Vote Buying

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Abstract

We examine the consequences of vote buying, assuming this practice were allowed and free of stigma. Two parties competing in a binary election may purchase votes in a sequential bidding game via up-front binding payments and/or campaign promises (platforms) that are contingent upon the outcome of the election. We analyze the role of the parties' budget constraints and voter preferences in determining the winner and the payments to voters.

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1 Introduction

The practice of vote buying appears in many societies and organizations, and in different forms. Obvious examples include direct payments to voters, donations to a legislator's campaign by special interest groups, the buying of the voting shares of a stock, and the promise of specific programs or payments to voters conditional on the election of a candidate. While we generally think of the trade of goods as being welfare improving, the buying and selling of votes is often (albeit not always) viewed as undesirable.

Our purpose in this paper is to explore the consequences of vote buying. The aim is both to enhance the understanding of those forms of vote buying that are widely practiced, such as lobbying legislators and making campaign promises, and to shed light on the hypothetical question of what might happen if vote buying were allowed where it is currently prohibited. The latter question can of course help us think about the rationale behind current social conventions. To do so we study how vote buying would function in an environment in which it is allowed and free of stigma. We focus on the following questions.

- How do voters' preferences (over outcomes and over how they vote) and bidders' budgets affect the outcome of the election?
- How does the institutional environment—whether parties can purchase votes with up-front payments or can only make campaign promises—affect the outcome?
- Is the outcome with vote buying efficient, and how is this affected by allowing bidders' budgets to be raised from donations?

We address these questions using the following model. There is a finite population of voters choosing between two competing parties. Each of the parties is interested in obtaining a majority of the votes while spending as little as possible, subject to not exceeding its budget. We examine two scenarios: one in which the parties only compete in campaign promises (that are contingent upon the outcome of the election, but not upon the actual vote); and the other where parties also compete in up-front vote buying (where the payment is contingent on the vote, but not on the outcome). In both scenarios the parties make offers in a sequential and alternating bidding process. Although voters are not formally modeled as players, their assumed behavior is motivated by considerations of utility maximization.

The answers to the first two questions raised above are intertwined. The identity of the winning party and the distribution of payments to voters depend not just on voter preferences and party budgets, but also critically on whether up-front vote buying is permitted or only campaign promises are allowed.

First, when parties compete only through campaign promises, the total payments received by voters tend to be substantially higher than under up-front vote buying. The intuition is that, with up-front buying, the party that knows it will lose in equilibrium will not wish to buy any votes since those must be paid for up-front, but will be happy to bid for votes with campaign promises that are only paid upon winning.[1]

Second, when parties compete only through campaign promises, the voters whose preferences matter are a specific subset of the voters near the median voter. In contrast, with up-front vote buying, the preferences of all voters are important in determining who wins.

Third, the trade-off between a party's budget and the intensity of the voters' preference for that party also differs across these two regimes. Roughly speaking, when up-front buying is allowed, increasing any voter's preference for voting for one party by the equivalent of \$2 has the same effect on whether that party will win as increasing its budget by \$1. In this sense, money is worth twice as much to a party as being liked by an equivalent amount. In contrast, when only campaign promises are allowed, a \$1 increase in a party's budget has the same effect as increasing a voter's preference for this party by the equivalent of \$1 if the voter is one of the key "near-median" voters, and \$0 otherwise (as such changes in other voters' preferences are inconsequential under campaign promises).

Fourth, in the equilibrium on which we focus, in the game with only campaign promises the winner ends up with a marginal majority. With up-front vote buying, the winner either maintains its initial majority (possibly a super- majority) or ends up with a just-marginal majority.

Fifth, under campaign promises voter preferences over final outcomes matter, while under up-front vote buying, in an important subset of the cases, voter preferences over final outcomes do not have a significant effect. Due to the way that we set up the model, this insight follows easily from our assumptions. But in fact it is an important insight that has broader validity.

The answer to the efficiency question is that, regardless of the vote buying protocol, the outcome of the election could generally be Pareto inefficient. This follows since in either situation voters' preferences are not fully accounted for in determining the winner

[1] Of course, as this follows from ex ante certainty about who wins and loses, it relies on the assumption that there is complete information about the budgets.

of the election. However, we argue that this depends on the source of the parties' budgets. If the parties' budgets are raised in an appropriate manner from voters' contributions, or for some other reason reflect voters' preferences, then it turns out that the party that maximizes the total utility of the voters is the winner. We argue that, more generally, the critical determinant of efficiency is whether each voter turns out to be pivotal in some non-trivial way, which occurs if voters can make campaign donations in a game that we describe.

Finally, we also consider situations where voters' preferences are unknown to the parties, in which case parties cannot "target" their vote buying. This also has an impact on the outcome. For instance, in the case of vote buying changing the distribution of preferences so that the (expected) *median* voter's preference for voting for one party increases by \$1 has the same effect on who wins as increasing that party's budget by $\$N/2$, where N is the number of voters.

There are several lines of related literature: the study of Colonel Blotto games, the political science literature on lobbying (e.g., Groseclose and Snyder (1996)), campaign promises (Myerson 1993), and vote buying (e.g., Buchanan and Tullock (1962) and Anderson and Tollison (1990)), and the finance literature on corporate control and takeover battles (e.g., Grossman and Hart (1988), Harris and Raviv (1988)). Discussing why our conclusions differ from some of these other lines of literature will be much easier after the presentation of our model and results, so we defer this discussion to Section 7.5.

2 A Model of Vote Buying

Two "parties," X and Y, compete in an election with an odd number, N, of voters. As mentioned in the introduction, we may think of these parties as candidates in the election, or in other applications as lobbyists or interest groups that compete for the votes of legislatures.

2.1 The Vote Buying Game

Prior to the election the parties try to influence the voting. Parties have two methods of influencing voters:

(1) Up-front payments: a binding agreement that gives the party full control of the vote in exchange for an up-front payment to the voter.

(2) Campaign promises: a promise that has to be honored by the party if it is elected; the voter maintains control of the vote.[2]

The parties alternate in making offers. Party k in its turn announces the up-front offer $p_i^k \geq 0$ to voter i for her vote, and the campaign promise $c_i^k \geq 0$ given to voter i if k is elected. A fresh offer (or promise) made to a voter cannot be lower than those previously made by the same party to the same voter. There is a smallest money unit $\varepsilon > 0$, so offers can only be made in multiples of ε.

The parties finance their up-front payments and campaign promises out of budgets denoted B^X and B^Y. The total of the up-front payments and campaign promises that a party would have to pay at any stage of the game, assuming that the game were to end at that stage and that party were to win, cannot exceed its budget. At each point in time, given the up-front offers and campaign promises, there is a unique party to which each voter will sell her vote (as we discuss in section 2.2 below). If party k's up-front offer p_i^k has been outbid by the other party, so that at that point voter i would sell her vote to the other party, then party k does not have to count this up-front offer against its budget. However, *all* campaign promises do need to be honored by the winner and thus count against the budget.

The budgets might derive from the state's resources that are controlled by the winner, or from donations.[3] Either interpretation is consistent with budgets used for financing campaign promises and up-front vote buying. We discuss these interpretations further in section 7.1.

When a party makes offers and promises, it observes the past offers and promises received by each voter. The preference of a party is to win at minimal cost. We can think of this as a situation where party k's utility of winning is $W^k - t$ and its utility of losing is $-t$, where $t \leq B^k$ is the total of all payments incurred by party k, and $W^k \geq B^k$ is k's value for winning. (If k wins then $t = \sum_i \left(p_i^k + c_i^k\right)$ as the winner makes good on campaign promises, while for the loser $t = \sum_i p_i^k$, where the p_i^k's are those counted against the budget as described above.) Without loss of generality, given that payments must be in multiples of ε, we round budgets down to the nearest multiple of ε as any remainder can never be bid. The bidding process ends when two rounds go by without

[2]Thus in case (1) payments are contingent on the individual's vote but not on the outcome of the election, and under (2) the opposite holds. Another natural strategy is where payments are contingent on both. This is discussed in Section 7.4.

[3]However, we do not allow parties to tax individuals if they win: campaign promises are positive and more generally state resources and the parties' access these resources is exogenous.

any change in who would win if the game ended at that point.[4] Once the bidding process ends, voters simultaneously tender their votes to the parties. The party that collects more than half the votes wins.

Initially, we consider the full information version of the game where the parties' budgets and the voters' preferences are known to the parties when they bid. Later, we relax those assumptions.

2.2 Voter Behavior

Voters are not formally modeled as players in this game, but instead are assumed to sell their votes according to the following simple rule. Each voter i is characterized by parameters U_i^X and U_i^Y that are interpreted as the utility she obtains from a victory of X and Y respectively. If voter i faces final payment promises p_i^k and final campaign promises c_i^k from party k, where $k = X$ or Y, she will sell her vote to X if and only if

$$p_i^X + \alpha\left(U_i^X + c_i^X\right) > p_i^Y + \alpha\left(U_i^Y + c_i^Y\right), \tag{1}$$

where α is a parameter in $(0,1]$.

Although the voters are not modeled here as players, the rule (1) is intended to capture their behavior in the decision problem they would face if they were modeled as players. The full decision problem of voter i would account for three terms: the up-front payments, a pivot calculation, and a direct utility that i obtains from casting a vote for party k, denoted V_i^k. Specifically, the voter would compare

$$\begin{aligned} & p_i^X + \Pr(X \text{ wins} \mid \text{vote } X)\left(U_i^X + c_i^X\right) + \Pr(Y \text{ wins} \mid \text{vote } X)\left(U_i^Y + c_i^Y\right) + V_i^X \\ \text{with } \; & p_i^Y + \Pr(X \text{ wins} \mid \text{vote } Y)\left(U_i^X + c_i^X\right) + \Pr(Y \text{ wins} \mid \text{vote } Y)\left(U_i^Y + c_i^Y\right) + V_i^Y \end{aligned} \tag{2}$$

and sell to X if the former expression is larger than the latter. Note that the probability of being pivotal is $\Pr(X \text{ wins} \mid \text{vote } X) - \Pr(X \text{ wins} \mid \text{vote } Y) = \Pr(Y \text{ wins} \mid \text{vote } Y) - \Pr(Y \text{ wins} \mid \text{vote } X)$. If this probability is negligible and if V_i^k is proportional to $U_i^k + c_i^k$, then (2) reduces to (1). Thus, using (1) to describe behavior is a simple way of encapsulating these two assumptions. The first assumption reflects our opinion that endogenous pivot probabilities do not play an important role in the situations we would

[4] Alternatively, one could end the game when two rounds go by with no bidding activity whatsoever. The equilibrium outcomes of that game would be the same, but there are many more equilibria in terms of the specifics of round-by-round bidding behavior, as for instance, a bidder might bid for voters a penny at a time, rather than bidding enough to gain a majority all at once.

like to consider. We explain this further in Section 7 by arguing that in more complete models pivot considerations are inconsequential in this setting even when voters are fully strategic. The assumption that V_i^k is proportional to $U_i^k + c_i^k$ is not required for the analysis. The main results can be derived using a general V_i^k instead of $\alpha(U_i^k + c_i^k)$ in (1). But it seems reasonable to assume that the utility from voting is influenced by the preferences over final outcomes. Expressing this relationship using a fixed proportionality factor α is just a convenient way to parameterize the strength of this influence.[5]

Allowing α to vary from 0 to 1 allows us to assess the role of voters' preferences for how they cast their vote, and we will see that this makes a substantial difference in the outcome of the vote-buying game. In large elections where votes are cast secretly α might be quite small. We still take it to be a positive parameter, so that preferences over final outcomes serve as a tie-breaker. In elections where votes are public, as in some legislative votes or committee votes, a voter (legislator) might care significantly about how the vote is cast regardless of outcome, and then α would be quite large.[6]

Without loss of generality we set $U_i^Y = 0$ and $U_i = U_i^X$, which of course can be positive or negative. To avoid dealing with ties, which add nothing of interest to the analysis, we assume that for all i, the values $U_i/2$ and $\alpha U_i/2$ are not multiples of ε.

2.3 Equilibrium

Strategies are defined in the obvious way, and the solution concept we use is subgame perfect equilibrium. There are several facts about these equilibria that we can easily deduce. Note that since the sum of payments guaranteed to all voters must go up by at least ε in any three consecutive rounds, the bidding process must end after a bounded number of rounds. This is thus a finite game with perfect information, and so a pure-strategy subgame-perfect equilibrium can be found by backward induction. Thus, equilibrium exists in pure strategies. Moreover, as ties never occur, the equilibrium outcome must be the same for all equilibria in any subgame. This means that there are well identifiable winners and losers.

[5]Another rationale for the form $\alpha(U_i^k + c_i^k)$ is that the voting preferences reflect a possibly incorrect subjective probability α of being pivotal. If we substitute $V_i^k = 0$ and $\Pr(X \text{ wins} \mid \text{vote } X) - \Pr(X \text{ wins} \mid \text{vote } Y) = \Pr(Y \text{ wins} \mid \text{vote } Y) - \Pr(Y \text{ wins} \mid \text{vote } X) = \alpha$ in (2) then it also reduces to (1).

[6]Specifically, in the case of voting in a legislature, X and Y are lobbyists; α is the weight the legislator assigns to constituents' preferences (say, for reelection considerations); p_i^k are the bribes offered to the legislator; and c_i^k are the lobbyists' promises for direct contributions (say, via projects) to the constituents.

PROPOSITION **1** *The vote-buying game has an equilibrium in pure strategies. In every equilibrium the same party wins, and the losing party never makes any payment.*

The proofs of all propositions appear in the appendix.

Another important observation is that the winner does at least as well when replacing her campaign promises by equal up-front payments because the latter are not impacted by α, can be re-allocated if the other party outbids her, and the winner is liable for the payments in either case.

PROPOSITION **2** *The winner in any equilibrium of the vote-buying game when both up-front payments and campaign promises are permitted, is the same as the winner in any equilibrium of a modified version of the game where only up-front payments are allowed.*

The implication of Proposition 2 is that, at least in terms of determining who wins the election, the relevant cases for us to consider are where only campaign promises are allowed and where only up-front payments are allowed. Nevertheless, although contingent payments are dominated as described above by up-front payments, the presence of campaign promises can still affect the total payments that the winner needs to make in equilibrium. This can be seen in the following example.

EXAMPLE **1** *Campaign Promises make a Difference in the Payments.*

Consider a three voter society where $\varepsilon = 1$, $U_i = 1/2$ for each i, and $B^X = 90$, while $B^Y = 30$. Let $\alpha = 1$. It is easy to see that X wins in each equilibrium. There is an equilibrium where Y sets $c_i^Y = 10$ for all i, and then X has to offer $p_i^X = 10$ to two voters in order to win.

If we rule out campaign promises and only allow up-front payments, then X would still win in all equilibria, but would never pay anything. That follows, since in order to get X to pay something in equilibrium, Y would need to make some promises of up-front payments. Once Y has bid, X's final purchase will involve the two cheapest voters and Y will end up buying at least one voter even though she does not win. This cannot be part of an equilibrium as Y could deviate and never make any payments and be better off. ■

3 Campaign Promises

We begin by studying the case where only campaign promises are permitted, and up-front vote buying is not. This serves as an important benchmark, as it is the case that applies to many election settings. Also, as we have seen from Proposition 2, it is the only case where campaign promises might have a significant impact on determining who wins the election (rather than just how much is paid).

The parameter α is now irrelevant and voter i will vote for X iff $c_i^X + U_i > c_i^Y$. We label voters so that U_i is non-increasing in i. Under this labeling, we refer to $m = (N+1)/2$ as the median voter. Without loss of generality, suppose that the median voter is a supporter of party X ($U_m > 0$). Let $n = |\{i : U_i > 0\}|$ be the number of a priori supporters of X. The analogous number for Y is simply $N - n$. Given a number z, let z^ε be the smallest multiple of ε greater than z.

Let $T = \sum_{i=m}^{n} U_i^\varepsilon > 0$ be the minimal sum that Y has to promise to voters in order to secure the support of a minimal majority, in case X does not promise anything. Thus T, depicted as the shaded part of Figure 1 (drawn assuming $U_i^\varepsilon = U_i$), is one possible measure of the preference advantage that X enjoys over Y.

Figure 1:

Proposition **3** *Y wins in any equilibrium if and only if $B^Y \geq B^X + T$.*

This can be deduced from the proof of Proposition 4 below.

The idea behind Proposition 3 is easily explained. Party Y must spend at least T in order to secure a majority. After that, X will try to obtain some of these votes back (or others, if Y has overspent on these marginal votes), and the competition back and forth will lead to the winner being the party with the largest budget once an expense of T has been incurred by Y.

Since the loser will not have to fulfill its promises, it is indifferent among all of its feasible promises. Therefore there will be many equilibria. However, in most of these equilibria the loser's behavior is optimal only because it is certain to lose. Thus, if there is any uncertainty about the relative strength of the parties, we expect that the range of equilibrium behaviors will narrow down dramatically. Indeed, Proposition 4 below establishes that the only equilibria that survive uncertainty over the relative size of the budgets involve "Least Expensive Majority" (LEM) strategies, in which the parties purchase the least expensive majority in their turn. Thus, any uncertainty over budgets rules out all the "implausible" equilibria.

PROPOSITION 4 *If B^X and B^Y are distributed with full support over $\{0, \varepsilon, ..., B\varepsilon\}$, then in any equilibrium:*

(i) Both parties play LEM strategies.

(ii) Y wins if $B^Y \geq B^X + T$ and ends up pledging exactly $B^X + T$, and X wins otherwise and ends up pledging exactly $\max\{B^Y - T + \varepsilon, 0\}$.

Let $\hat{n} = \{\min i : U_i > -\varepsilon\}$. If both parties use LEM strategies, then only voters between m and $\hat{n}$ can receive positive payments, and the total payments received are $\max\left\{0, B^Y - T + \varepsilon\right\}$ if X wins and $B^X + T$ if Y wins. That is, the winner commits ε more than the loser, who in turn commits its entire budget to a subset of these "near median" voters. (If $B^Y < T$ then any strategy by Y is an LEM strategy, and no payments are made, although Y might still make promises.)

While payments are concentrated among the voters between m and $\hat{n}$, the particulars of which voters get how much can differ across equilibria. For example, in one equilibrium using LEM strategies in a case where $B^Y > B^X + T$, the final outcome is that Party X ends up offering its entire budget B^X to a single voter, say voter m, and Party Y ends up winning by offering $U_m^\varepsilon + B^X$ to that voter and U_i^ε to all voters $i \in [m+1, n]$. This happens by having the parties repeatedly outbid each other by a minimal amount for voter m. In another equilibrium with LEM strategies, X's budget is spread equally over voters $i \in [m, n]$, and Y matches all those bids and tops them off by U_i^ε to compensate for these voters' initial preference for X.

4 Up-Front Vote Buying with Negligible Voting Preferences

We now consider the situation where up-front vote buying is permitted. We can then contrast that with the outcome where only campaign promises are allowed, to see the impact of up-front vote buying.

We first consider the case where voting preferences are negligible, that is, where α is small enough so that $|\alpha U_i| < \varepsilon$. This is a transparent case to analyze since voters view their vote as having no consequence on its own, and thus are happy to tender to the bidder with the highest offer. As a result, the party with the highest budget (up to a factor of ε) wins at a negligible cost.

PROPOSITION **5** *In the small α case, party X wins in (every) equilibrium if and only if $B^X + (n-m)\,\varepsilon \geq B^Y$, and then X's total payments are bounded above by $\frac{m\alpha B^Y}{m-1} + m\varepsilon$.*

The proof (in the appendix) shows that if $B^X+(n-m)\,\varepsilon \geq B^Y$ then the LEM strategy guarantees a victory to X against any bidding strategy that Y might adopt. This implies that, in equilibrium, Y will not enter the bidding except for possibly making offers that will be outbid or campaign promises that will never be paid.

Introducing up-front vote buying when α is negligible results in a winner determined purely by the relative size of the budgets. This contrasts with the case where only campaign promises are permitted, where the utility advantage of one candidate over another, T, effects significantly the identity of the winner. Also, the voters get lower payments under up-front vote buying than under competition in campaign promises. This is because the contingent nature of campaign promises allows the loser to make significant promises that need to be matched by the winner. In contrast, in up-front vote-buying competition with negligible α, the party destined to lose would just lose money if it made significant up-front bids. As with campaign promises alone, the loser may still make significant campaign promises, but when α is small the winner can compete against those with negligible up-front payments.

The conclusions of Proposition 5 are in contrast with the results of Groseclose and Snyder (1996) who analyzed a game where each party gets to move only once, and in sequence. Their model provides a significant second-mover advantage to one of the parties, which contrasts sharply with the open-ended sequential nature of our game. The small α case here corresponds to a case with small utilities in their model, where, with small utilities, the first mover would need a budget at least twice that of the second mover in order to win. The first mover needs to be able to bid in such a way that the second mover cannot afford to buy any majority. In a game without an exogenously determined last mover, as the one we analyze, if one party is (temporarily) outbid for some voter, it can remobilize those resources, which places parties on a more equal footing.

5 Up-front Vote Buying with Significant Voting Preferences.

We now study the case where α is significant. As mentioned earlier, the case of large αU_i's is relevant for a model of voting in a legislature in the presence of lobbying or where voters have nontrivial preferences over how they vote—regardless of their actual impact on the outcome.

Our main concern is which party wins the election. Therefore, appealing to Proposition 2, we study the scenario where only up-front payments are possible. This scenario is also interesting as a model of environments where campaign promises are not credible.

Besides the substantive interest in the case of significant α, it is also somewhat interesting from an analytical point of view. The identification of the winner turns out to be harder as it entails more complicated considerations that involve both the budgets and the preferences. Nevertheless, we can characterize the winners of this competition when the budgets are sufficiently large (as specified below).

In this case the winner is determined by comparing Y's advantage in the budgets $(B_Y - B_X)$ with (approximately) *one half* the *total* utility advantage of X over Y $(\Sigma_i \alpha U_i)$. To understand why the utilities of all voters matter, but only count half as much as the size of the budgets, it is useful to understand the structure of the winning strategies. The following example contrasts the optimal strategy for the winner with the LEM (least expensive majority) strategy, which previously seemed to be a good strategy.

Example **2** *Optimal versus Naive Strategies - Why Utility has a Shadow Price of 1/2.*

There are three voters with $\alpha U_1 = \alpha U_2 = 0.5$ and $\alpha U_3 = -30.5$. The grid size is $\varepsilon = 1$. Budgets are $B^X = 100$ and $B^Y = 80$.

Note that $B^X - B^Y = 20 < 29.5 = -\sum_i \alpha U_i$, so the total utility advantage for Y is greater than the absolute budget advantage of X. Nevertheless, as we show below in Proposition 6, X should win, because X's budget exceeds Y's budget plus half of the total utility difference. That is, basically what matters is the budget advantage relative to one half the total preference advantage (setting aside small corrections that are explained in the proof of the result). Let us see how X should play to win.

Suppose that X follows the naive LEM strategy of always spending the least amount necessary to guarantee a majority at any stage. Suppose (just for the purpose of illustration) that at the first stage Y makes offers of 55 to voter 1 and 25 to voter 3. The cheapest voter for X to buy back is voter 1 at a cost of 55. Assume Y now offers 55 to voter 2. At this point X has 45 left in her budget, and cannot afford to buy back either voter 2 or 3.

What was wrong with this strategy? The problem is that, while X bought the cheapest voter in response to Y's offer, X also freed up a large amount of Y's budget for Y to spend elsewhere, while X's budget was committed. X needs to worry not only about what X is spending at any given stage, but also about how much of Y's budget is freed up. Effectively, freeing up a unit of Y's budget is "just" as bad for X as spending an extra unit of X's budget.

So, instead of following the naive LEM strategy of buying the cheapest voters, let X always follow a strategy of measuring the "shadow price" of a voter as the amount that X must spend *plus the amount of Y's budget that is freed up*. If X had followed that strategy, then in response to Y's first stage offer above, X would have purchased voter 3 at a price of 56. Then Y would have 25 free, and could only spend it on voters 1 and 2. Regardless of how Y spends this budget, X can always buy voter 2 at the next stage at a price of at most 25, against which Y has no winning response. ■

The example shows that keeping track of the shadow price is a good strategy. In fact, for large budgets it guarantees a win for the winning candidate characterized in Proposition 6 below. Let us see how we get from this understanding of "shadow prices" to the expressions underlying Proposition 6.

Under the strategy suggested in the above example, X keeps track of the offer that X has to make to buy a voter given the current offer of Y, plus the amount of Y's budget that is freed up. The amount that X has to offer to buy a given voter i when Y has an offer of p_i^Y in place is $p_i^Y - \alpha U_i$. The amount of Y's budget that is freed up is p_i^Y. So the "shadow price" of buying voter i is $2p_i^Y - \alpha U_i$. Dividing through by 2 gives us $p_i^Y - \frac{\alpha U_i}{2}$. In the proof this translates into the "strength" of Y being Y's budget less the $\frac{\alpha U_i}{2}$'s of the majority of voters that are most favorable to Y. Similarly X's "strength" is X's budget plus the $\frac{\alpha U_i}{2}$'s of the majority of voters that are most favorable to X.

This is captured in Proposition 6 below, which includes some slight modifications to account for the grid size and some other details that are covered in the formal proof.

The result requires that budgets be sufficiently large as specified next.

$$B^X \geq \left|\frac{m\alpha U_1}{2}\right| - \frac{\sum_{i=m+1}^{N} \alpha U_i}{2} - \frac{\alpha U_N}{2} + m\varepsilon \tag{3}$$

$$B^Y \geq \left|\frac{m\alpha U_N}{2}\right| + \frac{\sum_{i=1}^{m-1} \alpha U_i}{2} + \frac{\alpha U_1}{2} + m\varepsilon. \tag{4}$$

PROPOSITION **6** *If the budgets are large enough so that (3) and (4) are satisfied, then X wins if*

$$B^X \geq B^Y - \sum_i \alpha U_i/2 - \alpha U_N/2 + m\varepsilon \tag{5}$$

and Y wins if

$$B^Y \geq B^X + \sum_i \alpha U_i/2 + \alpha U_1/2 + m\varepsilon. \tag{6}$$

The interesting feature is that, very roughly, increasing a voter's preference for a given party by $1 is equivalent, in terms of who wins, to increasing the budget of that party by $0.5. Thus money is worth much more to a party than being liked, as might be expected due to the use of funds being more flexible.

Note that the small α case of Proposition 5 is a special case of the above results. With small α, $\sum_i \alpha U_i$ is negligible relative to the budgets, and the comparison boils down to a comparison of the budgets. Moreover, then the optimal strategy simplifies to the LEM strategy, which is not optimal in general. Also note that, as the proof makes clear, in fact only one large-budget condition is needed in each case. That is wins if equations (4) and (5) hold, and wins if (3) and (6) are satisfied.

The next example shows that Proposition 6 is not valid without the assumption of large enough budgets.

EXAMPLE **3** *Large versus Small Budgets*

Let $B^Y = 0$, $B^X = 30.2$, $\varepsilon = 0.1$, $N = 3$, $\alpha U_1 = -10$, $\alpha U_2 = -20$, and $\alpha U_3 = -30$. Here X can win by buying voters 1 and 2 at prices of 10.1 and 20.1.

In this example

$$B^X + \frac{\sum_i \alpha U_i}{2} + \frac{\alpha U_1}{2} = -5 < B^Y - m\varepsilon = -.2,$$

and so if we applied the expressions from Proposition 6, we would mistakenly conclude that Y should win. ■

We close this section with an example showing that while voters preferences only count half as much as monetary budgets, having minority support that is very strong can be enough to help a candidate overcome having a smaller budget than the opposition.

Example 4 *The party with a smaller budget and minority support can win*

Let $B^X = 200$, $B^Y = 190$, $N = 3$, $\varepsilon = .1$, $U_1 = U_2 = 10$, $U_3 = -60$, and $\alpha = 1$. So X has a larger budget and starts with the support of the majority of voters. However, applying Proposition 6, we see that

$$B^X + \frac{\sum_i \alpha U_i}{2} + \frac{\alpha U_1}{2} = 185 < B^Y - m\varepsilon = 190 - .2.$$

Here, the strong support of the third voter for Y is a big asset. Very roughly, the game boils down to one where X has to win the support both voters 1 and 2, while Y needs only to get one of them.

Note that this example can be extended to the case of any number of voters, with only one voter liking Y, and with X having a larger budget than Y. So long as the one voter likes Y sufficiently more than any one of the others likes X, and the budgets are large enough, Y will win.■

6 Unknown preferences

Our analysis so far has focused on situations where the voting preferences are known. In many cases, this is a reasonable first approximation, as voters' preferences might be highly correlated with observable characteristics (and in such cases where parties are lobbies and voters are legislators with voting records and known constituencies). However, there are some cases where there may be significant uncertainty about voters' preferences and so it is worth understanding how our results are affected by the introduction of such uncertainty. In the case where α is small (with up-front vote buying), the introduction of uncertainty about voter's preferences will not have a significant impact, as the larger budget will still win. However, if either α is large, or up-front vote buying is ruled out and only campaign promises are possible, then uncertainty can matter.

We examine the case of up-front vote buying, as with the uncertainty introduced here, voters are essentially symmetric from the parties' viewpoint, and so now the analysis of the case where only campaign promises are permitted is similar to that of up-front vote buying.

Suppose that, for all i, αU_i is an independent draw from a continuous distribution F. We assume that F has a connected support and a continuous and positive density on its support, such that $z + F(z)/f(z)$ and $z + (F(z) - 1)/f(z)$ are both increasing on the support of F. There are many prominent distributions satisfying this, such as the uniform distribution. Let $\alpha\bar{U} = F^{-1}(0.5)$ be the median of the distribution F. In this environment we impose the constraint that parties' offers must in expectation be within their budgets at each point in the game, assuming it ends at that point.

Proposition 7 *For any $\delta > 0$, there is $N(\delta)$ and $\bar{\varepsilon}$ such that for all $N > N(\delta)$ and all grids with $\varepsilon \in (0, \bar{\varepsilon})$ the following hold.*

- *If $B^Y > B^X + \alpha\bar{U}N/2 + \delta$, then Y wins with probability of at least $1 - \delta$.*
- *If $B^X > B^Y - \alpha\bar{U}N/2 + \delta$, then X wins with probability of at least $1 - \delta$.*

The result is almost a complete characterization for large N, as the conditions cover budget differences except those that fall in an interval of size 2δ.

When δ is sufficiently small, the party who is likely to lose will not enter the bidding and the winning party will bid the minimum necessary to secure majority with sufficiently high probability. Thus, we again see a result that echoes the earlier result of minimal payment with up-front buying and small α, but now it obtains regardless of α.

As mentioned above, Proposition 7 extends readily to the analysis of campaign promises, where the main change is that α drops out and F is the distribution of U_i rather than the αU_i's. Again the uncertainty over preferences results in a significant change in equilibrium payments for this case.

7 Discussion

7.1 The Budgets

The analysis above treats the parties' budgets as exogenous. There are two main sources for payments that parties may make or promise: parties' own funds (e.g., donations and government funding) and government resources that the party controls if it wins (which may also differ across parties due to different abilities to generate or use these resources). Although the party takes control over government resources only after the election and only if it wins, these resources are not only relevant for funding campaign promises. One could also imagine parties taking loans that would be repaid using government resources once the party took over. (In fact, one might think of some donations as implicitly being of this nature.) The above analysis is consistent with the budget being drawn from either of these sources or both. Of course, in an expanded model in which such loans were determined endogenously, the availability of such loans and their terms would depend on the winning prospects of parties. We do not explicitly analyze this extended game.

7.2 Efficiency

In the absence of any mechanism for buying and selling votes, the outcome of voting will in general be inefficient. There is simply nothing to make voters take into account the effect of their vote on others. A natural hypothesis then might be that the opening of trade will lead to efficient outcomes. Our analysis shows that this is not always so. Even if we take the budgets of the parties to represent the utility of some unmodeled agents, the outcome of a vote-buying equilibrium is in general inefficient. In the small α case of the direct purchase scenario, essentially only the budgets matter: If voters strongly support X, but Y has a slightly larger budget, Y still wins. In the large α case, even if we consider the $\alpha = 1$ case in which the underlying preferences enter voting decisions fully, the effect of voters' preferences is only about half that of the budgets. Finally, in the campaign-promise scenario only the preferences of voters near the median group affect the outcome, and hence the outcome does not reflect the preferences of all voters.

Under what circumstances will vote buying result in efficiency? We answer this with respect to the game where up-front vote buying is allowed. Then, the equilibrium will be (approximately) efficient if for some reason the budgets are proportional to the true surpluses. That is, let U^X be X's support in terms of total utility of voters ($U^X = \sum_i [U_i]^+$), and U^Y be Y's support in terms of total utility of voters ($U^Y = \sum_i [-U_i]^+$), then the equilibria will be efficient if $B^X/U^X = B^Y/U^Y$. This would be the case, for example, if the budgets are raised through individual donations that are somehow proportional to values. More fundamentally, vote buying is capable of achieving efficiency, if every voter is made pivotal with respect to the decision. For example, if the voting mechanism requires unanimity (say, X is the status quo outcome that would be replaced by Y only upon unanimous approval), then when vote trading is allowed this is in fact an N-person bargaining problem with complete information for which a wide variety of trading procedures will result in efficiency.[7] But unanimity is not necessary. Even if the simple majority requirement is maintained, one can construct vote trading games that put a sufficient subset of the voters in a pivotal position so as to yield the efficient outcome. The vote-trading game outlined below does just that. It illustrates how efficiency is (almost always) attained when voters are pivotal.[8] In this game the parties' budgets are

[7] Both Buchanan and Tullock (1960) and Neeman (1999) make this point.

[8] The "almost always" caveat is needed since our characterizations have some slack. For small α, by Proposition 5, it is only up to a factor of $m\varepsilon$ that the winner is the party with the larger budget. Similarly for general α, by Proposition 6, a party that provides greater total utility and has a greater budget wins, but only up to a factor of $\max\{\alpha U_1, \alpha U_N\} + m\varepsilon$, and if the budgets are large enough. For

raised via a simple donation game that precedes the up-front vote-buying game analyzed above. While as before the voters are not pivotal in the voting stage, the sequential donation stage makes a sufficient subset of the voters pivotal in the donation game to guarantee efficiency.

The *Campaign-Donation Vote-Buying Game* is as follows.

(1) There is some ordering over voters, according to which voters sequentially choose an amount to donate to each party, where voter i's donations are denoted $(d_i^X, d_i^Y) \in [0, |U_i|]$. Donations are made in a series of rounds, and voters can increase their promised donations in any round. Any increase must be at least in multiples of ε, or the remaining budget that a voter has if that is smaller than ε. The donation part of the game ends when there is a round with no increases in donations.[9]

(2) The parties' budgets are $B^X = \sum_i d_i^X$ and $B^Y = \sum_i d_i^Y$.

(3) The parties play the vote-buying game.

We first consider the small α case.

PROPOSITION **8** *Party X wins in the campaign-donations vote-buying game if and only if $U^X \geq U^Y + (m-n)\varepsilon$.*

We omit the proof as it is fairly straightforward. An analogous result is available for the large α case (and is stated at the end of the appendix).

To understand the sense in which the donation game makes voters pivotal, consider a subgame in which Y's supporters have already donated U^Y and X's supporters have so far donated $D < U^Y + (m-n)\varepsilon$. Suppose that voter j is last in the sequence of X supporters, that $U_j > 0$, that j has not donated yet and that $D + U_j > U^Y + (m-n)\varepsilon$. Then this subgame has an equilibrium in which voter j donates at least an ε to X, which is the minimum required to keep the game moving. Thus, at this point j is made pivotal: if she does not donate the game will end with X's loss; if she donates only part of the

large populations this factor is small compared to total utility.

[9] As noted, we cap voters' donations at their total utility and require minimal increases. One could alternatively consider the infinite game where voters could make arbitrary increases in donations in any given period (and would have to assign a largely negative utility to the infinite path where the game never ends). In equilibrium, voters would never make payments exceeding their total utility in any case, although they might make higher payments off the equilibrium path. We have not explored whether this alternative leads to different outcomes.

sum, everybody will still expect her to complete her donation in the following round. In this manner the donation game guarantees efficiency by designating a sufficient subset of voters as pivotal in any subgame. Thus, if $U^X > U^Y + (m-n)\varepsilon$, there might be some slack and the equilibrium may place only some subset of X's supporters in pivotal positions, but if $U^X = U^Y + (m-n)\varepsilon$, every supporter of X will be made pivotal at least in some subgame (possibly off path).

Going back to the vote buying models we have analyzed, the main source of the inefficiency is now clear. In those models the voters are not pivotal. Notice, however, that this is not due to some peculiarity of those models. These models describe rather natural processes of vote trading; and other natural models (e.g., uniform restricted price offers[10]) would yield similar results with respect to efficiency. As we have just seen, it is possible to design vote-trading games, like the above campaign-donation-vote-buying game, that make everybody pivotal. But the artificial features of that game (such as the sequential donation process played by the voters) which are necessary to make everybody pivotal, just highlight the fact that natural processes of free bidding will not put every voter in a pivotal position and hence are inherently inefficient.

Does vote buying and selling entail greater welfare loss than would occur in its absence? Based on our results, we can see that it is easy to construct examples that generate higher or lower overall utility than straight voting. What we learn from our models is that budgets count for more than utilities. Thus, if we think of the budgets as being raised from donations of the voters and recognize that free riding would limit the donations of small anonymous individuals, the opening of trade is likely to give an advantage to groups of voters who are more capable of translating preferences to budgets. These might be small numbers of wealthy individuals who care intensely about the outcome or other groups organized in small cells with strong ties that manage to overcome free riding. (A donation game of the rough nature outlined above might be a reasonable model for a small non-anonymous groups.) The opening of vote trading will elevate the relative importance of such groups, but of course nothing can be said in general on whether these biases are likely to produce lower total utility than simple voting.

7.3 Voter Behavior

Modeling voters as individuals who sell to the party that offers the higher $p_i^k + \alpha\left(U_i^k + c_i^k\right)$ is a short-cut that embodies two assumptions: First, pivot probabilities play a negligible

[10]See the discussion of Harris and Raviv's work in 7.3.3 below.

role, and, second, voters have preferences over the voting itself that are influenced by the preferences over final outcomes.

There are two reasons why we think that pivot probabilities should not be an important consideration in the situations we would like to consider. First, in large elections there is inevitably sufficient noise to make the pivot probability of an individual voter insignificant. This can be modeled formally by introducing some "noise voters" into the model. The magnitude of such noise can be made small relative to the size of the electorate, hence leaving intact the essence of the analysis conducted above. At the same time, the noise can be significant enough to make the pivot probabilities negligible. To see this, suppose that, in addition to the N voters we consider, there are L noise voters each of whom votes randomly and independently with equal probability for each of the parties. Assume also that the L noise voters are not part of vote buying process. Let N and L be large, but L/N be very small. The large L implies a small pivot probability for each of the N voters who participate in the buying game. The small L/N implies that the analysis of the parties' competition over the N voters will be similar to the above analysis, except that the winner would typically purchase more than $(N+1)/2$ votes—though still close to 50% of the total vote.

Second, pivot considerations are negligible since the winning party can eliminate pivot considerations by buying more than the minimal majority. In particular, we argue that at a negligible increase in cost the winning party can always act as if it were playing a game where it needed a slight supermajority to win, and this puts a bound on the cost associated with the game where it does not need a slight supermajority. More precisely, suppose that the party that is to win in our model is Party X, and hypothetically consider a game where X needs $(N+1)/2+1$ votes to win rather than just $(N+1)/2$. Clearly, when N is large, the conditions under which X wins are close to those under which X wins when it needs $(N+1)/2$ votes. Suppose that X still wins in this nearby game, and let σ^X and σ^Y be corresponding equilibrium strategies such that Y offers no payments on the equilibrium path (obviously there is such an equilibrium). Now let us change to a game where X needs only $(N+1)/2$ votes to win, but where voters take into account the correct endogenous pivot probabilities. Observe that there must be an equilibrium in which X wins, and where the cost to X does not exceed the costs under σ^X, σ^Y. To see this let $\widetilde{\sigma}^X$ and $\widetilde{\sigma}^Y$ coincide with σ^X and σ^Y after all histories in which X has not deviated from σ^X; after any deviation by X from σ^X, let $\widetilde{\sigma}^X$ and $\widetilde{\sigma}^Y$ switch to (one of) the worst equilibrium for X from that point on. Clearly, $\widetilde{\sigma}^Y$ is **a** best response to $\widetilde{\sigma}^X$ and X wins with $\widetilde{\sigma}^X$. If $\widetilde{\sigma}^X$ is best response against $\widetilde{\sigma}^Y$ after any possible history, then we are done. If it is not, then X has a profitable deviation from $\widetilde{\sigma}^X$ after some history. Since by construction, $\widetilde{\sigma}^X$ is **a** best response off the equilibrium path, such **a** deviation might take place only on the path. Since Y makes no offers on the path, this means that X has a deviation in its first move that leads to a better outcome. But, by the construction of $\widetilde{\sigma}^X$ and $\widetilde{\sigma}^Y$, this means that there is an equilibrium in which X wins at a cost that is not more than the cost when using σ^X. To sum up this argument, when N is large, even when voters take into account pivot probabilities, there are equilibrium outcomes that are close to the equilibrium outcomes of our model.

In fact, one can go one step further. If the parties were allowed to offer payments that are contingent on the number of votes they end up getting, then in all equilibria the pivot probabilities will have no significant effect. The idea is as follows. Suppose again that X can win when it needs $(N+1)/2+1$ votes to win and voters make no pivot considerations, and let σ^X be a winning strategy. Let $\widehat{\sigma}^X$ be just like σ^X except that each voter is offered an additional bonus (slightly above her pivot utility difference) if she sells to X and it ends up with exactly $(N+1)/2$ votes. Now notice that $\widehat{\sigma}^X$ wins when the required majority is $(N+1)/2$ and voters take into account pivot considerations. Furthermore, X ends up paying exactly what it would pay with σ^X in the absence of pivot considerations. The "bonus" simply removes the pivot consideration for the voters and since $\widehat{\sigma}^X$ ends up buying $(N+1)/2+1$ votes, X never pays the bonuses (although X does pay for the extra vote purchased). This idea appears in Dal-Bo (2003).

The bottom line is that we think that, for the purposes of our analysis, it is appropriate to abstract away from pivot considerations. We chose to do so in a straightforward way. As the preceding paragraphs explain, this can be done in more sophisticated ways. However, if we were to adopt one of these approaches and carry it throughout, the complexity of the analysis would increase substantially without any gain in substance.

The second assumption embodied in the modeling of voter behavior is that they have preferences over the voting itself that are influenced by their preferences over final outcomes. As should be clear by now, this assumption is not needed for any of the analytical results. These could be stated in terms of general voting preference functions V_i^k. However, it seems reasonable to assume that such voting preferences are influenced by preferences over final outcomes. (Also, this assumption makes the comparison between the platform promises and up-front vote buying more meaningful as similar parameters appear in both.) Many of us vote in large elections, although we do not assign significant probability to being pivotal. When we do so, we probably often vote according to our true preferences over the outcomes. But since we realize that our vote carries very little weight, the voting preferences might be less intense than the actual preferences over the outcomes.[11] Thus, if we had to pay a thousand dollars to get to the polling station, we may skip the voting, although we might be willing to pay that sum to get the president we prefer. In this paper we do not try find a deep explanation for voting preferences and their rationale. We take this behavior as given and proceed to examine its implications.

As mentioned before, when we adopt the interpretation of vote buying of legislators by lobbyists, the prominent role of voting preferences is obvious and does not require any justification. The same may be true in other situations where votes are public. In large elections where votes are cast secretly, the voting preferences should probably be thought as less significant (i.e., α might be quite small).

7.4 Contingent Payments

Another natural form of strategy that the parties might use is one where an up-front promise is made and a vote purchased, but where the payment offered is contingent on winning. This is a sort of hybrid of campaign promises and up-front offers: the vote is explicitly purchased and controlled as in the case of an up-front payment, but the payment is contingent on winning as is a campaign promise. It is more complicated in terms of how voters value such contingent promises, as the value of the promise is endogenous to the equilibrium outcome.

Nevertheless, the consideration of such contingent payments *in addition to up-front purchases* has little impact on the outcome of the vote buying games studied above in the following sense. The winner has no benefit of using such purchases (and may have a cost if the voters value them less, e.g., by the factor α). For the loser, they do not cost anything, but still the promises made cannot exceed the budget. The consequence is that the equilibrium winner of the game where contingent payments are also allowed turns out to be the same as when they are not considered. The only modification is that the payments in equilibrium may change, as the loser might make some contingent promises that end up being costless for her, but the winner ends up having to outbid these promises in equilibrium.

Thus, all of the propositions extend to the *additional* consideration of contingent payments, modulo the fact that the payments by the winner might be larger in Proposition 5. (Note that Propositions 3 and 4 only consider campaign promises, and so no up-front

[11] Similarly, if we view α as a subjective (incorrect) prior of being pivotal we would expect it to be small in large elections. See also footnote 5.

promises would be considered, contingent or otherwise.) The idea of the proof is the following: suppose that the winner changed from X to Y due to the introduction of such contingent promises. Then in equilibrium, any of Y's promises turn out not to be contingent. By using non-contingent promises according to the original equilibrium strategy X can defeat Y's strategy. While this stops short of being a proof, it provides the essential ideas. Nevertheless, we do think it would be of interest to study such contingent vote buying on its own. As mentioned the main difficulty then is how to appropriately model voter behavior.

7.5 Related literature

As mentioned in the introduction, there are three literatures that have had something to say about vote buying. Having our results as a backdrop, we can discuss and contrast the results from those literatures with what we have shown here.

7.5.1 Colonel Blotto Games

A "Colonel Blotto Game" is one where two opposing armies simultaneously allocate forces among n fronts. Any given front is won by the army that committed a larger force to that front and the overall winner is the army that wins a majority of the fronts. This model can be readily interpreted as a model of electoral competition, where each party wins the voters to whom it made the larger promise and the overall winner of the election is the party that managed to win a majority of the votes. Indeed formal models of electoral competition with promises using this framework date back at least to Gross and Wagner's (1950) continuous version of a Colonel Blotto game.

One difficulty in using the Colonel Blotto Game to deduce anything about vote buying is that, even in the simplest setting with identical voters and candidates, such games are notoriously difficult to solve.[12] The existing analyses are of symmetric mixed strategy equilibria in which voters are treated identically (from an ex ante point of view) and the parties are equally likely to win.

In an important contribution Myerson (1993) circumvents some of the technical difficulties of Colonel Blotto games by allowing candidates to meet the budget constraint on average, rather than exactly, which renders the game much more tractable.[13] In particular, Myerson considers a simultaneous move game that is similar to the platform game we

[12]See Laslier and Picard (2002) and Szentes and Rosenthal (2003) for some characterizations of equilibria.

[13]We adopt a similar assumption in our analysis of the case with uncertainty about preferences.

analyze (where parties promise payments conditional on winning and not on individual voting behavior), but where parties' can offer random payments to each voter and the payments need only meet the budget in expectation. As in the previous Colonel Blotto literature, Myerson assumes voters and parties are symmetric, and derives a symmetric mixed strategy equilibrium in which parties exhaust their budgets. Our work differs from this in two (significant) ways. First, the sequential version of our game enables us to consider asymmetric voters and parties. This allows us to see how preferences and budgets matter in determining who wins in the vote buying game. If we just look at our campaign-promise game for the basis of comparison, then we can see how payments are distributed across voters as a function of their preferences. When voter preferences are known, then parties concentrate their competition completely on near-median voters. When voter preferences are unknown, payments are uniform across voters.[14] Second, we allow for two types of promises. It turns out that the up-front vote buying game has substantially different outcomes and intuitions than the campaign-promises game.

7.5.2 The Political Science Literature on Vote Buying

Groseclose and Snyder (1996) present a model of vote buying in a legislature. Their model is similar to the up-front vote buying version of our analysis, except for the distinction that their model ends after two rounds. This drastically alters the strategic quality of the game as in their analysis the second mover has a substantial advantage. The first mover has to purchase a supermajority of voters in order to successfully block the response of the second mover. Thus, for example, if all voters were indifferent between candidates, the first mover would need twice the budget of the second mover in order to win, since the second mover should not be able to purchase the least expensive 50%. As is evident from the above analysis, our more symmetric bidding process neutralizes the affect of the order of moves and consequently gets significantly different results both with respect to

See also Lizzeri (1999) who allows for asymmetries in the budgets to study why parties may create budget deficits, and Lizzeri and Persico (2001), who study games where candidates can choose whether or not to offer a public good in addition to a redistribution.

Our platform game and that of Myerson are also related to an earlier literature where parties compete in offering (simultaneously) redistributive platforms, where negative payments (taxation) is allowed. This literature includes, for example, Cox and McCubbins (1986), Dixit and Londregan (1996) and Lindbeck and Weibull (1987).

[14] In our case, this is ex post as well as ex ante: preference uncertainty substitutes for the random payments in Myerson's game.

the identity of the winner, and how much they pay and which voters they buy.[15] There is also theoretical and empirical literature on vote buying in popular elections. Anderson and Tollison (1990) claim that vote buying was wide spread (though never fully legal) in Britain and the USA prior to the introduction of secret ballots towards the end of the nineteenth and beginning of twentieth centuries. Their main hypothesis is that the elimination of vote buying contributed to the historical rise in government expenditures on redistributive policies.[16] Buchanan and Tullock (1962) discuss the rationale for the prohibition of vote buying. They observe that under unanimity voting rule, free trade in votes would lead to efficiency. They suggest however that this might not be the case when a simple majority rule is in force. They do not model the market for votes formally, but argue intuitively that a perfect market for votes would lead to efficiency, but that imperfections are likely to arise **and** might preclude efficiency. Our analysis provides in a sense a particular formal interpretation to these ideas. Neeman (1999) points out that, with some uncertainty over voters' behavior, pivot considerations are of marginal importance and hence vote buying (by a single buyer) need not result in efficiency.[17] Our own analysis of efficiency focuses on the next step—it inquires about the efficiency consequences of competition between vote buyers.

There are also related papers on lobbying that have roots in the common agency literature, such as Bernheim and Whinston (1986), Grossman and Helpman (1994), Dixit (1996), and Le Breton and Salanie (2003), among others. As such models generally look at a single voter (the politician or agent), the complete information solutions result in efficient outcomes (e.g., see Bernheim and Whinston (1986) and Le Breton and Salanie

[15]Baron (2001) analyzes a game where two competing lobbyists can make offers to legislators in repeated rounds. His game differs from ours in that he models agenda setting and the legislative game in much more detail (whereas we take two alternatives as fixed), and lobbyists pay to get their alternative proposed in addition to buying votes to get it passed. The agenda setting part of the game enriches the interaction substantially, but also makes it difficult to obtain general characterizations of equilibrium. Nevertheless, Baron obtains some interesting results on the pattern of the resulting majority and how it relates to the proposal process. Given the difference in game structure and focus, his work and ours are largely complementary.

[16]This observation can be related to our results concerning the comparison between competition in direct vote buying and competition in campaign promises when vote buying is not allowed. If payments for votes were small this would correspond to our results on large elections with and without up-front vote buying.

[17]This and the point made by Buchanan and Tullock regarding efficiency of vote trading under unanimity are just alternative statements of the observation we made above that trading results in efficiency when every voter is pivotal.

(2003)).[18] In particular the politician as well as each lobbyist ends up being pivotal; as if some lobbyist is making a payment that is not pivotal in swaying the politician, then they could lower their payment and not affect the outcome. This reinforces the idea that the inefficiencies that we uncovered are due to the fact that in many contexts at least some players end up not being pivotal in a vote buying game when the vote is not by unanimity.

There are other articles that are related in that they address the same considerations that motivate us. But those discussions are so distant in terms of their focus and framework that they should be considered largely complementary to our discussion, and it does not seem useful to try to relate them to our analysis. For example, Kochin and Kochin (1998) offer a logic for the prohibition of vote buying, which is based on the costs of buying votes and forming blocking coalitions. This, they argue, can lead to inefficient decisions depending on the source of costs and how they are distributed. They suggest that in the absence of any costs, vote buying will always lead to efficient decisions, although the specific vote buying process is not modeled.[19] Philipson and Snyder (1996) find Pareto improvements from vote buying. They model a specialist system for vote buying, and a one dimensional policy space, and find that, if the distribution of ideal points is skewed enough, then the equilibrium with vote buying differs from the equilibrium without vote buying (the median ideal point). This difference reflects the ability of an intense minority to obtain a policy it prefers in exchange for side payments.

7.5.3 Corporate control

The literature on corporate control (Harris and Raviv(1988), Grossman and Hart (1988)) is also related to our analysis. They examine settings in which two alternative management teams—an incumbent and a rival—are competing to gain control of a corporation through acquisition of a majority of the shareholders' votes. The alternative teams are the counterparts of our parties and the private benefits that these teams would extract from controlling the corporation are the counterparts of the parties' valuations for being elected. The model of Harris and Raviv[20] (henceforth HR) has one round of bidding

[18] As such, the focus of many of these models has been on various distributional issues such as taxation and redistribution, or the politics of protectionism and international trade.

[19] This idea is also implicit in arguments by Tobin (1970), who suggests that a market for votes would allow power to be concentrated among the rich.

[20] The related model of Grossman and Hart does not seem to have an explicit equilibrium model for the case that would be close to our model (what they call competition in restricted offers between parties with significant private benefits).

by each team (like that of Groseclose and Snyder), but with an additional difference that offers are not made to specific voters[21]—but to the public at large—with a cap on the number of shares that will be purchased, and then rationing if too many shares are tendered to one of the teams. Harris and Raviv characterize an equilibrium where the efficient team wins; that is, the team that maximizes the total shareholder value plus the private benefit of being the management team. However, that equilibrium relies critically on every voter believing that their tendering decision will be completely pivotal, and as such it is very fragile in the sense that any uncertainty about the number of shares, actions of other voters, or offers, etc., would destabilize the equilibrium.[22]

We believe the Harris and Raviv game has other equilibria which are stable and can be described as follows.[23] Voters do not believe they will be pivotal and tender so as to equate their expected revenue (price times probably of not being rationed) across the parties. In this equilibrium the party offering the higher price wins (with prices on a grid). Going backwards, this implies that in the overall equilibrium the party with the higher budget wins (with a payment that depends on whether it moves first or second, as in the analysis of Groseclose and Snyder (1996)).

As mentioned above, the instability of the efficient equilibrium that Harris and Raviv analyze seems to make it less plausible than our conjectured alternative, inefficient, equilibria. One question that comes to mind is why those equilibria differ from the Groseclose and Snyder (1996) equilibria which have a strong second mover advantage? The answer is that the Harris and Raviv model does not have targeted offers, but instead offers made to voters at large, with the possibility of some rationing. The absence of targeting effectively eliminates the second mover advantage (there are still some advantages of being second mover in the amount paid in equilibrium, but not in who wins). Thus, these conjectured equilibria would be closer to our model where there are repeated rounds and the largest budget has an advantage. Of course, given that the Harris and Raviv model does not have targeted offers, it would not permit an analysis of vote buying in the presence of

[21]In the corporate control model, all voters have identical preferences based on the difference in share value that will be generated under the two teams.

[22]Their model has a continuum of voters and so is not quite a closed game theoretic model. It appears that a large finite approximation to this equilibrium could be built, but the equilibrium would be unstable in that any shift in bidders' beliefs would lead to a change in their tendering strategies, and thus a movement to another equilibrium in the subgame (the one conjectured next in the text).

[23]These are equilibria that we conjecture, but are not mentioned by Harris and Raviv. We do not provide a formal analysis, as it would take a good deal of space to set up the model, for a relatively tangential point.

heterogeneous voters as we have analyzed here.

7.6 Minimal Payments

One fairly, straightforward prediction of our model is that with up-front vote-buying, no uncertainty, and small α, there will tend to be minimal spending in equilibrium. This is broadly consistent with some stylized facts that we see both in political elections and stock shares. For instance Ansolabehere, de Figueiredo and Snyder (2002) document the paucity of money being contributed to political campaigns and find that the largest part of the relatively small donations to campaigns comes from individuals and has little impact on legislator's votes (a puzzle first pointed out by Tullock (1972)). One could view the money contributed as attempts to "buy" votes. One also sees this in the price of stock shares, where the price of voting shares is generally similar to that of non-voting shares (Lamont and Thaler (2001)). While our highly stylized analysis is certainly not the only explanation for these observations, it does provide some intuition for them.

8 References

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9 Appendix

Proposition 1: *The vote-buying game has an equilibrium in pure strategies. In every equilibrium the same party wins, and the losing party never makes any payment (but may make contingent promises that do not result in payments).*

Proof of Proposition 1: The facts that the vote-buying game has an equilibrium in pure strategies follows from the fact that this is a finite game of perfect information, and hence we can find such an equilibrium via backwards induction.

The fact that in every equilibrium the same party wins, also follows from a backward induction argument. Each terminal node has a unique winner (as the αU_i's are not a multiple of ε and so voters are never indifferent), and parties prefer to win regardless of the payments necessary. Thus, in any subgame, working by induction back from nodes whose successors are only terminal nodes, there is a unique winner. It then follows directly that the losing party never makes any payments, as they could otherwise deviate to offer nothing and guarantee no payment. ∎

Proposition 2: *The winner in any equilibrium of the vote-buying game when both up-front payments and campaign promises are permitted, is the same as the winner in any equilibrium of a modified version of the game where only up-front payments are allowed.*

Proof of Proposition 2: By Proposition 1, we know that there is a unique winner in every equilibrium of the unmodified game. Without loss of generality, say that X is the winner and Y is the loser, of the game where both forms of promises are permitted. Consider a game where X is permitted to make both forms of promises and Y is only permitted to make up-front payments. As this only imposes a restriction on Y's strategies, X remains the winner of all equilibria of this game.[24] Next we note that there exists an equilibrium in this game where at any node X only makes up-front payment promises (or no promises), as at any point an up-front payment is at least as attractive to a voter as an equivalent campaign promise and is at least as flexible for X as it is no more binding.[25] This (properly trimmed) remains an equilibrium of the game (with the trimmed tree) where no campaign promises are permitted. ∎

Proposition 4: *If B^X and B^Y are distributed with full support over $\{0, \varepsilon, ..., B\varepsilon\}$, then in any equilibrium:*

1. *Both parties play LEM strategies.*
2. *Y wins if $B^Y \geq B^X + T$ and ends up pledging exactly $B^X + T$, and X wins otherwise and ends up pledging exactly* $\max\{B^Y - T + \varepsilon, 0\}$.

Proof of Proposition 4: The proof is based on three lemmas. First, we characterize the outcomes resulting when at least one player follows LEM strategies. Second, we conclude that there is an equilibrium in which both play LEM strategies. Third, we prove that in any equilibrium LEM strategies are played by both.

LEMMA **1** *1. If $B^Y \geq B^X + T$, then*

[24] More formally, start with an equilibrium in the larger game. Trim the tree so that we eliminate any actions of Y that result in campaign promises. By backward induction, in any subgame of the resulting tree if X won previously, X still wins, while if Y won, then either Y still wins or else X wins. As X won previously in the overall game, X still wins.

[25] To be careful, we need to keep track of Y's responses to X's actions. However, given that Y can only make up-front payments, using a backward induction argument we can establish that in any subgame X's chance of winning (which is either 0 or 1 in any subgame) can only go up by a switch from a campaign promise to an equivalent up-front payment.

(a) *If X uses an LEM strategy then with an LEM strategy Y wins and spends $B^X + T$.*

(b) *If X adopts an LEM strategy, then to win Y must spend at least $B^X + T$.*

(c) *If Y uses the LEM strategy then X cannot win.*

2. *If $B^Y < B^X + T$, then*

(a) *If Y uses an LEM strategy then with an LEM strategy X wins and spends $B^Y - T + \varepsilon$.*

(b) *If Y adopts the LEM strategy then to win X must spend at least $B^Y - T + \varepsilon$.*

(c) *If X uses the LEM strategy then Y cannot win.*

Proof of Lemma 1: 1a and 2a follow immediately from the nature of the LEM strategies: Y initially must buy (we use the term buy to indicate voters who are convinced by the platform to vote for the buying party) $n - m + 1$ of the voters from m to n at cost T; X then must buy one voter with an additional cost of ε (either one of those bought by Y or possibly $n + 1$ if $|U_{n+1}| < \varepsilon$); Y then must buy a voter back at additional cost ε; and so on. Iff $B^Y \geq B^X + T$ will this process end with Y winning. □

1b is proved by induction on B^X as follows. Clearly, 1b is true for $B^X = 0$ and any T. Suppose it is true for $B^X \leq K$ and for all T, and consider $B^X = K + \varepsilon$. Let $\hat{T}$ be the sum spent by Y in its first step. Clearly, $\hat{T} \geq T$. Following its LEM strategy X pays some S such that $\varepsilon \leq S \leq \hat{T} - T + \varepsilon$. If X's budget is such that it cannot purchase a majority then any payment more than T by Y in the first step is redundant. Otherwise, after X's purchase, the situation is equivalent to an initial configuration with $T' = \varepsilon$, $B^{Y\prime} = B^Y - \hat{T}$ and $B^{X\prime} = B^X - S \geq B^X - \left(\hat{T} - T + \varepsilon\right)$. Since $B^{X\prime} \leq K$, by the inductive assumption Y must spend from this point on at least $B^{X\prime} + \varepsilon$ and hence Y's overall expenditure will be $B^{X\prime} + \varepsilon + \hat{T}$. Now, this and $B^{X\prime} \geq B^X - \left(\hat{T} - T + \varepsilon\right)$ imply that Y's overall expenditure is at least $B^X - \left(\hat{T} - T + \varepsilon\right) + \varepsilon + \hat{T} = B^X + T$. So Y cannot benefit from spending more than T, and as noted above can lose. (Note that Y spending T initially is an LEM strategy for Y.) □

For all x, Part $2x$ is the counterpart of $1x$. In particular, $2b$ is analogous to $1b$. Finally, $1c$ follows from $2b$. This completes the proof of the lemma. ▮

Lemma **2** *LEM strategies for both parties constitute an equilibrium.*

Proof of Lemma 2: For $B^Y \geq B^X + T$, $1a$ and $1b$ of Lemma 1 imply that Y's LEM strategy is best response against X's LEM strategy. $1c$ implies that X's LEM strategy is best response against Y's LEM strategy. Analogously, $2a$–$2c$ of Lemma 1 imply that X's and Y's LEM strategies are mutual best responses when $B^Y < B^X + T$. ▮

Lemma **3** *All equilibria use LEM strategies.*

Proof of Lemma 3: The proof is by induction on B (the number of multiples of ε that bounds B^X and B^Y). For $B = 1$ the proposition is obviously true. Suppose that it is true for $B = K$; we now prove that it holds for $B = K + 1$.

If $B^Y < T$, then the claim follows immediately. Otherwise, in the first step Y promises some $\hat{T} \geq T$. The new situation then is $T' < 0$, $B^{X\prime} = B^X \leq (K+1)\varepsilon$ and $B^{Y\prime} = B^Y - \hat{T} \leq K\varepsilon$. If $B^X < |T'|$, then by definition the parties follow LEM strategies from that point on. Otherwise, to become the current winner X spends $S > |T'|$. This results in the configuration $T'' \in (0, S + T']$, $B^{X\prime\prime} = B^{X\prime} - S = B^{X\prime\prime} - S \leq K\varepsilon$ and $B^{Y\prime\prime} = B^{Y\prime} = B^Y - \hat{T} \leq K\varepsilon$. Notice that if X is playing a best response, then $T'' \leq K\varepsilon$, since if X makes $T'' = (K+1)\,\varepsilon$ then X wins at a cost that with positive probability is higher than necessary (recall that Y's budget was bounded by $(K+1)\,\varepsilon$). Therefore, X's best response would result in $T'' \leq K\varepsilon$.

Thus, following X's move, the inductive assumption applies and Y wins iff $B^{Y\prime\prime} \geq B^{X\prime\prime} + T''$ at incremental cost (from here on) of $B^{X\prime\prime} + T''$; X wins otherwise at incremental cost of $B^{Y\prime\prime} - T'' + \varepsilon$. Translating this to the original data, Y wins if $B^Y - \hat{T} \geq B^X - S + T''$, in which case its *overall* expenditure (from the start) will be $B^X - S + T'' + \hat{T}$, and X wins if and only if $B^Y - \hat{T} < B^X - S + T''$, in which case its overall expenditure will be $\max\left\{B^Y - T'', 0\right\} + S + \varepsilon$. Observe that, subject to the constraint $S \geq |T'|$, X's winning probability is maximized and its expected expenditure is uniquely minimized at $S = |T'|$, which is exactly what is required by an LEM strategy for X. Now, going back to Y's first move, this implies that Y will win iff $B^Y - \hat{T} > B^X + T'$, at overall expense of $\hat{T} + B^X - |T'| - \varepsilon$. Now, subject to the constraint $\hat{T} \geq T$, Y's winning probability is maximized and its expected expenditure is uniquely minimized at $\hat{T} = T$, which again corresponds only to LEM strategies for Y. ▮

This completes the proof of Proposition 4. ▮

Proposition 5: *In the small α case, party X wins in (every) equilibrium if and only if $B^X \geq B^Y + (m - n)\,\varepsilon$. In any equilibrium where X wins, its total payments are bounded above by $\frac{m\alpha B^Y}{m-1} + m\varepsilon$.*

Proof of Proposition 5: By Proposition 2, we can determine the winner by examining the game with only up-front payments. We then come back to bound the winner's payments in the game where campaign promises are also possible.

Suppose that $B^X \geq B^Y + (m-n)\,\varepsilon$. We show that then X has a strategy that guarantees a win. As a symmetric argument applies to show that Y wins if $B^X < B^Y + (m-n)\,\varepsilon$, this implies the if and only if statement. We show that the LEM strategy whereby in each stage of the bidding X acquires the least expensive available smallest majority (i.e., m voters), and purchases voters who prefer Y whenever the cost is the same, guarantees a victory to X against any bidding strategy that Y might adopt. This implies immediately that, in equilibrium, Y will only make offers if she expects X to overbid all her offers. As X bids for only the least expensive voters this can occur only if $n \leq m$. In this case X will have spend at least $\varepsilon\,(m-n)$ to purchase the majority. There are equilibrium in which X spends up to εm. In these equilibria Y bids ε for up to n voters i for which $\alpha U_i > 0$, and X buys them back.

We now argue that X wins with the LEM strategy above. A "current winner" at a point in the bidding process will refer to the party that would win if the process terminated at that point, and an "active offer" will refer to an offer that would be taken by a voter in the equilibrium of the selling game that would be played if the process were stopped at that point. Observe that if Y is the current winner and has a sum B committed in active offers, then X has to commit at most $B + (m-n)\,\varepsilon$ to become a current winner. To see this suppose that Y is the current winner, let p^Y be the m^{th} highest active offer that Y has outstanding, where we rank voters with identical offers from Y higher if they prefer Y to X, i.e., if $U_i < 0$. Let voter j be the target of that l^{th} highest offer. Let p^X be the highest active offer that X must have in order to become the current winner in the least expensive way, and let voter i be the target of that offer.

If $U_j > 0$ then $p^X \leq p^Y$ for otherwise, it would be cheaper for X to acquire j's vote instead of i's vote. (Recall that when faced with the same offers the voter sells to her preferred party.) Since to become current winner X needs only m active offers, it follows that its cost would be at most $p^X m \leq p^Y m \leq B$, where $p^Y m \leq B$ since to be a current winner Y must have at least m active offers with p^Y being the m^{th} highest offer.

If $U_j < 0$ the argument is similar, but requires a little care in counting. In this case assume that $k \leq n$ of the voters who prefer X have active offers from Y. By the ranking described above, these voters have an offer of at least $p^Y + \varepsilon$. Now consider those voters not receiving any of the m highest active offers from Y. These include $n-k$ voters who prefer X and whose offers from Y must be at most $p^Y - \varepsilon$. Therefore to purchase enough votes X needs at most $\left(p^Y+\varepsilon\right)m-(n-k)\varepsilon$, where $p^Y m+k\varepsilon \leq B$, since to be a current winner Y must have at least m active offers with p^Y being the m^{th} highest offer and at least k voters have active offers of $p^Y+\varepsilon$. Therefore $\left(p^Y+\varepsilon\right)m-(n-k)\varepsilon$ $=p^Y m+\varepsilon(m-(n-k)) \leq B-k\varepsilon+\varepsilon(m-(n-k))=B+\varepsilon(m-n)$.

This implies that, when X follows that LEM strategy, it can always outbid Y to become the current winner. Since the bidding process must end after a bounded number of rounds, X must win. Since X must buy $m-n$ votes, she must spend at least $\max\{(m-n)\varepsilon, 0\}$. If Y makes an offer to any of the votes that X purchased, then it would cost X more to repurchase that vote than to purchase a different one, and after X's purchase of a different vote Y will eventually lose and have to pay something, which is worse for Y than not purchasing in the first place (by hypothesis) so in equilibrium Y will not purchase back a vote that X purchased. If Y purchases a vote from i such that $U_i>0$ then X is indifferent between purchasing this vote back at cost ε and purchasing a different vote from j with $U_j<0$, so, as noted, there is an equilibrium where Y offers ε to some of the $n \leq m$ voters and X purchases them back, leading to total cost of up to $m\varepsilon$.

Now, let us come back to bound the payments that X makes when X wins in the game where both up-front payments and campaign promises are possible. X can still follow an LEM strategy, and that will still win. As Y surely loses, Y will not be making any binding up-front payments in equilibrium. Thus, consider the ending promises that are made by Y. It must that X has bought a least expensive majority, meaning that the maximum price paid for any voter in this majority is at most the minimum price of the voters not purchased. Any promises made by Y to the voters that X did not purchase must have been made in the form of campaign promises. The highest the minimum cost could be is then $\frac{\alpha B^Y}{m-1}+\varepsilon$. The claimed expression then follows directly. ∎

Proposition 6: *If the budgets are large enough so that (3) and (4) are satisfied, then X wins if*

$$B^X \geq B^Y-\sum_i \alpha U_i/2-\alpha U_N/2+m\varepsilon \tag{4}$$

and Y wins if

$$B^Y \geq B^X+\sum_i \alpha U_i/2+\alpha U_1/2+m\varepsilon. \tag{5}$$

Proof of Proposition 6

We prove the following result, which implies Proposition 6.

Party X wins if

$$B^X - B^Y \geq -\sum_i \alpha U_i/2 - \alpha U_N/2 + m\varepsilon \text{ and} \tag{4}$$

$$B^X \geq \left|\frac{m\alpha U_1}{2}\right| - \frac{\sum_{i=m+1}^{N} \alpha U_i}{2} - \frac{\alpha U_N}{2} + m\varepsilon \tag{2}$$

and party Y wins if

$$B^X - B^Y \leq -\sum_i \alpha U_i/2 - \alpha U_1/2 - m\varepsilon \text{ and} \tag{5}$$

$$B^Y \geq \left|\frac{m\alpha U_N}{2}\right| + \frac{\sum_{i=1}^{m-1} \alpha U_i}{2} + \frac{\alpha U_1}{2} + m\varepsilon. \tag{3}$$

We show that X has a strategy that guarantees that X wins if (3) and (5) are satisfied. The other case is analogous.

Party X can guarantee a win using the strategy we describe next. Have X allocate offers as follows. Let t be the period. X will identify a set of voters S_t to " buy" that has cardinality exactly m. X will make the minimal necessary offers to buy these votes.

To complete the proof we need only describe how X should select S_t, and then show that if X has followed this strategy in past periods, then X will have enough budget to cover the required payments regardless of the strategy of Y.

Let p_i^Y be the current offer that Y has to voter i. Set this to 0 in the case where Y has never made a viable offer to the voter, or in a case where X already has the best standing offer to the voter. Similarly define p_i^X.

X selects to whom to make offers by looking for those with that minimize the sum of what X has to offer, plus what offers of Y's that X frees up. In particular, let S_t be the set of voters than minimizes $\sum_{i \in S_t} 2p_i^Y - \alpha U_i$. This is equivalent to choosing the m voters that have the smallest values of

$$p_i^Y - \frac{\alpha U_i}{2}.$$

In the case where there are some i's that are tied under the above criterion, let X lexicographically favor voters with lower indices. To complete the proof, we simply need to show that this strategy is within X's budget in every possible situation, presuming that X has followed this strategy up to time t.[26]

[26]This implies the proposition, as it means that either Y will not respond and the game will end with X the winner, or else X will get to move again and can again follow the same strategy. As the game must end in a finite number of periods, this implies that X must win.

Notice that the cost of a voter $i \in S_t$ to X is at most

$$\left[p_i^Y - \alpha U_i\right]^+ + \varepsilon. \tag{7}$$

The expression $\left[p_i^Y - \alpha U_i\right]^+$ captures the fact that it could be that $p_i^Y < \alpha U_i$ in which case no offer is necessary.

The amount that must be offered to a voter can only rise or stay constant over time, and so if some voters were " purchased" by X in the past and have not been subsequently purchased by Y, then these voters are still among the cheapest m available in the current period time and would still be selected under X's strategy (including the lexicographic tie-breaking).

Let i^* denote the most "expensive" $i \in S_t$ in terms of the "adjusted price" $p_i^Y - \frac{\alpha U_i}{2}$. If there are several voters tied for this distinction, pick the one with the lowest index. So, $i^* \in \arg\max_{i \in S_t} \left\{p_i^Y - \frac{\alpha U_i}{2}\right\}$, and let $\overline{S}_t$ be the complement of S_t union $\{i^*\}$.

Given the algorithm followed by X, we know that

$$p_i^Y - \frac{\alpha U_i}{2} \leq p_{i^*}^Y - \frac{\alpha U_{i^*}}{2}$$

for every $i \in S_t$. This can be rewritten as

$$p_i^Y \leq p_{i^*}^Y - \frac{\alpha U_{i^*}}{2} + \frac{\alpha U_i}{2} \tag{8}$$

for each $i \in S_t$.

Equations (7) and (8) imply that the amount required by X to follow this strategy at this stage is at most

$$\sum_{i \in S_t} \left[p_{i^*}^Y - \frac{\alpha U_{i^*}}{2} - \frac{\alpha U_i}{2}\right]^+ + m\varepsilon \tag{9}$$

If we can get an upper bound on the expression $p_{i^*}^Y - \frac{\alpha U_{i^*}}{2}$, then we have an upper bound on how much X has to pay. So we want to maximize $p_{i^*}^Y - \frac{\alpha U_{i^*}}{2}$ subject to the following constraints:

(1) $p_i^Y - \frac{\alpha U_i}{2} \geq p_{i^*}^Y - \frac{\alpha U_{i^*}}{2}$ for every $i \notin S_t$,

(2) $p_i^Y \geq \alpha U_i + p_i^X$, and

(3) $\sum_{i \in \overline{S}_t} p_i^Y \leq B^Y$.

To get an upper bound, we ignore (2), and relax (3) by replacing B^Y with $B^Y = \max\left\{B^Y, \left|\frac{m\alpha U_1}{2}\right| + \frac{\sum_{i=1}^{m}\alpha U_i}{2}\right\}$. The solution then involves spending all of $\bar{B}^Y$ in a manner that equalizes $p_i^Y - \frac{\alpha U_i}{2}$ with $p_{i^*}^Y - \frac{\alpha U_{i^*}}{2}$ for each $i \notin S_t$. (This is feasible due to the lower bound imposed on B^Y; it is not necessarily feasible for B^Y, but still gives a bound). Thus, we end up with

$$p_i^Y = x^Y\left(\overline{S}_t\right) + \alpha U_i/2,$$

for each $i \in \overline{S}_t$, where

$$x^Y(\overline{S}_t) = \frac{\bar{B}^Y - \sum_{i\in\overline{S}_t}\frac{\alpha U_i}{2}}{m} \tag{10}$$

From (9), for X's strategy to be feasible it is sufficient that

$$B^X \geq \sum_{i\in S_t}\left[x^Y\left(\overline{S}_t\right) - \alpha U_i/2\right]^+ + m\varepsilon.$$

Substituting for x^Y from (10), this becomes

$$B^X \geq \bar{B}^Y - \sum_{i\in S_t\cup\overline{S}_t}\alpha U_i/2 + m\varepsilon.$$

This simplifies to

$$B^X \geq \bar{B}^Y - \sum_i \alpha U_i/2 - \alpha U_{i^*}/2 + m\varepsilon,$$

which has an upper bound when $i^* = N$, and which then yields the claimed expressions by substituting the definition of B^Y. ∎

Proposition 7: *For any $\delta > 0$, there is $N(\delta)$ and $\bar{\varepsilon}$ such that for all $N > N(\delta)$ and all grids with $\varepsilon \in (0, \bar{\varepsilon})$ the following hold.*

- *If $B^Y > B^X + \alpha\bar{U}N/2 + \delta$, then Y wins with probability of at least $1 - \delta$.*
- *If $B^X > B^Y - \alpha\bar{U}N/2 + \delta$, then X wins with probability of at least $1 - \delta$.*

Proof of Proposition 7:

Lemma 4 *Suppose that Party Y offers a constant price x to all voters, such that $1 > F(x) > 0$. The least expensive way for Party X to assure itself expected share $\sigma \in [0, 1]$ of the vote would be offering a constant price to all voters. The same is also true with the roles reversed.*

Note that we do not assume here that the constant price offered by X is a multiple of ε. If that constraint were added, then the cost to X of obtaining a share σ would be at least as high (and might involve a different strategy).

Proof of Lemma 4: The problem of finding bids p_i^X that Party X can make to assure expected share σ at minimum cost is

$$\min_{\{p_i^X\}} \sum_i p_i^X[1 - F(x - p_i^X)] \text{ s.t. } \sum_i 1 - F(x - p_i^X) \geq N\sigma, p_i^X \geq 0. \tag{11}$$

The first order conditions to (11) can be written as

$$p_i^X f(x - p_i^X) + 1 - F(x - p_i^X) - \frac{\lambda}{N} f(x - p_i^X) - \mu_i = 0. \tag{12}$$

where λ and μ_i are nonnegative multipliers.

Given that the support of F is connected and f is positive on F's support, we have three possible ranges for solutions to (12): one where $f(x - p_i^X) = 0$ and $F(x - p_i^X) = 0$, one where $f(x - p_i^X) > 0$ and $0 < F(x - p_i^X) < 1$, and one where $f(x - p_i^X) = 0$ and $F(x - p_i^X) = 1$. The first order conditions cannot be satisfied in the first case, unless $\mu_i = 1$ in which case the non-negativity constraint is binding and $p_i^X = 0$. However, by hypothesis, $0 < F(x - 0)$, which is a contradiction of the presumption of the case that $F(x - p_i^X) = 0$. In the third case, for $f(x - p_i^X) = 0$ and $F(x - p_i^X) = 1$ to hold, since $1 > F(x)$ it must be that $p_i^X < 0$. However, this cannot be a solution given the non-negativity constraint. Thus all possible solutions must fall in the second case. In the second case, in order to satisfy the first order conditions, it must be that $p_i^X \leq \frac{\lambda}{N}$. [If $\mu_i = 0$ then this is clear since $(1-F) > 0$. If $\mu_i > 0$, then the constraint that $p_i^X \geq 0$ must be binding, in which case $p_i^X = 0$ and again $p_i^X \leq \frac{\lambda}{N}$.] For this case, since $f(x - p_i^X) > 0$, we rewrite (12) as

$$x - p_i^X - \frac{1 - F(x - p_i^X)}{f(x - p_i^X)} - (x - \frac{\lambda}{N}) + \frac{\mu_i}{f(x - p_i^X)} = 0. \tag{13}$$

Suppose that there are two solutions, p_i^X and p_j^X to (13) in this range. Without loss of generality, letting $z^i = x - p_i^X > z^j = x - p_j^X$, we have

$$z^i - \frac{1 - F(z^i)}{f(z^i)} - (x - \frac{\lambda}{N}) + \frac{\mu_i}{f(z^i)} = 0 = z^j - \frac{1 - F(z^j)}{f(z^j)} - (x - \frac{\lambda}{N}) + \frac{\mu_j}{f(z^j)}.$$

Since $z - (1 - F(z))/f(z) = z + (F(z)) - 1)/f(z)$ is increasing (in this range where $f(z) > 0$), it follows that $0 = \mu_i < \mu_j$. (Note that μ_i takes on only two values.) But this implies $p_j^X = 0 < p_i^X$, which contradicts the fact that $z^i > z^j$.

Thus we have shown that any solution to (11) necessarily has identical prices offered to all agents.

The proof for Lemma 4 with the roles reversed for the parties has (11) replaced by

$$\min_{\{p_i^Y\}} \sum_i p_i^Y[F(p_i^Y - x)] \text{ s.t. } \sum_i F(p_i^Y - x) \geq N\sigma, p_i^X \geq 0,$$

with corresponding first order conditions

$$p_i^Y f(p_i^Y - x) + F(p_i^Y - x) - \frac{\lambda}{N} f(p_i^Y - x) - \mu_i = 0.$$

Working through similar cases as those above, and this time using the fact that $z + F(z)/f(z)$ is increasing on the support of F, yields the same conclusion. $\square$

LEMMA **5** *If* $(0.5+\eta)N[\frac{B^X}{(0.5-\eta)N} + F^{-1}(0.5-\eta)] < B^Y$, *then* Y *can obtain expected share* $(0.5+\eta)$ *of the vote at each stage. Similarly if,* $(0.5+\eta)N[\frac{B^Y}{(0.5-\eta)N} - F^{-1}(0.5+\eta)] < B^X$, *then* X *can obtain a share of* $(0.5+\eta)$ *at each stage.*

Proof of Lemma 5: We show the first claim, as the second is analogous. Suppose that it is Y's turn. If Y can offer all voters the same price $p = B^X/(0.5-\eta)N + F^{-1}(0.5-\eta)$, then Y can win in one step. This is so since, by the previous claim, X's least expensive way of getting at least $(0.5-\eta)N$ is by offering the same price to all voters. A constant price that suffices here is $B^X/(0.5-\eta)N$ which exactly exhausts X's budget (ignoring the constraint that X must make offers in multiples of ε, and more than exhausts it if the constraint is taken into account). Now, since $B^X\frac{0.5+\eta}{0.5-\eta} + (0.5+\eta)NF^{-1}(0.5-\eta) < B^Y$, the price p is feasible for Y when only $(0.5+\eta)N$ voters (or slightly more) accept it. Thus, if p is infeasible at that stage, then there are more than $(0.5+\eta)N$ voters who would prefer to sell to Y at that price. But this means that there is a lower price $p' < p$ that gives Y an expected majority of $(0.5+\eta)N$. Since $(0.5+\eta)Np' < (0.5+\eta)Np < B^Y$, the price p' is feasible. Clearly, if p' is not a multiple of ε then for any ε small enough there is a p'' that is slightly larger that also gives Y an expected majority of $(0.5+\eta)N$, and for a small enough grid size still more than exhausts X's budget. $\square$

We now show (1) and (2) of the proposition. We concentrate on (1), as the other case is analogous, given the lemmas above. For $\delta > 0$, there exists sufficiently small $\eta > 0$ such that $(0.5+\eta)N[\frac{B^X}{(0.5-\eta)N} + F^{-1}(0.5-\eta)] < B^X + \alpha\bar{U}N/2 + \delta$. Therefore, if η is sufficiently small, $B^Y > B^X + \alpha\bar{U}/2 + \delta$ together with Lemma 5 imply that Y can obtain an expected share of $(0.5+\eta)$. When N is made sufficiently large (here we mean that B^X and B^Y increase proportionately with N), an expected share of $(0.5+\eta)$ means an arbitrarily large probability of winning. Therefore, there exists $N(\delta)$ such that, for $N > N(\delta)$, Y's winning probability is above $1 - \delta$. ▮

This complete the proof of Proposition 7. ▮

Proposition 9 *Suppose that U^X satisfies (3) in the place of B^X, and U^Y satisfies (4) in the place of B^Y. In the large budget case, party X wins in the campaign donations vote-buying game if*

$$U^X - U^Y \leq -\alpha U_N/3 + \frac{2}{3}m\varepsilon$$

and Y wins if

$$U^X - U^Y \leq -\alpha U_1/3 - \frac{2}{3}m\varepsilon.$$

The proof of Proposition 9 is an easy extension of the proof of Proposition 8, and is again omitted, noting simply that the above equations follow from (5) and (6) and a maximum willingness to donate of U_i, and that $\sum_i \alpha U_i = \alpha(U^X - U^Y)$.