The Indeterminacy of Equilibrium City Formation under Monopolistic Competition and Increasing Returns*

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Revised April 27, 2005

Abstract

We study the indeterminacy of equilibrium in the Fujita-Krugman (1995) model of city formation under monopolistic competition and increasing returns. Both the number and the locations of cities are endogenously determined. Assuming smooth transportation costs, we examine equilibria in city-economies where a finite number of cities form endogenously. For any positive integer K, the set of equilibria with K distinct cities has a smooth manifold of dimension

*The authors thank an anonymous referee, Alex Anas, Glen Ellison, Masahisa Fujita, Jang-Ting Guo, John Nachbar, Esteban Rossi-Hansberg, Norman Schofield, Jacques-François Thisse, and the participants at the North American Meetings of the Regional Science Association International 2001 in Charleston, the 2003 North American Winter Meeting of the Econometric Society in Washington, D.C., the 13th European Workshop on General Equilibrium Theory in Venice, and a seminar at the University of Kansas for comments, but retain responsibility for any errors.

$K - 1$ as its interior for almost all parameter values in a regular parameterization. The disjoint union of these sets over all positive integers K constitutes the entire equilibrium set.

Keywords: city formation, smooth economies, increasing returns, indeterminacy of city systems.
JEL classifications: D51, F12, O18, R12, R14.

1. INTRODUCTION

"Why do cities emerge?" This has been one of the central questions in urban economics. Tracing back to the nineteenth century, von Thünen describes a central township on an agricultural plain as the following. The town is where all manufactured goods are produced; it supplies its production to and imports agricultural goods from the surrounding rural area (Wartenberg 1966). This points to the concentration of manufacturing firms in cities.

The phenomenon of production concentration cannot be explained in the classical general equilibrium framework. Starrett's (1978) Spatial Impossibility Theorem says that if the space is homogeneous, transport is costly, preferences and production technologies are independent of location, and there is a competitive market for each good in every location, then there is no equilibrium involving transportation in a closed economy (see also Fujita and Thisse 2002). That is, in a competitive equilibrium, production activities spread out over space. Cities cannot be equilibrium outcomes unless one or more of these classical assumptions are dropped. Many authors explain why firms concentrate at a location with imperfect competition.[1] Cities emerge because the production of firms exhibits increasing returns to scale and, hence, they engage in monopolistic competition. Abdel-Rahman (1988), Abdel-Rahman and Fujita (1990), and Krugman (1991, 1993a,b) introduce the Dixit-Stiglitz (1977) model of monopolistic competition and increasing returns into spatial economics. This approach is known as the "New Economic Geography." Fujita and Krugman (1995) use it to study the emergence of a monocentric city on a line. They find the parameter range within which a monocentric equilibrium exists and examine the comparative statics. Fujita and Mori (1997) investigate how a monocentric city evolves into multicity systems as population increases.

[1] Another prominent factor is technological externalities, which is another source of market imperfection (for example, Henderson 1974).

This framework successfully provides the necessary ingredients to explain the formation of cities. There are two types of pecuniary externalities that generate the agglomerative forces sustaining a city, resulting in positive feedback that comes from firms locating near each other. First, manufacturing production will concentrate where there is a large market, and the market is large where there are many workers that consume manufactured goods. Second, workers will move to where production concentrates because the manufactured goods are cheaper there. In spite of success in providing insights into the agglomerative mechanisms of monopolistic competition and increasing returns, the model is far from fully explored. Although only equilibria with cities, where a positive measure of firms locate[2], have been examined, there may exist equilibria where densities of firms distribute over pieces of land. For example, equilibria may involve no cities (firms spread out as density distributions) or a mixture of both cities and density distributions of firms. These equilibria are difficult to define and analyze. In line with previous work, we focus attention on equilibria with cities only. The problem of multiple equilibria, however, remains. The model lacks determinacy and it is difficult to characterize the equilibrium set. Consequently, previous work resorts to studying special cases of equilibrium. Examples of a monocentric city, systems of duocentric and tricentric cities, the symmetric formation of odd numbers of cities, and even a one-dimensional continuum of equilibria are found in the literature. They are derived under the assumptions of a fixed number of cities or fixed locations of cities or both. These illustrative examples are insightful and suggestive; nevertheless, a broader picture of the equilibrium set is still absent. *What kind of object is it? How many dimensions does it have?* The determination of dimension is important since it reveals the degree of indeterminacy in the model, the number of free variables for computational work, and the validity of comparative statics. Of particular importance is the question of local uniqueness of equilibrium, a necessary condition for (differential) comparative statics. This paper attempts to answer these questions in a general framework where both the number and locations of cities are endogenously determined.

We adopt the differentiable approach introduced into economics by Debreu (1970, 1976) and Smale (1973, 1974), and summarized by Dierker (1974) and Mas-Colell (1985). In this approach, the equilibria of an economy are defined as solutions to a system of smooth ($\mathcal{C}^r$, $r > 0$) equations. The system is regularly parameterized[3] in a

[2] More precisely, a city is an atom in the distribution of firms.

[3] A parameter space is "regular" if the system's Jacobian matrix with respect to endogenous

finite dimensional Euclidean space. The generic dimension of the solution manifold is then determined by the numbers of equations and endogenous variables. We can then conclude that for almost all economies, the equilibrium set is a smooth manifold[4] of known dimension. Berliant and Zenou (2002) is a recent application in urban economics. It studies the generic dimension of the equilibrium set in a city formation model with labor differentiation.

Our technique differs from the established smooth economy literature in that our equilibrium is defined by a system of equations and inequalities. The presence of inequalities causes difficulties. Consequently, we construct an "extended regular parameterization" and obtain a weaker result. Our main theorem is the following. For any positive integer K, the set of equilibria with K distinct cities has a smooth (boundariless) manifold of dimension $K-1$ as its interior for almost all parameter values. This result shows a great deal of indeterminacy in the model;[5] the equilibrium set with a given number of cities is generically a continuum of high dimension. Moreover, the equilibrium set of a typical economy is the disjoint union of equilibrium sets with any number of cities. This problem of indeterminacy results from the lack of equilibrium conditions: there are not enough equations to pin down every variable. In this general equilibrium framework where all agents choose locations simultaneously, the market does not provide equilibrium conditions that determine the linkage among city locations. There is a growing literature on indeterminacy discovered in economic models recently: for example, financial markets, endogenous growth models, and games. Our work adds to this literature by showing real indeterminacy

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variables and parameters has full rank at every equilibrium for all parameter values (Mas-Colell 1985).

[4] A smooth manifold is a set of which every element has a neighborhood that can be mapped to a piece of a Euclidean space by a smooth bijection with a smooth inverse. Thus, the set behaves "locally" like a Euclidean space. Formally, M is an n-dimensional $\mathcal{C}^r$-manifold if there is an open cover $\{U_i\}_{i\in\Lambda}$ of M such that for each $i \in \Lambda$, there is a $\mathcal{C}^r$-diffeomorphism $\varphi_i : U_i \to \Re^n$ which maps U_i to an open subset of $\Re^n$.

[5] Krugman (1993a) addresses the indeterminacy of the equilibrium location of a single city due to the neutrality of agents with respect to spatial shifts. His model, however, does not show the seriousness of this problem. If we replace the unit interval used in his model with a circle of land, then all equilibria become identical and the indeterminacy is trivial. Even with multiple cities, this type of indeterminacy is a simple parallel spatial shift of the locations of all agents in all cities. It is eliminated in Fujita and Krugman (1995) by introducing the use of land and normalizing a location to the origin. The indeterminacy presented in our paper is not trivial, since one equilibrium cannot be derived from another through a spatial shift. Moreover, the degree of indeterminacy rises with the number of cities.

can also be found in spatial models. Another type of indeterminacy, discovered by Ellison and Fudenberg (2003), comes from the market impact a firm generates when it moves into a region or market. This effect is commonly ignored or assumed to be zero. Although it renders comparative statics difficult, indeterminacy in itself does not invalidate a model, as illustrated in the modern literature on general equilibrium with incomplete asset markets. Our view is that there is a great deal of diversity in the world, and models with indeterminacies could help explain it.

The paper is organized as follows. Section 2 presents a generalization of the standard model of the new economic geography (à la Fujita and Krugman 1995, Fujita and Mori 1997, Fujita, Krugman, and Venables 1999, and Fujita and Thisse 2002) with a richer set of parameters. We study equilibria when the number and locations of cities are endogenously determined. There is a countable number of cities, each with a positive measure of firms. Cities import the agricultural good from a connected piece of land and locate within this land segment. Section 3 presents the main theorem. Section 4 concludes and contains our conjecture about the source of indeterminacy in this model as well as a detailed discussion of equilibrium selection in relation to the work of Fujita, Krugman and Venables (1999).

2. THE MODEL

We begin by introducing the benchmark model, which is quite standard in the literature. We need more parameters to provide a regular parameterization. To facilitate comparison, these parameters will be added later. The economy has a space of locations $Z = \Re$. Each location r in Z is endowed with one unit density of homogeneous land. Land is used for agricultural production only. The utilized land is denoted by $B \subset Z$. There are two types of commodities: a homogeneous agricultural A-good and differentiated manufactured goods. There is a continuum of manufactured goods of size $n \in \Re_+$, which is determined endogenously. Each manufactured good is denoted by $j \in [0, n]$. A-good is produced by farms that employ labor and rent land, while j-goods are produced by firms that employ labor only. Firms do not use or occupy land. Let $C \subset Z$ denote the set of firm locations. The delivered prices of A-good and j-good at location $r \in Z$ are denoted by $p^A(r)$ and $p^M(j, r)$ respectively where $p^A : Z \to \Re_{++}$ and $p^M : [0, n] \times Z \to \Re_{++}$ are measurable functions.

Consumers

There is a continuum of identical mobile workers of Lebesgue measure $N \in \Re_{++}$ who receive wages for their labor. Each worker is endowed with one unit of labor. They can work for the agricultural or the manufacturing sector. Each worker supplies labor and consumes goods at a chosen location. There is a continuum of immobile landlords distributed uniformly over Z with density one. Each landlord owns one unit density of land where she lives. Landlords receive rent if their land is utilized. Consumers are denoted by $i \in \{W, L\}$; workers are denoted by W and landlords by L. The wage and land rent at location r are denoted by $w(r)$ and $R(r)$ respectively, where $w : Z \to \Re$ and $R : Z \to \Re$ are measurable functions.

Let A^i and $m^i(j)$ be, respectively, the quantities of A-good and j-good consumed by consumer i where $A^i \in \Re_+$ and $m^i : [0, n] \to \Re_+$ is measurable. All workers and landlords have the same utility function

$$u\left(m^i, A^i\right) = \left(M^i\right)^{\mu}\left(A^i\right)^{1-\mu},$$

where $M^i = \left[\int_0^n m^i(j)^{\rho}\, dj\right]^{\frac{1}{\rho}}$, $0 < \mu, \rho < 1$. They enjoy no utility from leisure. Given prices p^A and p^M, a consumer who lives at location r with income Y (wage or rent) solves the following problem.

$$\begin{array}{c} \underset{A^i,\, m^i(j) \in \Re_+}{Max} \quad u\left(m^i, A^i\right), \\ s.t.\ p^A(r)\, A^i + \int_0^n p^M(j,r)\, m^i(j)\, dj = Y. \end{array} \tag{1}$$

This optimization problem yields the following demand functions.

$$\begin{array}{c} \hat{A}^i(r) = (1-\mu)\, Y / p^A(r), \\ \hat{m}^i(j,r) = \mu Y G(r)^{\frac{\rho}{1-\rho}} / p^M(j,r)^{\frac{1}{1-\rho}}, \end{array}$$

where $G(r) = \left[\int_0^n p^M(j,r)^{\frac{\rho}{\rho-1}}\, dj\right]^{\frac{\rho-1}{\rho}}$ is the manufacturing price index.

Substituting Y with $w(r)$ and $R(r)$ respectively in these functions, we get the worker's demand, $\hat{A}^W(r)$ and $\hat{m}^W(j,r)$, and the landlord's demand, $\hat{A}^L(r)$ and $\hat{m}^L(j,r)$. A worker's indirect utility at r is

$$v(r) = \mu^{\mu}(1-\mu)^{1-\mu}\, w(r)\, G(r)^{-\mu}\, p^A(r)^{\mu-1}.$$

Workers are freely mobile. They choose locations that offer the highest utility level. Landlords at locations with negative rent will not rent out their land. To simplify the analysis, we assume they will rent it out when the rent is zero. Thus,

$$r \in B \text{ if and only if } R(r) \geq 0. \tag{2}$$

All landlords in $Z\backslash B$ receive zero income; they do not consume anything.

Firms

Firms produce differentiated products. Labor is the only input required. All firms have the same inverse production function

$$L^M = F^M + c^M q^M.$$

L^M units of labor are required for q^M units of output, where $F^M \in \Re_{++}$ and $c^M \in \Re_{++}$ are the fixed and the marginal input requirements respectively. The production technology exhibits increasing returns to scale due to the fixed cost. There is free entry into the market. Because of increasing returns to scale, each j-good is produced by and is the only product of an operating firm. Operating firms choose locations and engage in Chamberlinian monopolistic competition. Each firm chooses a location and charges a uniform free on board (f.o.b.) price for its product. Firms make decisions simultaneously. Suppose a firm locates at r, charges price p^M, and sells output $q^M\left(p^M, r\right)$, where $q^M : \Re_+ \times Z \to \Re$ is the consumers' demand and is known to the firm. Then, its profit is

$$\pi^M\left(P^M, r\right) = p^M q^M\left(p^M, r\right) - w\left(r\right)\left[F^M + c^M q^M\left(p^M, r\right)\right].$$

Given a fixed location r, a firm solves the following problem.

$$\underset{p^M \in \Re_{++}}{Max}\ \pi^M\left(r, p^M\right). \tag{3}$$

Because of the assumed constant elasticity utility function and the iceberg transportation cost, which will be introduced later, the elasticity of demand facing a firm is independent of the locations of its consumers. (This is widely known; see Fujita and Krugman 1995, and Fujita, Krugman and Venables 1999.) A monopolistically competitive firm charges a price marked up from the marginal cost. The optimal f.o.b. price for a firm at r is $\hat{p}^M\left(r\right) = c^M w\left(r\right)/\rho$. Its maximized profit is

$$\hat{\pi}^M\left(r\right) = \frac{(1-\rho)}{\rho} c^M w\left(r\right)\left[q^M - \frac{F^M \rho}{(1-\rho)\, c^M}\right].$$

Note that the total number of operating firms, which is the same as the total variety of products, is determined endogenously.

Farms

The agricultural good is produced by perfectly competitive farms with a constant input-output ratio. There is free entry into the market. One unit of A-good requires one unit of land and $c^A \in \Re_{++}$ units of labor. Each of a unit density of operating farms has the capacity to produce one unit density of A-good. Each farm at location r pays wage $w(r)$ to the workers and rent $R(r)$ to the landlord. Its profit is

$$\pi^A(r) = p^A(r) - c^A w(r) - R(r).$$

TRANSPORTATION

Transportation costs take the Samuelson iceberg form. If one unit of A-good (respectively j-good) is shipped from location s to location r, then $t^A(s,r)^{-1}$ (respectively $t^M(s,r)^{-1}$) units arrive. The function t^τ, $\tau \in \{A, M\}$, satisfies the following assumptions for all $s, r \in Z$: (i) t^τ is $\mathcal{C}^\infty$.[6] (ii) $t^\tau(s,r) \geq 1$, $t^\tau(s,s) = 1$, and $\lim_{|s-r|\to\infty} t^\tau(s,r) = \infty$, and (iii) $(s-r)\frac{\partial}{\partial s} t^\tau(s,r) > 0$, $(s-r)\frac{\partial}{\partial r} t^\tau(s,r) < 0$. Assumption (ii) says that there is a positive cost when a good is transported to another location and no cost if it is transported to the same location, and the transport cost approaches infinity if the distance between locations approaches infinity. Assumption (iii) requires the transportation cost to increase as the distance between locations increases.

THE CITY ECONOMY

We restrict attention to economies with a countable number of distinct cities. A city is defined to be where a positive measure of firms locate[7]. The number and locations of cities are endogenous. Suppose there are K (a positive integer) cities. The set of city locations is denoted by $C = \{c_k\}_{k=1}^K$ where $c_k \in Z$ for all $k \in \{1, ...K\}$ and $c_1 < ... < c_K$. The list $\{N_k\}_{k=1}^K$, $0 < N_k < N$ for all $k \in \{1, ...K\}$, denotes the worker populations in cities. The real number N_A, $0 < N_A < N$, denotes the worker population on agricultural land. Since in equilibrium a worker will supply all of her labor endowment, we set local population equal to labor supply for simplicity. Note that a city accommodates a worker population of a positive measure, while each rural location hosts one unit density of landlords and c^A units density of farm workers. The list $\{n_k\}_{k=1}^K$, where $n_k \in \Re_{++}$ for all $k \in \{1, ...K\}$, denotes the number

[6] Note that the widely used exponential function $t(s,r) = e^{\tau|s-r|}$ is not differentiable at $r = s$ since $\lim_{r\to s^+} \frac{t(s,r)}{r-s} = 1$ and $\lim_{r\to s^-} \frac{t(s,r)}{r-s} = -1$.

[7] The most general setting is to characterize the distribution of workers and firms with a measure possessing atoms. Although this generalized setting allows both a countable number of cities and a continuum of firms spread out over pieces of land of positive measure, the equilibria cannot be reduced to the solutions to a system of a finite number of equations with a finite number of unknowns.

of firms locating in cities. Thus the total number of firms (and the total variety of manufactured goods) is $n = \sum_{k=1}^{K} n_k$. Since consumers (respectively firms) are identical and their equilibrium behavior differs only with location, we relabel them with their locations. Firms and manufactured goods are labeled by their cities. Thus, we replace $p^M(j, r)$ with $p^M(c_k, r)$ and $m^i(j)$ with $m^i(c_k)$ for all j-goods manufactured in city c_k. The utility functions are changed to $u\left(\{m^i(c_k)\}_{k=1}^{K}, A^i\right) = (M^i)^{\mu}(A^i)^{1-\mu}$ where $M^i = \left(\sum_{k=1}^{K} n_k m^i(c_k)^{\rho}\right)^{1/\rho}$, and the manufacturing price index is changed to $G(r) = \left(\sum_{k=1}^{K} n_k p^M(c_k, r)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$. Let $A^i(r)$, where $i \in \{W, L\}$ and $A^i : Z \to \Re_+$ is a measurable function, denote the consumption of A-good of a consumer at r. Let $m^i(c_k, r)$, where $i \in \{W, L\}$ and $m^i : C \times Z \to \Re_+$ is a measurable function, denote the consumption of goods manufactured in city c_k of a consumer at r. The number $q_k^M \in \Re_{++}$ denotes the output of firms in city c_k.

To simplify the analysis, we prohibit A-good resale. Note that in equilibrium, each rural location (in $B \backslash C$) has a surplus of A-good: the total local consumption is $(1-\mu)c^A w(r)/p^A(r) + (1-\mu)c^A\left(p^A(r) - c^A w(r)\right)/p^A(r) = (1-\mu)$ while the total local production is 1. Thus, every rural location exports A-good to cities and only cities import A-good. We further restrict allocations to satisfy the following conditions. (i) The utilized land B is a closed, connected interval. Then, $\int_B c^A dr \le N$ implies B has a finite length. We normalize the utilized land $B = [0, \beta]$. (ii) Cities locate inside the utilized land. (iii) Moreover, each city imports A-good from an interval of agricultural land[8] exclusive of other cities. These conditions are formally specified as follows.

Condition α.
(i) $B = [0, \beta]$.
(ii) $C \subset (0, \beta)$.
(iii) *There are* K *disjoint intervals* $I_k = [b_{k-1}, b_k)$ *for* $k \in \{1, ..., K-1\}$ *and* $I_K = [b_{K-1}, b_K]$, *where* $b_0 = 0$, $b_K = \beta$ *and* $b_{k-1} < b_k$, *such that city* c_k *imports* A*-good from and only from* I_k.

These restrictions are needed, for otherwise the equilibrium pattern of land use is indeterminate. In theory, B can be a collection of isolated pieces of land, and each city may import A-good from disconnected land pieces. With the exponential

[8] This assumption is in accord with Fujita and Mori (1997), Appendix A.

transportation cost function $t(s,r) = e^{|s-r|}$, (i) and (iii) can actually be derived in equilibrium from transportation cost minimization and a "no cheaper A-supply" condition, which means a city cannot find a location that can supply A-good cheaper than from where it imports. To study the equilibrium patterns of land use in general, however, involves a great number of endogenous variables describing the lengths and locations of all agricultural areas. This seems to be intractable.

The extended model

We add the following parameters to the benchmark model. These parameters constitute a regular parameter space so that the dimension of the equilibrium set can be examined. There are different ways to extend the benchmark model to a regular parameterization and these parameters are just an example. In the extended model, the urban wage at c_k may differ from agricultural wage $w(c_k)$ at the same location. This is because cities do not occupy land, and the density of land at c_k is employed for agricultural production. Let w_k denote urban wages at c_k; $w(r)$ is reserved for agricultural wage.

(i) City-specific fixed input, $\nu \in \Re^K_{++}$. There are differences among cities that affect a firm's production function. They can be city specific transaction costs or the costs to use cities' infrastructure. For each firm in city c_k, the fixed labor input is $F^M + \nu_k$. This creates differences among firms in different cities. Hence, there is a separate profit function for firms in each city. With output q^M, the profit of a firm in city c_k is

$$\hat{\pi}^M_k(c_k) = \frac{(1-\rho)}{\rho} c^M w_k \left[q^M - \frac{\left(F^M + \nu_k\right)\rho}{(1-\rho)\, c^M} \right].$$

(ii) Immobile worker population, $l \in \Re^{K-1}_{++}$. There are two types of labor: immobile workers of size $\sum_{k=1}^{K-1} l_k$ and mobile workers of size $N - \sum_{k=1}^{K-1} l_k$. l_k denotes the population of immobile workers in city c_k for $k \in \{1, ..., K-1\}$. The total worker population of city c_k is $N_k + l_k$ for $k \in \{1, ..., K-1\}$. (For convenience, we define $l_K = 0$ in summations of city populations. We could add the parameter l_K to the model, but it is not needed to generate a regular parameterization.)

(iii) Urban amenity factor, $\gamma \in \Re^K_{++}$. Workers have preferences over either the natural advantages of a location (e.g. weather) or over some fixed man-made amenities (e.g. the symphony). If a worker lives in the rural area, her utility function is the same as the benchmark case. If she lives in city c_k, her utility level is factored up by $1/\gamma_k$. The new utility function of city workers is $\frac{1}{\gamma_k} u\left(\left\{m^W(c_k)\right\}_{k=1}^K, A^W\right)$. Note that γ does not affect consumers' demand; it plays a role in their location choices only. The indirect utility of an urban worker in city c_k is

$$v_k^M(c_k) = \frac{1}{\gamma_k}\mu^\mu (1-\mu)^{1-\mu} w_k G(c_k)^{-\mu} p^A(c_k)^{\mu-1}.$$

For simplicity, we assume the workers employed by a type c_k firm face the same urban amenity factor γ_k at a rural location (this prevents a firm from locating slightly away from the city and profiting from a discontinuous wage drop).

(iv) Land development cost, $\delta_1, \delta_2 \in \Re_{++}$. It takes a development cost to utilize the boundary and the idle land (e.g. putting up fences). Landlords in $(-\infty, 0]$ pay a fixed cost δ_1 to utilize their land, and landlords in $[\beta, \infty)$ pay a fixed cost δ_2 to utilize theirs. The development cost is deducted from landlords' rent income. So, the net income of a landlord at r is $R(r) - \delta_1$ for $r \in (-\infty, 0]$ and $R(r) - \delta_2$ for $r \in [\beta, \infty)$.

(v) Moving cost, $e \in \Re_{++}^K$. If a firm is to move from city c_k to $r \in Z \setminus \cup_{h=1}^K \{c_h\}$, then e_k units of labor at the new location is required for relocation. Note that e does not affect firms' production decisions; it plays a role only when a firm is considering relocation. In addition, we assume that the production function of a relocating firm does not change. With output q^M, the potential profit of a firm in city c_k relocating to $r \in Z \setminus \cup_{h=1}^K \{c_h\}$ is

$$\hat{\pi}_k^M(r) = \frac{(1-\rho)}{\rho} c^M w(r) \left[q^M - \frac{\left(F^M + \nu_k + e_k\right)\rho}{(1-\rho) c^M} \right].$$

(vi) Farm tax, $\lambda \in \Re_{++}$ and $\delta_1, \delta_2 \geq \lambda$. A uniform tax λ is levied on all operating farms in $(0, \beta)$. The tax revenue is then transferred to the landlords at the corresponding locations. So, the after-tax farm profit is $\hat{\pi}^A(r) = p^A(r) - c^A w(r) - \lambda - R(r)$, while the landlord receives rent $\bar{R}(r) = R(r) - \lambda$ plus the transfer.

Notice that if parameters of this extended model are chosen appropriately, the benchmark model is a special case. Let $\Theta \subset \Re_{++}^{4K+2}$ be a bounded, open parameter space with elements $\theta = (\nu, l, \gamma, \delta_1, \delta_2, e, \lambda)$. Thus, the 9-tuple $\left(N, \mu, \rho, F^M, c^M, c^A, t^A, t^M, \theta\right)$ represents a K-city economy. The benchmark model is parameterized at $\nu_k = 0$, $l_k = 0$, $\gamma_k = 1$, $\delta_1 = 0$, $\delta_2 = 0$, $e_k = 0$, and $\lambda = 0$. An *allocation* in a K-city economy is a 9-tuple

$$\left(\{c_k\}_{k=1}^K, \{N_k\}_{k=1}^K, \{b_k\}_{k=1}^{K-1}, \beta, N_A, \left\{A^i(r)\right\}_{i=W,L}, \left\{m^i(c_k, r)\right\}_{i=W,L}, \left\{q_k^M\right\}_{k=1}^K, \{n_k\}_{k=1}^K\right).$$

A *feasible allocation* satisfies the following constraints.

$$\sum_{k=1}^K N_k + \sum_{k=1}^{K-1} l_k + N_A = N. \tag{4}$$

$$\begin{array}{c}\sum_{k=1}^{K} m^{W}\left(c_h, c_k\right) t^{M}\left(c_h, c_k\right)\left(N_k+l_k\right) \\ +\int_0^{\beta}\left[m^{W}\left(c_h, r\right) c^{A}+m^{L}\left(c_h, r\right)\right] t^{M}\left(c_h, r\right) dr-q_h^{M}=0 \text{ for } h \in\{1, \ldots, K\} .\end{array} \tag{5}$$

$$\begin{array}{c}\int_{b_{k-1}}^{b_k}\left[1-c^{A} A^{W}(r)-A^{L}(r)\right] t^{A}\left(r, c_k\right)^{-1} dr=A^{W}\left(c_k\right) N_k, \\ \text{for all } k \in\{1, \ldots, K\} \text{ where } b_0=0,\, b_K=\beta .\end{array} \tag{6}$$

Equation (4) balances the total demand for workers and total worker population. Equation (5) balances the demand for each manufactured good and its supply. Equation (6) balances A-good exports and surplus at each rural location and balances each city's A-good demand and imports.

Equilibrium

Facing prices $\left\{p^{M}\left(c_k, r\right)\right\}_{k=1}^{K}$, $p^{A}(r)$, $w(r)$, $\left\{w_k\right\}_{k=1}^{K}$, and $R(r)$, the following conditions are satisfied in equilibrium:

Workers are freely mobile and identical, so their utility levels are the same wherever they locate.

$$\begin{array}{c}v(r)=v(0) \text{ for all } r \in[0, \beta], \\ v_k^{M}\left(c_k\right)=v(0) \text{ for } k \in\{1, \ldots, K\} .\end{array} \tag{7}$$

Because of free entry, if there are positive profits, new firms enter the market until the profits are brought down to zero. Thus, operating firms earn zero profits.

$$\hat{\pi}_k^{M}\left(c_k\right)=0 \text{ for all } k \in\{1, \ldots, K\} . \tag{8}$$

The location choices of firms constitute a Nash equilibrium. Together with free entry, it means the potential profit a firm can earn at any location outside cities is nonpositive.

$$\hat{\pi}_k^{M}(r) \leq 0 \text{ for all } r \in Z \backslash C . \tag{9}$$

Free entry will drive farms' profits to zero at any operating location.

$$\hat{\pi}^{A}(r)=0 \text{ for all } r \in[0, \beta] . \tag{10}$$

There is no arbitrage in the transportation sector. The transportation costs fully account for the price differences of a good at different locations. The transportation of a manufactured good is always from the producing firm's location (this is the only location that can export) to a buyer's location. The following condition determines the price of manufactured goods at different locations.

$$p^M(c_k, r) = \hat{p}^M(c_k)\, t^M(c_k, r) \text{ for all } k \in \{1, ..., K\}, r \in Z. \tag{11}$$

The transportation of agricultural good is determined by a list of city locations and A-supply intervals. Then, the price of the agricultural good at every location is determined by the no arbitrage condition. This is discussed in detail in Appendix A.

An *equilibrium* is a list of prices and a feasible allocation such that conditions (1), (2), (3), and (7) to (11) are satisfied.

The next lemma shows that in equilibrium, landlord income is zero at the two edges of the utilized land, and that the potential landlord income is negative on idle land.

Lemma 1. *In equilibrium,*

$$R(0) - \delta_1 = R(\beta) - \delta_2 = 0, \tag{12}$$

and $R(r) - \delta_1 < 0$ for $r \in (-\infty, 0)$, $R(r) - \delta_2 < 0$ for $r \in (\beta, \infty)$.

Proof: See Appendix C.

Thus, we replace the landlords' optimization condition (2) with (12) plus

$$R(r) - \lambda \geq 0 \text{ for all } r \in (0, \beta). \tag{13}$$

We examine equilibria with K distinct cities. The equilibrium set of a general city-economy is the disjoint union of K-equilibrium sets for all positive integers K. Before proceeding to the formal definition, we need to make some modifications. First, the prices belong to an infinite dimensional space. In order to determine the dimension of the equilibrium set, we fit the system into a finite dimensional Euclidean space by eliminating variables. Second, the price of A-good is not smooth. We approximate it with a smooth function in Appendix B.

Definition 1. A K-*equilibrium* is a list $\left(\{c_k\}_{k=1}^K, \{N_k\}_{k=1}^K, \{b_k\}_{k=1}^{K-1}, \{w_k\}_{k=0}^K\right) \in \Re_{++}^{4K}$ that satisfies the following system of equations and inequality constraints. (Note that $\beta = \frac{N - \sum_{k=1}^K N_k - \sum_{k=1}^K l_k}{c^A}$, $\bar{q}_k^M = \frac{(F^M + \nu_k)\rho}{c^M(1-\rho)}$; $G\left(c_k, \{w_k\}_{k=1}^K\right)$ and $\bar{Q}(r)$ are defined in Appendix B.)

$$\sum_{k=1}^{K} m^W (c_h, c_k)\, t^M (c_h, c_k)\, (N_k + l_k) + \int_0^{\beta} \left[m^W (c_h, r)\, c^A + m^L (c_h, r) \right] t^M (c_h, r)\, dr$$
$$-\bar{q}_k^M = 0 \text{ for all } h \in \{1, ..., K\}. \tag{14}$$

$$\int_{b_{k-1}}^{b_k} \left[1 - c^A A^W (r) - A^L (r) \right] t^A (r, c_k)^{-1}\, dr - A^W (c_k)\, (N_k + l_k) = 0,$$
$$\text{for all } k \in \{1, ..., K\} \text{ where } b_0 = 0,\ b_K = \beta. \tag{15}$$

$$\frac{w_k}{\gamma_k} - \frac{w_0 G\left(c_k, \{w_k\}_{k=1}^K\right)^{\mu} p^A (c_k)^{1-\mu}}{G\left(0, \{w_k\}_{k=1}^K\right)^{\mu}} = 0 \text{ for } k \in \{1, ..., K\}. \tag{16}$$

$$1 - c^A w_0 - \delta_1 = 0. \tag{17}$$

$$p^A (\beta) - c^A \frac{w_0 G (\beta)^{\mu} p^A (\beta)^{1-\mu}}{G (0)^{\mu}} - \delta_2 = 0. \tag{18}$$

$$c_1 < ... < c_K < \beta,\ N - \sum_{k=1}^{K} N_k > 0,\ b_1 < ... < b_{K-1} < \beta. \tag{19}$$

$$\bar{Q}^k (r) \leq 0 \text{ for all } r \in (-\bar{z}, \bar{z}) \setminus \cup_{h=1}^{K} \{c_h\} \text{ for all } k \in \{1, ..., K\}. \tag{20}$$

$$\bar{R} (r) \geq 0 \text{ for all } r \in (0, \beta). \tag{21}$$

Equations in (14) and (15) are linearly dependent because of Walras' law as presented in the next lemma. Label the lefthand side functions in (14) $f_1, ..., f_K$ and those in (15) $f_{K+1}, ..., f_{2K}$.

LEMMA 2. *In any economy, for any integer K,*

$$\sum_{k=1}^{K} n_k p^M (c_k)\, f_k - \sum_{k=1}^{K} p^A (c_k)\, f_{k+K} = 0.$$

Proof. See Appendix C. ∎

3. THE GENERIC DIMENSION OF THE EQUILIBRIUM SET

To determine the dimension of the equilibrium set defined by smooth equations, we need a regular parameterization where the system's Jacobian matrix with respect to endogenous variables and parameters has full rank at every equilibrium for all parameter values. Then, by the Transversality Theorem, the Jacobian matrix with respect to endogenous variables has full rank at every equilibrium for almost all parameter values. When it has full rank, the Implicit Function Theorem implies that the equilibrium set is a smooth manifold of dimension equal to the number of endogenous variables minus the number of equations. There are complications when applying these results to urban models. First, it is difficult to check regularity in the benchmark model. This is the reason why we add more parameters. Second, our system contains inequality constraints. Strict inequalities do not cause a problem. The presence of weak inequalities, however, changes the topological properties of the equilibrium set. To resolve this, we include weak inequalities as equalities and parameterize the augmented system of equations. We deal with Θ, a bounded open subset of the parameters. The results hold for the whole parameter space, that contains vectors of parameters of the form $\left(N, \mu, \rho, F^M, c^M, c^A, t^A, t^M, \theta\right)$, and any other regular parameterizations.

THEOREM 1. *For a regular parameterization Θ, the set of K-equilibria for an economy has a $\mathcal{C}^\infty$-manifold of dimension $K-1$ as its interior for almost all $\theta \in \Theta$.*[9]

Proof. See Appendix C. ∎

This raises the issue of indeterminacy: for a given number of cities, the equilibrium set is generically a continuum of rather high dimension. The equilibrium set of a general city economy is the disjoint union of K-equilibrium sets for all integers K. Existence of city equilibria is demonstrated in the literature for a wide range of parameters by solving out for them explicitly. Our result shows that there are many equilibria. To understand how far the equilibrium set extends beyond the manifold, we further investigate it in Appendix D and show that for almost all $\theta \in \Theta$, the set of K-equilibria for an economy is approximately[10] contained in the closure of its

[9] We use the following phrases interchangeably: "for almost all $\theta \in \Theta$," and "except for θ in a set of Lebesgue measure zero in Θ."

[10] We use the following notion of approximation. Let $f_i : X \to \Re^n$ and $\hat{f}_i : X \to \Re^n$ be families of continuous maps, where $i \in I_1 \cup I_2 \cup I_3$ and I_1, I_2, I_3 are finite. If sets $A = \{x | \, f_i(x) = 0, \forall i \in I_1; \, f_i(x) > 0, \forall i \in I_2; f_i(x) \geq 0, \forall i \in I_3\}$ and $B =$

interior.

The formation of symmetric cities draws much attention in the literature. By symmetry, we mean that there is a geographic center at $\beta/2$ and any parameters and endogenous variables on one side of the center has a mirror image on the other side. Symmetry reduces the difference between the numbers of endogenous variables and equations; this renders equilibrium sets of lower dimension.

Corollary 1. *When K is even (respectively odd), the set of symmetric K-equilibria has a $\mathcal{C}^\infty$-manifold of dimension $K/2$ (respectively $(K-1)/2$) as its interior for almost all parameter values in a regular parameterization.*

Proof. See Appendix C. ∎

When $K = 2$ or 3, the set of symmetric K-equilibria is generically one-dimensional. This is confirmed in Fujita and Mori (1997, Appendix F), where one-dimensional continua of equilibria are computed in cases of symmetric equilibrium with two and three cities. When $K = 1$, the equilibrium set is of dimension zero, which is the union of isolated points. Fujita and Krugman (1995) study symmetric equilibria with one city in the benchmark model and find a unique equilibrium, consistent with Corollary 1. We show, in the following example, that there is no asymmetric equilibrium with one city in the benchmark model. Therefore, the symmetric monocentric equilibrium is the unique equilibrium in this case, consistent with Theorem 1.

Example. Consider a city in the middle of linear agricultural land. The land stretches on both sides of the city. The transportation costs have the following form: $t^\tau(s,r) = \bar{t}^\tau(|s-r|)$. The functions $\bar{t}^\tau$ are determined by distance only, not by location.

Let $[0,\beta]$ denote the utilized agricultural land. Suppose there is an asymmetric equilibrium with the city at location c, $0 < c < \beta$, $c \neq \beta/2$. Without loss of generality, suppose the city is closer to β and the mirror image of β to c is α $(= 2c - \beta)$. Then $p^A(0) < p^A(\alpha) = p^A(\beta)$, $p^M(c,0) > p^M(c,\alpha) = p^M(c,\beta)$, and $G(0) > G(\alpha) = G(\beta)$. Note that $\bar{t}^\tau(|c-\beta|) = \bar{t}^\tau(|c-\alpha|)$. By $R(0) = 0 = R(\beta)$,

$\left\{x \mid \hat{f}_i(x) = 0, \forall i \in I_1;\ \hat{f}_i(x) > 0, \forall i \in I_2; \hat{f}_i(x) \geq 0, \forall i \in I_3\right\}$, then B *approximates* A if $\hat{f}_i$ approximates f_i in the C^0-Whitney topology for all $i \in I_1 \cup I_2 \cup I_3$.

we have $w(0) = p^A(0)/c^A$, $w(\beta) = p^A(\beta)/c^A$. Thus,

$$v(0) = \frac{\mu^\mu (1-\mu)^{1-\mu} p^A(0)^\mu}{c^A G(0)^\mu} < \frac{\mu^\mu (1-\mu)^{1-\mu} p^A(\beta)^\mu}{c^A G(\beta)^\mu} = v(\beta).$$

which violates (7). ∎

4. CONCLUSION

This paper takes a step further towards characterizing the equilibrium set with cities: its generic dimension is determined. The result reveals a great deal of indeterminacy in the model. The equilibrium set is generically the disjoint union of sets of arbitrarily high dimension. This restricts the model's predictive power, and differential comparative statics analysis is not valid.

There are at least three causes of indeterminacy in economic models. The first type is discussed in footnote 5 and is rather trivial; it involves parallel spatial shifts of variables and can be eliminated by normalization of the location of one city. The second type involves models whose parameterizations are not regular. Such examples can be found in game theory. Finally, there is indeterminacy even when one has a regular parameterization, caused by more unknowns than equations. This is the third type and where our work lies. In standard models with complete markets, the numbers of endogenous variables and equations are equal. In our model, as in models with incomplete asset markets, this is not true. There are not enough equations to pin down every variable. More precisely, there are four endogenous variables *for each city* ($k = 1, ..., K$): location, population, wage, and A-good supply interval, but only three equations: market clearance of city exports (manufactured goods), equal utility of workers, and market clearance of city imports (agricultural good). The use of agricultural land brings one extra condition independent of the number of cities and this is why the equilibrium set does not contain a K-dimensional continuum, but rather a $K-1$ dimensional manifold. As we will discuss below, this can be viewed as an indeterminacy in the locations of $K-1$ cities relative to the location of a given city whose location has been normalized.

It is natural to think of "remedies" for indeterminacy[11] that add more markets into the model; for example, we can add land markets in cities if firms employ land as

[11] We wish to remind the reader that as argued in the introduction, we do not think that indeterminacy in a model is necessarily undesirable.

an input. Such modifications, however, will not reduce indeterminacy since they add the same number of equations (market clearing conditions) and endogenous variables (prices).[12] Indeterminacy is persistent in competitive spatial economies as in other areas of economics. In many cases, we know that the most basic and commonly used model suffers from indeterminacy. The question of how model specification affects indeterminacy warrants further research: What general features of a model generate indeterminacy? How does the way in which firms compete, whether perfectly or not, affect it? How do other agglomerative factors, such as natural advantages, public goods, and production externalities, affect it?

Where does our indeterminacy come from? In games where an agent's strategy impacts their utility, the optimization conditions of an agent usually pin down their choice of strategy (given that we have a smooth system with a regular parameterization), so the number of endogenous variables is equal to the number of first order conditions. This is not true in the New Economic Geography. One agent's choice of strategy will have a negligible impact on the aggregates in a competitive economy. If there are not enough markets to pin down these aggregate variables, there is indeterminacy.

In our context, the New Economic Geography, these variables are the relative distances between the locations of cities. Due to the trivial indeterminacy caused by parallel spatial shifts, we can normalize the location of one city. This leaves $K-1$ variables, the locations of all the other cities, indeterminate, and not choice variables of any agent.

Determinacy might be obtained by sacrificing competitiveness. We need to allow some agents to choose the variables in question, but without adding any new endogenous variables to the system. One way to proceed is to make the number of firms finite. Then when one firm moves away from a location to another, its impact is non-negligible and it knows this. However, there will be problems with existence of equilibrium. Firm reaction correspondences might not be convex valued, as they could take on values at several locations. Land developers and governments can play roles similar to large firms.

The number of equilibria can be reduced if we restrict attention to equilibria that are stable with respect to a dynamic adjustment process that describes how equilibrium is reached. Adjustment dynamics define a system of equations that yields

[12] Masahisa Fujita points out that a few models with land in the cities still have a continuum of equilibria.

steady states coinciding with the static equilibria. A common example of adjustment dynamics used in urban economics can be found in Fujita and Mori (1997); Fujita, Krugman and Mori (1999); and Fujita, Krugman and Venables (1999), where population changes are proportional to the differences between local utility levels and the average utility level. We conjecture that except for a set of measure zero, our equilibria are all stable under this stability criterion. The exceptional set consists of parameters generating equilibria where firms' potential profit is exactly zero at some location outside of cities in equilibrium.

The morphogenesis approach developed by Alan Turing is another possible treatment for indeterminacy. The growth of an economy from a near flat distribution of firms is simulated in Krugman (1996) and Fujita, Krugman and Venables (1999). It generates surprising regularities and seems to be a very natural approach to resolving indeterminacy. There are evenly spaced clusters of firms, and the number of clusters is a divisor of the number of total sites. Since the distribution is decomposed into Fourier series and only one dominating frequency survives, the outcome will be evenly spaced clusters. In a circular land model with a continuum of locations, only integer frequencies can survive. This restricts cities to be at equally spaced locations on the circle. In a circular land model with a finite number of potential sites, the surviving frequency has to divide total number of sites. This means only frequencies that are factors of the total number of sites can be equilibrium outcomes. There is a need for less restrictive and less mechanical dynamics that can help us understand the evolution of cities and how one equilibrium is realized from the continuum of possibilities.[13]

Another interesting method for selecting equilibria is proposed by Fujita and Mori (1997) and Fujita, Krugman and Mori (1999). They simulate the evolution of a city system as population grows using an approach that emphasizes inertia and continuity. It has the following features: at any time, the system is stable with respect to migration; new cities appear near the frontier of agricultural land; old cities do not change locations unless the current locations cannot sustain any firms. The Fujita and Mori (1997) paper uses the same model as ours, and it is useful to contrast their results with ours.

[13]It is conceptually interesting to envision using Turing dynamics to examine stability of an equilibrium. However, it is apparent that an equilibrium with cities, or spikes in the density of agents, will remain stable under the kinds of perturbations to which the flat distribution is subjected in the cited literature.

Consider all equilibrium sets along the dimension of the population parameter. Our result characterizes a slice at a particular population value to be the union of, roughly speaking, manifolds (their precise structure is found in our Theorems 1 and 2). Manifolds of equilibria with the same number of cities are connected along the population dimension. The economic model itself yields no clear conclusion on which equilibrium will materialize for a given parameter value. Starting with a small population, Fujita and Mori (1997) select the symmetric monocentric city; it is always an equilibrium when the population is small. There are also other equilibria with different numbers of cities and different spatial patterns. As we move to larger population values, the monocentric city equilibrium is selected for every parameter until it becomes unstable in the sense that a small measure of firms can profit by moving to either of two frontier critical locations. At this parameter value, Fujita and Mori (1997) select an equilibrium where there are three stable cities, one each at the original and the two critical locations. There are other equilibria as well; for example, new cities can be at a positive distance from the critical locations, the central city can be at a different location, and there can be more than three cities at a stable equilibrium. This process traces a path inside the connected manifold of equilibria with one city and then jumps to a path inside the manifold of equilibria with three cities. The selection continues with cities fixed at original locations until old locations cannot sustain them as cities. Then two more new cities appear at critical locations. The traced equilibrium path passes through manifolds of one, three, five, and seven cities, and so on.

A more elaborate model is presented by Fujita, Krugman and Mori (1999), where there are three industries with different critical distances (the distance between the central city and the critical locations). Thus, when some firms of an industry move to frontier critical locations, there are still industries with all firms staying at central locations. When a frontier city emerges, it can change its location gradually to maintain stability. This new model generates an urban hierarchy where firms of an industry with the smallest critical distance appear in all cities, and firms of an industry with a larger critical distance appear in fewer and bigger cities. Bigger cities are also farther apart from each other.

Our results are general enough to apply to this hierarchy model as well (our theorems still work with more industries). The hierarchy generating process is also, as described above, a selection inside manifolds of equilibria with odd numbers of cities. At the same time, there are many possible stable equilibria. For example, the central industry can split into two cities.

This evolutionary approach displays a very interesting process of birth and death of cities. The urban hierarchy model captures important aspects of the historical development of US cities. It is, however, only one of many possible selection processes. Although the evolutionary approach is very insightful, there is still a need for an equilibrium selection method that might also satisfy Zipf's law.

APPENDIX A. Determination of the Agricultural Price

The direction of A-good transportation is determined by a list of city locations and A-supply intervals, $\left(\{c_k\}_{k=1}^{K}, \{b_k\}_{k=1}^{K-1}, \beta\right)$. The agricultural price $p^A(r)$ is then determined as follows: $p^A(r)$ has peaks at cities and troughs at the end points of A-supply intervals. When a city c_k is inside its A-good supply interval, A-good is transported to c_k from both sides: $p^A(c_k)$ is at a peak. When a left (respectively right) end point of an A-supply interval b_k is between its city and the next city to the left (respectively right), A-good is transported away from b_k on both sides: $p^A(b_k)$ is at the bottom of a trough. Also, b_0 and b_K are troughs since A-good will not be transported from outside $[0, \beta]$. To sum up,

$$c_k \text{ is a peak if and only if } b_{k-1} < c_k < b_k;$$
$$b_k \text{ is a trough if and only if } c_k < b_k < c_{k+1} \text{ or } k \in \{0, K\}.$$

Note that not every city is a peak, and not every end point of an A-supply interval is a trough. Let $\{b_l\}_{l=0}^{L}$ be the set of troughs. Between each pair of adjacent troughs b_{l-1} and b_l, there is a peak city c_l. The price of A-good at any location between b_{l-1} and b_l is determined relative to the peak city price $p^A(c_l)$ since all A-good is transported towards c_l. More precisely, by no arbitrage,

$$p^A(r) = p^A(c_l)\, t^A(r, c_l)^{-1} \text{ for } r \in [b_{l-1}, b_l) \text{ for } l = \{1, ..., L\},$$
$$p^A(\beta) = p^A(c_L)\, t^A(\beta, c_L)^{-1}.$$

Note that p^A has domain Z; yet, the formulae above determine $p^A(r)$ only for $r \in B$. To complete the determination of p^A over Z, we have to specify the "potential prices" on the idle land $Z \backslash B$. There are two potential prices at each location: one for the farms and one for the firms. First, a potential farm may rent at an idle location in $(-\infty, 0)$ and compete with farms at $r = 0$ by selling to c_1, or rent at an idle location in (β, ∞) and compete with farms at $r = \beta$ by selling to c_K. The potential supply price determined by no arbitrage is

$$p^A(r) = p^A(c_1)\, t^A(r, c_1)^{-1} \text{ for } r \in (-\infty, 0),$$
$$p^A(r) = p^A(c_K)\, t^A(r, c_K)^{-1} \text{ for } r \in (\beta, \infty).$$

Second, a firm may choose to locate on the idle land, and the workers' A-good consumption has to come from one of the edges of B. The potential demand price is

$$p^{A,M}(r) = p^A(0)\, t^A(0, r) \text{ for } r \in (-\infty, 0),$$
$$p^{A,M}(r) = p^A(\beta)\, t^A(\beta, r) \text{ for } r \in (\beta, \infty).$$

This price will be used to determine the potential wage at $r \in Z \backslash B$.

APPENDIX B. Reduction and Smoothing of the System

An equilibrium is a 14-tuple $\left(\begin{array}{c} p^M(c_k, r), p^A(r), w(r), \{w_k\}_{k=1}^K, R(r), \{c_k\}_{k=1}^K, \{N_k\}_{k=1}^K, \{b_k\}_{k=1}^{K-1}, \\ \beta, N_A, \{A^i(r)\}_{i=W,L}, \{m^i(c_k, r)\}_{i=W,L}, \{q_k^M\}_{k=1}^K, \{n_k\}_{k=1}^K \end{array}\right)$ that satisfies (1) and (3) to (13). We simplify the system by eliminating variables. This is possible because of the separability generated by the functional forms we use (also see Helpman and Krugman 1989). Given a list of city locations, populations, A-supply intervals, and wages, $\left(\{c_k\}_{k=1}^K, \{N_k\}_{k=1}^K, \{b_k\}_{k=1}^{K-1}, w(0), \{w_k\}_{k=1}^K\right) \in \Re_{++}^{4K}$, where $c_1 < ... < c_K < \beta$, $N - \sum_{k=1}^K N_k > 0$, $b_1 < ... < b_{K-1} < \beta$, and $\beta = \frac{N - \sum_{k=1}^K N_k - \sum_{k=1}^K l_k}{c^A}$, the rest of the endogenous variables can be determined uniquely as follows. First, by (4), $N_A = N - \sum_{k=1}^K N_k - \sum_{k=1}^K l_k$, and by (3), $\hat{p}^M(c_k) = c^M w_k / \rho$ for all $k \in \{1, ..., K\}$. Then, by (8), the zero-profit output level is $\bar{q}_k^M = \frac{(F^M + \nu_k)\rho}{c^M(1-\rho)}$ and $n_k = (N_h + l_h) / \left(F^M + \nu_k + c^M \frac{(F^M + \nu_k)\rho}{c^M(1-\rho)}\right) = \frac{(N_h + l_h)(1-\rho)}{F^M + \nu_k}$ for all $k \in \{1, ...K\}$. The prices of the manufactured goods are determined by (11):

$$p^M(c_k, r) = t^M(c_k, r)\, c^M w_k / \rho \text{ for all } k \in \{1, ..., K\}, r \in Z.$$

We normalize the A-good price at the origin: $p^A(0) = 1$. Then, $p^A(r)$ is determined by $\{c_k\}_{k=1}^K$ and $\{b_k\}_{k=1}^K$ as in Appendix A. $R(r)$ can be expressed as a function of $p^A(r)$ and $w(r)$ by (10): $R(r) = p^A(r) - c^A w(r)$ for $r \in (0, \beta)$. Next, $\{A^i(r)\}_{i=W,L}$ and $\{m^i(c_k, r)\}_{i=W,L}$ are determined by (1) as follows.

$$\begin{aligned} A^W(r) &= \hat{A}^W(r), A^L(r) = \hat{A}^L(r), \\ m^W(c_k, r) &= \hat{m}^W(c_k, r), m^L(c_k, r) = \hat{m}^L(c_k, r). \end{aligned}$$

The wage function $w(r)$ can be determined by a list of city wages, $\{w_k\}_{k=1}^K$, and the agricultural wage at $r = 0$, $w(0)$. To simplify notation, let $w(0) = w_0$. Note that $G(r)$ is a function of $\left\{\hat{p}^M(c_k)\right\}_{k=1}^K$ and hence a function of $\{w_k\}_{k=1}^K$. Expand $G(r)$ to show all arguments as $G\left(r, \{w_k\}_{k=1}^K\right)$. By (7) a consistent list $\{w_k\}_{k=0}^K$ satisfies the following K equations.

$$\frac{w_k}{\gamma_k} = \frac{w_0 G\left(c_k, \{w_k\}_{k=1}^K\right)^\mu p^A(c_k)^{1-\mu}}{G\left(0, \{w_k\}_{k=1}^K\right)^\mu} \text{ for } k \in \{1, ..., K\}. \tag{22}$$

Then, $w(r)$ in $[0, \beta]$ is determined by condition (7) as

$$w(r) = \frac{w_0 G\left(r, \{w_k\}_{k=1}^K\right)^\mu p^A(r)^{1-\mu}}{G(0)^\mu} \text{ for all } r \in [0, \beta].$$

The potential wage for farms at location s outside $[0, \beta]$ is determined by the same formula, whereas the potential wage for firms at location s outside $[0, \beta]$ is

$$w^M(s) = \frac{w_0 G\left(s, \{w_k\}_{k=1}^K\right)^\mu p^{A,M}(s)^{1-\mu}}{G(0)^\mu}.$$

An equilibrium is now reduced to $4K$ endogenous variables, $\left(\{c_k\}_{k=1}^K, \{n_k\}_{k=1}^K, \{b_k\}_{k=1}^{K-1}, \{w_k\}_{k=0}^K\right)$, that satisfy $3K+2$ equations, (5), (6), (12) and (22), and inequality constraints, $c_1 < ... < c_K < \beta$, $N - \sum_{k=1}^K N_k > 0$, $b_1 < ... < b_{K-1} < \beta$, (9) and (13). We replace (12) with the following two equations:

$$1 - c^A w_0 - \delta_1 = 0,$$

$$p^A(\beta) - c^A \frac{w_0 G(\beta)^\mu p^A(\beta)^{1-\mu}}{G(0)^\mu} - \delta_2 = 0.$$

The following lemma shows that firms have negative potential profits at locations far enough away from the utilized land $[0, \beta]$.

LEMMA 3. $\lim_{s \to \pm\infty} \hat{\pi}_k^M(s) < 0$.

Proof. Note that the profit of a firm at location s outside $[0, \beta]$ is

$$\hat{\pi}_k^M(s) = \frac{(1-\rho)}{\rho} c^M w^M(s) \left[Q^k(s) - \frac{\left(F^M + \nu_k + e_k\right)\rho}{c^M(1-\rho)}\right]$$

where $Q^k(s)$ is the demand for a type c_k manufactured good produced at s. Then $\hat{\pi}_k^M(s) \leq 0$, if and only if $Q^k(s) \leq \bar{q}_k^M + \frac{e_k \rho}{c^M(1-\rho)}$. Without loss of generality, suppose $s > \beta$. Each consumer at location r has demand $Q^k(s,r)$ for the firm's product.

$$
\begin{aligned}
Q^k(s,r) &= \hat{m}(s,r)\, t^M(s,r) = \mu Y(r)\, G(r)^{\frac{\rho}{1-\rho}}\, t^M(s,r) / p^M(s,r)^{\frac{1}{1-\rho}} \\
&= \frac{\mu Y(r) G(r)^{\frac{\rho}{1-\rho}}}{\left(\frac{c^M}{\rho} w(s)\right)^{\frac{1}{1-\rho}} t^M(s,r)^{\frac{\rho}{1-\rho}}} = \frac{\mu Y(r) G(r)^{\frac{\rho}{1-\rho}}}{\left(\frac{c^M w_0 p^A(\beta)^{1-\mu}}{\rho G(0)^{\mu}}\right)^{\frac{1}{1-\rho}} G(s)^{\frac{\mu}{1-\rho}} t^A(\beta,s)^{\frac{1-\mu}{1-\rho}} t^M(s,r)^{\frac{\rho}{1-\rho}}}
\end{aligned}
$$

Note that the potential wage $w^M(s)$ for s outside $[0,\beta]$ is determined by the manufacturing price index G and the A-good potential demand price $p^{A,M}(s)$.

$$
w^M(s) = \frac{w_0 G(s)^{\mu}\, p^{A,M}(s)^{1-\mu}}{G(0)^{\mu}} = \frac{w_0 p^A(\beta)^{1-\mu}}{G(0)^{\mu}} G(s)^{\mu}\, t^A(\beta,s)^{1-\mu}.
$$

Note that $\lim_{s\to\infty} t^A(\beta,s) = \infty$ and $\lim_{s\to\infty} t^M(s,r) = \infty$. Also, $\lim_{s\to\infty} G(s) = \lim_{s\to\infty}\left(\sum_{k=1}^{K} n_k \left(c^M w_k t^M(c_k,s)/\rho\right)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} = \infty$ since $\lim_{s\to\infty} t^M(c_k,s) = \infty$. Therefore, $\lim_{s\to\infty} Q^k(s,r) = 0$ and $\lim_{s\to\infty} Q^k(s) = 0 < \bar{q}_k^M + \frac{e_k\rho}{c^M(1-\rho)}$.

The case for $s < 0$ follows the same argument and $\lim_{s\to-\infty} Q^k(s) = 0 < \bar{q}_k^M + \frac{e_k\rho}{c^M(1-\rho)}$. ∎

Consequently, the potential locations for firms can be bounded in $(-\bar{z}, \bar{z})$ for a large $\bar{z} > 0$. Letting $\bar{Q}^k(r) = Q^k(r) - \left(\bar{q}_k^M + \frac{e_k\rho}{c^M(1-\rho)}\right)$, we replace condition (9) with

$$
\bar{Q}^k(r) < 0 \text{ for all } r \in (-\bar{z}, \bar{z}) \setminus \cup_{k=1}^{K} \{c_k\}.
$$

Given city locations and A-supply intervals $\left(\{c_k\}_{k=1}^{K}, \{b_k\}_{k=1}^{K-1}\right)$, the transportation cost t^A determines a p^A function that is not smooth. The following example illustrates this problem: Suppose there is a p^A trough at location r_2 and two peak cities at r_1 and r_3 where $r_1 < r_2 < r_3$. Then $p^A(r) = p^A(r_1)\left[t^A(r_2,r_1)\right]^{-1}$ for $r \in [r_1, r_2)$, and $p^A(r) = p^A(r_3)\left[t^A(r_2,r_3)\right]^{-1}$ for $r \in [r_2, r_3)$. Thus, $\lim_{r\to r_2^+} \frac{d}{dr} p^A(r) > 0$ and $\lim_{r\to r_2^-} \frac{d}{dr} p^A(r) < 0$; p^A is not differentiable at r_2.

There are two ways to fix this. We can approximate $p^A(r)$ by a $\mathcal{C}^\infty$ function. Let $C^r(M,N)$ denote the set of $\mathcal{C}^r$ maps from manifold M to manifold N, and $C_S^r(M,N)$ denote the strong topology on $C^r(M,N)$ (see Hirsch 1976). The following theorem shows such an approximation exists.[14]

[14] Similar smoothing techniques can be found in, for example, Mas-Colell (1974) and Kehoe (1980).

(Theorem 2.6, Hirsch 1976, ch.2): Let M and N be C^s manifolds, $1 \leq s \leq \infty$. Then $C^s(M,N)$ is dense in $C^r_S(M,N)$, $0 \leq r < s$.

Alternatively, we can define a "structure-dependent" transportation cost function. More precisely, we do not use a fixed transportation cost function but rather assume the function depends on the list $\left(\{c_k\}_{k=1}^{K}, \{b_k\}_{k=1}^{K-1}\right)$. $t^{\tau}(s,r)$ is determined only when city locations and A-supply intervals are determined. We write $t^A\left(s,r;\{c_k\}_{k=1}^{K}, \{b_k\}_{k=1}^{K-1}\right)$ and assume it to be smooth in $\left(s,r;\{c_k\}_{k=1}^{K}, \{b_k\}_{k=1}^{K-1}\right)$. Moreover, it needs to result in a smooth $p^A\left(r;\{c_k\}_{k=1}^{K}, \{b_k\}_{k=1}^{K-1}\right)$. The restriction on t^A is that it keeps $p^A(s,r)$ smooth in (s,r) when $\left(\{c_k\}_{k=1}^{K}, \{b_k\}_{k=1}^{K-1}\right)$ changes. At first glance, this type of transportation cost is not intuitive. But in the real world, transportation costs depend on the spatial structure. If a city appears at a location, the transportation costs in the vicinity are bound to change.

APPENDIX C. Proofs

Proof of Lemma 1. First, we prove $R(0) - \delta_1 = R(\beta) - \delta_2 = 0$. The landlord at 0 has a net income $R(0) - \delta_1 \geq 0$ since $0 \in B$. Suppose $R(0) - \delta_1 = \delta > 0$. For any small $\epsilon > 0$, the potential supply price is $p^A(0-\epsilon) = p^A(0)\, t^A(0,-\epsilon)^{-1}$. By (7), $w(r) = \frac{w(0)G(r)^{\mu}p^A(r)^{1-\mu}}{G(0)^{\mu}p^A(0)^{1-\mu}}$ and it is continuous in r since $p^M(c_k,r)$ is continuous in r for all c_k. So for any $\delta > 0$ there is an $\epsilon > 0$ small enough such that $\left|p^A(0-\epsilon) - p^A(0)\right| < \delta/2$ and $c^A\left|w(0-\epsilon) - w(0)\right| < \delta/2$. By free-entry, $R(r) = p^A(r) - c^A w(r)$. Thus,

$$
\begin{aligned}
R(0-\epsilon) - \delta_1 &= \left[p^A(0-\epsilon) - c^A w(0-\epsilon)\right] - \left[p^A(0) - c^A w(0)\right] + R(0) - \delta_1 \\
&= \left[p^A(0-\epsilon) - p^A(0)\right] - c^A\left[w(0-\epsilon) - w(0)\right] + \delta > 0
\end{aligned}
$$

But $0 - \epsilon \notin B$; this is a contradiction. The same argument applies to β.

Next, we show $R(r) - \delta_1 < 0$ for all $r \in (-\infty, 0)$. Note that $R(r) = p^A(r) - \frac{c^A w(0)G(r)^{\mu}p^A(r)^{1-\mu}}{G(0)^{\mu}p^A(0)^{1-\mu}} = p^A(r)\left(1 - \frac{c^A w(0)G(r)^{\mu}p^A(r)^{-\mu}}{G(0)^{\mu}p^A(0)^{1-\mu}}\right)$. Let $O(r) = \frac{c^A w(0)G(r)^{\mu}p^A(r)^{-\mu}}{G(0)^{\mu}p^A(0)^{1-\mu}}$; then, $\frac{d}{dr}O(r) = \frac{c^A w(0)}{G(0)^{\mu}p^A(0)^{1-\mu}}\left(\frac{d}{dr}G(r)^{\mu}p^A(r)^{-\mu} + G(r)^{\mu}\frac{d}{dr}p^A(r)^{-\mu}\right)$. For $r \in (-\infty, 0)$,

$$
\begin{aligned}
\tfrac{d}{dr}G(r) &= -\tfrac{1-\rho}{\rho}G(r)^{-\frac{1}{\rho}}\tfrac{d}{dr}\textstyle\sum_{k=1}^{K} n_k p^M(c_k,r)^{\frac{\rho}{\rho-1}} \\
&= -\tfrac{1-\rho}{\rho}G(r)^{-\frac{1}{\rho}}\textstyle\sum_{k=1}^{K} n_k \tfrac{-\rho}{1-\rho}p^M(c_k,r)^{\frac{1}{\rho-1}}\tfrac{c^M w(c_k)}{\rho}\tfrac{d}{dr}t^M(c_k,r) < 0,
\end{aligned}
$$

and $\frac{d}{dr}p^A(r) = p^A(c_1)\frac{d}{dr}t^A(r,c_1)^{-1} > 0$. This follows from $\frac{d}{dr}t^M(c_k,r) < 0$ and $\frac{d}{dr}t^A(r,c_1) < 0$ since $r < c_k$ for all k. So, $\frac{d}{dr}O(r) < 0$ for $r \in (-\infty, 0)$. Thus, $\frac{d}{dr}R(r) > 0$ and we have $R(r) - \delta_1 < 0$ for $r \in (-\infty, 0)$.

For $r \in (\beta, \infty)$, $\frac{d}{dr} t^M (c_k, r) > 0$ and $\frac{d}{dr} t^A (r, c_K) > 0$, since $r > c_k$ for all k. Thus, $\frac{d}{dr} G(r) > 0$, $\frac{d}{dr} p^A (r) < 0$, and $\frac{d}{dr} O(r) > 0$. Therefore, $\frac{d}{dr} R(r) < 0$ and $R(r) - \delta_2 < 0$ for $r \in (\beta, \infty)$. ∎

Proof of Lemma 2. Summing up the budget constraint of each consumer over $[0, \beta]$ including both rural and urban locations, we have

$$\begin{array}{c}\sum_{h=1}^{K} \int_{b_{h-1}}^{b_h} p^A (r) \left[A^L (r) + c^A A^W (r)\right] dr \\ + \sum_{h=1}^{K} \int_{b_{h-1}}^{b_h} \sum_{k=1}^{K} n_k p^M (c_k, r) \left[m^W (c_k, r) + c^A m^L (c_k, r)\right] dr \\ + \sum_{h=1}^{K} p^A (c_h) A^W (c_h) (N_h + l_h) + \sum_{h=1}^{K} \sum_{k=1}^{K} n_k p^M (c_k, c_h) m^W (c_k, c_h) (N_h + l_h) \\ - \sum_{h=1}^{K} \int_{b_h - 1}^{b_h} \left[c^A w (r) + R(r)\right] dr - \sum_{h=1}^{K} w_h (N_h + l_h) = 0.\end{array}$$

The first plus the fifth term is

$$\begin{array}{c} - \sum_{h=1}^{K} \int_{b_{h-1}}^{b_h} p^A (r) \left[1 - A^L (r) - c^A A^W (r)\right] dr \\ = - \sum_{h=1}^{K} p^A (c_h) \int_{b_{h-1}}^{b_h} t^A (r, c_h)^{-1} \left[1 - A^L (r) - c^A A^W (r)\right] dr. \end{array}$$

This plus the third term is

$$\begin{array}{c} - \sum_{h=1}^{K} p^A (c_h) \left[A^W (c_h) (N_h + l_h) - t^A (r, c_h)^{-1} \left[1 - A^L (r) - c^A A^W (r)\right] dr\right] \\ = - \sum_{h=1}^{K} p^A (c_h) f_{h+K}. \end{array}$$

The second plus the fourth term is

$$\begin{array}{c} \sum_{k=1}^{K} n_k \int_0^{\beta} p^M (c_k, r) \left[m^W (c_k, r) c^A + m^L (c_k, r)\right] dr \\ + \sum_{k=1}^{K} \sum_{h=1}^{K} n_k p^M (c_k, c_h) m^W (c_k, c_h) (N_h + l_h) \\ = \sum_{k=1}^{K} n_k p^M (c_k, c_k) \left[\begin{array}{c} \int_0^{\beta} t^M (c_k, r) \left[m^W (c_k, r) c^A + m^L (c_k, r)\right] dr \\ + \sum_{h=1}^{K} t^M (c_k, c_h) m^W (c_k, c_h) (N_h + l_h) \end{array} \right]. \end{array}$$

This plus the sixth term, which is

$$- \sum_{h=1}^{K} w_h (N_h + l_h) = \sum_{h=1}^{K} \frac{p^M (c_h, c_h) \rho}{c^M} \frac{\left(F^M + \nu_h\right) n_h}{1 - \rho} = - \sum_{h=1}^{K} n_k p^M (c_h, c_h) \bar{q}_h^M,$$

becomes

$$\sum_{k=1}^{K} n_k p^M (c_k, c_k) f_k.$$

Therefore,

$$\sum_{k=1}^{K} n_k p^M (c_k, c_k) f_k - \sum_{h=1}^{K} p^A (c_h) f_{h+K} = 0.$$

∎

Proof of Theorem 1. Before proving Theorem 1, we present a few lemmas. Let M be a boundariless $\mathcal{C}^{\infty}$-manifold of dimension m, (a,b) be an open interval in $\Re$, and $M \times \Re$ be endowed with the product topology. Let int denote interior taken in M. We have the following results.

LEMMA 4. *(Milnor 1965, Section 2, Lemma 3) If $g : M \to \Re$ is a $\mathcal{C}^{\infty}$-map with 0 as a regular value, then the set $\{x \in M | g(x) \geq 0\}$ is a $\mathcal{C}^{\infty}$-manifold of dimension m with $g^{-1}(0)$ as its boundary.*

LEMMA 5. *Suppose $g : M \times \Re \to \Re$ is continuous and there is $\bar{\epsilon} > 0$ such that $g(x,y) > 0$ for all $(x,y) \in [M \times (a, a+\bar{\epsilon})] \cup [M \times (b-\bar{\epsilon}, b)]$. Let $U = \{x \in M | g(x,y) \geq 0, \forall y \in (a,b)\}$ and $V = \{x \in M | g(x,y) > 0, \forall y \in (a,b)\}$. Then (i) U is closed and V is open in M. (ii) If g is $\mathcal{C}^{\infty}$ with 0 as a regular value, then $intU = V$.*

Proof. (i) Note that $U = \{x \in M | g(x,y) \geq 0, \forall y \in [a+\bar{\epsilon}/2, b-\bar{\epsilon}/2]\}$ and $V = \{x \in M | g(x,y) > 0, \forall y \in [a+\bar{\epsilon}/2, b-\bar{\epsilon}/2]\}$. Let $h(x) = \min_{y \in [a+\bar{\epsilon}/2, b-\bar{\epsilon}/2]} g(x,y)$. $h(x)$ is continuous by the Maximum Theorem since we minimize a continuous function $g(x,y)$ over y, constrained by a correspondence $\Phi(x) = [a+\bar{\epsilon}/2, b-\bar{\epsilon}/2]$ that is compact valued and continuous. Note that $U = \{x \in M | h(x) \geq 0\}$ and $V = \{x \in M | h(x) > 0\}$. Thus, U is closed and V is open by the continuity of h.

(ii) Obviously, $V \subset U$ and V is open, so $V \subset IntU$. We show that $IntU \subset V$. Take a point $\hat{x} \in IntU$. There is an open neighborhood $N_{\hat{x}} \subset M$ such that $x \in U$ for all $x \in N_{\hat{x}}$. This means for all $(x,y) \in N_{\hat{x}} \times (a,b)$, $g(x,y) \geq 0$ and $N_{\hat{x}} \times (a,b) \subset O = \{(x,y) \in M \times (a,b) | g(x,y) \geq 0\}$. We claim $g(\hat{x},y) > 0$ for all $y \in (a,b)$. Suppose not; there is $\bar{y} \in (a,b)$ such that $g(\hat{x},\bar{y}) = 0$. By Lemma 4, O is an $m+1$ dimensional $\mathcal{C}^{r}$-manifold with boundary $g^{-1}(0)$. And $(\hat{x},\bar{y}) \in g^{-1}(0)$. Since $(\hat{x},\bar{y})$ is on the boundary of O, any open set $S \subset M \times (a,b)$ such that $(\hat{x},\bar{y}) \in S$ contains both the inside and the outside of O. So, $S \cap \tilde{O} \neq \emptyset$ ($\tilde{O}$ is the complement of O). Since $N_{\hat{x}} \times (\bar{y}-\epsilon, \bar{y}+\epsilon)$ is an open set in $M \times (a,b)$ that contains $(\hat{x},\bar{y})$, $[N_{\hat{x}} \times (\bar{y}-\epsilon, \bar{y}+\epsilon)] \cap \tilde{O} \neq \emptyset$. Yet $N_{\hat{x}} \times (\bar{y}-\epsilon, \bar{y}+\epsilon) \subset N_{\hat{x}} \times (a,b) \subset O$; a contradiction. ∎

Let $x = \left(\{c_k\}_{k=1}^{K}, \{N_k\}_{k=1}^{K}, \{b_k\}_{k=1}^{K-1}, \{w_k\}_{k=0}^{K}\right) \in \Re_{++}^{4K}$. Label the left-hand side functions in (14) $f_1, ..., f_K$. Because of Walras' law, we take the last equation in (15) as redundant and label the left-hand side of the rest $f_{K+1}, ..., f_{2K-1}$. Label those in (16) $f_{2K}, ..., f_{3K-1}$, in (17) f_{3K}, and in (18) f_{3K+1}. Let $f = (f_1, ..., f_{3K+1})$; the equations above constitute the system $f(x, \theta) = 0$ where $f : \Re^{4K}_{++} \times \Theta \to \Re^{3K+1}$. For the analysis, we add an extra dimension $\Re$ to the domain of f, which does not affect its values. Let $\bar{f}(x, r, \theta) = f(x, \theta)$ with the dummy argument $r \in \Re$ where $\bar{f} : \Re^{4K}_{++} \times \Re \times \Theta \to \Re^{3K+1}$. Let $F_k = \bar{f}_k$ for $k \in \{1, ..., 3K+1\}$, $F_{k+3K+1} = \bar{Q}^k$ for $k \in \{1, ..., K\}$, and $F_{4K+2} = \bar{R}$. We have $F = \left(\bar{f}, \bar{Q}^1, ..., \bar{Q}^K, \bar{R}\right)$ where $F : \Re^{4K}_{++} \times \Re \times \Theta \to \Re^{4K+2}$. Next, we show that Θ is a regular parameterization; i.e., $D_\theta F$ has full rank at every equilibrium for all $r \in (-\bar{z}, \bar{z})$ and all $\theta \in \Theta$. This means $\left\{D_\theta \bar{f}_k\right\}_{k=1}^{3K+1}$, $\left\{D_\theta \bar{Q}^k\right\}_{k=1}^{K}$ and $D_\theta \bar{R}$ are always lineally independent at an equilibrium.

LEMMA 6. *$D_\theta F$ has rank $4K+2$ at every equilibrium x for all $r \in (-\bar{z}, \bar{z})$ for all $\theta \in \Theta$.*

Proof. (i) Differentiate F partially with respect to ν. $\partial \nu_k$ causes changes only in the output levels of firms in city c_k. The other endogenous variables do not change. So, only (14) and (20) are affected. Thus, $\frac{\partial F_k}{\partial \nu_k} = \frac{-\rho}{(1-\rho)c^M} \neq 0$ for all $k \in \{1, ..., K\}$, $\frac{\partial F_h}{\partial \nu_k} = 0$ for all $h \neq k$, $k, h \in \{1, ..., K\}$, and also $\frac{\partial F_h}{\partial \nu_k} = 0$ for all $h \in \{K+1, ..., 3K+1, 4K+2\}$, $k \in \{1, ..., K\}$. This means $D_\nu F_{(1,...,K)}$ is always a $K \times K$ diagonal matrix with nonzero elements.

(ii) Differentiate with respect to l. This does not affect prices, so $D_l F_{(2K,...,3K+1,4K+2)} = 0$. Moreover, $D_l F_{(K+1,...,2K-1)}$ is a $(K-1) \times (K-1)$ diagonal matrix with nonzero elements, since $\frac{\partial F_h}{\partial l_k} = -A^W(c_k) \neq 0$ in equilibrium for all $h = k+K$, $k \in \{1, ..., K-1\}$, and also $\frac{\partial F_h}{\partial l_k} = 0$ for all $h \neq k+K$, $h \in \{K+1, ..., 2K-1\}$, $k \in \{1, ..., K-1\}$.

(iii) Differentiate with respect to γ. Note that all w_k are held fixed, so γ does not affect prices. We have $\frac{\partial F_h}{\partial \gamma_k} = -\frac{w_k}{\gamma_k^2} \neq 0$ in equilibrium for all $h = k+2K-1$, $k \in \{1, ..., K\}$, and also $\frac{\partial F_h}{\partial \gamma_k} = 0$ for all $h \neq k+2K-1$, $h \in \{2K, ..., 3K-1\}$, $k \in \{1, ..., K\}$. Moreover, $D_\gamma F_{(1,...,2K-1,3K,3K+1,4K+2)} = 0$.

(iv) Differentiate with respect to δ_1 and δ_2. δ_1 affects (17) and δ_2 affects (18) only: $\frac{\partial F_{3K}}{\partial \delta_1} = -1$, $\frac{\partial F_{3K+1}}{\partial \delta_2} = -1$. None of the other functions changes.

(v) Differentiate with respect to e. Only $\bar{Q}^k$ are affected. We have $\frac{\partial F_h}{\partial e_k} = \frac{-\rho}{(1-\rho)c^M} \neq 0$ for all $h = k+3K+1$, $k \in \{1, ..., K\}$, and also $\frac{\partial F_h}{\partial \nu_k} = 0$ for all $h \neq k+3K+1$, $h \in \{3K+2, ..., 4K+1\}$, $k \in \{1, ..., K\}$. Moreover, $D_e F_{(1,...,3K+1,4K+2)} = 0$.

(iv) Differentiate with respect to λ. λ affects $\bar{R}$ only and $\frac{\partial F_{4K+2}}{\partial \lambda} = -1$. None of the other functions change.

Thus, $D_\theta F =$

$$\begin{pmatrix} D_\nu F_{(1,...,K)} & A2 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_l F_{(K+1,...,2K-1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_\gamma F_{(2K,...,3K-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{\delta_1} F_{3K} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{\delta_2} F_{3K+1} & 0 & 0 \\ A1 & A3 & A4 & 0 & 0 & D_e F_{(3K+2,...,4K+1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D_\lambda F_{4K+2} \end{pmatrix}$$

$D_\theta F$ always has rank $4K+2$ at an equilibrium since $A1$ to $A4$ can be eliminated by elementary operations and what remains is a diagonal matrix with nonzero elements. ∎

Next, we break down the equilibrium conditions into the following sets; the equilibrium set is the intersection of these sets. Let $E(\theta) = \left\{x \in \Re^{4K}_{++} | f(x, \theta) = 0\right\}$ denote the solution set to equalities (16), (17), (18), (14), and (15). Let

$$\begin{aligned} H_1(\theta) &= \left\{x \in \Re^{4K}_{++} | c_1 < ... < c_K < \beta,\ N - \sum_{k=1}^{K} N_k > 0,\ b_1 < ... < b_{K-1} < \beta\right\}, \\ E'(\theta) &= E(\theta) \cap H_1(\theta), \\ H_{2,k}(\theta) &= E'(\theta) \cap \left\{x \in \Re^{4K}_{++} | \bar{Q}^k(x, r, \theta) \le 0, \forall r \in (-\bar{z},\ \bar{z}) \setminus \cup_{k=1}^{K} \{c_k\}\right\}, \\ H_3(\theta) &= E'(\theta) \cap \left\{x \in \Re^{4K}_{++} | \bar{R}(x, r, \theta) \ge 0, \forall r \in (0, \beta)\right\}. \end{aligned}$$

Then, $E^*(\theta) = \cap_{k=1}^{K} H_{2,k}(\theta) \cap H_3(\theta)$ is the K-equilibrium set for parameter value θ.

The proof relies much on the following version of the Transversality Theorem (see Guillemin and Pollack 1974, p. 68, and Mas-Colell 1985, p. 320). A few definitions are in order. For a $\mathcal{C}^r$ map $f: M \to N$ between manifolds, $y \in N$ is a *regular value* if $Df(x)$ has full rank for all $x \in f^{-1}(y)$; and $y \in N$ is a *critical value* if not.

Transversality Theorem: *Suppose that $f: X \times S \to \Re^m$ is a $\mathcal{C}^r$ map where X, S are $\mathcal{C}^r$ boundariless manifolds with $r > \max\{0, \dim(X) - m\}$; let $f_s(x) = f(x, s)$, $f_s: X \to \Re^m$. If $y \in \Re^m$ is a regular value for f, then except for s in a set of measure zero in S, y is a regular value for f_s.*

By Lemma 6, $D_\theta f$ has full rank $3K+1$ at every equilibrium for all $\theta \in \Theta$; hence, $D_{(x,\theta)} f = (D_x f, D_\theta f)$ has full rank whenever $f(x,\theta) = 0$. So, 0 is a regular value of $f(x,\theta)$. Obviously, f is $\mathcal{C}^\infty$. The Transversality Theorem says that except for θ in a set of measure zero, $f_\theta(x) = f(x,\theta)$, where $f_\theta : \Re^{4K}_{++} \to \Re^{3K+1}$, has 0 as a regular value. The preimage of a regular value of f_θ is a $\mathcal{C}^\infty$-manifold of dimension $K-1$ (see Guillemin and Pollack 1974, p. 28; see also the Implicit Function Theorem in Mas-Colell 1985, p. 38). Therefore, $E(\theta) = f_\theta^{-1}(0)$ is generically a $\mathcal{C}^\infty$-manifold of dimension $K-1$. Moreover, $H_1(\theta)$ is an open subset of $\Re^{4K}_{++}$ and a $\mathcal{C}^\infty$-manifold of dimension $4K$ (codimension zero). It is transversal to $E(\theta)$ in $\Re^{4K}_{++}$, which has codimension $3K+1$ in $\Re^{4K}_{++}$. So, $E'(\theta) = E(\theta) \cap H_1(\theta)$ is generically a $\mathcal{C}^\infty$-manifold of codimension $3K+1$ in $\Re^{4K}_{++}$; this means it has dimension $K-1$. The next lemma shows that this manifold is bounded, which will be useful later.

LEMMA 7. *$E'(\theta)$ is bounded in $\Re^{4K}$.*

Proof. Note that for a given $\theta \in \Theta$, $\{c_k\}_{k=1}^K$, $\{N_k\}_{k=1}^K$, and $\{b_k\}_{k=1}^{K-1}$ are bounded by $H_1(\theta)$. We show that $\{w_k\}_{k=0}^K$ are bounded as well. For all $k \in \{1, ..., K\}$,

$$\frac{w_k}{w_0} = \frac{\gamma_k G(c_k)^\mu p^A(c_k)^{1-\mu}}{G(0)^\mu} = \frac{\gamma_k \left(\sum_{k=1}^K n_k \frac{c^M w_k}{\rho} t^M(c_k, r)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\mu} p^A(c_k)^{1-\mu}}{\left(\sum_{k=1}^K n_k \frac{c^M w_k}{\rho} t^M(c_k, 0)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\mu}}.$$

So, $\lim_{w_0 \to \infty} \frac{w_k}{w_0} > 0$, $\lim_{w_k \to \infty} \frac{w_k}{w_0} > 0$, and these limits are bounded. This implies if one of $\{w_k\}_{k=0}^K$ is unbounded, then all of them are unbounded. If all wages are unbounded, then workers' demand $A^w(r) = (1-\mu) w(r) / p^A(r)$ is unbounded. Since $\beta = \frac{N - \sum_{k=1}^K N_k}{c^A}$ is bounded, this violates (15). Therefore, $E'(\theta)$ is bounded in $\Re^{4K}$. ∎

We define the following functions for a given θ by restricting the domains of $\bar{Q}^k(., \theta)$ and $\bar{R}(., \theta)$ to be $E'(\theta) \times \Re$. Let $\bar{Q}^{*k}_\theta : E'(\theta) \times \Re \to \Re$ be such that $\bar{Q}^{*k}_\theta(x, r) = \bar{Q}^k(x, r, \theta)$ for all $(x, r, \theta) \in E'(\theta) \times \Re \times \Theta$. Let $\bar{R}^*_\theta : E'(\theta) \times \Re \to \Re$ be such that $\bar{R}^*_\theta(x, r) = \bar{R}(x, r, \theta)$ for all $(x, r, \theta) \in E'(\theta) \times \Re \times \Theta$. Note that

$$\begin{aligned} H_{2,k}(\theta) &= \left\{x \in E'(\theta) \,|\, \bar{Q}^{*k}_\theta(x, r) \leq 0, \forall r \in (-\bar{z}, \bar{z}) \setminus \cup_{k=1}^K \{c_k\}\right\}, \\ H_3(\theta) &= \left\{x \in E'(\theta) \,|\, \bar{R}^*_\theta(x, r) \geq 0, \forall r \in (0, \beta)\right\}. \end{aligned}$$

Next, we present an equivalent condition to $D_{(x,r)}\bar{Q}^{*k}_{\theta}(x,r)$ (respectively $D_{(x,r)}\bar{R}^{*}_{\theta}(x,r)$) having full rank, and show that generically 0 is a regular value of $\bar{Q}^{*k}_{\theta}(x,r)$ (respectively $\bar{R}^{*}_{\theta}(x,r)$).

LEMMA 8. *Given (x,r,θ) such that $D_{(x,r)}\bar{f}(x,r,\theta)$ has full rank, (i) $D_{(x,r)}\bar{Q}^{*k}_{\theta}(x,r)$ has full rank if and only if $D_{(x,r)}\bar{Q}^{k}(x,r,\theta)$ and $\left\{D_{(x,r)}\bar{f}_k(x,r,\theta)\right\}_{k=1}^{3K+1}$ are linearly independent; (ii) $D_{(x,r)}\bar{R}^{*}_{\theta}(x,r)$ has full rank if and only if $D_{(x,r)}\bar{R}(x,r,\theta)$ and $\left\{D_{(x,r)}\bar{f}_k(x,r,\theta)\right\}_{k=1}^{3K+1}$ are linearly independent.*

Proof. (i) Let $T_x(M)$ denote the tangent space of a manifold M at $x\in M$. $D_{(x,r)}\bar{Q}^{*k}_{\theta}(x,r)$ maps from $T_{(x,r)}(E'(\theta)\times\Re)$ to $\Re$. It has full rank if its range equals $\Re$. Therefore, we need $D_{(x,r)}\bar{Q}^{k}$ not to carry all elements in $T_{(x,r)}(E'(\theta)\times\Re)$ to zero. That is, there is $u\in T_{(x,r)}(E'(\theta)\times\Re)$ such that $D_{(x,y)}\bar{Q}^{k}(x,r,\theta)\,u\neq 0$. This is true if and only if $D_{(x,y)}\bar{Q}^{k}(x,r,\theta)$ does not belong to the orthogonal complement of $T_{(x,r)}(E'(\theta)\times\Re)$. Since $T_{(x,r)}(E'(\theta)\times\Re)=\left\{u\in\Re^{4K}_{++}\times\Re | D_{(x,r)}\bar{f}(x,r,\theta)\,u=0\right\}$, its orthogonal complement is the linear space spanned by $\left\{D_{(x,r)}\bar{f}_k(x,r,\theta)\right\}_{k=1}^{3K+1}$. This means $D_{(x,y)}\bar{Q}^{k}(x,r,\theta)$ is linearly independent of $\left\{D_{(x,r)}\bar{f}_k(x,r,\theta)\right\}_{k=1}^{3K+1}$. (ii) The argument for $D_{(x,r)}\bar{R}^{*}_{\theta}(x,r)$ is exactly the same. ∎

LEMMA 9. *For almost all $\theta\in\Theta$, $\bar{Q}^{*k}_{\theta}(x,r)$ and $\bar{R}^{*}_{\theta}(x,r)$ have 0 as a regular value.*

Proof. Let $G^k=\left(\bar{f}_1,...,\bar{f}_{3K+1},\bar{Q}^k\right):\Re^{4K}_{++}\times\Re\times\Theta\to\Re^{3K+2}$. $D_{\theta}G^k$ always has full rank $3K+2$ at every equilibrium for all $r\in(-\bar{z},\,\bar{z})$ and all $\theta\in\Theta$ by Lemma 6. Hence, $D_{(x,r,\theta)}G^k=\left(D_{(x,r)}G^k,D_{\theta}G^k\right)$ has full rank whenever $G^k(x,r,\theta)=0$, and 0 is a regular value of $G^k(x,r,\theta)$. By the Transversality Theorem, $G^k_{\theta}:\Re^{4K}_{++}\times\Re\to\Re^{3K+2}$, where $G^k_{\theta}(x,r)=G^k(x,r,\theta)$, has 0 as a regular value except for θ in a set of measure zero. This means for almost all $\theta\in\Theta$, for all (x,r) such that $G^k_{\theta}(x,r)=0$, $D_{(x,r)}G^k(x,r)$ has full rank, and this means $D_{(x,r)}\bar{Q}^k(x,r,\theta)$ and $\left\{D_{(x,r)}\bar{f}_k(x,r,\theta)\right\}_{k=1}^{3K+1}$ are linearly independent. Note that $G^k_{\theta}(x,r)=0$ if and only if $x\in E'(\theta)$ and $\bar{Q}^k(x,r,\theta)=0$. So, by Lemma 8, $D_{(x,r)}\bar{Q}^{*k}_{\theta}(x,r)$ has full rank whenever $\bar{Q}^{*k}_{\theta}(x,r)=0$ for almost all θ. Letting $G^R=\left(\bar{f}_1,...,\bar{f}_{3K+1},\bar{R}\right)$, the argument for $\bar{R}^{*}_{\theta}(x,r)$ follows in the same way. ∎

Next, we proceed to show that the equilibrium set $E^{*}(\theta)$ has a $K-1$ dimensional manifold as its interior. Let

$$
\begin{aligned}
H_{2,k}^{0}(\theta) &= \left\{ x \in E'(\theta) \,|\, \bar{Q}_{\theta}^{*k}(x,r) < 0, \forall r \in (-\bar{z},\ \bar{z}) \setminus \cup_{k=1}^{K} \{c_k\} \right\}, \\
H_{3}^{0}(\theta) &= \left\{ x \in E'(\theta) \,|\, \bar{R}_{\theta}^{*}(x,r) > 0, \forall r \in (0,\beta) \right\}.
\end{aligned}
$$

LEMMA 10. *For almost all $\theta \in \Theta$, (i) $H_{2,k}^{0}(\theta)$ and $H_{3}^{0}(\theta)$ are open subsets of $E'(\theta)$; (ii) in $E'(\theta)$, $intH_{2,k}(\theta) = H_{2,k}^{0}(\theta)$ for all $k \in \{1,...,K\}$ and $intH_{3}(\theta) = H_{3}^{0}(\theta)$.*

Proof. We deal with $H_3(\theta)$ and $H_3^0(\theta)$ first. $\bar{R}_{\theta}^{*}$ is $\mathcal{C}^{\infty}$ and Lemma 9 ensures that 0 is a regular value of $\bar{R}_{\theta}^{*}$ for almost all $\theta \in \Theta$. Take a regular $\theta \in \Theta$. In order to apply Lemma 5, we need to find a suitable $\bar{\epsilon}$ for each θ. Note that for all $r \in [0,\beta]$, $\frac{\partial}{\partial r}\bar{R}_{\theta}^{*}(x,r)$ is bounded for all $x \in E'(\theta)$, since $\bar{R}_{\theta}(x,r,\theta)$ is smooth and $E'(\theta)$ is bounded. So, $\sup_{x \in E'(\theta)} \left| \frac{\partial}{\partial r}\bar{R}_{\theta}^{*}(x,r) \right|$ is bounded. Condition (17) requires $1 - c^{A}w_0 - \delta_1 = 0$. This and the fact that $\delta_1 > \lambda$ imply that $\bar{R}_{\theta}^{*}(x,0) = 1 - c^{A}w_0 - \lambda = \delta_1 - \lambda > 0$, a constant, for all $x \in E'(\theta)$. So, there is an $\epsilon_0 > 0$ such that $\bar{R}_{\theta}^{*}(x,r) > 0$ for all $r \in (0,\epsilon_0)$ for all $x \in E'(\theta)$. In the same fashion by (18), $\bar{R}_{\theta}^{*}(x,\beta) = p^{A}(\beta) - c^{A}w(\beta) - \lambda = \delta_2 - \lambda > 0$ for all $x \in E'(\theta)$. So there is an $\epsilon_{\beta} > 0$ such that $\bar{R}_{\theta}^{*}(x,r) > 0$ for all $r \in (\beta - \epsilon_{\beta}, \beta)$ for all $x \in E'(\theta)$. Take $\bar{\epsilon}^{R} = \min\{\epsilon_0, \epsilon_{\beta}\}$. Lemma 5 applies with R_{θ}^{*} as g, $E'(\theta)$ as M, and $(0,\beta)$ as (a,b). Therefore, $H_3^0(\theta)$ is an open set in $E'(\theta)$, and $intH_3(\theta) = H_3^0(\theta)$.

Next, we deal with $H_{2,k}(\theta)$ and $H_{2,k}^{0}(\theta)$. For any $k \in \{1,...K\}$, $\bar{Q}_{\theta}^{*k}$ is $\mathcal{C}^{\infty}$ and Lemma 9 ensures that 0 is a regular value of $\bar{Q}_{\theta}^{*k}$ for almost all $\theta \in \Theta$. Take a regular $\theta \in \Theta$. Because $\bar{Q}^{k}(x,r,\theta)$ is smooth and $E'(\theta)$ is bounded, $\sup_{x \in E'(\theta)} \left| \frac{\partial}{\partial r}\bar{Q}_{\theta}^{*k}(x,r) \right|$ is bounded for all $r \in (-\bar{z},\ \bar{z})$. We can choose $\bar{z}$ such that $\bar{Q}_{\theta}^{*k}(x,-\bar{z}) = \bar{Q}_{\theta}^{*k}(x,\bar{z}) = \alpha < 0$ for all $x \in E'(\theta)$. So there are $\epsilon', \epsilon'' > 0$ such that $\bar{Q}_{\theta}^{*k}(x,r) < 0$ for all $r \in (-\bar{z},\epsilon') \cup (\epsilon'',\bar{z})$ for all $x \in E'(\theta)$. Also, by condition (14), $\bar{Q}_{\theta}^{*k}(x,c_h) = -\frac{\rho e_k}{(1-\rho)c^{M}} < 0$ for all $h \in \{1,...,K\}$. So, for all $h \in \{1,...,K\}$, there is ϵ_h such that $\bar{Q}_{\theta}^{*k}(x,c_h) < 0$ for all $r \in (c_h - \epsilon_h, c_h) \cup (c_h, c_h + \epsilon_h)$ for all $x \in E'(\theta)$. Take $\bar{\epsilon}^{k} = \min\{\epsilon', \epsilon'', \epsilon_1, ..., \epsilon_K\}$. Although we work with $r \in (-\bar{z},\ \bar{z}) \setminus \cup_{k=1}^{K}\{c_k\}$ instead of one interval, the argument in Lemma 5 applies with $-\bar{Q}_{\theta}^{*k}(x,c_h)$ as g, $E'(\theta)$ as M, and $(-\bar{z},\ \bar{z}) \setminus \cup_{k=1}^{K}\{c_k\}$ as (a,b). So, $H_{2,k}^{0}(\theta)$ is an open subset of $E'(\theta)$ and $intH_{2,k}(\theta) = H_{2,k}^{0}(\theta)$. ∎

By Lemma 10, $H_{2,k}^{0}(\theta)$ and $H_{3}^{0}(\theta)$ are open subsets of $E'(\theta)$ and hence $\mathcal{C}^{\infty}$-manifolds of dimension $K-1$ for almost all $\theta \in \Theta$. Each of them has zero codimension in $E'(\theta)$. Let $E^{0}(\theta) = \left(\cap_{k=1}^{K} H_{2,k}^{0}(\theta)\right) \cap H_{3}^{0}(\theta)$ denote their intersection. Since these manifolds are transversal to each other in $E'(\theta)$, their intersection $E^0(\theta)$ is a $\mathcal{C}^\infty$ manifold in $E'(\theta)$ of codimension equal to the sum of the codimension of all, which is zero. So, for almost all $\theta \in \Theta$, $E^0(\theta)$ is a $\mathcal{C}^\infty$-manifold of dimension $K-1$. Therefore, for almost all $\theta \in \Theta$,

$$\begin{aligned} E^0(\theta) &= \left(\cap_{k=1}^K intH_{2,k}(\theta)\right) \cap intH_3(\theta) \\ &= int\left(\cap_{k=1}^K H_{2,k}(\theta) \cap H_3(\theta)\right) = intE^*(\theta). \end{aligned}$$

So, $E^*(\theta)$ has $E^0(\theta)$, a $K-1$ dimensional $\mathcal{C}^\infty$-manifold, as its interior (taken in $E'(\theta)$). ∎

Proof of Corollary 1. First, we restrict the model to be symmetric. Some of the parameters and equations are redundant; they will be eliminated later. Note that the rent conditions (17) and (18) mean $R(0) - \delta_1 = 0$ and $R(\beta) - \delta_2 = 0$ respectively. One of them is redundant and is eliminated since $\delta_1 = \delta_2$ and symmetry implies $R(0) = R(\beta)$.

When K is even, symmetry reduces the numbers of free variables c_k and N_k to $K/2$ respectively, the number of variables b_k to $(K/2) - 1$ (since $b_{K/2} = \beta/2$ is fixed at the middle), and the number of variables w_k (this includes w_0) to $(K/2) + 1$. The number of equations in each of (16), (14), and (15) is reduced to $K/2$. Walras' law renders one equation redundant. Adding one equation for the rent, the total number of independent equations is $3K/2$. $2K$ variables and $3K/2$ equations generate a $\mathcal{C}^\infty$ solution manifold of generic dimension $K/2$. Note that inequalities (19), (21), and (20) do not affect generic dimension (their numbers are reduced accordingly). Eliminate parameters in Θ if their associating equalities or inequalities are eliminated. Then, the remaining parameters constitute a regular parameterization for the symmetric economy. As argued in the proof of Theorem 1, for almost all parameter values, the equilibrium set has the solution manifold for (16), (17), (18), (14), and (15) as its interior.

When K is odd, the numbers of free variables c_k and b_k are both $(K-1)/2$ (note that $c_{(K+1)/2} = \beta/2$) and those of variables N_k and w_k are $(K+1)/2$ and $((K+1)/2) + 1$ respectively. Each of (16), (14), and (15) has $(K+1)/2$ equations, and Walras' law renders one redundant. Adding one equation for the rent, the total number of independent equations is $3(K+1)/2$. $2K+1$ variables and $(3K+1)/2$ equations generate a $\mathcal{C}^\infty$ solution manifold of dimension $(K-1)/2$. The conclusion is reached in the same way as above. ∎

APPENDIX D. Approximating the Equilibrium Set

Sets $H_{2,k}$ and H_3 involve minimization, which does not preserve smoothness. We resort to their approximations in order to further utilize the differentiable approach. Theorem 2 illustrates how far the equilibrium set extends beyond its interior.

THEOREM 2. *For almost all $\theta \in \Theta$, the set of K-equilibria of an economy is approximately contained in the closure of its interior.*

Proof. Define

$$h^R(x,\theta) = \min_{r \in [\bar{\epsilon}^R/2, \beta - \bar{\epsilon}^R/2]} \bar{R}(x,r,\theta)$$

where $h^R(x,\theta) : \Re^{4K} \times \Theta \to \Re$ ($\bar{\epsilon}^R$ is defined in the proof of Lemma 10). Restricting its domain to $E'(\theta)$ for a given θ, we have

$$h^R_\theta(x) = \min_{r \in [\bar{\epsilon}^R/2, \beta - \bar{\epsilon}^R/2]} \bar{R}^*_\theta(x,r)$$

where $h^R_\theta(x) : E'(\theta) \to \Re$. Define

$$h^k(x,\theta) = \min_{r \in \left[-\bar{z}+\bar{\epsilon}^k/2,\, \bar{z}-\bar{\epsilon}^k/2\right] \setminus \cup_{k=1}^K \left[c_k - \bar{\epsilon}^k/2, c_k + \bar{\epsilon}^k/2\right]} -\bar{Q}^k(x,r,\theta)$$

where $h^k(x,\theta) : \Re^{4K} \times \Theta \to \Re$ ($\bar{\epsilon}^k$ is defined in the proof of Lemma 10). Restricting its domain to $E'(\theta)$ for a given θ, we have

$$h^k_\theta(x) = \min_{r \in \left[-\bar{z}+\bar{\epsilon}^k/2,\, \bar{z}-\bar{\epsilon}^k/2\right] \setminus \cup_{k=1}^K \left[c_k - \bar{\epsilon}^k/2, c_k + \bar{\epsilon}^k/2\right]} -\bar{Q}^{*k}_\theta(x,r)$$

where $h^k_\theta(x) : E'(\theta) \to \Re$. Thus,

$$\begin{aligned} H_{2,k}(\theta) &= \left\{ x \in E'(\theta) \mid h^k_\theta(x) \geq 0 \right\}, \\ H_3(\theta) &= \left\{ x \in E'(\theta) \mid h^R_\theta(x) \geq 0 \right\}. \end{aligned}$$

Functions h^R and h^k are continuous by the Maximum Theorem since we minimize continuous functions over constraint correspondences that are compact-valued and continuous. They are not, however, necessarily smooth. We analyze their smooth approximations in order to further study the equilibrium set. We want to use $\mathcal{C}^\infty$ approximations of h^R and h^k that preserve their first order derivatives with respect to (e,λ).

First, note that $D_{(e,\lambda)}h^R(x,\theta) = D_{(e,\lambda)}\bar{R}(x,r,\theta) = (0,...,0,-1)^T$ for all (x,θ) since $D_{(e,\lambda)}\bar{R}(x,r,\theta)$ is constant over r. Let $h^R\left(x,\theta_{-(e,\lambda)},0,...,0\right)$ denote the function obtained from $h^R(x,\theta)$ by holding $(e,\lambda)=0$. Approximate $h^R\left(x,\theta_{-(e,\lambda)},0,...,0\right)$ with a $\mathcal{C}^\infty$ function $\hat{h}^R\left(x,\theta_{-(e,\lambda)},0,...,0\right)$ (Hirsch's theorem, see Appendix B). Then function $\hat{h}^R(x,\theta) = \hat{h}^R\left(x,\theta_{-(e,\lambda)},0,...,0\right) - \lambda$ is a $\mathcal{C}^\infty$ approximation of $h^R(x,\theta)$. Next, approximate $h^k\left(x,\theta_{-(e,\lambda)},0,...,0\right)$ with a $\mathcal{C}^\infty$ function $\hat{h}^k\left(x,\theta_{-(e,\lambda)},0,...,0\right)$. Note that for all k, $D_{e_k}h^k(x,\theta) = -\frac{\partial}{\partial e_k}\bar{Q}^k(x,r,\theta) = \frac{\rho}{(1-\rho)c^M}$ and $D_{(e_{-k},\lambda)}h^k(x,\theta) = -D_{(e_{-k},\lambda)}\bar{Q}^k(x,r,\theta) = 0$ (because $D_{(e,\lambda)}\bar{Q}^k(x,r,\theta)$ is constant over r). Thus function $\hat{h}^k(x,\theta) = \hat{h}^k\left(x,\theta_{-(e,\lambda)},0,...,0\right) + \frac{\rho}{(1-\rho)c^M}e_k$ is a $\mathcal{C}^\infty$ approximation of $h^k(x,\theta)$. We have $\mathcal{C}^\infty$ maps $\hat{h}^R$ and $\hat{h}^k$ such that $D_{(e,\lambda)}\hat{h}^R(x,\theta) = D_{(e,\lambda)}h^R(x,\theta)$ and $D_{(e,\lambda)}\hat{h}^k(x,\theta) = D_{(e,\lambda)}h^k(x,\theta)$.

For a given θ, define maps $\hat{h}^R_\theta : E'(\theta) \to \Re$ and $\hat{h}^k_\theta : E'(\theta) \to \Re$ by restricting the domains of $\hat{h}^R$ and $\hat{h}^k$ to be $E'(\theta)$ respectively. That is, $\hat{h}^R_\theta(x) = \hat{h}^R(x,\theta)$ and $\hat{h}^k_\theta(x) = \hat{h}^k(x,\theta)$ for all $(x,\theta) \in E'(\theta)\times\Theta$. Let

$$\begin{aligned}
\hat{H}_{2,k}(\theta) &= \left\{x \in E'(\theta) \,|\hat{h}^k_\theta(x) \geq 0, \forall r \in (-\bar{z},\, \bar{z}) \setminus \cup_{k=1}^K \{c_k\}\right\}, \\
\hat{H}_3(\theta) &= \left\{x \in E'(\theta) \,|\hat{h}^R_\theta(x) \geq 0, \forall r \in (0,\beta)\right\}, \\
\hat{H}^0_{2,k}(\theta) &= \left\{x \in E'(\theta) \,|\hat{h}^k_\theta(x) > 0, \forall r \in (-\bar{z},\, \bar{z}) \setminus \cup_{k=1}^K \{c_k\}\right\}, \\
\hat{H}^0_3(\theta) &= \left\{x \in E'(\theta) \,|\hat{h}^R_\theta(x) > 0, \forall r \in (0,\beta)\right\}.
\end{aligned}$$

These sets are the approximations of $H_{2,k}(\theta)$, $H_3(\theta)$, $H^0_{2,k}(\theta)$ and $H^0_3(\theta)$ respectively, that we will use. Thus $\left(\cap_{k=1}^K \hat{H}_{2,k}(\theta)\right) \cap \hat{H}_3(\theta)$ approximates $E^*(\theta)$, and $\left(\cap_{k=1}^K \hat{H}^0_{2,k}(\theta)\right) \cap \hat{H}^0_3(\theta)$ approximates $E^0(\theta)$. Lemmas D1 and D2 imply that $E^*(\theta)$ is approximately contained in the closure of $E^0(\theta)$. ∎

We will show $\left(\cap_{k=1}^K \hat{H}_{2,k}(\theta)\right) \cap \hat{H}_3(\theta) \subset cl\left(\left(\cap_{k=1}^K \hat{H}^0_{2,k}(\theta)\right) \cap \hat{H}^0_3(\theta)\right)$ where cl denotes closure taken in $\Re^{4K}$. Let cl' denote closure taken in $E'(\theta)$.

Lemma 11. *For almost all $\theta \in \Theta$, $cl'\hat{H}^0_{2,k}(\theta) = \hat{H}_{2,k}(\theta)$ for all $k \in \{1,..K\}$ and $cl'\hat{H}^0_3(\theta) = \hat{H}_3(\theta)$.*

Proof. Take $\hat{H}^0_3(\theta)$ and $\hat{H}_3(\theta)$ for example. $cl'\hat{H}^0_3(\theta) \subset \hat{H}_3(\theta)$ is implied by Lemma 10, we show $\hat{H}_3(\theta) \subset cl'\hat{H}^0_3(\theta)$ for almost all $\theta \in \Theta$. Take a $\theta \in \Theta$ such that $\hat{H}_3(\theta) \setminus cl'\hat{H}^0_3(\theta) \neq \emptyset$. Note that $x \in \hat{H}_3(\theta)$ if and only if $\hat{h}^R_\theta(x) \geq 0$, and $x \in \hat{H}^0_3(\theta)$ if and only if $\hat{h}^R_\theta(x) > 0$. Therefore, for all $x \in \hat{H}_3(\theta) \setminus cl'\hat{H}^0_3(\theta)$, we have $\hat{h}^R_\theta(x) = 0$

and there is a neighborhood $N \subset E(\theta)$ around x such that $\hat{h}_\theta^R(x') \leq 0$ for all $x' \in N$. So, x is a local maximum of $\hat{h}_\theta^R(x)$ in $E'(\theta)$ and $D_x\hat{h}_\theta^R(x) = 0$. This means 0 is a critical value of $\hat{h}_\theta^R$.

Next, we show that for almost all $\theta \in \Theta$, 0 is a regular value of $D_x\hat{h}_\theta^R$ and hence $\hat{H}_3(\theta) \backslash cl'\hat{H}_3^0(\theta) = \emptyset$. Note that $D_x\hat{h}_\theta^R$ maps from $T_x(E'(\theta))$ to $\Re$. Therefore, it has full rank if $D_x\hat{h}^R(x,\theta)$ and $\{D_x f_k(x,\theta)\}_{k=1}^{3K+1}$ are linearly independent (as argued in Lemma 8). Since $D_{(e,\lambda)}\hat{h}^R(x,\theta) = D_{(e,\lambda)}\bar{R}(x,r,\theta)$, $D_\theta\hat{h}^R(x,\theta)$ and $\{D_\theta f_k(x,\theta)\}_{k=1}^{3K+1}$ are always linearly independent (Lemma 6). Therefore, for almost all $\theta \in \Theta$, $D_x\hat{h}^R(x,\theta)$ and $\{D_x f_k(x,\theta)\}_{k=1}^{3K+1}$ are linearly independent whenever $x \in E'(\theta)$ and $D_{(e,\lambda)}\hat{h}^R(x,\theta) = 0$ (by the Transversality Theorem as argued in Lemma 9). This means 0 is a regular value of $D_x\hat{h}_\theta^R$ for almost all θ. Following the same argument and noting that $D_{(e,\lambda)}\hat{h}_\theta^k(x) = D_{(e,\lambda)}\hat{h}^k(x,\theta) = D_{(e,\lambda)}\bar{Q}^k(x,r,\theta)$ for all k, we can show $\hat{H}_{2.k}(\theta) \subset cl'\hat{H}_{2,k}^0(\theta)$ for almost all θ. ∎

Lemma 12. $\left(\cap_{k=1}^K cl'\hat{H}_{2,k}^0(\theta)\right) \cap cl'\hat{H}_3^0(\theta) = cl'\left(\left(\cap_{k=1}^K \hat{H}_{2,k}^0(\theta)\right) \cap \hat{H}_3^0(\theta)\right)$, *for almost all* $\theta \in \Theta$.

Proof. Apparently $cl'\left(\left(\cap_{k=1}^K \hat{H}_{2,k}^0(\theta)\right) \cap \hat{H}_3^0(\theta)\right) \subset \left(\cap_{k=1}^K cl'\hat{H}_{2,k}^0(\theta)\right) \cap cl'\hat{H}_3^0(\theta)$. We show the converse is also true for almost all θ. Let $\hat{H}_\theta^S = \left(\hat{h}_\theta^i\right)_{i\in S}$ where $S = \{1, ..., K, R\}$. First we show that for a given $\theta \in \Theta$, if $\left\{D_x\hat{h}_\theta^i(x)\right\}_{i\in S}$ are linearly independent whenever $\hat{H}_\theta^S(x) = 0$ then $\left(\cap_{k=1}^K cl'\hat{H}_{2,k}^0(\theta)\right) \cap cl'\hat{H}_3^0(\theta) \subset cl'\left(\left(\cap_{k=1}^K \hat{H}_{2,k}^0(\theta)\right) \cap \hat{H}_3^0(\theta)\right)$. Take any $\bar{x} \in \cap_{k=1}^K cl'\hat{H}_{2,k}^0(\theta) \cap cl'\hat{H}_3^0(\theta)$. If $\bar{x}$ belongs to the interior of all $cl'\hat{H}_{2,k}^0$ and $cl'\hat{H}_3^0$ then it belongs to the closure of the intersection. Suppose $\bar{x}$ belongs to the boundary of some sets and the interior of the others. That is, there is $S' \subset S$ such that $\hat{h}_\theta^i(\bar{x}) = 0$ for $i \in S'$ and $\hat{h}_\theta^i(\bar{x}) > 0$ for $i \in S\backslash S'$. Since $\left\{D_x\hat{h}_\theta^i(\bar{x})\right\}_{i\in S'}$ are linearly independent (note that $\hat{H}_\theta^{S'}(\bar{x}) = 0$), there exists a vector $u \in T_x(E'(\theta))$ such that $D_x\hat{h}_\theta^i(x)u > 0$ for all $i \in S'$. (There exists a solution to $D_x\hat{H}_\theta^{S'}(x)u >> 0$ because $D_x\hat{H}_\theta^{S'}(x)$ has full rank). Let $u(\epsilon)$ denote the projection of $\bar{x} + \epsilon u$ onto $E'(\theta)$. For sufficiently small $\epsilon > 0$, we have $\hat{h}_\theta^i(u(\epsilon)) > 0$ for all $i \in S'$ and $\hat{h}_\theta^i(u(\epsilon)) > 0$ for all $i \in S\backslash S'$ by continuity. This means $\bar{x}$ is a limit point of the intersection, so $x \in cl'\left(\left(\cap_{k=1}^K \hat{H}_{2,k}^0(\theta)\right) \cap \hat{H}_3^0(\theta)\right)$.

Let $\hat{H}^S = \left(\hat{h}^i\right)_{i\in S}$. $D_\theta\hat{H}^S(x,\theta)$ always has full rank (by Lemma 6 and that $D_{(e,\lambda)}\hat{h}^k(x,\theta) = D_{(e,\lambda)}\bar{Q}^k(x,r,\theta)$ for all k and $D_{(e,\lambda)}\hat{h}^R(x,\theta) = D_{(e,\lambda)}\bar{R}(x,r,\theta)$).

Then, $\{D_x f_k(x,\theta)\}_{k=1}^{3K+1}$ and $\left\{D_x \hat{h}^i(x,\theta)\right\}_{i\in S}$ are linearly independent whenever $x \in E'(\theta)$ and $\hat{H}^S(x,\theta) = 0$ except for θ in a set of measure zero (by the Transversality Theorem). Thus, $\left\{D_x \hat{h}^i_\theta(x)\right\}_{i\in S}$ are linearly independent whenever $\hat{H}^S_\theta(x) = 0$ except for θ in a set of measure zero. ∎

References

[1] H. Abdel-Rahman, Product differentiation, monopolistic competition and city size, Regional Science and Urban Economics 18 (1988), 69-86.

[2] H. Abdel-Rahman, M. Fujita, Product variety, Marshallian externalities and city size, J. Regional Science 30 (1990), 165-183.

[3] M. Berliant, Y. Zenou, Labor differentiation and agglomeration in general equilibrium, mimeo, Washington University and University of Southampton, 2002.

[4] G. Debreu, Economies with a finite set of equilibria, Econometrica 38 (1970), 387-392.

[5] G. Debreu, The application to economics of differential topology and global analysis: regular differentiable economies, American Economic Review 66 (1976), 280-287.

[6] E. Dierker, Topological Methods in Walrasian Economics, in Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, New York, 1974.

[7] A. Dixit, J. Stiglitz, Monopolistic competition and optimum product diversity, American Economic Review 67 (1977), 297-308.

[8] G. Ellison, D. Fudenberg, Knife-edge or plateau: when do market models tip?, Quarterly Journal of Economics 118 (2003), 1249-1278.

[9] E. Helpman, P. Krugman, Trade Policy and Market Structure, MIT Press, Cambridge, MA, 1989.

[10] M. Fujita, P. Krugman, When is the economy monocentric?: von Thünen and Chamberlin unified, Regional Science and Urban Economics 25 (1995), 505-528.

[11] M. Fujita, P. Krugman, T. Mori, On the evolution of hierarchical urban systems, European Economic Review 34 (1999), 209-251.

[12] M. Fujita, P. Krugman, A.J. Venables, The Spatial Economy: Cities, Regions, and International trade, MIT Press, Cambridge, MA, 1999.

[13] M. Fujita, T. Mori, Structural stability and evolution of urban systems, Regional Science and Urban Economics 27 (1997), 399-442.

[14] M. Fujita, J-F. Thisse, Economics of Agglomeration: Cities, Industrial Location, and Regional Growth, Cambridge Univ. Press, Cambridge, 2002.

[15] V. Guillemin, A. Pollack, Differential Topology, Prentice-Hall, Englewood Cliffs, NJ, 1974.

[16] J.V. Henderson, The sizes and types of cities, American Economic Review 64 (1974), 640-656.

[17] M.W. Hirsch, Differential Topology, Springer-Verlag, New York, 1976.

[18] T. Kehoe, An index theorem for general equilibrium models with production, Econometrica 48 (1980), 1211-1232.

[19] P. Krugman, Increasing returns and economic geography, J. Political Econ. 99 (1991), 483-499.

[20] P. Krugman, First nature, second nature, and metropolitan location, J. Regional Science 33 (1993a), 129-144.

[21] P. Krugman, On the number and location of cities, European Economic Review 37 (1993b), 293-298.

[22] P. Krugman, The Self-Organizing Economy, Blackwell Publishers, Cambridge, MA, 1996.

[23] A. Mas-Colell, Continuous and smooth consumers: approximation theorems, J. Econ. Theory 8 (1974), 305-336.

[24] A. Mas-Colell, The Theory of General Economic Equilibrium: A Differential Approach, Cambridge Univ. Press, Cambridge, 1985.

[25] J.W. Milnor, Topology from the Differentiable Viewpoint, Univ. Press Virginia, Charlottesville, SC, 1965.

[26] S. Smale, Global analysis and economics I: Pareto optimum and a generalization of Morse theory, in Salvador Symposium on Dynamical Systems, Academic Press, New York, 1973.

[27] S. Smale, Global analysis and economics IIA: extension of a theorem of Debreu, J. Math. Econ. 1 (1974), 1-14.

[28] D. Starrett, Market allocations of location choice in a model with free mobility, J. Econ. Theory 17 (1978), 21-37.

[29] C. M. Wartenberg, von Thünen's Isolated State, Pergamon, London, 1966.