady-state equilibrium is non-degenerate*): $\hat{\delta} > 0$.

Notice that there is an ***ex ante*** optimal level of idea heterogeneity, $\bar{\delta}$, between any two matched parties. Based on parameters of the model such as matching rates, agents establish the maximal distance $\hat{\delta}$ away from $\bar{\delta}$ to match. As illustrated in Figure 1, this determines the equilibrium range of agents with whom agent *k* exchanges ideas. Upon deriving the range of individuals for whom agents match, we can pin down the degree of diversity of knowledge exchange in this economy in steady-state equilibrium. Once the equilibrium knowledge spread, $\hat{\delta}$, is determined, the steady-state populations of matched and unmatched agents can be derived using (12). Symmetry is also important here. Under the matching rule, only a zero measure of matches take place in which one individual gains more from matching than his partner. Therefore, in virtually all matches that occur in equilibrium, if an agent of type k wants to match with an agent of type k', then type k' also wants to match with type k.

For the remainder of the paper, we assume: $\mu(U) = \alpha U$.

Theorem 1: (Existence and Uniqueness) *Suppose that Assumption 1 holds and s_0 and a_1 are strictly positive. Then the (non-degenerate) symmetric, steady-state equilibrium exists and is unique, where the steady-state equilibrium knowledge spread, $\hat{\delta}$, is given by:*

$$N = \frac{r+\eta}{\alpha\eta\hat{\delta}}\left(\frac{q_0 + s_0 a_0 - s_0 a_1 \hat{\delta}}{s_0 a_1 \hat{\delta}}\right)^2 + \frac{r+\eta}{2\alpha}\left(\frac{q_0 + s_0 a_0 - s_0 a_1 \hat{\delta}}{s_0 a_1 \hat{\delta}^2}\right) \tag{13}$$

Upon establishing existence of the steady-state equilibrium for the economy, we seek to understand how the pattern of information flows, as exhibited by the knowledge spread, responds to the extent of agglomeration, as measured by the exogenous population size in the basic model. We can show:

Proposition 1: (Effect of the Extent of Agglomeration on the Pattern of Knowledge Exchange and Per Capita Flow of Matches) *Suppose that Assumption 1 holds and s_0 and a_1 are strictly positive. In an SSE,*

(i) *a higher population mass lead to a smaller equilibrium knowledge spread but a greater per capita flow of matches ($\beta U/N$);*

(ii) *an increase in the degree of matching efficacy (higher α) or a decrease in the rate of matching detachment (lower η) reduces the equilibrium knowledge spread;*

(iii) *narrower knowledge (higher a_1) yields a smaller equilibrium knowledge spread, while changes in the overall level of technology (A) have no effect on knowledge exchange.*

Intuitively, this occurs because the probability of finding other unmatched agents is higher in economies with a higher population mass. As a consequence, agents are more selective in knowledge exchange. It can be shown that the effect of the higher population mass dominates the effects of the smaller knowledge spread which implies that the flow of matches for each individual agent is higher when N is higher. Thus, our result lends formal theoretical support to the claim by Pred (1966), "[i]t is logical that the larger the city, the larger the number of intentionally and unintentionally overlapping information fields of laborers and other industrial personnel, the larger the volume of influential short-distance information flows" (pp. 128-9). Our model also provides an important testable hypothesis – cities or other economic units with a higher population mass will also have a higher per capita measure of innovative activity (such as patents).[12]

We next turn to the various effects of matching frictions. An increase in the efficacy of the local economy's matching technology is associated with a smaller knowledge spread. When it is easier for unmatched agents to find potential partners for knowledge exchange, they can concentrate on finding more productive matches. The effects of the discount rate are similar. If the discount rate is higher, agents are less patient and therefore less finicky when trying to locate partners. Additionally, the exogenously given detachment rate is positively associated with more diverse patterns of knowledge exchange. This occurs because higher values of η imply that matches do not last as long on average, resulting in a smaller opportunity cost to taking part in relatively less effective knowledge exchange.

It may be noted that as knowledge itself becomes narrower, knowledge exchange with agents in other fields becomes less effective. This is captured in our model by a higher penalty for heterogeneity, which induces agents to become more selective in matching. However, the overall level of technology has no effect on the equilibrium knowledge spread. A higher level of technology raises the productivity of each match, but it also raises the costs of a higher knowledge spread because agents forfeit production that would have been obtained from more effective knowledge exchange with other agents.

IV. Endogenous Migration

In this section, we discuss equilibrium determination of the population size along with the steady-state equilibrium knowledge spread. In contrast with the benchmark case investigated in Section III, where the

[12]Moreover, it is important to note that the per capita flow *value* of matches would also be higher in economies with a larger population since the knowledge spread would be lower. However, this richer hypothesis would be more difficult to test since it requires a measure of the commercial value of patents across geographic units.

population mass is exogenously given, we pin down the equilibrium knowledge spread along with the endogenous population mass. This framework provides numerous insights into the interactions between the patterns of knowledge exchange and the process of agglomeration. We demonstrate that these considerations identify new sources of inefficiency associated with population migration.

In regard to migration decisions, individuals must account for costs associated with residing in the city under consideration. These costs (which are asserted to depend positively on N) may, for example, be seen as the (present-discounted) value of city taxes imposed on each immigrant. A higher N may be associated with congestion costs or destruction of existing structures, thus requiring government infrastructure spending to increase at least proportionately (i.e., a higher N leads to an increase in real per capita infrastructure costs).[13] In short, we specify $v=v(N)$ as the (average) entry cost function, where v is strictly increasing (and convex) in N. For comparative statics exercises, we view any increase in costs associated with residing in the city as an increase in 'entry' costs and represent them as a shift in the entry cost function.

The reader is referred to Figure 3 to better understand how the endogenous population mass is determined. The horizontal axis of Figure 3 represents different values of the population mass, N, and the vertical axis provides the different values for an unmatched agent's expected lifetime utility and entry costs for each value of N. In our analysis, an agent's unmatched value function is expressed in terms of N by substituting for U from the steady-state population condition (12). Individuals will continue to migrate as long as $V_U(\delta;N) \geq v(N)$.[14] For values of N less than $\tilde{N}$, $V_U(\delta;N) > v(N)$, providing an incentive for agents to move to the area. If N is greater than $\tilde{N}$, agents would obtain higher expected lifetime utility by choosing not to migrate to the area. Thus when the extent of agglomeration is such that $V_U(\delta;\tilde{N}) = v(\tilde{N})$, migration will no longer occur. For these reasons, we refer to the condition that $V_U(\delta;\tilde{N}) = v(\tilde{N})$ as the equilibrium entry condition. It may also be useful to think of the condition $V_U(\delta;\tilde{N}) = v(\tilde{N})$ as the endogenous population condition. From the equilibrium entry condition, we can obtain a locus of δ and N where individuals are

[13]See Chapter 6 of Fujita (1989) for a discussion of congestible city goods.

[14]Our model of endogenous migration is an open city model. We could close the model using a system of cities approach where migration across locations may occur until equilibrium is achieved.

indifferent between migrating and not migrating. Through this migration choice, the population mass is pinned down for each possible value of the knowledge spread. In combination with the knowledge spread locus (to be defined shortly), we are able to obtain a steady-state equilibrium allowing for endogenous migration.

IV.A. Steady-State Equilibrium with Endogenous Migration

The elements above forge our definition of a steady-state equilibrium with endogenous migration.

Definition 2: (Steady-State Equilibrium with Endogenous Migration) *A* ***non-degenerate, symmetric steady-state equilibrium with endogenous migration*** *(SSEEM) is a SSE* $\left\{\{R(k)\}_{k\in\mathbf{K}}, \hat{\delta}, \hat{U}\right\}$ *together with a population mass* $\hat{N}$ *satisfying the following additional conditions:*

(E-6) *equilibrium entry:* $V_U(\hat{\delta}_k;\hat{U}) = v(\hat{N})\ \forall k \in \boldsymbol{K}$;

(E-7) *population agglomeration occurs* (*the steady-state equilibrium is non-degenerate*): $\hat{N} > 0$.

We illustrate our solution algorithm through the use of Figure 4 where the horizontal axis represents the different values of an agent's knowledge spread and the vertical axis lists values of *N*. We first derive the ***knowledge spread (KS) locus*** which determines the choice of the knowledge spread δ by each agent for a given size of population mass *N*. We next determine the values of δ, *U*, and *N* that keep agents indifferent between migrating and not migrating to the area, i.e., equilibrium entry of agents from Figure 3. For tractability, we begin our analysis by focusing our attention on the case where the entry cost function is linear in the population size ($v(N)=v_0 N$).[15] We refer to the ***equilibrium entry (EE) locus*** as the relationship between the total population, the mass of unmatched agents and the knowledge spread such that individuals are indifferent between migrating or not migrating to the city. A steady-state equilibrium with endogenous migration occurs for values of the knowledge spread and population mass where the equilibrium entry and knowledge spread loci intersect.

In order to generate a city that is nondegenerate (i.e., with positive mass of population) in equilibrium,

[15]Our results can be generalized to the case of increasing, convex functional forms for the entry cost function. We present the linear case here because it is the most tractable.

the benefits of entry must exceed the cost when no one lives there. This is guaranteed by,

Assumption 2. (Nondegenerate City) $\frac{2\alpha}{r+\eta}\frac{A}{r}\frac{(q_0+s_0a_0)^2}{s_0a_1} > v_0$.

Theorem 2: (Existence of Steady-State Equilibrium under Endogenous Migration) *Suppose that Assumptions 1 and 2 hold and s_0 and a_1 are strictly positive. Then a non-degenerate steady-state equilibrium with endogenous migration (SSEEM) exists and is unique.*

We continue by outlining some interesting connections between knowledge exchange and endogenous agglomeration (additional comparative statics are available from the authors upon request).

Proposition 2: (Interactions Between the Pattern of Knowledge Exchange and the Extent of Agglomeration) *Suppose that Assumptions 1 and 2 hold and s_0 and a_1 are strictly positive. In an SSEEM,*

(i) *an increase in the degree of matching efficacy (higher α) or a decrease in the rate of matching detachment (lower η) reduces the equilibrium knowledge spread ($\hat{\delta}$) and the equilibrium per capita flow of matches ($\beta U(\hat{N})/\hat{N}$) but raises the equilibrium population mass ($\hat{N}$);*

(ii) *better technology (higher A) or narrower knowledge (higher a_1) yields a smaller equilibrium knowledge spread, a lower equilibrium per capita flow of matches ($\beta U(\hat{N})/\hat{N}$) and a higher equilibrium population mass .*

Proposition 2 is an extension of Proposition 1 to the case of endogenous migration. A higher degree of matching efficacy implies that agents can concentrate on finding more effective opportunities for knowledge exchange and hence choose a smaller knowledge spread -- this is reflected by the shift of the *KS* locus to the left (see Figure 4). In addition, greater matching efficacy raises the gains from migrating because there is less delay between matches. This results in a higher steady-state equilibrium population mass which further encourages agents to favor more effective collaborative efforts. In response to the higher value of α, the *EE* locus will also shift up, thereby reinforcing the effects of the backward shift of the *KS* locus. The effects of a lower matching detachment rate are qualitatively the same.

In contrast with the closed-city model in Section IV, we find that more technologically advanced cities (cities with higher A) will have more selective patterns of information sharing. This occurs because the higher level of technology has the direct effect of inducing more population agglomeration. Because of the beneficial aspects of population density for matching, agents in turn decide to become more selective in their matching. Similar results emerge in regard to the effect of different parameter values concerning the effectiveness of knowledge exchange. For example, narrower knowledge captured by a higher penalty for heterogeneity still induces agents to become more selective in matching, despite the presence of a negative effect of the penalty for migration incentives that tends to lower population mass and encourage agents to increase their knowledge spread in equilibrium.[16] Additionally, one may easily derive that an increase in the return to knowledge exchange (q_0) or the cost of entry (ν_0) makes agents less selective and reduces the equilibrium population mass, thereby raising their equilibrium knowledge spread.

While the finding of a positive effect of technology on population agglomeration corroborates with that in conventional urban models with a Lucas-Romer production externality, our paper generates a number of new insights. First, we provide theoretical predictions of the pattern of knowledge exchange, which cannot be conducted by previous studies in which knowledge spillovers are mechanically presumed. Second, we are able to characterize how an array of matching and knowledge exchange parameters affect population agglomeration and knowledge flows. In particular, we show that in response to changes in matching, technology and knowledge exchange parameters, the equilibrium population mass and the equilibrium knowledge spread change in opposite directions. Finally, our results suggest that in response to changes in matching, technology and knowledge exchange parameters, the equilibrium population mass and the equilibrium per capita flow of matches change in the same direction. Recent work by Carlino et al (2004) establishes empirical support for the predictions of our model -- they find that the number of patents per capita (measuring the local per capita flow of matches) is positively correlated with the employment density of metropolitan areas.

[16]In Appendix B, we show that this entry effect is dominated by the direct selectivity effect.

IV.B. Socially Optimal Knowledge Spread and Population Mass

In Section IV.A., we analyzed the various two-way interactions between the endogenous knowledge transmission mechanism and the process of population agglomeration. In this section, we demonstrate that these interactions lead to new sources of inefficiency associated with population migration.

In a decentralized equilibrium, individuals choose their knowledge spread to maximize their unmatched value function given the population size, and continue to migrate until the net utility from migrating is equal to zero. In a social planner's problem, we assume that the city planner (or the immigration officer) seeks to maximize the net welfare of a representative city resident (i.e., social welfare maximization in the spirit of J.S. Mill) over (i) the knowledge spread and (ii) the population mass (or mass of unmatched agents) to be established when all residents simultaneously move to the city.[17] Since an agent enters the economy as unmatched, his/her expected value of potential future matches is captured by $V_U(\delta;U)$. Since both δ and U may vary over time, a true social optimum would involve solving the entire path of $\{\delta, U, N\}$. For simplicity as well as for comparison with the steady-state equilibrium analysis, we instead restrict our attention to a "steady-state" social welfare maximization with invariant values of $\{\delta, U, N\}$.[18] Formally, define:

Definition 3: (Social Optimum) *A **symmetric social optimum** (SO) is a triple* $\{\delta^*, U^*, N^*\}$ *satisfying:*

(*S-1*) *optimal knowledge spread and population size:* $\{\delta^*, N^*\} \in \left\{ \underset{\delta, N}{argmax} \; [V_U(\delta;U) - \nu(N)] \right\}$ *where* V_U *is defined as in* (10);

(*S-2*) *steady-state population*: (12).

[17]We make two clarifying remarks. First, we use our definition of a social optimum instead of a Pareto optimum as our welfare criterion. In particular, the latter concept need not be well-defined in an "open city" model with migration, since the set of agents present in the model is ill-defined. Conceivably, one could close the model using a system of a large number of cities and show that the Pareto and social optima are the same; that is beyond the scope of this paper. Second, we consider the case where a city planner chooses to maximize the net welfare of only the individuals residing in the city; the city does not yet exist when the planner solves the optimization problem. One may also consider the case in which the city does exist and the social planner maximizes the welfare of current residents and potential immigrants. This entails consideration of redistribution issues which detract from our principal interest – the interactions between the social inefficiencies from knowledge exchange and congestion externalities.

[18]Conceptually, this means that at time 0, all agents that will enter the city are unmatched, but at any time greater than zero in this continuous time model, steady state is presumed to occur.

Theorem 3: (Existence of a Social Optimum) *Suppose that Assumption 1 holds and that* s_0 *and* a_1 *are strictly positive. Then a social optimum exists and is unique.*

Thus, the social planner chooses δ and N simultaneously to maximize the net welfare of a representative resident. Because the arrival rate depends on the mass of unmatched agents, it is convenient to transform the problem such that the social planner chooses δ and U to maximize the net utility of a potential representative resident. Then, by applying the steady-state population condition, (12), we can solve for N.[19] As we demonstrate below, the situation where μ is a function of U implies that a matching externality occurs in equilibrium. This, in turn, distorts the economy's extent of population agglomeration.

We can compare the decentralized equilibrium with the social optimum to conclude:

Proposition 3: (Social inefficiency) *Suppose that Assumptions 1and 2 hold and that* s_0 *and* a_1 *are strictly positive. It is possible that a decentralized equilibrium SSEEM is under-populated or over-populated relative to the social optimum (SO); it is also possible that an SSEEM is under-selective or over-selective in knowledge exchange compared to the SO.*

We shall illustrate this result graphically after some discussion. We first present the equilibrium conditions for the endogenous knowledge spread and population mass:

$$\beta(\delta;U) = 4\alpha\delta U = 2(r+\eta)\left(\frac{q_0+s_0a_0}{s_0a_1\delta} - 1\right) \equiv B(\delta), \tag{15}$$

$$v_0(\eta+B(\delta))(r+\eta+B(\delta)) = \eta\left(\frac{A}{r}\right)(4\alpha\delta)\left(q_0+s_0a_0 - \frac{1}{2}s_0a_1\delta\right) \tag{16}$$

In contrast, the planner's choices of the knowledge spread and population mass are governed by:

$$(r+\eta)(q_0+s_0a_0 - s_0a_1\delta) - 2\alpha U s_0a_1\delta^2 = \delta(4\alpha U)\left(q_0+s_0a_0 - \frac{1}{2}s_0a_1\delta\right)\frac{r+\eta}{\eta+8\alpha U\delta} \tag{17}$$

---

[19]Mathematically, this is isomorphic to solving for δ and N first, which provides a solution for U in a recursive manner, though the transformed problem is more tractable.

$$v_0(\eta+8\alpha U\delta)(r+\eta+4\alpha U\delta)^2 = \eta(r+\eta)(\frac{A}{r})(4\alpha\delta)(q_0+s_0a_0 - \frac{1}{2}s_0a_1\delta) \quad (18)$$

where the LHS is the marginal social benefit whereas the RHS is the marginal social cost. Straightforward comparison suggests that the equilibrium and social optimum solutions are generally different.

In our model, social inefficiency arises for two reasons. First, the city may feature an equilibrium population mass larger than the social optimum because individuals do not consider their impact of their migration decision on the city utility level. By not taking into account their effect on the overall population mass, the city may be over-populated relative to the social optimum. This result occurs in much of the conventional urban economics literature – we show, however, this ***congestion externality effect*** leads to an additional distortion in the pattern of information flows in an economy. In particular, we demonstrate that the congestion externality may cause individuals to under-search for potential collaborators by becoming over-selective in knowledge exchange.[20]

Second, there is another possible inefficiency when the gains from a higher population density are sufficiently large. The selection of the knowledge spread induces a matching externality in the economy. This occurs because agents fail to account for the fact that accepting matches with more types of individuals lowers the mass of unmatched agents in the economy, rendering it more difficult for everyone to meet other unmatched agents. This ***matching externality effect*** potentially results in a decentralized equilibrium that is under-selective in its patterns of knowledge exchange and under-populated relative to the social optimum.[21]

Specifically, the first-order condition for the social planner's choice of the knowledge spread is:

$$\left.\frac{\partial V_U(\delta,U)}{\partial\delta}\right|_{\bar{U}} + \frac{\partial V_U(\delta,U)}{\partial U}\frac{\partial U}{\partial\delta} = 0 \quad (14)$$

---

[20]See chapter 6 of Fujita (1989) for details of the conventional model that results in cities that are overpopulated in equilibrium relative to the social optimum due to a congestion externality. There are some models that generate cities that are underpopulated in equilibrium relative to the optimum. In the presence of either a fixed set-up cost [Abdel-Rahman (1990)] or a free-rider effect [Palivos and Wang (1997)], the equilibrium city size may be too small. Within our general equilibrium search-theoretic framework, channels for either over-population or under-population are present.

[21]It should be noted that when the function representing the aggregate number of meetings (m) exhibits constant returns (i.e., $\gamma = 1$ and thus μ is a constant), the matching externality effect is absent.

Note that the first term in (14) corresponds to the choice of the individual's knowledge spread in a decentralized equilibrium. Individuals, taking the mass of unmatched agents as given, choose the knowledge spread to maximize their expected lifetime utility. The social planner, however, takes into account that a larger knowledge spread lowers the mass of unmatched agents in the economy (which we refer to as the matching externality effect). This is the second term in equation (14).

As stated in Proposition 3, it is possible that compared to the social optimum, a decentralized equilibrium is under-populated or over-populated and under-selective or over-selective in knowledge exchange. To illustrate, we discuss the two most interesting cases (i) a decentralized equilibrium is over-populated and over-selective relative to the optimum (Figure 5); (ii) a decentralized equilibrium is under-populated and under-selective relative to the optimum (Figure 6). To introduce these arguments graphically, we need to introduce some additional notation. Denote $V_U(\hat{\delta}, N)$ as an unmatched agent's expected lifetime utility given the private choice of the knowledge spread $\hat{\delta}$ and $V_U(\delta^*, N)$ as an unmatched agent's expected lifetime utility under the planner's choice of the knowledge spread, δ^*.

Next, we refer to Figure 5 where the horizontal axis gives the population mass, N, and the vertical axis gives the entry cost (ν), and expected lifetime utilities for unmatched agents under the private and planner's choice of the knowledge spread ($V_U(\hat{\delta}, N)$ and $V_U(\delta^*, N)$). Consider the case where the matching externality is minimal so that the planner's choice of the knowledge spread is not too much smaller than in equilibrium. Since the matching externality in this case is not too strong, an agent's unmatched value function ($V_U(\hat{\delta}, N)$) will not lie much below the lifetime utility that would occur (for each value of N) under the planner's choice of the knowledge spread ($V_U(\delta^*, N)$). Recall that under endogenous migration, the steady-state equilibrium population level is pinned down where the unmatched value function intersects with the entry cost function, $\hat{N}$. For the social optimum, however, the population level is found where the slope of the unmatched value function with respect to N has the same slope as the entry cost function, N^*. In this case, we obtain the standard over-population result in decentralized equilibrium as in the urban economic literature. Moreover, individuals are over-selective in knowledge exchange and this under-searching behavior is consistent with that obtained in the endogenous growth literature.

Now refer to Figure 6. In contrast with the analysis above, when the matching externality effect is strong, the equilibrium value of the knowledge spread is large relative to the optimum. As a consequence, $V_U(\delta^*, N)$ may lie far above $V_U(\hat{\delta}; N)$ and the optimal population N^* exceeds the equilibrium population $\hat{N}$. Interestingly, in a decentralized equilibrium, cities are under-populated and individuals are over-searching to yield an under-selective knowledge exchange pattern.[22] This finding contrasts with conventional results in both the urban and growth literatures.

V. Knowledge Accumulation and Economic Growth

In this section, we demonstrate how to extend the basic model of knowledge exchange to allow for endogenous knowledge accumulation and endogenous growth. Specifically, we regard knowledge exchange as a process to create "new" knowledge. Thus, the effectiveness of matching affects the rate of knowledge accumulation, which in turns fosters the growth of the economy. The main message here is to illustrate the possibility of embedding our micro structure into the canonical frameworks of endogenous growth that rely on knowledge or human capital spillovers.

In order to pursue this objective in the most tractable manner, we assume that all matches between agents are ***permanent***.[23] That is, we do not allow for breakups to occur in the economy. Upon matching, both agents exit the search pool forever and are replaced by an identical pair of unmatched individuals. We assume that the new knowledge created between an individual k and another individual k' (previously denoted as $S(k,k')$ in Section II) now adds to k's stock of knowledge or human capital, $H(k)$. In order to ensure that the value created by knowledge exchange is bounded, we consider that new knowledge may partially duplicate the existing knowledge and can be subject to obsolescence or internal depreciation. Assuming that the depreciation rate of human capital is denoted by π, we require that $S(k,k') < r + \pi$ in order for the value of human capital

[22] It is evident that a mixed force of matching and congestion externalities may lead to under-population with over-selectivity or over-population with under-selectivity in a decentralized equilibrium.

[23] Alternatively, if we allow for matches to breakup, there will be a non-uniform distribution of individuals with different levels of human capital due to randomness in meeting partners.

to be bounded.[24] We can then specify the capitalized value of the human capital stock to individual k from time t and on (to infinity) as:

$$H(k,t) = H_0 e^{[S(k,k')-\pi]t} \int_t^{\infty} e^{-[r+\pi-S(k,k')](s-t)} ds = \frac{H_0 e^{[S(k,k')t-\pi]}}{r+\pi-S(k,k')} \tag{19}$$

where H_0 is the initial stock of human capital.[25] In order to express an agent's expected lifetime utility in stationary terms, we normalize the value of the human capital stock by its growth factor. Therefore, the ***effective human capital stock*** of an individual k permanently matched with an individual k' is simply:

$$h(k) = H(k,t)/e^{[S(k,k')-\pi]t} = H_0/[r+\pi-S(k,k')]. \tag{20}$$

We next determine the level of utility through the impact of knowledge exchange on production. In addition to the effects of matching, we also examine the distinct roles of general and specific human capital for population agglomeration and growth. In particular, we posit that general human capital in the economy improves the productivity of each match by envisioning that the productivity factor, A, depends on the economy's average knowledge spread such that $A = A(\tilde{\delta})$ and $A'(\tilde{\delta}) \leq 0$. This hypothesis is analogous to the specification of the externality of general knowledge and human capital as in Romer (1986) and Lucas (1988). Since a smaller knowledge spread implies that all matches lead to more knowledge creation, the overall stock of knowledge in the economy will be higher when $\tilde{\delta}$ is smaller. Thus, while higher values of $A(\tilde{\delta})$ can be regarded an increase in general knowledge, $h(k)$ represents the role of individual or match-specific knowledge. Our model also captures the distinction between general and specific human capital in the conventional dynamic labor and human capital literature. In this manner, we will be able to show how an increase in general knowledge enhances the value of match-specific knowledge and therefore encourages a

[24]Layard, Nickell, and Jackman (1991) document loss of human capital due to obsolescence or internal depreciation of skills. It is also possible that the obsolescence of human capital is a consequence of intergenerational competition, as in Laing, Palivos and Wang (2003).

[25]See Laing, Palivos and Wang (1995) for a detailed discussion on establishing a well-behaved model of search and matching with a perpetually growing stock of capital.

higher level of population agglomeration. Therefore, more general knowledge will be associated with a higher population size and a higher rate of growth.

In order to express utility in stationary terms, we note that (2) can be modified as:

$$y(k) = A(\tilde{\delta})\,h(k) = A(\tilde{\delta})H_0/[r+\pi-S(k,k')]\,. \tag{21}$$

This is the value of surplus accrued from matching for a forward-looking individual. Notably, since *y(k)* is still strictly increasing (and concave) in $S(k,k')$, the equilibrium properties established in Section III and IV will still apply, with an exception that it is now a ***balanced growth equilibrium*** rather than the previously defined zero-growth steady state.[26] Since the human capital stock grow unboundedly in this nonstationary environment, we must ensure bounded value for any match outcomes by imposing,

Assumption 3. (Bounded Value) $r+\pi > q_0+s_0a_0$.

Proposition 4: (Endogenous Growth) *In a closed economy without migration and under Assumptions 1 and 3, the balanced growth equilibrium features a growth rate satisfying:*

$$g = q_0+s_0a_0-s_0a_1\hat{\delta}-\pi \tag{22}$$

and possesses the following property: a higher population size causes greater selectivity in knowledge exchange and higher growth. When endogenous migration is allowed, the balanced growth equilibrium is still socially inefficient and satisfies the properties given in Proposition 3.

VI. The Evolution of Knowledge

We now consider that the creation of new knowledge can lead to the introduction of new *types* of ideas and hence the knowledge space of an economy may expand over time.

VI.A. Expansion of the Knowledge Space

A natural and simple way to permit the knowledge space to expand is to link patterns of knowledge

[26]Of course, to accept the balanced growth equilibrium, the migration cost must now be assumed to grow at the common rate S - π (that is, the effective cost of migration is constant over time).

exchange in an economy to its ***overall measure of diversity***. We let b represent the circumference of the knowledge space $\boldsymbol{K}$ in the economy (initially set at one) and assume that b is endogenously determined. We attempt to study the interactions between patterns of knowledge exchange and an economy's overall measure of diversity by envisioning that the economy-wide diversity depends on the economy's average knowledge spread $\tilde{\delta}$: b=b($\tilde{\delta}$). In contrast to the previous section on endogenous growth, we consider that the overall base of knowledge in the economy is higher when the knowledge spread is higher. As depicted in Figure 7, a particular agent k's knowledge location moves along a fixed ray from the center O.

Our focus is merely on exploring the determinants of an economy's overall measure of diversity. We thus restrict our attention to examining this setup within the context of a closed-city model as in Section III. As before, we derive the knowledge spread of a representative individual. As long as the economy-wide diversity is increasing and sufficiently convex in the average knowledge spread, multiple equilibria emerge for the ***same*** set of parameters as a result of ***self-fulfilling prophecies***. We find there exists both a specialized equilibrium and a diversified equilibrium. The specialized equilibrium emerges because when agents believe that the overall economy is very specialized, they expect a higher density of agents across ideas and can thus concentrate on more effective matching. The resultant knowledge spread is therefore more selective and the economy-wide diversity turns out to be narrow--a self-fulfilling prophecy. By similar arguments, the diversified equilibrium emerges when agents believe that the economy is more diverse. These equilibria are Pareto-rankable as welfare is higher in the specialized equilibrium than in the diversified equilibrium. This provides a role for city planners to induce the economy to select a more specialized set of industries in an effort to help encourage more effective means of informal learning from others.

Our conclusion that a specialized equilibrium is Pareto superior depends on the form of knowledge exchange. One could extend our knowledge-distance framework with an optimal level of idea-diversity ($\bar{\delta}$) to allow for "industries" by partitioning the knowledge space into clusters of types of knowledge. For example, industry i may encompass knowledge types along the interval [i, $i + c_i$] where c_i represents the Lebesgue measure of cluster (or industry) i. Because interaction with individuals from the same industries contains knowledge that is too similar to promote highly effective knowledge exchange, economies with more industrial diversity may have higher welfare since they allow for the possibility of meeting those from a greater variety of industries. In this manner, the model would allow for a ***diversified*** equilibrium to be associated with more effective knowledge exchange and higher city growth as documented empirically by Glaeser et al. (1992).

VI.B. The Evolution of Individual Knowledge

While the previous sections have demonstrated how knowledge exchange can lead to the creation of new types of knowledge, it is also possible that an individual's inherent base of knowledge will evolve as a result of matching. In particular, upon matching, an individual's type of knowledge may become closer to his partner so that both individuals will become more alike. In order to model the evolution of an individual's type of knowledge, we need to carefully distinguish the initial date of a match from calendar time t. To begin, consider a particular individual who is currently matched. In particular, denote n as agent k's n^{th} successful match and τ_n as the corresponding calendar time at which the match began. In order to emphasize how an individual's type of knowledge will depend on his previous matches, let the agent's knowledge type be denoted as $k_n \in \mathcal{K}$. Further, let k_n' denote the knowledge type of agent k_n's current partner in exchange.

Next, recall that individual meetings and separations follow a Poisson process. Given the constant break-up rate η, the pair (k_n, k_n') will (on average) remain matched for the length of time $1/\eta$. Further, in a steady state, the expected time at which the individual will begin his $(n+1)^{\text{th}}$ successful match is:

$$\tau_{n+1} \ = \ \tau_n + \frac{1}{\eta} + \frac{1}{\mu(U)} . \tag{23}$$

It remains to specify the process of knowledge evolution. In particular, individual knowledge locations are no longer time invariant. After completing their collaboration, the pair (k_n, k_n') will find that knowledge locations have evolved towards each other. To put the idea in perspective, consider a match between an economist and a mathematician. Knowledge exchange enables the economist to learn more rigorous mathematical analyses and the mathematician to learn more practical economic applications. Such an exchange can be simply captured by a convex combination of their knowledge:

$$k_{n+1} = \lambda k_n + (1-\lambda)k'_{n'}$$
$$k'_{n'+1} = \lambda' k'_{n'} + (1-\lambda')k_n \qquad (24)$$

where $\lambda \in [0, 1]$ and the case of $\lambda = 1$ captures the benchmark model presented in Sections II-IV. The evolution of knowledge of agents (k_n, k_n') from $\tau = \tau_{n'}$ to $\tau_n+(1/\eta)$ can be best illustrated by Figure 8.

Since we study a symmetric equilibrium in which all agents choose the same knowledge spread, the density of agents across the knowledge space $\mathcal{K}$ will remain stationary due to the law of large numbers. Therefore, the non-degenerate, symmetric, steady-state equilibrium is well-defined. As a consequence, our main findings established in previous sections remain qualitatively unchanged.

VII. Concluding Remarks

This paper develops a search-theoretic model to examine the patterns of knowledge exchange and their interaction with agglomerative activity. It illustrates that heterogeneity in agents' knowledge types plays a crucial role in the transmission of ideas. This occurs because agents in our model face a trade-off between selective, highly beneficial knowledge exchange and a higher number of matches.

We believe there are a number of interesting issues which may be pursued in this research. The first objective is to explore the interactions between knowledge exchange and agglomerative activity in environments where individuals have different levels of human capital. This would allow for a rich array of possible interactions among agents due to 'horizontal' and 'vertical' aspects of knowledge, and their consequences for agglomeration. Second, one may seek to investigate the relationships between knowledge exchange and agglomerative activity when individuals make human capital investments prior to engaging in the exchange of information. In this manner, the benefits of agglomeration due to lower costs of communication in dense environments will affect initial human capital decisions.

An important objective of our research is examining the implications of horizontal differences in knowledge for patterns of information exchange and agglomerative activity. In particular, our model demonstrates that larger cities should have more selective patterns of information flows due to lower costs of communication in dense economic environments. With these insights, we believe it would be interesting to further examine the evidence on knowledge spillovers using patent data as in Jaffe et al (1993). An attempt at such an endeavor has been made recently by Carlino et al (2004). Interestingly, they find that the number of patents per capita is positively correlated with the employment density of metropolitan areas, thus lending empirical support to our theoretical predictions (see our Proposition 2). Further quantitative studies may be conducted to study knowledge exchange and knowledge production across cities of different size.

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Figure 1: Knowledge Space and Selection of Knowledge Spread

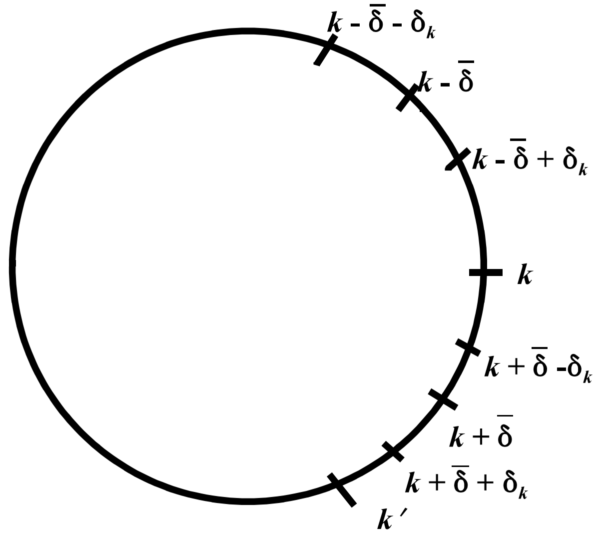


Figure 2: Role of Heterogeneity for Knowledge Creation

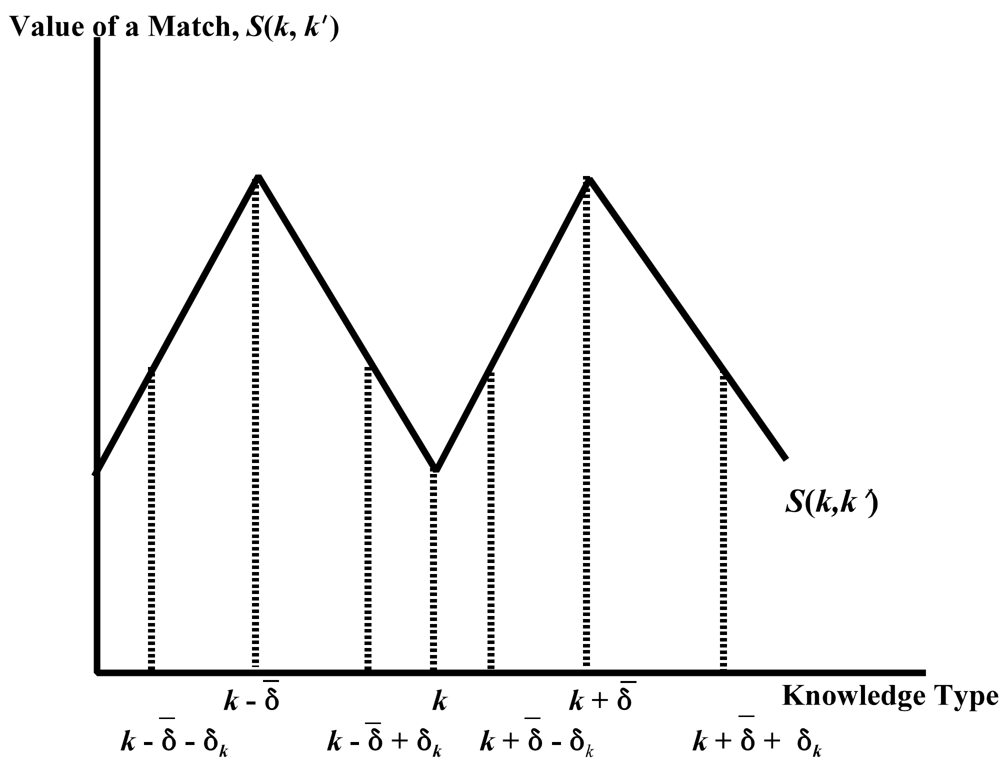


Figure 3: Equilibrium Entry (Migration) Decisions

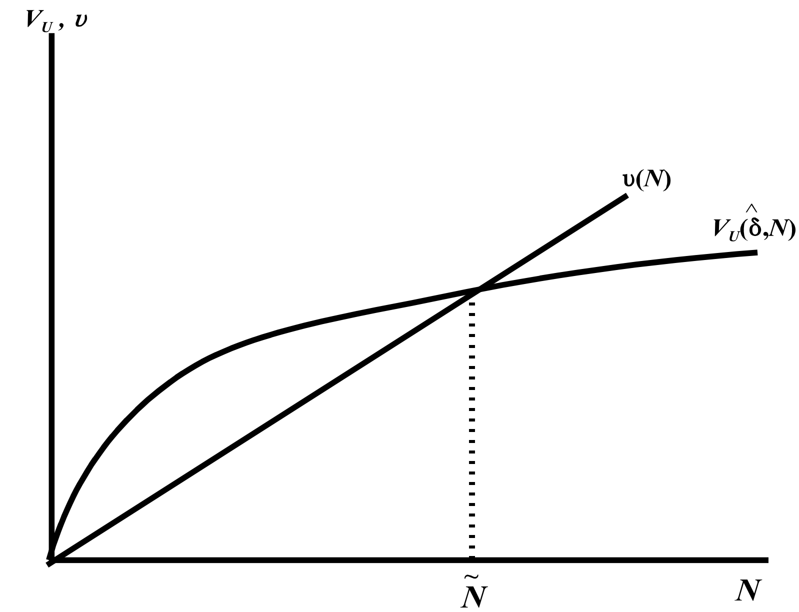


Figure 4: Steady-State Equilibrium under Endogenous Migration ($\mu(U)=\alpha U$)

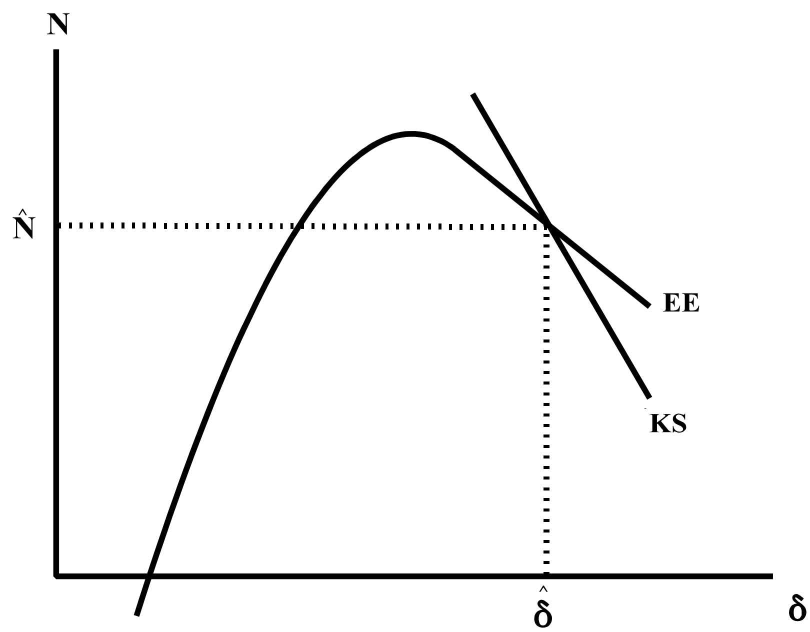


Figure 5: Comparison Between Social Optimum and Equilibrium with Strong Congestion Externality

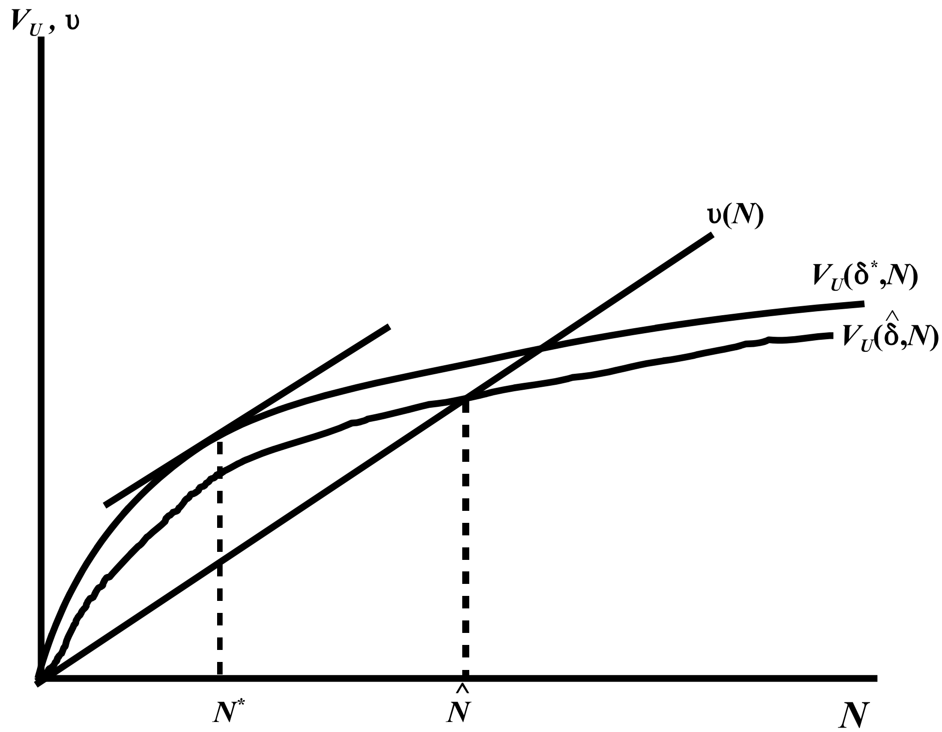


Figure 6: Comparison Between Social Optimum and Equilibrium with Strong Matching Externality

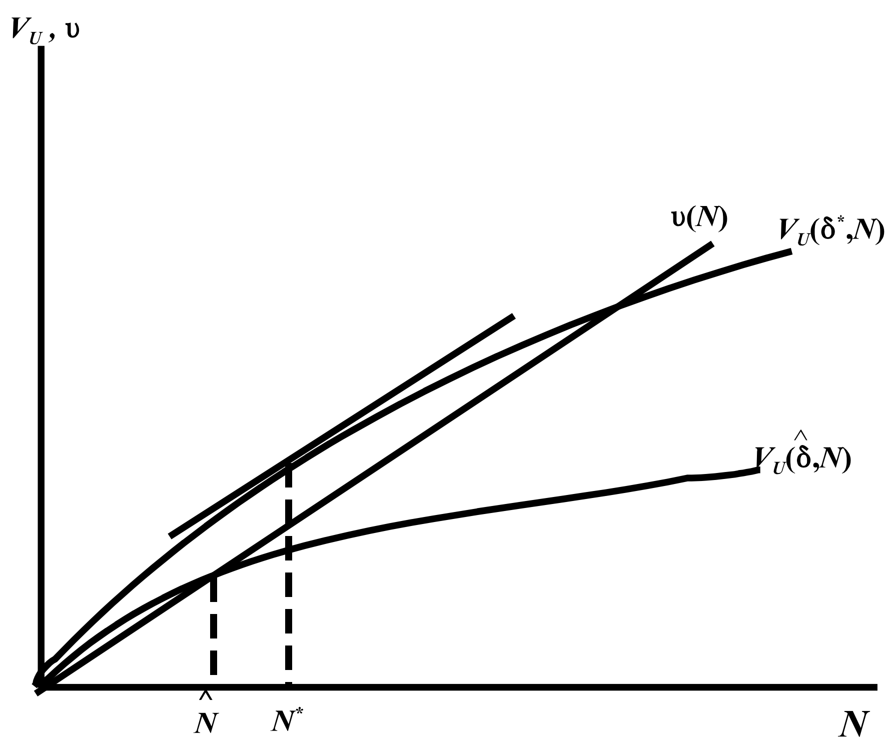


Figure 7: Knowledge Exchange and Evolution of Knowledge Types

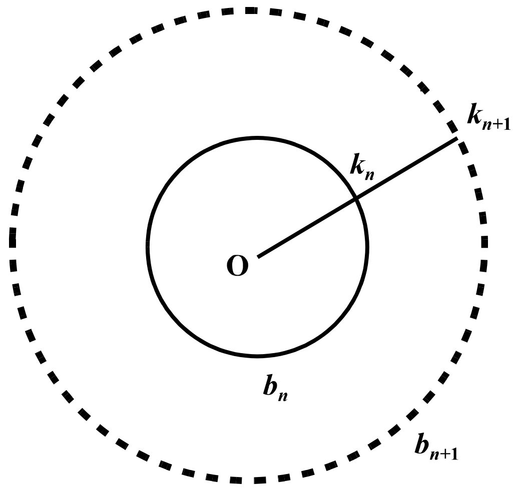
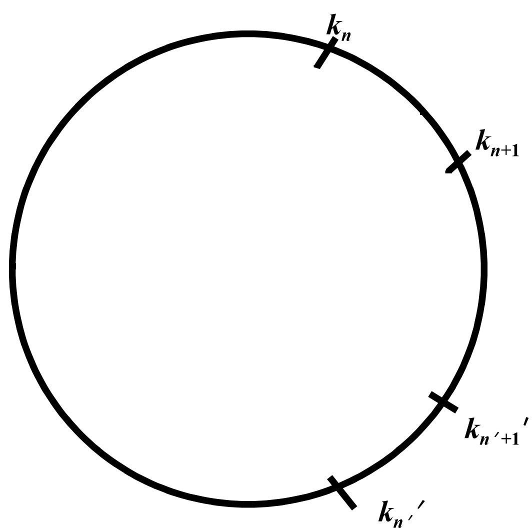


Figure 8: Knowledge Exchange and Evolution of Knowledge Types



Appendix A: Extension – Knowledge Exchange Most Effective when Agents are Alike

Throughout the paper, we have explored an agent's pattern of interaction with others while acknowledging knowledge exchange depends on differences among agents in terms of types of ideas. In that version of our knowledge distance structure ($\bar{\delta}>0$), we assume that the exchange of knowledge is not very effective when agents possess very similar types of knowledge or when agents have little in common.

We now consider an alternative view by assuming knowledge exchange is most effective when agents are alike ($\bar{\delta} = 0$). Thus, this is a special case of the set of parameters where Assumption 1 does ***not*** hold. Recalling (10), for the case where $\bar{\delta} = 0$, the value function is:

$$V_U(\delta;U) = \left[\frac{2\mu(U)\delta}{r+\eta+2\mu(U)\delta}\right]\left(\frac{A}{r}\right)\left(q_0+s_0a_0-\frac{1}{2}s_0a_1\delta\right) \tag{B1}$$

Although the hypothesis concerning knowledge exchange is different than the one we pursue in Section III, the underlying determinants of the knowledge spread in the economy remain the same. As before, agents choose their selection strategy recognizing tradeoffs between higher probabilities of matches (a contact rate effect) and more effective matches (a knowledge efficacy effect). Because the underlying costs and benefits of matching are the same as in Section III, the properties of the steady-state equilibrium are also similar.

There is an intermediate case where $\bar{\delta}$ satisfies neither Assumption 1 nor $\bar{\delta}=0$. This case does not admit a closed-form solution for the relationship between the individual's knowledge spread and the population mass. Our analysis does seem to suggest that, for a given mass of unmatched agents, a higher value of $\bar{\delta}$ is associated with a smaller knowledge spread. The smaller value of the knowledge spread implies that individuals are more selective of their partners for collaboration which leads to a higher mass of unmatched agents in the steady-state. A higher steady-state mass of unmatched agents further encourages agents to be more selective since the arrival rate of matches is higher when there is a higher mass of unmatched agents in the economy.

Appendix B: Proofs

This Appendix is devoted to deriving two fundamental equilibrium relationships, the equilibrium knowledge spread locus (KS) and the equilibrium entry locus (EE), as well as proving Lemma 1 and Theorems 1 and 2.

1. *Proof of Lemma 1:*

We can substitute (7) into (8) under symmetry ($\delta_k = \delta_{k'}$) and $f_0 \to 0$, transform the unmatched value in terms of the match-specific knowledge spread δ_k, and then find V_U over the region where $\delta_k < \bar{\delta}$ by integrating from zero to δ_k:

$$V_U(\delta;U) = \left(\frac{4\mu(U)\delta}{r+\eta+4\mu(U)\delta}\right)\frac{A}{r}\left(q_0+s_0a_0-\frac{1}{2}s_0a_1\delta\right), \text{ for } \delta<\bar{\delta}. \tag{A1}$$

Repeating the same exercise by integrating over the region where $\delta_k \geq \bar{\delta}$ yields (10). Q.E.D.

2. *Derivation of the Equilibrium Knowledge Spread and Proof of Theorem 1:*

We begin by checking that in equilibrium, if $f_0 > 0$, the strategy $\hat{\delta}$ is the unique dominant strategy (for all players). Then we select this equilibrium in the case $f_0 = 0$ by taking the (rather trivial) limit as $f_0 \to 0$ (as all elements of the sequence are $\hat{\delta}$). To see this, recall the matching rule $f(k,k',\delta_k,\delta_{k'})$ with $f_0 \in (0,1)$: when agent k wants a match and the other doesn't, the probability of a match is strictly positive but less than one. Recall that $\hat{\delta}_k$ is the global optimum of the unmatched agent k's optimization problem if agent k is allowed to choose his matches unilaterally. Thus, if agent k expands the utility-maximizing δ_k, his payoff decreases as he increases the probability for a match to occur with less desirable agents; if he contracts δ_k, his utility by construction of $V_U(\delta_k;U)$ can only go down. This implies that $\hat{\delta}$ is the unique dominant strategy when $f_0 > 0$, so all agents playing $\hat{\delta}$ is the unique pure strategy Nash equilibrium.

Next, we exploit what we have just proved: that $\hat{\delta}$ can be derived from the optimization problem of an unmatched agent who can choose matches unilaterally. Differentiating $V_U(\delta;U)$ with respect to δ provides the first-order condition for obtaining the equilibrium knowledge spread. Denote $\Delta_{\max} \equiv 2(q_0+s_0a_0)/(s_0a_1)$; for any value of $\delta > \Delta_{\max}$, $V_U(\delta;U) < 0$. Thus, under Assumption 1, the kink in the value function occurs over a region where an agent's unmatched value is negative and is not important for determining the individual's knowledge spread. Simple algebra yields the following quadratic equation that can be used to derive $\hat{\delta}$:

$$\delta^2 + \frac{r+\eta}{2\alpha U}\delta - \frac{r+\eta}{2\alpha}\frac{q_0+s_0a_0}{s_0a_1U} = 0 \tag{A2}$$

This can be used to solve for the mass of unmatched agents, which can then be substituted into the steady-state population condition (12) to obtain:

$$N^{KS} = \frac{r+\eta}{\alpha\eta\delta}\left(\frac{q_0+s_0a_0-s_0a_1\delta}{s_0a_1\delta}\right)^2 + \frac{r+\eta}{2\alpha}\left(\frac{q_0+s_0a_0-s_0a_1\delta}{s_0a_1\delta^2}\right) \tag{A3}$$

The knowledge spread locus is a downward-sloping curve for any $\delta > 0$, as depicted in Figure 4. Setting $N = N^{KS}$, we obtain Theorem 1. Q.E.D.

3. *Derivation of the Equilibrium Entry Locus:*

Recall the equilibrium entry locus is a curve of values of the knowledge spread and the population mass where $V_U(\delta;U) = \nu(N)$. This implies:

$$V_U(\delta;U) = \left(\frac{4\mu(U)\delta}{r+\eta+4\mu(U)\delta} \right) \frac{A}{r} \left(q_0 + s_0 a_0 - \frac{1}{2} s_0 a_1 \delta \right) = \upsilon_0 N \tag{A4}$$

Substituting the steady-state population condition (12) for the mass of unmatched agents in (A4) above generates the equilibrium entry locus.

4. ***Proof of Proposition 1:*** From (13), N is decreasing in δ. The first part of the proposition thus follows directly from Theorem 1. For the second part, note that we can divide both sides of (A2) by δ^2 to obtain:

$$U\delta = \frac{r+\eta}{2\alpha} \left(\frac{q_0 + s_0 a_0}{s_0 a_1 \delta} - 1 \right)$$

which is decreasing in δ. This together with the result that $\beta(\delta;U) = 4\alpha U\delta$ implies that a higher N is associated with a higher $U\delta$ and hence a higher β. Utilizing (11) and (12), we can rewrite the per capita match rate as: $\beta U/N = \eta(1-U/N) = \eta\beta/(\eta+\beta)$, which is increasing in β and thus positively related to N as well. Q.E.D.

5. ***Proof of Theorem 2:***

Define $\delta_{max} \equiv (q_0 + s_0 a_0)/(s_0 a_1) = \Delta_{max}/2$. We claim there exists a unique value of (δ, U) satisfying both the knowledge spread (KS) and the equilibrium entry (EE) relationships and such that $\delta \in (0, \delta_{max})$. Using (A2),

$$\beta(\delta;U) = 4\alpha\delta U = 2(r+\eta)\left(\frac{\delta_{max}}{\delta} - 1\right) \equiv B(\delta) \tag{A5}$$

which is strictly decreasing in δ for $\delta \in (0, \delta_{max})$ with $\lim_{\delta \searrow 0} B(\delta) = \infty$ and $B(\delta_{max}) = 0$. Substituting (A5) into (A4) and using (11) and (12), the equilibrium entry condition becomes:

$$v_0(\eta + B(\delta))(r + \eta + B(\delta)) = \eta\left(\frac{A}{r}\right)(4\alpha\delta)\left(q_0 + s_0 a_0 - \frac{1}{2} s_0 a_1 \delta\right)$$

which can be further simplified by applying the definition of δ_{max} and $B(\delta)$ and cancelling out the common term, $r+\eta+B(\delta)$,

$$v_0(\eta + B(\delta)) = \frac{\eta}{r+\eta} \frac{A}{r} 2\alpha s_0 a_1 \delta^2 \tag{A6}$$

This can be rewritten as:

$$\Lambda(\delta) \equiv v_0(\eta + B(\delta)) - \frac{\eta}{r+\eta} \frac{A}{r} 2\alpha s_0 a_1 \delta^2 = 0 \tag{A7}$$

We prove the existence of the steady-state equilibrium knowledge spread using the following mean value argument. Note that $\Lambda(\delta)$ is continuous and strictly decreasing in δ over the range of $(0, \delta_{max})$. Moreover, it is clear that $\lim_{\delta \searrow 0} \Lambda(0) = \infty$ and that $\Lambda(\delta_{max}) = v_0\eta - \frac{\eta}{r+\eta}\frac{A}{r}(2\alpha s_0 a_1)(\delta_{max})^2 < 0$ under Assumption 2. Since individuals will never choose a knowledge spread which yields zero utility, δ must be chosen below the upper bound δ_{max}. Therefore, there is a value of $\delta \in (0, \delta_{max})$ such that $\Lambda(\delta)=0$, which demonstrates that a steady-state equilibrium under endogenous migration exists (as we can show that the second-order condition holds). Since $\Lambda(\delta)$ is a monotone decreasing function of δ over the range of $(0, \delta_{max})$, there can only be one value of δ solving the steady-state equilibrium under endogenous migration. Q.E.D.

6. ***Proof of Proposition 2:*** Under the conditions imposed in Theorem 2, the equilibrium is determined at the intersection of the KS and EE loci where both are downward sloping in (δ, *N*) space, as shown in Figure 4. It is obvious that any (local) shift in the KS locus changes equilibrium values of δ and *N* in opposite direction along the downward-sloping portion of the EE locus. Similarly, any (local) shift in the EE locus changes equilibrium values of δ and *N* in opposite directions along the KS locus. Concerning the second part, since (11), (12) and (A2) continue to hold when *N* is endogenous, the proof of Proposition 1 still applies. Q.E.D.

7. ***Proof of Comparative Statics:***

The comparative statics under endogenous migration can be easily obtained by differentiating (A7). From (A5), $dB/da_1 < 0$ and $dB/dq_o > 0$. Thus, we have: $d\Lambda/da_1 = v_o(dB/da_1) - \eta A 2\alpha s_0\delta^2/[r(r+\eta)] < 0$ and $d\Lambda/dq_o = v_o(dB/dq_o) > 0$; moreover, $d\Lambda/dv_o = \eta + B > 0$. Since $d\Lambda/d\delta < 0$, by the implicit function theorem, one obtains: $d\hat{\delta}/da_1 < 0$, $d\hat{\delta}/dq_o > 0$ and $d\hat{\delta}/dv_o > 0$.

8. ***Proof of Theorem 3:***

Using the relationship $\beta(\delta; U) = 4\alpha U\delta$ and (12), the first-order conditions for the knowledge spread and population mass facing the social planner are::

$$(r+\eta)(q_0+s_0a_0 - s_0a_1\delta) - 2\alpha U s_0 a_1\delta^2 = \delta(4\alpha U)(q_0+s_0a_0 - \frac{1}{2}s_0a_1\delta)\frac{r+\eta}{\eta+8\alpha U\delta} \tag{A8}$$

$$v_0(\eta+8\alpha U\delta)(r+\eta+4\alpha U\delta)^2 = \eta(r+\eta)(\frac{A}{r})(4\alpha\delta)(q_0+s_0a_0 - \frac{1}{2}s_0a_1\delta) \tag{A9}$$

where the LHS is the marginal social benefit whereas the RHS is the marginal social cost of δ and *N*, respectively. Manipulation of (A8) implies:

$$\beta^2 + \frac{1}{2}\left\{(r+\eta)\left[3-2\left(\frac{\delta_{max}}{\delta}\right)\right]+\eta\right\}\beta - \eta(r+\eta)\left(\frac{\delta_{max}}{\delta}-1\right) = 0, \tag{A10}$$

which yields a unique positive root $\beta = F(\delta)$. Define $\delta_u = \frac{4(r+\eta)}{5r+8\eta}\delta_{max} < \delta_{max}$. Then *F* is a strictly decreasing function of δ over the range of $(0, \delta_u)$. Substituting this into (A9) gives a fixed point map in δ. Following similar arguments as in the Proof of Theorem 2, we can show the existence of a unique fixed point δ as long as η and r are both strictly positive. This implies that a social optimum exists when $v(N)=v_0 N$ (as we can show that the second-order condition holds). Q.E.D.

9. ***Proof of Proposition 3:*** Rather than deriving conditions on the parameter space for the various cases of Proposition 3, it is sufficient to verify the proposition by construction. Let $v(N) = v_1 + v_0 N$, under which three out of four possible cases can be established:

Case 1 (Under-Selectivity and Over-Population). Consider: q_0=1, s_0=1, a_0=.5, a_1=5, A=1, r=.2, η=.2, α=.2, v_0 =.2, v_1=0. In this case, the steady-state equilibrium (with endogenous migration) allocation is $(\hat{\delta}, \hat{N}) = (.175, 15.6)$ and the socially optimal allocation is $(\delta^*, N^*) = (.171, 4.08)$. So the equilibrium allocation is *under-selective* and the economy is now *over-populated* in equilibrium compared to the social optimum.

Case 2 (Over-Selectivity and Over-Population). Maintain all values of the parameters except α=.5. In this case, the steady-state equilibrium (with endogenous migration) allocation is $(\hat{\delta}, \hat{N}) = (.136, 20.5)$ whereas the socially optimal allocation is $(\delta^*, N^*) = (.139, 4.33)$. Thus, the equilibrium allocation is *over-selective* and the economy is now *over-populated* in equilibrium compared to social optimum. Further, higher values of α lead to further deviations between the equilibrium and the socially optimal allocation.

Case 3 (Under-Selectivity and Under-Population). Consider: q_0=1, s_0=1.5, a_0=.5, a_1=9.5, A=1, r=.2, η=.22, α=.31, v_0=.2145, v_1=0.7. In this case, the steady-state equilibrium (with endogenous migration) allocation is $(\hat{\delta}, \hat{N}) = (.113, .690)$ and the socially optimal allocation is $(\delta^*, N^*) = (.101, .718)$. Now, the equilibrium allocation is *under-selective* and the economy is *under-populated* in equilibrium compared to the social optimum.

One may establish the remaining case with more complex polynomial function of $v(N)$ and a general arrival intensity function $\mu(U) = \alpha_0 + \alpha U$. Q.E.D.

10. ***Proof of Proposition 4:*** By symmetry, the rate of economic growth, g, must be equal to each individual's human capital growth rate. Utilizing (1), we obtain the endogenous growth rate as in (22). Thus, given a bounded value guaranteed by Assumption 3 and under the particular human capital evolution setup, (20), we can follow the techniques developed by Laing et al. (1995) to establish a balanced growth equilibrium. The property then follows directly from (22). Since the channels of social inefficiency described in the steady-state model with endogenous migration are still present, the results in Proposition 3 must apply. Q.E.D.