Taxes on Buildings and Land in a Dynamic Model of Real Estate Markets

by

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ABSTRACT

Henry George's *single tax on land* is an elusive concept to implement, because land is occupied by a variety of buildings or is undeveloped. *Land value* is undefined since the value of the land lying under buildings is difficult to estimate and does not correspond to real market value. Therefore, it is hard to find taxes that are accurately related to land value and, hence, to the *ability to pay* and still satisfy George's axiom. Static models unrealistically pretend that all the land is available in the market at all points in time. To properly treat dynamics, a generalized perfect-foresight model of real estate markets solvable by simulation is presented. Using a version of this model stripped-down to its bare essentials, the effects of the conventional ad-valorem property tax and of an ad-valorem tax on undeveloped land are analyzed. We show a new result that the conventional tax speeds up the demolition-reconstruction cycle, shortening the life span of buildings and thus resulting in excessive use of structural capital over time, while a tax on undeveloped land has the opposite effects. We then turn to the application of the dynamic simulation model to the optimal taxation problem adapted to real estate markets. In this problem a different tax rate is levied on each type of undeveloped land and each type of building to meet a desired revenue goal, recognizing the different price elasticities of demand and supply for these assets. The formulation is designed to calculate deadweight losses associated with such optimal taxation schemes.

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1. Introduction

The Henry George (1879) *single tax* is a tax on land. But how should the tax be levied? The simplest example would be a lump sum tax on each unit of land to be paid regardless of what is to be done with that land and disregarding whether it is currently developed or not. Such a tax system is generally presumed to be neutral as George had envisioned. And, it is presumed, one could vary the tax from one unit of land to the other: the implied tax rate as a proportion of land value would not have to be the same everywhere to achieve neutrality. But such a lump sum tax system – while probably deserving a lot more attention than it has gotten – would be considered inequitable unless it is related either to the *benefits received* by the owners of the land or to the landowner's *ability to pay*. Arguably, in an efficient capital market, the best measure of a landowner's ability to pay is his land value.

But can tax authorities or econometricians accurately measure the value of land covered with buildings? Mills (1998) has argued that they cannot. The consequences of inaccurate measurement could be quite severe. Consider the example of an owner of a building who is planning to demolish his building and sell the land because that is the most profitable action. Suppose that the tax authority, not knowing the land value, sets

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the lump sum tax on this owner so high that the after-tax value of the land that will be released by demolition becomes negative.[2] In a free society, if the owner were fully rational, he would abandon the building if all other alternatives yielded a negative return and if – by doing so – the tax could be avoided. Inducing abandonment in this way would be distorting and, hence, inefficient. To avoid the distortion and restore efficiency, owners could be forced to pay the tax even if abandoning. Knowing that they could so be forced, they would not abandon because they would reduce their losses by demolishing and selling the land. Such a prohibition of abandonment is a form of fascism. Alternatively, society could also restore efficiency by taking over and demolishing the abandoned building. Arguably, this is a form of socialism. To avoid both extremes, the tax authority and the building's owner could negotiate the tax down to some reasonable level. That, of course, is neither fascism nor socialism. But, at worst, it opens the door to corrupt dealings between landowners and tax authorities. At best, it increases the transaction costs involved in determining a reasonable tax. So we should look for land tax instruments that are easy to administer at arms length and are based on observable measures of value.[3]

The modern literature on land taxation, implicitly recognizing the importance of tax schemes related to the ability to pay, has focused on ad-valorem taxes that maintain proportionality between the tax paid and the "true land value" so that all landowners faced the same tax rate. But all these modern attempts run into the basic question: "what is the true land value that should be taxed?" Looking at the world, we see at least two

[2] This is more likely to be the case if land value is a relatively small part of property value as is the case in the United States.
[3] Of course, the same pitfalls exist for taxes proportional to assessed values if those values cannot be accurately calculated as they are unlikely to be for land occupied by buildings.

types of land. The first is land that is undeveloped or vacant in the sense that there is no building on it. The second type of land is land that is occupied by a building of some sort or having minimal infrastructure improvements on it or near it, making it suitable for supporting buildings at a later date. Further scrutiny reveals additional complexity. The types of buildings as well as the types of vacant land that we see vary enormously in their underlying economics. There are, for example, tall buildings (e.g. skyscrapers) that are, for economic reasons, extremely unlikely to be demolished any time soon. Hence, the land that they occupy is virtually locked out from the active market for land. There are, as well, shorter buildings in poor structural condition that are very likely to be demolished and could make their underlying land available for some other type of building to be constructed. Buildings can also be changed (from apartments to offices for example) without affecting the underlying land or their structural density. To make a long story short, virtually every piece of land is a different type of land with a different propensity to remain in its present use or to become recycled into some other.

Looking at things in this way gives rise to a situation that is far different from George's idealized view. While it is still true that the aggregate supply of land is fixed for all practical purposes, the supply of vacant land at any one place is not at all fixed. More vacant land can be created by demolition of buildings and the supply of it is far from inelastic in general. Meanwhile, the supply of a particular type of building at a location can be very elastic or very inelastic depending – among other things – on the costs of construction, demolition, conversion without demolition, as well as the availability of vacant land nearby. These elasticities also depend on time horizons.

Given the above realistic way of looking at what we mean by a "land market" what exactly is to be understood by a *single rate ad-valorem tax on land value*? It is perhaps best to approach this question gradually by making a brief review of recently published modeling exercises by several urban economists who treated the response of an *entire* land market to different tax structures. These are the recent *static* models by Mills (1998), Brueckner (1999) and Brueckner and Kim (2000) in which an idealized, homogeneous and instantly available land market exists by assumption. These authors examined the land tax in the context of the monocentric city of urban economics. They drew conclusions about the effects on physical city size and land use within such a city arising from a revenue-neutral switch from a conventional ad-valorem property tax falling on land and structures at the same rate, to a hypothetical pure ad-valorem land tax falling on land only. Mills examined a monocentric city containing exporting businesses only and open to the in and out-migration of these businesses. He showed that the switch to the land tax increased the capital per acre (structural density) that businesses would employ throughout the city. Because this increases the productivity of each acre, the rent-distance function rises and the city expands in radius and in total output. Brueckner examined a city of housing consumers closed in population and showed that the switch to a land tax causes the after-tax cost of capital to fall and thus the structural density of housing to rise on each acre. Dwelling sizes and total population being constant, the city shrinks in radius (less urban sprawl occurs). In Brueckner and Kim, it is shown that this result can be reversed if dwelling sizes are not constant. The lower after-tax cost of structural capital, arising from the switch to the single tax, increases the dwelling size demanded by each consumer while also increasing the structural density of buildings. If the dwelling-size effect dominates, there could be fewer households on each acre even with taller buildings on each acre and a city of a given total household population could expand in radius causing more not less sprawl.

The above models being static, they cannot shed light on the dynamic effects of taxes. Dynamic analyses that can treat conversions involving buildings and land is needed. A paper by Arnott (1998) is a step in this direction and a good summary of earlier literature.[4] He considered how to devise a neutral tax on a single developer/landowner rather than devise such a tax system for a whole land market. The developer in question has a unit of vacant land to build on and he decides, under perfect foresight, when to build and at what structural density. However, once he builds, the building remains undemolished forever. Arnott looks for a neutral tax in this context, a tax that is neutral with respect to the timing (when to develop the unit vacant land) as well as the density of development, and also raises the desired revenue. Arnott shows how to achieve such neutrality by the coordinated setting of three separate taxes all related to some concept of value. The first tax is on the *pre-development value of the land*, defined as “what the land is worth in its vacant state before it is developed.” The second tax is on the *post-development residual site value*, defined as “property value minus the depreciated cost of the structure on the site.” The third tax is a subsidy on the structural capital employed on the site.

The purpose of the current paper is to present a more general and complete

[4] Earlier papers by Skouras (1978), David Mills (1981, 1982), Tideman (1982) and Wildasin (1982) debated the neutrality of alternative land taxes. Bentick (1979), Kanemoto (1985) and Turnbull (1988) focused on the effects of taxation on the timing and efficiency of development. But none of these authors considered demolition as we do in this article.

theoretical framework and empirically useful modeling tool for analyzing alternative tax structures on buildings and land over an entire real estate market. Using such a framework in a dynamic context, it should be possible to devise alternative tax structures and make revenue-neutral comparisons among them, quantifying the deadweight losses of these alternative tax structures. Clearly, a dynamic model properly grounded in economic theory is the appropriate approach for such a research program. To this end, I will use the model that I have developed in earlier work with Richard Arnott, and I will modify it to deal with problems of property taxation and optimal property taxation.[5]

The paper is organized as follows. Section 2 presents the structure of the model and the solution properties in the case where there are no taxes on buildings and land. (Appendix A is a technical appendix that explains how some of the relationships used in the model are derived.) In Appendix B, I describe a computational algorithm (Anas and Choi (2001)) that we have designed to solve the dynamic simulation model with exogenously specified taxes on building submarkets (or building types) and on land. Then, in Section 3, I strip the model down to its bare essentials and investigate several simple properties of it in the presence of alternative taxation schemes for stationary state dynamics. By performing comparative statics on such a stationary state, I compare a tax on undeveloped land to a conventional property tax that falls on land and buildings at the same rate. I show that the conventional tax causes inefficient use of structural capital, because it speeds up the construction-demolition cycle shortening the lives of buildings. In the *intensive margin* of building replacement, this results in excessive use of structural capital over time on each unit amount of land. But the conventional tax also works in the

[5] See Anas and Arnott (1991, 1993a,1993b, 1993c, 1994, 1997) and Anas, Arnott and Yamazaki (2000) for the earlier publications on this model and its empirical application.

extensive margin, resulting in a lower stock of buildings (fewer units of land are developed). Hence, unlike what is commonly believed, the total amount of structural capital used over time would decrease only if the extensive margin effect dominated the intensive margin effect. I also show that the tax on undeveloped land works in the opposite way, offsetting – even though imperfectly – the inefficiencies of the conventional tax. (These results are summarized in the text and the details of the analysis are in Appendix C.) The Mills or Brueckner models discussed earlier are static models and thus ignore the intensive margin of building replacement although they treat the other intensive margin of structural density. The intensive margin of building replacement is also absent from Arnott's model who also treats structural density but does not consider demolition. In his model, development is clay-putty: once buildings are constructed they remain in place forever. In Section 4, I present a generalized optimal taxation problem that can be solved using an extended formulation of the dynamic model of Section 2. In this formulation, there is a different tax rate on vacant land and on each building type and these taxes can be optimized for every year over a planning period. The "optimal taxation problem" is a well-known textbook problem in economics. In this textbook version, it is recognized that the price elasticity of demand varies a great deal from commodity to commodity. Efficiency requires levying a higher tax rate on those commodities for which the demand is relatively price inelastic and a lower tax rate on those for which the demand is relatively price elastic. How does the setting of this textbook problem differ from that of the real estate market with land? It should be natural to view buildings in different sub-markets and vacant land in different locations as being different commodities. As explained earlier, we also know that the price elasticities of demand for these buildings and land will differ as will also the price elasticities of supply. In that sense, our problem is similar to the textbook problem but somewhat more complex. I leave solution of this optimal taxation problem to future work. Part of the research program is to embed the simulation procedure described in Section 2 into a more general algorithm that can determine the optimal tax rates taking the interdependent demand and supply elasticities into account.

2. Structure of Model

We now turn to a description of the simulation model.[6] The description presented here follows closely that in Anas, Arnott and Yamazaki (2000).[7]

2.1 Basic Assumptions

Time consists of periods of equal length (years). Time t denotes the start of year t and time $t+1$, the end of year t. $t = 0$ denotes the present (initial) year. Variables change only at the start or end of each year. The model incorporates idiosyncratic uncertainty on both the demand and the supply sides. For example, at the start of each year, consumers of buildings learn the idiosyncratic components of their tastes, earn incomes, choose their most preferred submarket and pay rents, while investors receive

rents and bid on vacant land or building assets determining asset prices, prior to learning their idiosyncratic costs.[8] Idiosyncratic costs of maintenance for buildings for the year, are revealed a bit after the start of a year, while costs of construction, structural

[6] The model is consistent with theory. But I refer to it as a "simulation model" because simulation is the only practical way to solve it in the general case.

[7] The model follows discrete choice theory. Despite my efforts over the years, the approach is still considered *unorthodox* within urban economics. But its enormous advantage is that it lends itself to direct empirical and numerical implementation of the theoretical models without sacrificing the theoretical form of the model equations (see Anas and Arnott (1993c, 1994, 1997). This contrasts with *orthodox* urban economics where, with few exceptions, authors develop a theoretical model and then have to switch to some reduced form, vaguely related to the theoretical model, to test it empirically.

conversion or demolition are revealed just before the end of a year. The timeline in Table 1 illustrates the sequence in investor and consumer actions and in the revelation of information within a year. The left column shows when a particular item of information is revealed and the right side shows the decisions that follow.

The building stock is divided into $k=1...K$ building types or submarkets. Each submarket represents a different combination of size (e.g. floor space), physical quality, structural density or location. k=0 represents vacant land.[9] Buildings consist of structure and land. Buildings can be created from land via construction or from buildings of other types via structural conversions. Land is created by demolitions of buildings. m_{0k} is the lot size (or land needed per unit building) in submarket k and $\frac{1}{m_{0k}}$ is *construction density*. $m_{kk'}$, for $k > 0$, is the number of building units of type k, used up in the conversion process, to create one unit of type k' and $\frac{1}{m_{kk'}}$ is the $k \to k'$ *conversion density*. Of course, $m_{kk} = 1$ for all k >0. The vector $\mathbf{S}_t = [S_{0t}, S_{1t}, ..., S_{Kt}]$ is the stock (number of unit buildings[10]) in each submarket (or land for k=0) in year t. The total amount of land (vacant plus occupied by buildings) is A_t and is exogenous for each t. We will normally assume that $A_t = A$ for each year t. $\mathbf{V}_t = [V_{0t}, V_{1t}, ..., V_{Kt}]$ is the vector of asset prices for buildings and land in year t, and $\mathbf{R}_t = [R_{0t}, R_{1t}, ..., R_{Kt}]$ is the vector of land and building

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[8] By “building” we refer to any type of building such as single family home, apartment building, office building etc. Consumers of buildings can be households or business establishments.

[9] The model is here presented for a single land market, but in Anas and Arnott (1993c, 1994, 1997) it has been applied to metropolitan housing markets (e.g. Chicago) with two (city and suburban) land markets.

[10] A “unit building” is the quantity a building consumer wants to consume. In the case of housing consumers, a “unit building” is a whole housing unit. In the case of business establishments it could be viewed, for example, as a unit amount of floor space in a building.

rents (per unit building) in year t. The asset price of a unit building includes the value of the land on which the unit is built. The rent for vacant land, R_{0t}, is exogenously given for each t and the initial stock vector $\mathbf{S}_0$, is exogenous as an initial condition.[11] All other elements of the rent, stock and asset price vectors are endogenous for each t.

As noted, a unique feature of the model is that it treats stochastic heterogeneity at the level of individual agents on both the demand and the supply sides of the market. The chief reason for doing so is to achieve empirical realism in applied studies (e.g. Anas and Arnott (1993c,1994, 1997) as well as smooth computational solutions. On the demand side, consumers who belong to the same group differ in the idiosyncratic taste constants they attach to building submarkets. On the supply side, building units differ in idiosyncratic costs of maintenance for occupied and vacant units as well as in the costs of converting those units to land or to other units. Similarly, there are idiosyncratic costs in converting land to buildings. We assume that each idiosyncratic utility or cost is a draw each year from the following double exponential distribution, known as the Gumbel and given by (1) below. We assume that all agents know the distribution of utility or cost and its mean for each alternative, but learn the value of their idiosyncratic deviation from the mean only after it is realized.

(1) $$G(x < z) = \exp-[\exp-\gamma(z-\eta)] \ , \gamma > 0,$$

where x stands for a random realization of idiosyncratic utility or cost. The distribution has mode η. We assume $E[x] = \eta + \frac{g}{\gamma}$ =0 (by imposing $\eta = -\frac{g}{\gamma}$) where $g = 0.5772$ is

[11] That the vacant land rent is exogenous is analogous to the assumption of exogenous agricultural rent in the models by Mills (1998), Brueckner (1999), Brueckner and Kim (2000) or Arnott (1998) discussed earlier. Of course, there is no loss of generality in assuming that the rent for vacant land is zero.

Euler's constant. The variance is $Var[x]=\frac{\pi^2}{6\gamma^2}$. γ^2 is inversely proportional to the variance, and is called the *dispersion parameter* or the *heterogeneity coefficient*. As $\frac{1}{\gamma^2}\to 0$, idiosyncratic heterogeneity vanishes and all decision makers (consumers or investors) most-prefer the same choice. And as $\frac{1}{\gamma^2}\to\infty$, idiosyncratic heterogeneity swamps nonrandom effects, making choices extremely heterogeneous. Then, all choices are most preferred with the same probability.

Closure Property of (1)**:** *The Gumbel distribution given by (1) has the property that it is closed under the maximization operation. Hence, if n random variates* $(x_1, x_2, x_3, ..., x_n)$ *are each i.i.d. with means* $(X_1, X_2, X_3, ..., X_n)$ *and dispersion parameter γ according to (1), then the random variate* $max(x_1, x_2, x_3, ..., x_n)$ *is also distributed according to (1) with mode* $\frac{1}{\gamma}\ln\sum_{i=1...n}\exp(\gamma X_i)+\text{constant}$ *and dispersion parameter γ.* The proof is in Appendix A.

This closure property implies, for example, that the distribution of a maximized objective in a population of maximizing agents has the same distribution as the *un*-maximized objective has in the same population of agents. Thus, aggregation across maximizing agents does not affect the *ex post* distribution of the maximized objective. This further implies that welfare comparisons can be made knowing that two different policies which affect individual objectives will not affect the variance of the maximized objective in the population of agents.

While the model will be stochastic at the level of consumers and investors as explained above, there is no uncertainty at the aggregate level. Hence, rents, asset prices and stocks are all obtained as deterministic variables. Asset prices are deterministic because risk-neutral investors bid on buildings and on land before the uncertainty in costs is realized. Hence, at the time of bidding, investors are *ex ante* identical and make the same bids. The model solves for the expected stock distribution as a function of deterministic asset prices and for the expected allocation of households among submarkets as a function of deterministic rents.

2.2 Demand Side: Consumers

Consumers view submarkets as internally homogeneous. Hence, they are indifferent about choice within a submarket. After learning his idiosyncratic realization of utilities for each submarket for that year, a consumer chooses the most-preferred submarket and randomly selects a unit building to rent within that submarket. Each consumer reevaluates his choice at the start of each period and can costlessly relocate. We treat consumers as myopic and we assume that they neither borrow nor save. We divide them into $h = 1 \ldots H$ demand groups according to socioeconomic types (income, family size, age of household head or race). $\mathbf{N}_t = [N_{1t},...,N_{Ht}]$ is the exogenous vector of the number of consumers per year in each demand group. $\mathbf{y}_t = [y_{1t}, \ldots, y_{Ht}]$ is the exogenous income of a consumer in demand group h in year t.[12] $\boldsymbol{Y}_t = [Y_{1t},...,Y_{Kt}]$ is a vector of submarket qualities and β_h is the marginal utility of quality. Then, $\hat{U}_{hkt} = U_{hkt} + u_{hkt}$ is the utility a consumer enjoys from renting a unit building in submarket k in year t. $U_{hkt} = y_{ht} - R_{kt} + \beta_h Y_{kt}$, is the utility of submarket k which is common to all consumers of type h. u_{hkt} measures an idiosyncratic submarket-specific utility varying around mean utility U_{hkt} for consumers of type h. For each consumer of type h, idiosyncratic utilities $\mathbf{u}_h = [u_{h1t}, \ldots, u_{hKt}]$ are drawn from (1) at the start of each year with dispersion parameter

[12] For businesses, we may think of this as the net income before rent is paid, for example.

δ_{ht} (i.e. $\gamma \equiv \delta_{ht}$ in (1)). Then, the probability that a type *h* consumer selects submarket *k* is $P_{hkt} = \Pr ob.[U_{hkt} + u_{hkt} > U_{hst} + u_{hst}; \forall s \neq k]$, with $\sum_{k=1...K} P_{hkt} = 1$. Appendix A presents the proof that imposing (1) on each u_{hkt} (with mean U_{hkt}) yields the multinomial logit choice probabilities:

$$(2) \qquad P_{hkt}(\mathbf{R}_t) = \frac{\exp(\delta_{ht} U_{hkt})}{\sum_{z=1...K} \exp(\delta_{ht} U_{hzt})}; \quad \sum_{z=1...K} P_{hzt} = 1.$$

Appendix A also proves that, under the closure property of (1) described above, the *ex ante* expected value of the maximized utility level (or expected consumer surplus) of a consumer in group *h* at the start of year *t* is:

$$(3) \qquad \Psi_{ht}(\mathbf{R}_t) \equiv E[\max(U_{hkt} + u_{hkt}; k = 1...K)] = \frac{1}{\delta_{ht}} \ln \sum_{z=1...K} \exp(\delta_{ht} U_{hzt}).$$

2.3 Supply Side: Investors

Investors are risk neutral and perfectly competitive. As shown in Table 1, investors value buildings or land at the beginning of each year, knowing only the probability distribution of the idiosyncratic costs they will encounter later. Let $\mathbf{C}_t$=[$C_{kk't}$] be the (*K+1*)×(*K+1*) matrix of expected *k* to *k′* conversion costs for *t>0*, and let $\mathbf{c}_t$=[$c_{kk't}$] be the corresponding (*K+1*)×(*K+1*) matrix of idiosyncratic conversion costs per unit of type *k′* building for *t>0*, measured as a deviation from the expected cost. These are revealed right before the end of year *t*. Also, let $\mathbf{D}_{ot}$=[$D_{1ot},..., D_{Kot}$], $\mathbf{D}_{vt}$=[$D_{1vt},..., D_{Kvt}$], be the expected maintenance costs for type *k* ($k > 0$) occupied and vacant units common to investors, with $\mathbf{d}_{ot}$=[$d_{1ot},..., d_{Kot}$], $\mathbf{d}_{vt}$=[$d_{1vt},..., d_{Kvt}$], the idiosyncratic deviations from the respective expected costs. These are revealed right after the start of year *t*. For each investor, the $-c_{kk't}$ and the $-d_{kot}, -d_{kvt}$ are drawn from (1) for each submarket and for land with dispersion parameters Φ_{kt} and ϕ_{kt}, and means $-\frac{C_{kk't}}{(1+r)m_{kk'}}$ and $-D_{kot}, -D_{kvt}$ respectively. Let $\pi_{kot} = R_{kt} - D_{kot} - d_{kot}$ ($\pi_{kvt} = -D_{kvt} - d_{kvt}$) be the profits from keeping a type *k* unit (*k* >0) occupied (vacant) during year *t*. Then, the probability that a unit in submarket *k*>0 will be let to a consumer for year *t* is $q_{kot} = \Pr ob.[\pi_{kot} > \pi_{kvt}]$. The procedure of Appendix A, gives the binomial logit:

$$(4) \qquad q_{kot}(R_{kt}) = \frac{\exp\phi_{kt}(R_{kt} - D_{kot})}{\exp\phi_{kt}(R_{kt} - D_{kot}) + \exp\phi_{kt}(-D_{kvt})}; \; q_{kvt}(R_{kt}) = 1 - q_{kot}(R_{kt}).$$

From Appendix A, under the closure property, the expected profit of this occupancy decision at the start of year *t*, is

$$(5) \qquad \omega_{kt}(R_{kt}) \equiv E[\max(\pi_{kot}, \pi_{kvt})] = \frac{1}{\phi_{kt}} \ln[\exp\phi_{kt}(R_{kt} - D_{kot}) + \exp\phi_{kt}(-D_{kvt})].$$

For vacant land, we assume that it can always be rented for the exogenous land rent. Hence, $\omega_{0t}(R_{0t}) \equiv R_{0t}$. Now note that year-end conversion profits from type *k* to type *k′* asset, discounted to the start of year *t* are $\Pi_{kk't} = \frac{V_{k't+1} - C_{kk't}}{(1+r)m_{kk'}} - c_{kk't}$. The probability that an investor who holds asset type *k = 0,1,…,K* at the start of year *t* will choose to convert to type *k′* just before the end of year *t* is $Q_{kk't} = \Pr ob.[\Pi_{kk't} > \Pi_{kst}; \forall s \in B(k)]$ for all *k′*∈ *B(k),* where *B(k)* is the set of asset types to which a type *k* asset can be converted and $Q_{kk't} = 0$ for all *k′*∉*B(k)*.

Figure 1 illustrates two (of many) possible ways of defining the sets *B(k),* for three building types and land. We will refer to these as alternative *conversion technologies*. For realism, it should be assumed that *k*∈ *B(k),* for each *k,* so that it is always possible to keep a building at its current submarket state by incurring expenditure C_{kkt}. Figure 1a shows a quality hierarchy of buildings. Supposing that structural densities are the same for all these buildings, they differ only in structural quality. Buildings are constructed at the highest quality (quality 3 in the example of the figure) and – depending on the cost shocks experienced – can either stay in the same quality level or deteriorate only one quality interval per time period. Only the lowest quality buildings can be demolished. Thus, a quality cycle is implied where buildings deteriorate gradually (and some faster than others depending on idiosyncratic cost shocks experienced) until they are demolished and the new highest quality buildings are built on the land released by the demolition. So, the owner of the lowest quality building can change his asset to a highest quality building over at least two periods: he demolishes in period 1 and builds the highest quality building in period 2. Figure 1b illustrates a situation in which there are three buildings that differ in structural density (unlike the first part of the figure) but not in quality. Each structural density can be demolished or constructed. To change from one structural density to another on the same land requires at least two periods: the existing structural density is demolished in the current period and the desired structural density is constructed in the next period.

The procedure of Appendix A now yields the following multinomial logit for year-end conversion probabilities in year t (i.e. the probability that an owner of a type k asset will convert it to a type k' in year t):

$$(6) \qquad Q_{kk't}(\mathbf{V}_{t+1}) = \frac{b(k',k)\exp\Phi_{kt}\dfrac{V_{k't+1}-C_{kk't}}{(1+r)m_{kk'}}}{\sum_{s=0,1\ldots K} b(s,k)\exp\Phi_{kt}\dfrac{V_{st+1}-C_{kst}}{(1+r)m_{ks}}}\ ; \quad \sum_{k'=0,1\ldots K} Q_{kk't} = 1.$$

where $b(k',k) \equiv 1$ if $k' \in B(k)$, and $b(k',k) \equiv 0$ if $k' \notin B(k)$. Those owners who undertake the conversion are those who draw (are shocked by) a low idiosyncratic conversion cost $c_{kk't}$. From Appendix A, applying the closure property, the expected discounted *ex ante* conversion profit from a type *k* unit at the start of *t* is:

$$(7)\quad \Omega_{kt}(\mathbf{V}_{t+1}) = E[\max(\Pi_{kk't}; \forall k' \in B(k))] = \frac{1}{\Phi_{kt}} \ln \sum_{s=0,1\ldots K} b(s,k) \exp \Phi_{kt} \frac{V_{st+1} - C_{kst}}{(1+r)m_{ks}}.$$

2.4 Dynamic Market Equilibrium

We will now define the dynamic market equilibrium problem for a real estate market as consisting of two phases. To do so, we must specify how the exogenous variables driving the market such as incomes, demand group populations and costs will be changing over time. Suppose that these variables change in some arbitrary pattern for an extended period with each variable eventually settling on a stationary value thereafter. Then, the response of the real estate market will consist of two phases. In phase 1, the real estate market adjusts to the arbitrary time profile of the exogenous variables, by evolving from the given initial stocks of buildings and vacant land to an eventual stationary state of stocks, rents and asset prices. This adjustment requires stocks, rents and asset prices to change year by year until they all become stationary at some terminal year *T* and remain stationary thereafter. This stationary phase is phase 2.

Solving the dynamic equilibrium problem requires doing three things, illustrated in Figures 2 and 3. First, one must solve for the stationary (phase 2) stocks, rents and asset prices and this is independent of any initial conditions. This phase 2 solution can be obtained conditional on the stationary values of the exogenous variables only. Second, one must solve for the non-stationary (phase 1) stocks, rents and asset prices given only the stationary asset prices obtained from phase 2 and the initial year stocks. Third, one must pin down a reasonable approximation for the value of the terminal year *T* (i.e. for the length of phase 1), so that stocks, rents and asset prices for *T* are sufficiently close to their stationary (phase 2) values. This is done simply by extending the non-stationary phase 1 until the difference between the year *T* and corresponding stationary variables is as small as possible (Figure 3). The algorithm that we have devised (Anas and Choi (2001)) and which I describe in Appendix B in fact implements this solution procedure. In this section, we will just focus on the formal statement of this two-phase dynamic equilibrium problem so that the solution procedure's basic structure can be seen and discussed in more detail.

PHASE 1 (*Finite Horizon Non-stationary Dynamic Equilibrium*): *Given the initial stock vector* $\mathbf{S}_0$*, the exogenous vacant land rent series* R_{0t}*, t=0,1,2…T, all other exogenous vectors, matrices, sets B(k) and a vector of end-of-terminal-year asset prices*, $\mathbf{V}_{T+1}$, *a dynamic equilibrium* $[\mathbf{S}_t, \mathbf{V}_t, \mathbf{R}_t]_{t=0}^{T}$ *satisfies (8),(9) and (10)*:

(8) $$\sum_{h=1\ldots H} N_{ht} P_{hkt}(\mathbf{R}_t) - S_{kt} q_{kot}(R_{kt}) = 0\text{; } k=1,\ldots,K;\ t=0,1,\ldots,T.$$

(9) $$V_{kt} - \Omega_{kt}(\mathbf{V}_{t+1}) - \omega_{kt}(R_{kt}) = 0;\ k = 0,1,\ldots,K;\ t=0,1,\ldots,T.$$

(10) $$S_{kt+1} - \sum_{z=0,1,\ldots,K} \frac{1}{m_{zk}} S_{zt} Q_{zkt}(\mathbf{V}_{t+1}) = 0;\ k = 0,1,\ldots,K;\ t=0,1,\ldots,T.\ \square$$

PHASE 2 (*Infinite Horizon Stationary Dynamic* Equilibrium): *Removing the time subscripts, and letting* **R** *and* **V** *denote the rents and year-end asset prices respectively and letting* ***S*** *be the stationary stocks, the dynamic equilibrium conditions in the stationary state (t > T) are written as*:

(11) $$\sum_{h=1\ldots H} N_h P_{hk}(\mathbf{R}) - S_k q_{ko}(R_k) = 0\text{; } k=1,\ldots,K.$$

(12) $$V_k - \Omega_k(\mathbf{V}) - \omega_k(R_k) = 0;\ k = 0,1,\ldots,K.$$

(13) $$S_k - \sum_{z=0,1,\ldots,K} \frac{1}{m_{zk}} S_z Q_{zk}(\mathbf{V}) = 0;\ k = 0,1,\ldots,K.\ \square$$

Anas, Arnott and Yamazaki (2000) have used a concave mathematical programming approach to prove that the above two-phase problem has a unique solution and that this solution is a welfare maximum (i.e. Pareto efficient). We now turn to a sketch of the solution procedure. Note that there are *3K+2* equations for each $t < T$ and for the stationary state. These equations have a block recursive structure as shown in Figure 2. (8) (and (11)) are the *market clearing* conditions and state that the quantity of building units demanded equals the supply of building units offered for rent in each submarket and each year. Given some sequence of stocks, $[\mathbf{S}_t]_{t=0}^{T}$, these can be solved simultaneously for tentative equilibrium market-clearing rents $[\mathbf{R}_t]_{t=0}^{T}$. (9) are the *zero-profit* conditions or *asset bid-price* equations. They state that the asset prices are determined by competitive bidding such that zero *ex ante* economic profits accrue to each investor at the start of each year (or, equivalently, a normal expected rate of return equal to the exogenous interest rate, *r*, is earned.) Given the terminal asset prices $\mathbf{V}_{T+1}$ and the rents $[\mathbf{R}_t]_{t=0}^{T}$ from the previous step, equations (9) can be solved simultaneously for each *t* by backward recursion, for *t= T, T-1, T-2,…,1,0*, to find tentative equilibrium asset price series $[\mathbf{V}_t]_{t=0}^{T}$. (10) are the Markovian *stock adjustment* equations. For each *t*, there are *K+1* such equations, but one becomes redundant by the assumption that the total (vacant plus built-up) land is a time-invariant constant, *A*: $\sum_{z=0,1,\ldots,K} m_{0z} S_{zt} = A$.[13] Given the asset prices $[\mathbf{V}_t]_{t=1}^{T}$, (10) are solved by forward recursion for *t=1,…,T-2, T-1,T* to calculate new stocks $[\mathbf{S}_t]_{t=1}^{T}$. In an iterative solution algorithm, one revisits (8) with these new stocks, repeating the loop, (8)→(9)→(10)→(8)→… (see Figure 2). The process continues until (8), (9) and (10) are jointly satisfied for all *t*.

3. Comparison of the conventional tax with a tax on undeveloped land

It is helpful to examine the role of property taxation within the model presented in section 2. In order to do so, it is helpful to initially analyze a version of the model stripped-down to its bare essentials. Suppose for example that K=1 (only one type of building exists). Then the model simplifies to a simple housing-land cycle [as in Anas and Arnott, (1993a) or its neoclassical version in Anas and Arnott (1993b)]. In any one year, some buildings are constructed on vacant land (if the investor owning the land is shocked by a low construction cost) while other buildings are demolished to create vacant land (if the owner is shocked by a low demolition cost or a high maintenance cost). At stationary state, the flow of constructions must equal the flow of demolitions in each year so that the stock of vacant land and the stock of buildings is constant over time. Given this situation, which is the simplest version of the model, I can analyze the effects of a variety of tax schemes on the market. I will here examine two cases. One is the conventional ad-valorem property tax that is levied at the same rate on both assets (vacant land and buildings). The other is a tax on vacant land only. What are the effects of such taxes on the *land-to-building-to-land* conversion cycle? What are their effects on the rent for buildings and on the asset prices of land and of buildings?

3.1 The conventional property tax

With only one building type and only one land market the model equations simplify as follows. I also assume that all buildings are fully occupied (no vacancy rate). The tax is assumed paid at the beginning of each year:

(14) $$D(R_1) - S_1 = 0,$$

(15) $$(1+\theta)V_0 - \frac{1}{\Phi_0}\ln\left(e^{\Phi_0\frac{V_0 - C_{00}}{1+r}} + e^{\Phi_0\frac{V_1 - C_{01}}{1+r}}\right) - R_0 = 0,$$

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[13] Lemma 2 in Anas, Arnott and Yamazaki (2000)) proves this redundancy of one of (10) for each t.

(16) $$(1+\theta)V_1 - \frac{1}{\Phi_1}\ln\left(e^{\Phi_1\frac{V_1-C_{11}}{1+r}} + e^{\Phi_0\frac{V_0-C_{10}}{1+r}}\right) - R_1 = 0\,,$$

(17) $$S_0Q_{01} - S_1Q_{10} = 0\,,$$

(18) $$S_0 + S_1 - A = 0\,.$$

Let us review the notation. $D(R_1)$ is any downward sloping aggregate demand function for building units as a function of the rent for buildings. As explained earlier, R_0, the rent on vacant land, is exogenous and could be taken as zero without any loss of generality. A is the total amount of land available with S_0 and S_1 the stocks of buildings and vacant land respectively. The structural density of buildings is assumed to be unity and there is no loss of generality since there is only one building type. V_0 and V_1 are the asset prices for vacant land and for a unit building respectively. Q_{01} and Q_{10} are the construction and demolition probabilities respectively (namely, the probabilities that in any one year a unit building will be demolished releasing a unit amount of land and the probability that a unit amount of vacant land will be built on creating a unit building.) These probabilities are given by the following binary logit models:

(19) $$Q_{01} = \frac{e^{\Phi_0\frac{V_1-C_{01}}{1+r}}}{e^{\Phi_0\frac{V_1-C_{01}}{1+r}} + e^{\Phi_0\frac{V_0-C_{00}}{1+r}}} \quad \text{and} \quad Q_{10} = \frac{e^{\Phi_1\frac{V_0-C_{10}}{1+r}}}{e^{\Phi_1\frac{V_0-C_{10}}{1+r}} + e^{\Phi_1\frac{V_1-C_{11}}{1+r}}}\,.$$

It is useful to rewrite these as follows, by dividing numerator and denominator with the second exponential in the denominators:

(19′) $$Q_{01} = \frac{e^{\Phi_0\frac{V_1-V_0-(C_{01}-C_{00})}{1+r}}}{1+e^{\Phi_0\frac{V_1-V_0-(C_{01}-C_{00})}{1+r}}} \quad \text{and} \quad Q_{10} = \frac{e^{\Phi_1\frac{V_0-V_1-(C_{10}-C_{11})}{1+r}}}{1+e^{\Phi_1\frac{V_0-V_1-(C_{10}-C_{11})}{1+r}}}\,.$$

This second way of writing things shows explicitly that the probability of construction on vacant land is an increasing function of the difference between the asset price of a building and the asset price of land, and that the probability of demolition is an increasing function of the difference between the asset price of land and the asset price of a building. $C_{00}, C_{10}, C_{01}, C_{11}$ are the average (non-idiosyncratic) costs of vacant land maintenance, demolition, construction and unit building maintenance respectively. Φ_0 and Φ_1 are the dispersions of the idiosyncratic costs associated with land and buildings respectively. r is the interest rate and θ is the property tax rate.

Equation (14) is the market clearing condition insuring that the demand for buildings equals the stock. (15) and (16) are the after-tax asset price equations for land and buildings respectively. These equations state that assets (vacant land and buildings) are valued in such a way that investors make only a normal rate of return after taking into account taxes, rents and expected profits from future conversions. Equation (17) is the stationary state condition stating that the expected quantity of buildings that are demolished and the expected quantity of vacant land that is developed are equal. Finally (18) insures that buildings and vacant land do not exceed the given total land units, *A*.

Equations (14)-(18) comprise a five equation system that must be solved for R_1, V_0, V_1, S_0, S_1. This problem easily yields to conventional comparative static analysis. Appendix C provides the details. The key for deriving the result is to first show that $\frac{dR_1}{d\theta} > 0.$ It states that an increase in the ad-valorem tax rate increases the rent on buildings. Using this in equation (14), it follows that $\frac{dS_1}{d\theta} < 0$: the property tax results in a lower stock of buildings. Hence, from (18), $\frac{dS_0}{d\theta} > 0$ which establishes that more land is held vacant. Now note from (17) that $\frac{S_0}{S_1} = \frac{Q_{10}}{Q_{01}}$. Since the left side is increased by a higher θ, so the right side must increase also. It is also easy to establish the intuitive results that $\frac{dV_0}{d\theta} < 0$ and $\frac{dV_1}{d\theta} < 0$: the property tax reduces the value of vacant land and of buildings. More important for the result we are about to establish, $\frac{d(V_1 - V_0)}{d\theta} < 0$. This means that the tax increases the value of vacant land relative to the value of a unit building. Therefore the rate of construction falls and the rate of demolition rises as one can see by inspection of (19′) and this is consistent with the stock of vacant land increasing at the expense of the stock of buildings. Because the rate of demolition increases while the rate of construction falls, the average age of standing buildings falls. Hence, as pointed out earlier, the conventional property tax – operating in the intensive margin – speeds up the demolition-construction cycle shortening the life span of buildings while, in the extensive margin, the property tax reduces the building stock.

This finding enriches the conventional view of the property tax. It is widely recognized that the property tax increases the cost of structural capital relative to the cost of land. This fact has been widely touted by observing that developers would use less structural capital relative to land when constructing buildings on a given amount of land. But this perspective comes from models in which demolition and reconstruction are ignored. Our finding says that *ceteris paribus* buildings would not last as long with a conventional property tax than without it. Hence, because the conventional property tax encourages more demolitions and subsequent reconstruction, it causes an excessive use of structural capital over time. If this excessive use of capital over time in the intensive margin outweighs the reduced use of capital due to fewer buildings in the extensive margin (or due to lower structural density), then the property tax increases rather than decreases the total use of capital over time.

3.2 A tax on vacant land

Next we will assume that there is only one tax and it is levied on vacant land. This is same as the *ad-valorem tax on predevelopment land value* encountered in the literature. Note first, that this will change only equations (15) and (16). In the case of (16) the tax rate θ disappears (set it to zero) since buildings are not taxed. In (15) there is actually no change and θ is now replaced with θ_0 which now stands for the tax rate levied on vacant land value. The effects of this tax on vacant land value are the opposite of the effects of the conventional tax analyzed above. It is easy to grasp intuitively and easy to prove analytically that the tax decreases the stock of vacant land because it increases the cost of holding vacant land. Hence, the tax increases the stock of buildings: $\frac{dS_0}{d\theta_0} < 0$ and $\frac{dS_1}{d\theta_0} > 0$. Since the stock of buildings increases, rent falls: $\frac{dR_1}{d\theta_0} < 0$. Both building and vacant land asset prices fall: $\frac{dV_0}{d\theta_0} < 0$ and $\frac{dV_1}{d\theta_0} < 0$, but the value of a building is increased relative to that of land: $\frac{d(V_1 - V_0)}{d\theta_0} > 0$. Therefore the rate of construction rises and the rate of demolition falls. Thus, this unconventional tax on vacant land slows down the demolition-reconstruction cycle lengthening the life span of buildings. The average age of standing buildings increases. Hence, because this unconventional tax on vacant land encourages fewer demolitions and subsequent reconstruction, it discourages an excessive use of structural capital over time. The effects of the tax on vacant land are then the opposite of those of the conventional property tax.

4. An optimal taxation problem for real estate markets

How would the dynamic equilibrium formulation studied in Section 2 change in the presence of taxes on buildings and land?

Note first that the formulation captures most key variables that are active in real estate markets. It is possible, within this framework, to introduce a variety of tax/subsidy instruments on the following, for example: (a) rents; (b) asset prices; (c) costs of construction, demolition, maintenance and other conversions; (d) net revenues of investors in buildings and land (i.e. profit taxes); (e) option values. Furthermore, in each case, taxes can be lump-sum or ad-valorem. (For example, a lump sum tax becomes due when a particular conversion is made or an ad-valorem tax on conversion profits is levied.) Thus, it is possible – in principle – to pose a very general problem in which optimal tax policies are selected by picking and choosing from a large menu of such tax instruments that can be treated within the model.

Special assumptions that are built into the model will limit conclusions we may draw about the effects of some taxation schemes. For example, the model treats real estate consumers as having linear-in-income utility functions.[14] Hence, there would be no income effects from certain taxes/subsidies on consumers (such as income taxes). The same result holds on the supply side as well, because we have modeled investors as risk neutral.

To keep things simple, I will here focus only on ad-valorem taxes on asset prices. Thus, suppose that θ_{kt} is the tax rate on a type *k* asset (vacant land or building) in year *t*.

Then, each asset of type *k* has a tax-cost of $\theta_{kt} V_{kt}$. Equation (9), the asset-bid price (or asset valuation) equation now becomes modified by deducting the cost of the tax (assumed paid at the beginning of each year). This modified equation is:

$$(9') \qquad V_{kt} - \frac{1}{1+\theta_{kt}}[\Omega_{kt}(\mathbf{V}_{t+1}) + \omega_{kt}(R_{kt})]=0.$$

The revenue raised in year t is then $\sum_{k=0}^{K} \theta_{kt} V_{kt} S_{kt}$. We can now set up a welfare optimization problem over the infinite time horizon *t*=0,1,2…*T*, *T*+1, *T*+2,…∞, where *T* is the terminal time *T* of phase one.

$$(20) \quad Maximize\ \tilde{Z} = \sum_{t=0}^{T} \frac{1}{(1+r)^t} \sum_{h=1}^{H} N_{ht} \Psi_{ht}(\mathbf{R}_t) + \frac{1}{(1+r)^T r} \sum_{h=1}^{H} N_{hT} \Psi_{hT}(\mathbf{R}_T)$$
$$+ \sum_{k=0}^{K} V_{k0} S_{k0}$$

with respect to $[S_{0t}, S_{1t}, ..., S_{Kt}]$ for *t=1,…,T*, $[V_{0t}, V_{1t}, ..., V_{Kt}]$ for *t=0, 1,…,T,* $[R_{1t}, ..., R_{Kt}]$ for *t=0, 1,…,T* and $[\theta_{1t}, ..., \theta_{Kt}]$ for *t=0, 1,…,T* given all exogenous variables including the initial stocks S_{k0} for *k=0,…,K* subject to the constraints:

$$(8) \qquad \sum_{h=1...H} N_{ht} P_{hkt}(\mathbf{R}_t) - S_{kt} q_{kot}(R_{kt}) = 0;\ k=1,...,K;\ t=0, 1,...,T.$$

$$(9') \qquad (1+\theta_{kt})\, V_{kt} - \Omega_{kt}(\mathbf{V}_{t+1}) - \omega_{kt}(R_{kt}) = 0;\ k = 0,1,...,K;\ t=0, 1,...,T.$$

$$(10) \qquad S_{kt+1} - \sum_{z=0,1,...,K} \frac{1}{m_{zk}} S_{zt} Q_{zkt}(\mathbf{V}_{t+1}) = 0;\ k = 0,1,...,K;\ t=0,1,...,T.$$

$$(11) \qquad \sum_{h=1...H} N_{hT} P_{hkT}(\mathbf{R}_T) - S_{kT} q_{koT}(R_{kT}) = 0;\ k=1,...,K.$$

$$(12') \qquad (1+\theta_{kT})\, V_{kT} - \Omega_{kT}(\mathbf{V}_T) - \omega_{kT}(R_{kT}) = 0;\ k= 0;\ 1,...,K.$$

$$(13) \qquad S_{kT} - \sum_{z=0,1,...,K} \frac{1}{m_{zk}} S_{zT} Q_{zk}(\mathbf{V}_T) = 0;\ k= 0,1,...,K.$$

[14] It is easy to change utility functions and allow income effects. See Anas and Arnott (1993c, 1994))

$$(21) \quad \sum_{k=0}^{K} [\sum_{t=0}^{T} \frac{1}{(1+r)^t} \theta_{kt} V_{kt} S_{kt} + \frac{1}{r(1+r)^{T+1}} \theta_{kT} V_{kT} S_{kT}] - \Re = 0 .$$

The objective function, which measures social welfare, consists of the three additive terms. Of these, the first is the non-stationary consumer surplus series of phase one discounted to the initial point in time. The second is the stationary-phase (phase 2) consumer surplus series discounted to the initial point in time. Investors make zero profit in each year and, because this holds by the equations (9′) and (12′) of the dynamic market equilibrium, their profits need not be included in the social welfare function. However, the introduction of taxes will alter initial asset prices and, under the assumption that the tax policy is unanticipated, windfall gains or losses will accrue to asset holders at time t=0. Hence, the level of *initial* asset prices must be included as the third additive term in (20). The constraints (11), (12′) and (13) are the stationary market equilibrium conditions modified for taxes. They insure that the stationary state is a market equilibrium conditional on the tax rates. Similarly, (8), (9′) and (10) insure that the non-stationary market equilibrium also holds conditional on the tax rates. The last constraint (21) requires that a present value tax revenue of $\Re > 0$ is raised. Note that there is no restriction on the signs of the tax rates. Some can be negative (subsidies) while others are positive, but clearly at least one must be positive.

An alternative sub-optimal formulation would require that pre-specified revenue constraints must be met each year by a myopic tax planner: $\sum_{k=0}^{K} \theta_{kt} V_{kt} S_{kt} - \Re_t = 0$ for year t. The sub-optimality arises from the fact that the myopic tax planner cannot shift funds between periods but must balance his budget every period. This problem is considerably easier to solve because a present value budget need not be balanced across time periods. One could first solve the stationary state problem with the budget for that year imposed and would find the stocks, rents, asset prices and tax rates for the stationary state which maximized the stationary state's consumer surplus plus aggregate asset values. This would determine terminal asset prices conditional on terminal-year optimal taxes. Then one would begin a loop of backward-in-time and forward-in-time recursions to find stocks and asset prices together with tax rates for each year by maximizing that year's objective and still meeting the revenue target for that year. This solution procedure is similar to the one without taxes discussed earlier (see Figure 2) except that taxes are also calculated at each step.

A comment is in order here about welfare comparisons (deadweight loss measurement) for dynamic optimal taxation problems such as those discussed above. Figure 4 illustrates the key point. Consider first curve I, the path of the discounted welfare level each year when there is no tax policy in place. This welfare path is Pareto efficient. Now introduce any tax policy such as those discussed above. To capture the uses of tax revenue, the present value tax revenue $\Re$ can simply be added to the optimized value of the consumer surplus plus initial asset values since, with linear utilities and risk-neutral investors, the redistribution of tax revenue does not affect the present value welfare level. Clearly, the optimal tax policy is distortionary (has a deadweight loss). Therefore, curve II is either below curve I for each t, or – in the case shown in Figure 4 – curve II cuts curve I potentially several times (only once in the figure). In the case of the figure, area A must be larger than area B since the optimal tax policy must be distorting in present value terms. (An alternative possibility is that curve II starts above curve I but cuts it from above. In that case, the corresponding area B would have to be larger than the corresponding area A.) The case illustrated in the figure is interesting because the distortionary tax policy yields improved welfare sometime in the future and, in particular, in the eventual stationary state. This possibility appears counterintuitive at first. But the reason for it has to do with the fact that the path of the building stock is changed by the tax policy. At any point in time, the building stocks on curve I and curve II will not be the same. One can think up the following scenario that would be a real example of this. Suppose that the optimal tax policy calls for high taxes on vacant land relative to taxes on buildings. This increases the investor's cost of holding vacant land which, in turn, causes buildings to be built early on so that the stock on curve II eventually exceeds the stock on curve I for the corresponding later time periods. This creates an inefficient abundance of buildings later on, causing rents to be lower and consumer surplus to be higher. Although consumer surplus is eventually higher on curve II, we know that the optimal tax policy causes a distortion in present value terms. Hence, area B must be smaller than area A. The example of the figure illustrates the pitfalls of looking only at long run benefits, ignoring the transient benefits along the adjustment path. It underscores that policies must be compared in terms of net present value benefits.

5. Closing comments

The algorithm described in Appendix B has been designed to examine the effects of ad-valorem taxes on types of buildings and vacant land within the simulation model discussed in Section 2. It calculates the deadweight losses arising from such tax systems with exogenously specified tax rates. The optimal tax policy problem posed in Section 3 has not yet been implemented as an algorithm. When so implemented, it would allow direct derivation of the optimal tax policy and, hence, would allow us to investigate how optimal tax policies consisting of taxes on buildings and land should vary according to the circumstances of particular metropolitan land markets. Given that truly neutral and efficient taxes on land are controversial on the basis of equity or simply difficult to implement, it is important to be able to compare the efficiency of alternative tax systems.

Appendix A: The Multinomial Logit Calculus

A number of references are available detailing equivalent derivations of the multinomial logit model and the associated welfare measures: for example, McFadden (1974) or Anderson et. al. (1992). This Appendix follows the approach of the latter. Part (b), below, can also be proved by the approach of Small and Rosen (1981) who integrated the choice probability function to show that the expression in **(b)** is the (consumer or producer) surplus measure.

Derivation of the Multinomial Logit Model: *Suppose that the payoffs (utilities or profits) of i = 1,…,n discrete alternatives are measured by* $X_i + x_i$ *where each* $-\infty < x_i < +\infty$ *is distributed i.i.d. among the decision making agents according to the cumulative density G(•) of the Gumbel given by (1), with dispersion parameter γ, and mode* $\eta = -\frac{g}{\gamma}$ *(mean zero). Then*:

(a) *The probability that an agent most-prefers alternative i is* $\Pr(i) = \frac{\exp \gamma X_i}{\sum_{j=1}^{n} \exp \gamma X_j}$;

(b) $E[\max(X_i + x_i)] = \frac{1}{\gamma} \ln \sum_{j=1}^{n} \exp \gamma X_j$.

***Proof of (a)*:** The density function of (1) is $f(z) = \gamma[\exp-(\gamma z + g)]\{\exp-[\exp-(\gamma z + g)]\}$.
$\Pr(i) = \Pr ob.[X_i + x_i > \max_{\forall j \neq i}(X_j + x_j)] = \int_{-\infty}^{+\infty} f(x_i) \prod_{j \neq i} G(x_j \leq X_i - X_j + x_i) dx_i$. We evaluate this, using the substitutions $a_i = \exp[-(\gamma x_i + g)]$ and $b_j = \exp(\gamma X_j)$. Then,

$$\Pr(i) = \int_0^{\infty} \exp(-a_i) \prod_{j \neq i} \left[\exp\left(-\frac{a_i b_j}{b_i} \right) \right] da_i = \int_0^{\infty} \exp\left[-a_i \left(\sum_{j=1}^{n} \frac{b_j}{b_i} \right) \right] da_i$$

$$= \frac{-b_i}{\sum_{j=1}^{n} b_j} \left\{ \exp\left[-a_i \left(\sum_{j=1}^{n} \frac{b_j}{b_i} \right) \right] \right\}_0^{\infty} = \frac{\exp(\gamma X_i)}{\sum_{j=1}^{n} \exp(\gamma X_j)}. \blacksquare$$

***Proof of (b)*:** The procedure outlined in Anderson at. al. (1992) is to derive the cumulative density, $H(w)$, of the random variable $w = \max_{\forall i}(X_i + x_i)$ and then calculate its expected value as $E[w] = \int_{-\infty}^{+\infty} w H'(w) dw.$

$$H(w) = \prod_{i=1}^{n} G(x_i < w - X_i) = \prod_{i=1}^{n} \exp[-k \exp(-\gamma w + \gamma X_i)]$$

$= \exp[-kF \exp(-\gamma w)]$, where $k = \exp(\gamma \eta)$ and $F = \sum_{i=1}^{n} \exp(\gamma X_i)$. The density is, $H'(w) = \gamma k F H(w) \exp(-\gamma w)$. Before integrating we make the substitution $\overline{w} = \exp(-\gamma w)$.

Then, $E[w] = \frac{1}{\gamma}\int_{\infty}^{0}\exp\left(-kF\overline{w}\right)kF\ln\overline{w}\,d\overline{w} = -\frac{1}{\gamma}kF\int_{0}^{\infty}\exp\left(-kF\overline{w}\right)\ln\overline{w}\,d\overline{w}$. Rename $kF = s$ and use the Laplace transformation $\int_{0}^{\infty} e^{-st}\ln t\,dt = -\frac{\ln s + g}{s}$ to integrate. Then, $E[w] = \frac{1}{\gamma}\left(\ln kF + g\right) = \eta + \frac{g}{\gamma} + \frac{1}{\gamma}\ln\sum_{i=1}^{n}\exp\left(\gamma X_i\right) = \frac{1}{\gamma}\ln\sum_{j=1}^{n}\exp\left(\gamma X_j\right)$, because $\eta + \frac{g}{\gamma} = 0$. Note that $H(w) = \exp\left[-kF\exp\left(-\gamma w\right)\right]$ is a Gumbel with mode $\eta + \frac{1}{\gamma}\ln\sum_{j=1}^{n}\exp\left(\gamma X_j\right)$.■

The choice probabilities (2), (4) and (6) are obtained by applying the procedure of part (a). To get (2), the consumer choice probabilities, define $X_i \equiv U_{hit}$ and $x_i \equiv u_{hit}$, *i* = *1*,…,*K* for each *ht*. To get (4), the occupancy/vacancy probabilities, define $X_1 \equiv R_{kt} - D_{kot}$, $X_2 \equiv -D_{kvt}$, $x_1 \equiv -d_{kot}$, $x_2 \equiv -d_{kvt}$ for each *kt*. To get (6), the investor's conversion probabilities, define $X_i \equiv \rho\frac{V_{it+1} - C_{kit}}{m_{ki}}$ and $x_i \equiv -c_{kit}$, all $i \in B(k)$ and each *kt*. The expected values (3), (5) and (7) are derived by applying the procedure of part (b).

<u>Appendix B</u>: FORTRAN code for solving dynamic simulation model of real estate markets with taxes on buildings and land

Anas and Choi (2001) describes the FORTRAN code we have developed to solve dynamic real estate market problems described by equations (8)-(13), with exogenously specified tax rates. The algorithm is designed to solve problems conforming either to the commodity hierarchy cycle of Figure 1a or to the pattern of Figure 1b in which buildings differ according to structural densities. In each of these two cases, multiple land markets can be included, each land market containing potentially all building types.

Our algorithm first solves the Phase 2 stationary equilibrium given the exogenous variables and calibrated parameters. Then, the algorithm solves the first phase non-stationary dynamic equilibrium including an accurate time horizon truncation which determines *T*, the time at which the non-stationary phase converges to the stationary phase. The algorithm computes the deadweight losses of the pre-specified tax schemes.

Under a different operating option, the user specifies key data and key elasticities of demand and supply. Given these inputs, the dynamic simulation model calibrates itself and is then poised to perform simulations with these calibrated coefficients. A variety of exogenous inputs including taxes can be altered to explore their inter-temporal effects on the real estate market.

<u>APPENDIX C:</u> Comparative Static Analysis of the Effects of the Taxes

The comparative static analysis of (14)-(18) with respect to θ is as follows:

$$\begin{bmatrix} D'(R) & 0 & 0 & 0 & -1 \\ 0 & 1+\theta-\dfrac{Q_{00}}{1+r} & -\dfrac{Q_{01}}{1+r} & 0 & 0 \\ -1 & -\dfrac{Q_{10}}{1+r} & 1+\theta-\dfrac{Q_{11}}{1+r} & 0 & 0 \\ 0 & -B & B & Q_{01} & -Q_{10} \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} dR_1/d\theta \\ dV_0/d\theta \\ dV_1/d\theta \\ dS_0/d\theta \\ dS_1/d\theta \end{bmatrix} = \begin{bmatrix} 0 \\ -V_0 \\ -V_1 \\ 0 \\ 0 \end{bmatrix},$$

where $B = \left[S_0\Phi_0 Q_{01}Q_{00} + S_1\Phi_1 Q_{10}Q_{11}\right]/(1+r)$. The determinant of the system is:

$$DET = (-B)(1+\theta-\frac{1}{1+r}) + (Q_{10}+Q_{01})D'(R)\left[(1+\theta)(1+\theta-\frac{1}{1+r}) + \frac{Q_{10}(Q_{11}-Q_{01})}{(1+r)^2}\right] < 0,$$

assuming that $Q_{11} > Q_{01}$ or $Q_{10}+Q_{01} < 1$: the sum of the construction and demolition probabilities is not extremely large. Calculating the effect on rent we get:

$\dfrac{dR_1}{d\theta} = \left(\dfrac{B}{-DET}\right)(V_1 - V_0)\left(1+\theta-\dfrac{1}{1+r}\right) > 0$ assuming that a developed piece of land (land plus building) is worth more than an undeveloped one. From (14), it follows that $\dfrac{dS_1}{d\theta} < 0$ and from (18) that $\dfrac{dS_0}{d\theta} > 0$. Since, we have established that $\dfrac{d\frac{S_0}{S_1}}{d\theta} > 0$ it follows from (17) that $\dfrac{d\frac{Q_{10}}{Q_{01}}}{d\theta} > 0$. Hence, from (19′), we can see by inspection that $\dfrac{d(V_0-V_1)}{d\theta} > 0$ must be true. Next, solving for $\dfrac{dV_1}{d\theta}$ and $\dfrac{dV_0}{d\theta}$ we find that,

$$\frac{dV_0}{d\theta} = D'(R)\frac{Q_{01}+Q_{10}}{DET}\left[-V_0(1+\theta-\frac{Q_{11}}{1+r}) - V_1\frac{Q_{01}}{1+r}\right] + \frac{BV_0}{DET} < 0 \text{ and}$$

$$\frac{dV_1}{d\theta} = D'(R)\frac{Q_{01}+Q_{10}}{DET}\left[-V_1(1+\theta-\frac{Q_{00}}{1+r}) - V_0\frac{Q_{10}}{1+r}\right] + \frac{BV_1}{DET} < 0.$$ And from these:

$$\frac{d(V_0-V_1)}{d\theta} = \frac{D'(R)(Q_{01}+Q_{10})}{DET}\left(1+\theta-\frac{1}{1+r}\right)(V_1-V_0) > 0$$ which was inferred earlier.

The comparative statics with respect to the unconventional tax on land only follows a similar procedure and is intuitive given the above results. The matrix equation is:

$$\begin{bmatrix} D'(R) & 0 & 0 & 0 & -1 \\ 0 & 1+\theta_0-\frac{Q_{00}}{1+r} & -\frac{Q_{01}}{1+r} & 0 & 0 \\ -1 & -\frac{Q_{10}}{1+r} & 1-\frac{Q_{11}}{1+r} & 0 & 0 \\ 0 & -B & B & Q_{01} & -Q_{10} \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} dR/d\theta_0 \\ dV_0/d\theta_0 \\ dV_1/d\theta_0 \\ dS_0/d\theta_0 \\ dS_1/d\theta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -V_0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$DET = (Q_{01}+Q_{10})D'(R)\left\{\left[\frac{(1+\theta_0)(1+r)-Q_{00}}{1+r}\right]\left[\frac{1+r-Q_{11}}{1+r}\right]-\left(\frac{1-Q_{00}}{1+r}\right)\left(\frac{1-Q_{11}}{1+r}\right)\right\}+B\left(\frac{1}{1+r}-(1+\theta_0)\right)<0$$

$\frac{dR_1}{d\theta_0}=\left(\frac{B}{-DET}\right)(-V_0)\left(\frac{1}{1+r}-1\right)<0.$ From (14), it follows that $\frac{dS_1}{d\theta_0}>0$ and from (19) that $\frac{dS_0}{d\theta_0}<0$. Since, we have established that $\frac{d\frac{S_0}{S_1}}{d\theta_0}<0$ it follows from (18) that $\frac{d\frac{Q_{10}}{Q_{01}}}{d\theta_0}<0$. Hence, from (19′), we can see by inspection that $\frac{d(V_0-V_1)}{d\theta_0}<0$ must be true. Next, solving for $\frac{dV_1}{d\theta_0}$ and $\frac{dV_0}{d\theta_0}$ we find that,

$\frac{dV_0}{d\theta_0}=D'(R)\frac{Q_{01}+Q_{10}}{DET}(-V_0)\left(1-\frac{Q_{11}}{1+r}\right)+\frac{BV_0}{DET}<0$ and

$\frac{dV_1}{d\theta_0}=D'(R)\frac{Q_{01}+Q_{10}}{DET}(-V_0)\frac{Q_{10}}{1+r}+\frac{BV_0}{DET}<0$. And from these:

$\frac{d(V_0-V_1)}{d\theta_0}=\frac{D'(R)(Q_{01}+Q_{10})}{DET}\left(\frac{1}{1+r}-1\right)V_0<0$ which was inferred earlier.

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Time	State of information	Investor and consumer actions
t	$\mathbf{R}_t, \mathbf{V}_{t+1}$ are revealed. Investors know the expected values $\mathbf{C}_t$, $\mathbf{D}_t$ of conversion and maintenance costs and their dispersions $\mathbf{\Phi}_t$ and ϕ_t. Consumers earn their income $\boldsymbol{y}_t$ and know their taste premium values of housing submarkets $\mathbf{Y}_t$.	Risk neutral investors bid on housing units and land under perfect foresight on prices and under uncertainty about costs, determining asset prices $\mathbf{V}_t$, on the basis of $\mathbf{R}_t, \mathbf{V}_{t+1}, \mathbf{C}_t$, $\mathbf{D}_t$, $\mathbf{\Phi}_t$ and ϕ_t.
t+ε	Idiosyncratic maintenance costs $\mathbf{d}_t$ for vacancy and occupancy are revealed for each housing unit as a draw from the double exponential with dispersions ϕ_t. Idiosyncratic tastes $\mathbf{u}_t$ are revealed to consumers as draws from the double exponential with dispersions δ_t.	Investors decide, based on rents, $\mathbf{R}_t$, and revealed maintenance costs, $\mathbf{D}_t + \mathbf{d}_t$, whether to keep a unit vacant or let it to a tenant. Consumers choose to rent in the most-preferred submarket, on the basis of net income $\boldsymbol{y}_t - \boldsymbol{R}_t$, taste premia, $\mathbf{Y}_t$, and the revealed idiosyncratic utilities $\mathbf{u}_t$.
t+1−ε	Idiosyncratic conversion costs, $\mathbf{c}_t$, are revealed for each feasible $k \to k' \in B(k)$ conversion of a unit, as a draw from the double exponential with dispersions $\mathbf{\Phi}_t$.	Investors undertake the most profitable conversion on the basis of the revealed conversion costs $\mathbf{C}_t + \mathbf{c}_t$ and $\mathbf{V}_{t+1}$.
t+1	$\mathbf{R}_{t+1}, \mathbf{V}_{t+2}$ are revealed……	Risk neutral investors bid on housing units and land……

<u>TABLE 1</u>: Timeline indicating flow of information and actions of market agents within one year: from time $t \to t+1$. (<u>Note</u>: ε>0 is a very small constant.)

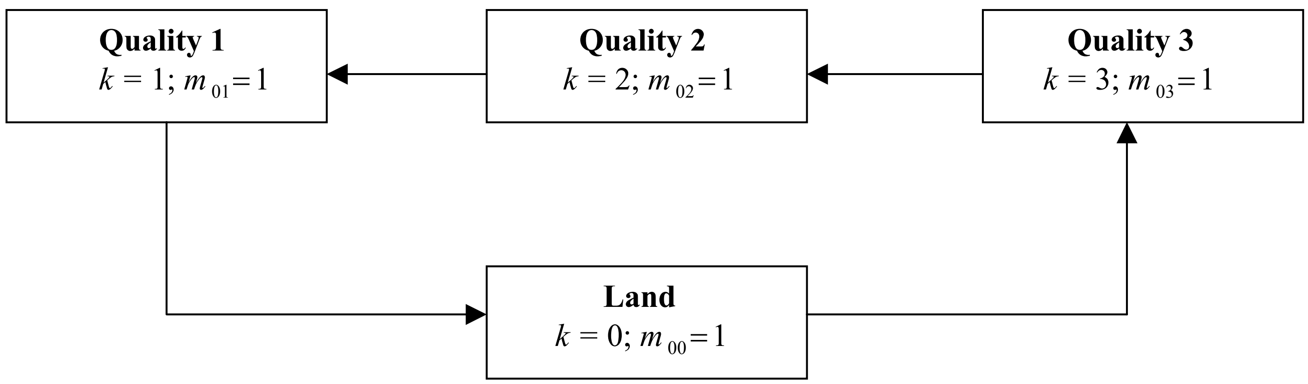

FIGURE 1a: Buildings are of the same structural density but differ only in quality. They deteriorate in quality, become demolished at the lowest quality and new housing is built at the highest quality. $B(0) = \{0,3\}$, $B(1) = \{0,1\}$, $B(2) = \{1,2\}$, $B(3) = \{2,3\}$.

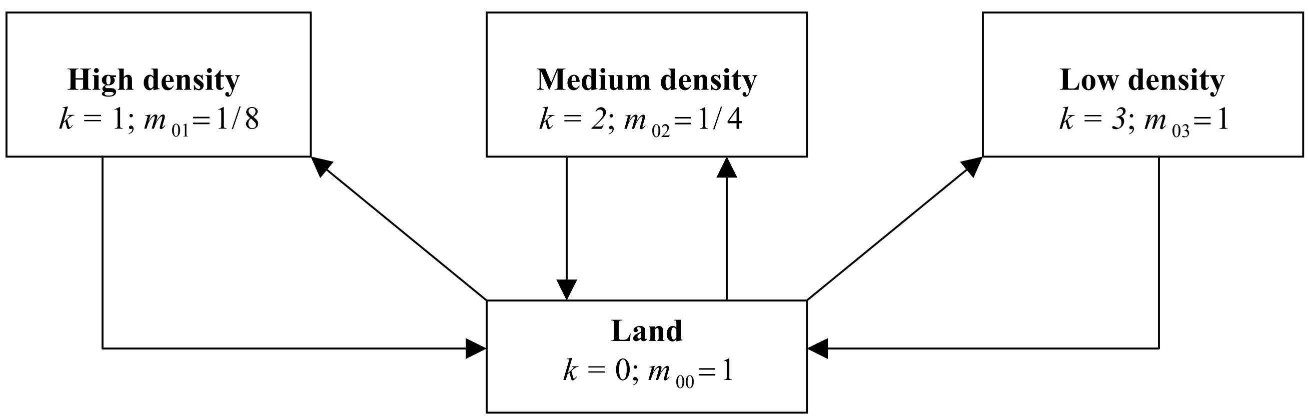

FIGURE 1b: Buildings do not deteriorate in quality. Buildings are of different structural densities that cannot be directly converted to one another. Each housing type can be demolished and any of the three types can be rebuilt in its place. $B(0) = \{0,1,2,3\}$, $B(1) = \{0,1\}$, $B(2) = \{0,2\}$, $B(3) = \{0,3\}$.

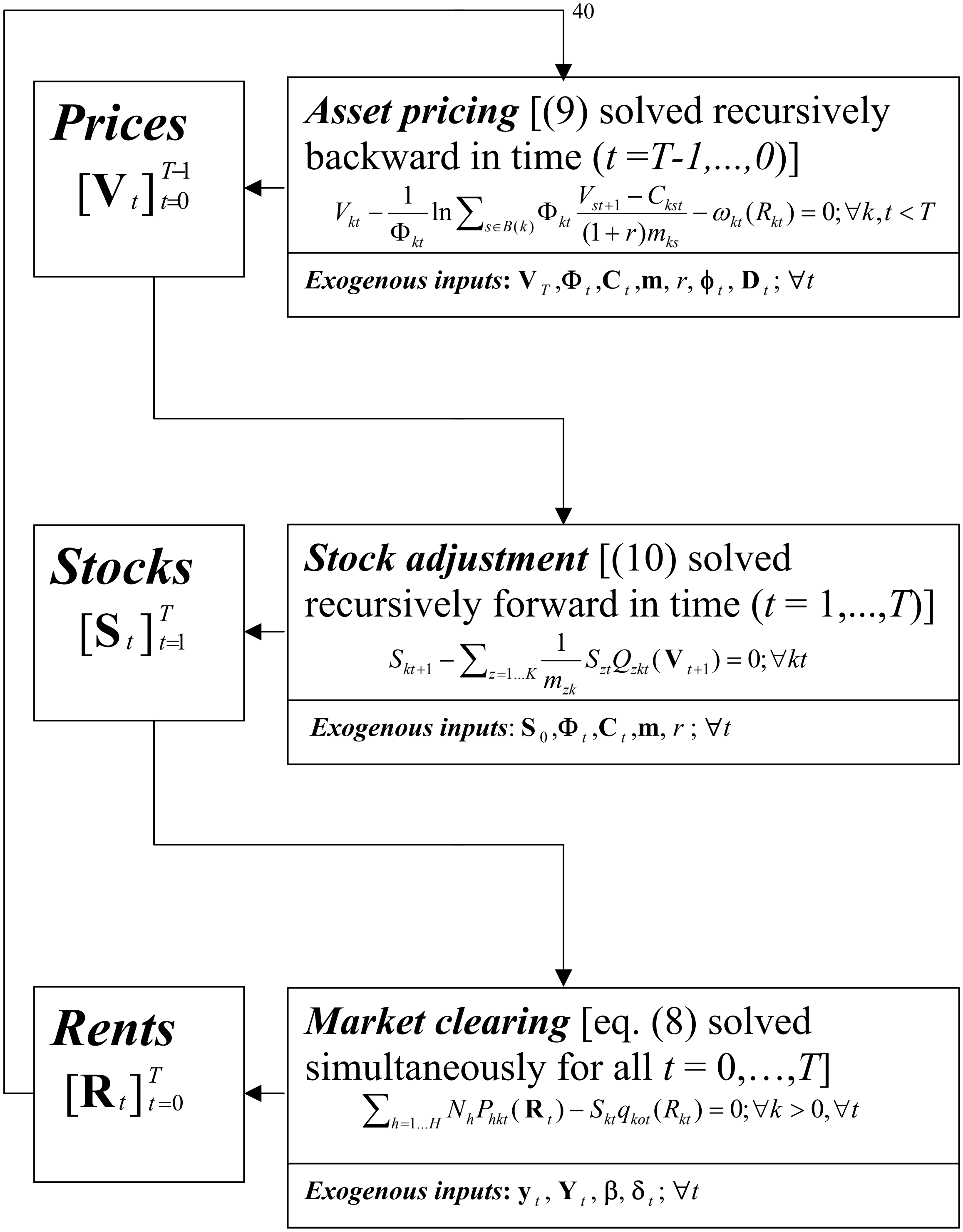

FIGURE 2: Block-recursive structure of dynamic housing market equilibrium.

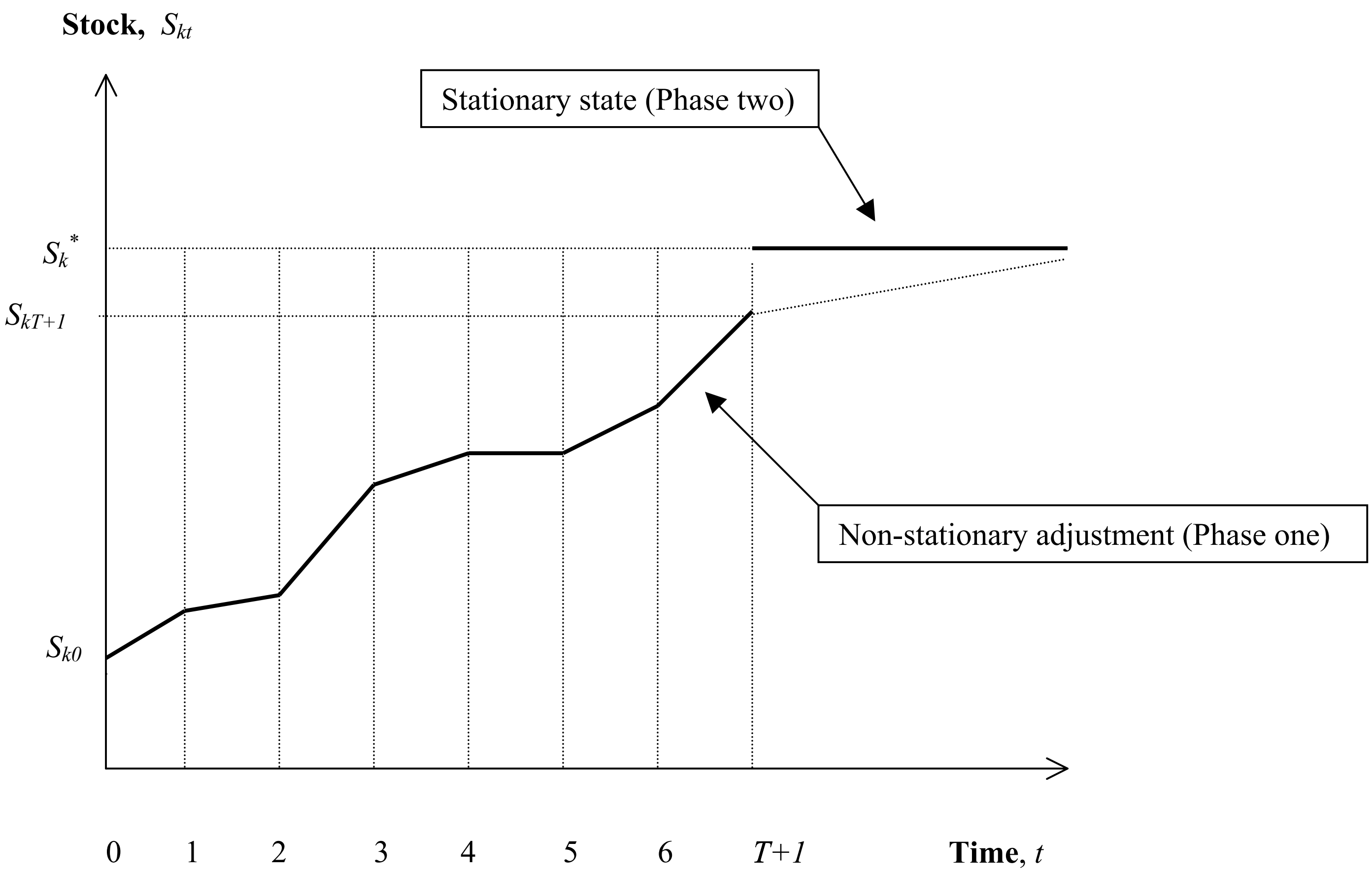

FIGURE 3: The terminal period gap in stock of type k with predetermined terminal period T, is $|S_k^* - S_{kT+1}|$, where S_k^* is the stationary equilibrium stock. The longer the choice of terminal time period, T, the smaller the gap becomes.

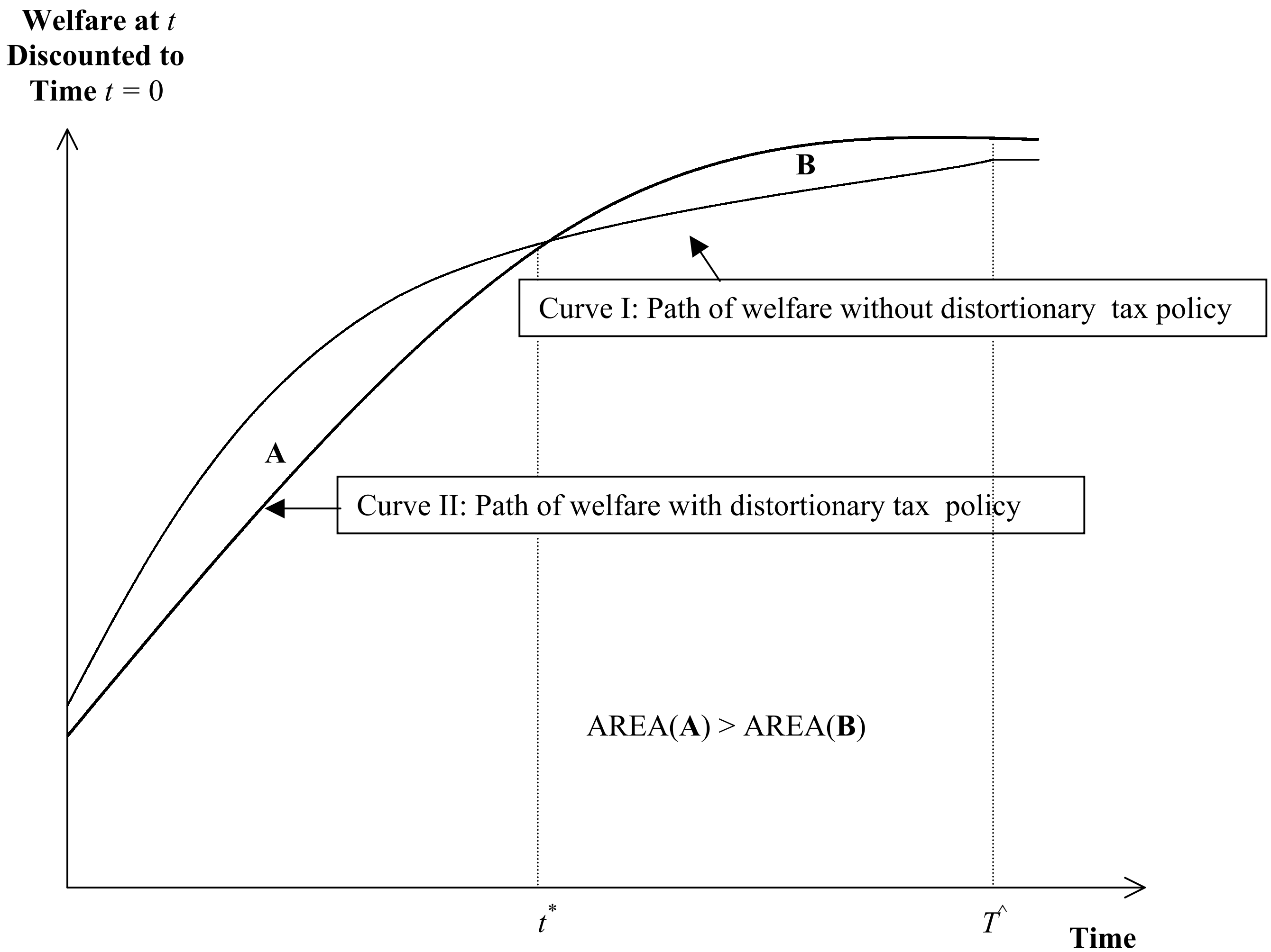

FIGURE 4: Negative net benefits of a distortionary tax policy in the dynamic model with convergence to stationary state at time *T*^. In this example, positive net benefits occur after time t^* including *T*^ at which time stationary state is reached. But these positive net benefits are outweighed by the negative net benefits which occur in $0 < t < t^*$.