A New Asymptotically Non-Scale Endogenous Growth Model

Taiji HARASHIMA
University of Tsukuba
Cabinet Office of Japan

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Abstract

The paper explores an endogenous growth model in which scale effects asymptotically vanish and an economy grows without population growth. The key mechanism behind these features is substitution between investing in capital and in knowledge when firms face growing uncompensated knowledge spillovers. The model shows that firms invest more in capital than in knowledge and thus scale effects asymptotically evaporate as the number of population and thus uncompensated knowledge spillovers increase, and an economy grows without population growth.

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Correspondence: Taiji HARASHIMA:
Graduate School of Systems and Information Engineering, University of Tsukuba,
1-1-1 Tenoudai, Tsukuba, Ibaraki 305- 8573, Japan
Email: tharashm@sk.tsukuba.ac.jp
t-harashima@mve.biglobe.ne.jp

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I. INTRODUCTION

Scale effects have been the central issue in the field of endogenous growth models over the last decade. The early endogenous growth models, e.g. Romer (1986, 1987) or Lucas (1988), had the nature of scale effects. However, this nature is not supported by existing empirical evidence, as e.g. Jones (1995a) shows. The source of scale effects lies in the assumption of a linear relation between K_t and A_t. The familiar Euler condition in case of a Harrod neutral production function such that $y_t = \frac{Y_t}{L_t} = A_t{}^{\alpha} k_t{}^{1-\alpha} = A_t{}^{\alpha}\left(\frac{K_t}{L_t}\right)^{1-\alpha}$ is $\frac{\dot{c}_t}{c_t} = \frac{(1-\alpha)\left(\frac{A_t}{k_t}\right)^{\alpha} - n_t - \theta}{\varepsilon}$ where y_t is outputs per capita, c_t is consumption per capita, k_t is capital inputs per capita, Y_t is outputs, K_t is capital inputs, L_t is labor inputs, A_t is knowledge/technology/idea, $n_t = \frac{\dot{L}_t}{L_t}$ is the growth rate of population, θ is the rate of time preference, ε is the coefficient of relative risk aversion, and α is a constant. Hence, if $\frac{A_t}{k_t}\left(= \frac{A_t L_t}{K_t}\right)$ and n_t are constant, then the growth rate of consumption can be constant. To make $\frac{A_t}{k_t}$ be constant, it is necessary that $\frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} = \varphi_1 \frac{\dot{A}_t}{A_t}$ where φ_1 is a constant.

The simplest solution to construct a model that satisfies $\frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} = \varphi_1 \frac{\dot{A}_t}{A_t}$ is to assume that there is a linear relation between K_t and A_t while $\frac{\dot{L}_t}{L_t} = 0$.[1] Early endogenous growth models like the familiar "*AK*" model adopt this strategy explicitly or implicitly.[2] Assuming a

[1] See e.g. Romer (1990), Grossman and Helpman (1991), or Aghion and Howitt (1992).

[2] Early human capital-based endogenous growth models are also categorized to this class of models.

linear relation between A_t and $K_t\left(=k_t L_t\right)$ means that $\frac{A_t}{k_t}=\frac{\varphi_2 K_t}{k_t}=\varphi_2 L_t$ where φ_2 is a constant. Hence, L_t plays an important role that is called scale effects.

Jones (1995b) adopts a completely different strategy.[3] This strategy focuses on the relation between L_t and A_t instead of the linear relation between K_t and A_t and assumes that there is a linear relation between $\frac{\dot{A}_t}{A_t}$ and $\frac{\dot{L}_t}{L_t}$ such that $\frac{\dot{A}_t}{A_t}=\varphi_3\frac{\dot{L}_t}{L_t}$ where φ_3 is a constant, and the only one case such that $\frac{\dot{K}_t}{K_t}=\left(1+\varphi_1\varphi_3\right)\frac{\dot{L}_t}{L_t}=\frac{\dot{Y}_t}{Y_t}$ is selected to be relevant because only this case satisfies both the relation $\frac{\dot{K}_t}{K_t}-\frac{\dot{L}_t}{L_t}=\varphi_1\frac{\dot{A}_t}{A_t}$ and a "balanced growth path."[4] This model can eliminate scale effects because there is no linear relation between K_t and A_t. Nevertheless, the growth rate of population $\frac{\dot{L}_t}{L_t}$, instead, plays a crucial role because of the linear relation between $\frac{\dot{A}_t}{A_t}$ and $\frac{\dot{L}_t}{L_t}$. In this sense, Jones' (1995b) model may not still appear perfectly successful as the endogenous growth model.

To eliminate the influence of population growth, Young (1998), Peretto (1998), Aghion and Howitt (1998), and Dinopoulos and Thompson (1998) propose the third approach. They assume a relation between $\frac{\dot{A}_t}{A_t}$ and L_t such that $\frac{\dot{A}_t}{A_t}=\varphi_4 L_t^{1-\varphi_5}$ where φ_4 and φ_5 are constants.

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[3] See also Kortum (1997), Segerstrom (1998), or Eicher and Turnovsky (1999).

[4] Another important feature of Jones' (1995) model is that it firstly limits the study to "balanced growth path" that is defined as all variables being growing at constant (exponential) rates, although it is not explained what forces are at work behind sustaining "balanced growth path." That is, this model keeps away from investigating the mechanism behind the linear relation $\frac{\dot{K}_t}{K_t}-\frac{\dot{L}_t}{L_t}=\varphi_1\frac{\dot{A}_t}{A_t}$ but from the beginning assumes a linearity among $\frac{\dot{L}_t}{L_t}$, $\frac{\dot{K}_t}{K_t}$ and $\frac{\dot{A}_t}{A_t}$.

Hence, $\frac{\dot{K}_t}{K_t} = \frac{\dot{L}_t}{L_t} + \varphi_1\varphi_4 L_t^{1-\varphi_5} = \frac{\dot{Y}_t}{Y_t}$ if the relation $\frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} = \varphi_1 \frac{\dot{A}_t}{A_t}$ holds and an economy is on a balanced growth path, and thus if $\varphi_5 = 1$ then even though $\frac{\dot{L}_t}{L_t} = 0$ an economy can grow at a constant rate $\varphi_1\varphi_4$. This type of models can eliminate the influence of population growth as well as scale effects, however, Jones (1999) shows that it crucially depends on a very special assumption such that $\varphi_5 = 1$.

Peretto and Smulders (2002) take the fourth approach. They assume that $A_t L_t$, instead of A_t, and K_t are positively linked and $\lim_{L_t \to \infty} A_t L_t = \varphi_6 K_t$ where φ_6 is a constant. Hence $\frac{A_t}{k_t} = \frac{A_t L_t}{K_t}$ and $\lim_{L_t \to \infty} \frac{A_t}{k_t} = \lim_{L_t \to \infty} \frac{A_t L_t}{K_t} = \varphi_6$, and thus asymptotically scale effects vanish. An important feature of this type of models is that they do not need the growth of population for endogenous growth contrary to non-scale models developed initially by Jones (1995b).

The basic strategy of the model in the paper is this fourth approach.[5] However, the model in the paper is fundamentally different from the model in Peretto and Smulders (2002) with regard to the mechanism how the relation such that $\lim_{L_t \to \infty} A_t L_t = \varphi_6 K_t$ emerges. The key assumption in the model of Peretto and Smulders (2002) is that uncompensated knowledge spillovers diminish as the number of firms and thus the number of population increases, thereby the relation such that $\lim_{L_t \to \infty} A_t L_t = \varphi_6 K_t$ emerges. However, this assumption appears problematic. The theories of uncompensated knowledge spillovers are, broadly speaking, divided to two categories: one is the theory of intra-sectoral knowledge spillover, which was

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[5] Hence, the model in the paper is different from early endogenous growth models like the familiar "*AK*" model because $A_t \neq \varphi_2 K_t$ in the model where φ_2 is constant, and is also different from non-scale models initially presented by Jones (1995b) because $\frac{\dot{A}_t}{A_t} \neq \varphi_3 \frac{\dot{L}_t}{L_t}$ and $\frac{\dot{A}_t}{A_t} \neq \varphi_4 L_t^{1-\varphi_5}$ in the model where φ_3, φ_4 and φ_5.

developed by Marshall (1890), Arrow (1962) and Romer (1986), abbreviated as MAR, and the other is the theory of inter-sectoral knowledge spillover, which was developed by Jacobs (1969). The former theory assumes that knowledge spillovers between homogenous firms work out most effectively and thus spillovers primarily emerge within one sector. As a result, uncompensated knowledge spillovers will be more active if the number of firms within one sector is larger. The latter Jacobs' (1969) theory contends that knowledge spillovers are most effective among firms that practice different activities, and hence diversification, i.e. variety of sectors, is important for spillovers. As a result, uncompensated knowledge spillovers will be more active if the number of sectors is larger in an economy. Hence, both theories equally predict that if the number of firms increases, uncompensated knowledge spillovers becomes more active, which contradicts to the key assumption of Peretto and Smulders (2002). The problem of this assumption may arise primarily because they neglect Jacobs externalities and focus only on the negative side of MAR externalities, i.e. as the number of sectors increases, knowledge spillovers work out less effectively. Many empirical researches favor Jacobs externalities, therefore neglecting Jacobs externalities may heavily bias the result of the model.[6]

The model in the paper, contrary to the model in Peretto and Smulders (2002), assumes that uncompensated knowledge spillovers become more active when the number of firms increases as the theories of knowledge spillovers predict. However, there will be a natural question: won't this reverse of assumption make scale effects much worse? The answer is "no," if we consider substitution between accumulations of capital and knowledge. An intuitive explanation behind this result is that a firm will invest more in K_t than in A_t if firms that invest in A_t are less compensated due to more active uncompensated knowledge spillovers.

The model in the paper, to begin with, focuses on the behavior of a firm with respect to whether investing in K_t or in A_t. The decision whether to invest in K_t or in A_t is made by a firm by equaling returns on investing in K_t and in A_t. A linear relation between K_t and A_t implies

[6] See e.g. Glaeser et al (1992), Chen (2002) or Stel and Nieuwenhuijsen (2002).

that returns on investing in K_t and in A_t have a relation such that $\frac{\partial Y_t}{\partial K_t} = \varphi_7 \frac{\partial Y_t}{\partial A_t}$ where φ_7 is a constant. Hence, a firm that invests in A_t can obtain a constant share of $\frac{\partial Y_t}{\partial A_t}$ for any time. It suggests that for scale effects to exist, constancy of φ_7 is crucial. However, the constancy of φ_7 does not have a strong theoretical foundation. The parameter φ_7 indicates the degree of uncompensated knowledge spillovers, i.e. how much a firm that invents a new technology can obtain from $\frac{\partial Y_t}{\partial A_t}$ as the compensation for the technology. The theories of uncompensated knowledge spillovers predict that if the number of firms increases, uncompensated knowledge spillovers becomes more active. According to the theories, the variable φ_7 will not be constant but will be a function of the number of firms and will decreases as the number of firms increases. The paper incorporates this feature of φ_7 in the model in which substitution between investing in K_t and in A_t can be tractable. The model that has these features shows that if the number of firms increases and thus uncompensated knowledge spillovers becomes more active, each firm tends to invest more in K_t rather than in A_t while the economy wide returns on investing in A_t increases, hence the relation such that $\lim_{L_t \to \infty} A_t L_t = \varphi_6 K_t$ emerges. This is the key mechanism of the model in the paper that results in asymptotically diminishing scale effects. As a result, the model can eliminate both scale effects and the influence of population growth.

Asymptotically diminishing scale effects indicate that if the number of firms is very small, scale effects have significant influence on growth rates, however, if the number of firms becomes sufficiently large, scale effects vanish. This result suggests that in the early history of civilizations, scale effects may have been a crucial factor for economic growth, but in modern day industrialized economies, scale effects may not have to be seen as an important factor for economic growth. Hence, although the model in the paper does not escape from scale effects completely, it escapes from scale effects virtually without influence of population growth.

The paper is organized as follows. In section II, the basic framework of the model is explained, particularly how the nature of uncompensated knowledge spillovers is incorporated into the model that can track substitution between investing in K_t and in A_t is explained in detail. In section III, the optimization problem for a central planer based on this new model is solved. The result shows that the model has a feature of asymptotically diminishing scale effects, which suggests that scale effects may not play an important role in modern day industrialized economies. In section IV, a decentralized model with a patent system is examined and it is shown that the model also has the feature of asymptotically diminishing scale effects. Finally some concluding remarks are offered in section V.

II. THE MODEL

1. The production function

The production function is assumed to be $Y_t = F(A_t, K_t, L_t)$, where Y_t (≥ 0) is outputs, K_t (≥ 0) is capital inputs, L_t (≥ 0) is labor inputs, and A_t (≥ 0) is knowledge/technology/idea inputs in period *t*. Knowledge/technology/idea is produced with capital inputs, labor inputs and knowledge/technology/idea inputs, and is purchased in markets just as consumer goods and capital goods are. Each goods, whichever it is consumer goods, capital goods or "knowledge/technology/idea goods," is produced by a unique combination of capital inputs, labor inputs and knowledge/technology/idea inputs, but as an aggregated function they can be expressed by the above production function. Hence, outputs Y_t consist of consumption C_t (≥ 0), the increase in capital $\dot{K}_t$, and the increase in knowledge/technology/idea $\dot{A}_t$. This expression is standard in the literature of endogenous growth. Therefore in the paper, accumulations of capital and knowledge/technology/idea are modeled as follows:

Assumption:

(A1) Accumulations of capital and knowledge/technology/idea are $\dot{K}_t = Y_t - C_t - v\dot{A}_t$, where $v(>0)$ is a constant and a unit of K_t and $\frac{1}{v}$ of a unit of A_t are produced using the same amounts of inputs.[7]

Hence, unlike most idea-based endogenous growth models, the paper does not assume any special production function for "knowledge/technology/idea goods" in the model, which is possible because, in the model, the driving force behind constant endogenous growth rates, i.e., $\frac{A_t}{k_t}$ = constant, is not a special production function for "knowledge/technology/idea goods," but arbitrage between investing in K_t and in A_t that will be explained in the following sections.

More specifically, the production function is assumed to have the following functional form; $Y_t = F(A_t, K_t, L_t) = A_t^\alpha f(K_t, L_t)$, where α $(1 > \alpha > 0)$ is a constant. Let $y_t = \frac{Y_t}{L_t}$, $k_t = \frac{K_t}{L_t}$ and $c_t = \frac{C_t}{L_t}$, and assume that $f(K_t, L_t)$ is homogenous of degree one. Thereby $y_t = A_t^\alpha f(k_t)$, and $\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t$.

2. Uncompensated knowledge spillovers

The following assumption is another key assumption in the model.

Assumption: Every firm is identical and has the same size, and for any period,

[7] Hence, like Jones' (1995b) non-scale model, A_t, as well as K_t, is produced less as A_t and L_t increase if the usual production function of homogeneous of degree one is assumed.

(A2) $m = \frac{M_t^{\rho}}{L_t} = \text{constant}$, where M_t is the number of firms and $\rho(>1)$ is a constant,

(A3) $\frac{\partial Y_t}{\partial K_t} = \frac{1}{M_t^{\rho}} \frac{\partial Y_t}{\partial (vA_t)}$ and thus $\frac{\partial y_t}{\partial k_t} = \frac{1}{mv} \frac{\partial y_t}{\partial A_t}$.

Firstly, assumption (A2) simply assumes that the number of population and the number of firms in an economy are positively related, which seems intuitively natural. In assumption (A3), the paper assumes that returns on investing in K_t and investing in A_t for a firm are kept equal.[8] The driving force behind this relation is that rational entrepreneurs consider all the opportunities at any time and select the most profitable investments, and thus, through arbitrages, returns on investments in K_t and returns on investments in A_t should be equal in any period. However it is also assumed in assumption (A3) that a firm that invents a new technology can not obtain all the returns on investing in A_t. This means that investing in A_t increases Y_t but returns of an individual firm that invests in A_t is only a fraction of the increase of Y_t such that $\frac{1}{M_t^{\rho}} \frac{\partial Y_t}{\partial (vA_t)} = \frac{1}{mL_t} \frac{\partial Y_t}{\partial (vA_t)}$. The reason why only a fraction of the increase in Y_t the returns of an individual firm is, is uncompensated knowledge spillovers to other firms.

Broadly speaking, there are two types of uncompensated knowledge spillovers: one is the intra-sectoral knowledge spillover, i.e. MAR externalities, and the other is the inter-sectoral knowledge spillover, i.e. Jacobs externalities. The theory of MAR assumes that knowledge spillovers between homogenous firms work out most effectively and thus spillovers primarily emerge within one sector. As a result, uncompensated knowledge spillovers will be more active if the number of firms within one sector is larger. On the other hand, Jacobs (1969) contends that knowledge spillovers are most effective among firms that practice different

[8] Remind that a unit of K_t and $\frac{1}{v}$ of a unit of A_t are produced using the same amounts of inputs by assumption (A1). Hence, a unit of K_t and a unit of V_t have the same value as an investment where $V_t = vA_t$.

activities, and hence diversification, i.e. variety of sectors, is important for spillovers. As a result, uncompensated knowledge spillovers will be more active if the number of sectors is larger in an economy.

If it is assumed that all the sectors have the same number of firms, an increase of the number of firms in an economy results in more active knowledge spillovers due to either of MAR externalities or Jacobs externalities. That is, if an increase of the number of firms in an economy is a result of an increase of the number of firms in each sector, uncompensated knowledge spillovers will become more active by MAR externalities, and if an increase of the number of firms in an economy is a result of an increase of the number of sectors, uncompensated knowledge spillovers will become more active by Jacobs externalities. In either case, an increase of the number of firms in an economy leads to more active uncompensated knowledge spillovers.

Furthermore more active uncompensated knowledge spillovers will reduce the returns of a firm that invests in A. $\frac{\partial Y_t}{\partial A_t}$ indicates the over all increase in Y in an economy by an additional A, that consists of both the increase in production in the firm that invented the new technology and the increase in production in other firms that use the newly invented technology that the firms obtained either compensating for it to the firm or by uncompensated knowledge spillovers. If the number of firms becomes larger and thus uncompensated knowledge spillovers becomes more active, the compensated fraction in $\frac{\partial Y_t}{\partial A_t}$ that the firm can obtain will become smaller and thus the returns of the firm will become also smaller. The assumption (A3) simply describes this mechanism.

By assumptions (A2) and (A3), $A_t = \frac{\alpha f(k_t)}{mvf'(k_t)}$ because $\frac{\partial y_t}{mv\partial A_t} = \frac{\partial y_t}{\partial k_t} \Leftrightarrow \frac{\alpha}{mv} A_t^{\alpha-1} f(k_t) = A_t^{\alpha} f'(k_t)$. Since $A_t = \frac{\alpha f}{mvf'}$, then $y_t = A_t^{\alpha} f = \left(\frac{\alpha}{mv}\right)^{\alpha} \frac{f^{1+\alpha}}{f'^{\alpha}}$ and $\dot{A}_t = \frac{\alpha}{mv}\dot{k}_t\left(1 - \frac{f f''}{f'^2}\right)$.

Hence, $\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t = \left(\frac{\alpha}{mv}\right)^{\alpha} \frac{f^{1+\alpha}}{f'^{\alpha}} - c_t - \frac{\alpha}{mL_t}\dot{k}_t\left(1 - \frac{f f''}{f'^2}\right) - n_t k_t$, and thus

$$\dot{k}_t = \frac{\left(\frac{\alpha}{mv}\right)^{\alpha} \frac{f^{1+\alpha}}{f'^{\alpha}} - c_t - n_t k_t}{1 + \frac{\alpha}{mL_t}\left(1 - \frac{f f''}{f'^2}\right)}.$$

In the paper, the case of Harrod neutral technological progress such that $y_t = A_t^{\alpha} k_t^{1-\alpha}$ and thus $Y_t = K_t^{1-\alpha}(A_t L_t)^{\alpha}$ is examined.[9] As Barro and Sala-i-Martin (1995) argue, technological progress must take the labor-augmenting form in the production function if the models are to display a steady state. The model in the paper also can not achieve a stable growth path if technological progress is not Harrod neutral (see Appendix).

If the production function is Harrod neutral such that $y_t = A_t^{\alpha} k_t^{1-\alpha}$ and thus $Y_t = K_t^{1-\alpha}(A_t L_t)^{\alpha}$, then $A_t = \frac{\alpha}{mv(1-\alpha)} k_t$ and $\frac{f f''}{f'^2} = -\frac{\alpha}{1-\alpha}$, which lucidly indicates that the model has the feature $\frac{A_t}{k_t} = \text{constant}$, therefore the model can be an endogenous growth model. At the same time, clearly the model in the paper is not a type of early endogenous growth models like the familiar "*AK*" model because $A_t \neq \varphi_2 K_t$ in the model where φ_2 is constant, nor a type of non-scale models initially presented by Jones (1995b) because $\frac{\dot{A}_t}{A_t} \neq \varphi_3 \frac{\dot{L}_t}{L_t}$ and $\frac{\dot{A}_t}{A_t} \neq \varphi_4 L_t^{1-\varphi_5}$ in the model where φ_3, φ_4 and φ_5 are constants, and of course it is not a type of human capital-based endogenous growth models.

III. THE CENTRAL PLANNER'S PROBLEM

[9] As is well known, only Harrod neutral technological progress matches the stylized facts presented by Kaldor (1961).

1. Growth rates

The optimization problem of a central planner is

Max $E_0 \int_0^\infty u(c_t)\exp(-\theta t)dt$

subject to

$$\dot{k}_t = \frac{\left(\frac{\alpha}{mv}\right)^\alpha \frac{f^{1+\alpha}}{f'^\alpha} - c_t - n_t k_t}{1 + \frac{\alpha}{mL_t}\left(1 - \frac{f f''}{f'^2}\right)}.$$

Let Hamiltonian H be

$$H = u(c_t)\exp(-\theta t) + \lambda_t \left[\frac{\left(\frac{\alpha}{mv}\right)^\alpha \frac{f^{1+\alpha}}{f'^\alpha} - c_t - n_t k_t}{1 + \frac{\alpha}{mL_t}\left(1 - \frac{f f''}{f'^2}\right)}\right]$$

where λ_t is a costate variable. The optimality conditions for the problem are

$$\frac{\partial u(c_t)}{\partial c_t}\exp(-\theta t) = \frac{\lambda_t}{1 + \frac{\alpha}{mL_t}\left(1 - \frac{f f''}{f'^2}\right)}, \tag{1}$$

$$\dot{\lambda}_t = -\frac{\partial H}{\partial k_t}, \tag{2}$$

$$\dot{k}_t = \frac{\left(\frac{\alpha}{mv}\right)^\alpha \frac{f^{1+\alpha}}{f'^\alpha} - c_t - n_t k_t}{1 + \frac{\alpha}{mL_t}\left(1 - \frac{f f''}{f'^2}\right)}, \tag{3}$$

$$\lim_{t\to\infty} \lambda_t k_t = 0. \tag{4}$$

To begin with, three lemmas are proved to show that the growth rates of output, knowledge/idea, consumption and capital converge at the same rate. First, to make the model more easily tractable, it is assumed that the growth rate of population n_t is constant and non-negative and the utility function is a CRRA type.

Lemma 1: The growth rate of consumption is $\dfrac{\dot{c}_t}{c_t} = \dfrac{\dfrac{mL_t(1-\alpha)\left[\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{mL_t(1-\alpha)+\alpha} - \theta}{\varepsilon}$.

Proof: Because the production function is $y_t = A_t^{\alpha} k_t^{1-\alpha}$, the Hamiltonian is

$$H = u(c_t)\exp(-\theta t) + \lambda_t \left[\frac{\left(\dfrac{\alpha}{mv}\right)^{\alpha} \dfrac{f^{1+\alpha}}{f'^{\alpha}} - c_t - n_t k_t}{1 + \dfrac{\alpha}{mL_t}\left(1 - \dfrac{f f''}{f'^2}\right)}\right]$$

$$= u(c_t)\exp(-\theta t) + \lambda_t \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} k_t - c_t - n_t k_t\right].$$

Then condition (2) is

$$\dot{\lambda}_t = -\lambda_t \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] \tag{5}$$

and condition (3) is

$$\dot{k}_t = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} k_t - c_t - n_t k_t\right]. \tag{6}$$

Hence, by condition (1) and equation (5),

$$\frac{\dot{c}_t}{c_t} = \frac{\dfrac{mL_t(1-\alpha)\left[\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{mL_t(1-\alpha)+\alpha} - \theta}{\left(-\dfrac{c_t u''}{u'}\right)} = \frac{\dfrac{mL_t(1-\alpha)\left[\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{mL_t(1-\alpha)+\alpha} - \theta}{\varepsilon}.$$

Q.E.D.

Therefore the growth rate of consumption is constant if the utility function is a CRRA type.

Lemma 2: The transversality condition (4) $\lim_{t\to\infty}\lambda_t k_t = 0$ is not satisfied if and only if $\frac{c_t}{k_t} = 0$.

Proof: By equation (6), $\dot{k}_t = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_t - c_t - n_t k_t\right]$, and thus $\frac{\dot{k}_t}{k_t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t - \frac{c_t}{k_t}\right]$. If $\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t > \frac{c_t}{k_t}$ then $\frac{\dot{k}_t}{k_t} > 0$.

On the other hand, by equation (5), $\dot{\lambda}_t = -\lambda_t \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right]$ and thus $\frac{\dot{\lambda}_t}{\lambda_t} = -\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right]$.

Here, $\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{k}_t}{k_t} = -\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] + \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t - \frac{c_t}{k_t}\right]$ $= -\frac{mL_t(1-\alpha)}{[mL_t(1-\alpha)+\alpha]}\frac{c_t}{k_t}$. Thereby if $\frac{c_t}{k_t} > 0$, then $\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{k}_t}{k_t} < 0$. Hence, the transversality condition (4) $\lim_{t\to\infty}\lambda_t k_t = 0$ is not satisfied if and only if $\frac{c_t}{k_t} = 0$ (Because $c_t \geq 0$ and $k_t \geq 0$).

Q.E.D.

Lemma 3: If and only if $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$, all the conditions are satisfied.

Proof:

(Step 1) Since the growth rate of population n_t is constant by assumption (A3), L_t increases exponentially, and thus for any n_t $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \text{constant}$ because for $n_t > 0$

$$\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dfrac{mL_t(1-\alpha)\left[\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{mL_t(1-\alpha)+\alpha} - \theta}{\varepsilon} = \frac{\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t - \theta}{\varepsilon} = \text{constant} \text{ , and}$$

for $n_t = 0$

$$\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dfrac{mL_t(1-\alpha)\left[\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{mL_t(1-\alpha)+\alpha} - \theta}{\varepsilon} = \frac{\dfrac{mL_t(1-\alpha)\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}}{mL_t(1-\alpha)+\alpha} - \theta}{\varepsilon} = \text{constant}.$$

On the other hand, for $n_t > 0$

$$\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} = \lim_{t\to\infty}\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv(1-\alpha)}\right)^{\alpha} - n_t - \frac{c_t}{k_t}\right] = \left(\frac{\alpha}{mv(1-\alpha)}\right)^{\alpha} - n_t - \lim_{t\to\infty}\frac{c_t}{k_t} \text{, and for } n_t = 0,$$

$$\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} = \lim_{t\to\infty}\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t - \frac{c_t}{k_t}\right] = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \lim_{t\to\infty}\frac{c_t}{k_t}\right].$$

(Step 2) If $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} > \lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$, then $\frac{c_t}{k_t}$ diminishes as time passes, then $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$ increases. Hence, eventually $\frac{c_t}{k_t}$ diminishes to zero. Therefore, by lemma 2, the transversality condition (4) is not satisfied.

If $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} < \lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$, then $\frac{c_t}{k_t}$ increases as time passes, then $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$ diminishes and eventually becomes negative. Hence, k_t decreases and eventually violates equation (6) since $k_t \geq 0$ and thus k_t can not be negative.

However, if $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$, then $\frac{c_t}{k_t}$ is constant and thus $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$ and $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$ continue to be constant and identical.

Q.E.D.

By lemma 1, lemma 2 and lemma 3, it is proved that the growth rates of output, knowledge/idea, consumption and capital converge at the same rate, if the central planner behaves as the following assumption.

Assumption:

(A4) Given the initial A_0 and k_0, the central planner sets the initial consumption so as to achieve a growth path that satisfies all the conditions, i.e. a growth path of $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$ while adjusts k_t so as to achieve $\frac{\partial Y_t}{\partial K_t}=\frac{1}{M_t^{\rho}}\frac{\partial Y_t}{\partial(vA_t)}$.[10]

Proposition 1: $\lim_{t\to\infty}\frac{\dot{y}_t}{y_t}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$.

Proof:

(Step 1) As for y_t, because $y_t=A_t^{\alpha}k_t^{1-\alpha}$,

$$\dot{y}_t=\left(\frac{A_t}{k_t}\right)^{\alpha}\left[(1-\alpha)\dot{k}_t+\alpha\frac{k_t}{A_t}\dot{A}_t\right]. \tag{7}$$

Since, $\dot{A}_t=\frac{\alpha}{mv}\dot{k}_t\left(1-\frac{f f''}{f'^2}\right)=\frac{\alpha}{mv(1-\alpha)}\dot{k}_t$, then $\dot{y}_t=\dot{k}_t\left(\frac{A_t}{k_t}\right)^{\alpha}\left[(1-\alpha)+\frac{\alpha^2}{mv(1-\alpha)}\frac{k_t}{A_t}\right]$, and thus $\frac{\dot{y}_t}{y_t}=\frac{\dot{k}_t}{k_t}\left[(1-\alpha)+\frac{\alpha^2}{mv(1-\alpha)}\frac{k_t}{A_t}\right]$. Because $A_t=\frac{\alpha}{mv(1-\alpha)}k_t$, $\frac{\dot{y}_t}{y_t}=\frac{\dot{k}_t}{k_t}[(1-\alpha)+\alpha]=\frac{\dot{k}_t}{k_t}$. Hence

[10] Because $A_t=\frac{\alpha}{mv(1-\alpha)}k_t$, it is assumed that $A_0=\frac{\alpha}{mv(1-\alpha)}k_0$.

$\lim_{t\to\infty}\frac{\dot{y}_t}{y_t}=\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$.

(Step 2) As for A_t, by equation (7) and $\dot{A}_t=\frac{\alpha}{mv(1-\alpha)}\dot{k}_t$, $\dot{y}_t=\dot{A}_t\left(\frac{A_t}{k_t}\right)^{\alpha}\left[\frac{mv(1-\alpha)^2}{\alpha}+\alpha\frac{k_t}{A_t}\right]$,

and thus $\frac{\dot{y}_t}{y_t}=\frac{\dot{A}_t}{k_t}\frac{mv(1-\alpha)^2}{\alpha}+\alpha\frac{\dot{A}_t}{A_t}$. Because $\dot{A}_t=\frac{\alpha}{mv(1-\alpha)}\dot{k}_t$, then $\frac{\dot{y}_t}{y_t}=(1-\alpha)\frac{\dot{k}_t}{k_t}+\alpha\frac{\dot{A}_t}{A_t}$.

Hence, $\frac{\dot{y}_t}{y_t}=\frac{\dot{k}_t}{k_t}=(1-\alpha)\frac{\dot{k}_t}{k_t}+\alpha\frac{\dot{A}_t}{A_t}$ and thus $\frac{\dot{k}_t}{k_t}=\frac{\dot{A}_t}{A_t}$. Therefore $\lim_{t\to\infty}\frac{\dot{y}_t}{y_t}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$.

Q.E.D.

Hence, if the central planner set the initial consumption so as to achieve $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$ that leads to a growth path satisfying all the conditions, the growth rates of y_t, c_t, k_t and A_t asymptotically converge at the same rate. This growth path can be seen as a natural extension of "steady state" in the conventional Ramsey models with exogenous technology growth and may be called as "steady state growth path." Like "steady state," "steady state growth path" is the only path that satisfied all the conditions and achieved by setting the initial consumption at a unique appropriate level.

In the special case such that $n_t=0$, the growth rates are equal at any time.

Corollary 1: If $n_t=0$, then $\frac{\dot{y}_t}{y_t}=\frac{\dot{A}_t}{A_t}=\frac{\dot{c}_t}{c_t}=\frac{\dot{k}_t}{k_t}=\text{constant}$.

Proof: $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\frac{mL_t(1-\alpha)\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t\right]-\alpha n_t}{mL_t(1-\alpha)+\alpha}-\theta}{\varepsilon}=\frac{\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta}{\varepsilon}$

$$= \frac{\dot{c}_t}{c_t} = \text{constant}, \text{ and}$$

$$\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} = \lim_{t\to\infty}\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t - \frac{c_t}{k_t}\right] = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \lim_{t\to\infty}\frac{c_t}{k_t}\right].$$

Therefore, if and only if $\frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t}$, $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$. Hence, by the proof of proposition 1,

$$\frac{\dot{y}_t}{y_t} = \frac{\dot{A}_t}{A_t} = \frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t} = \text{constant}.$$

Q.E.D.

This case is important since it indicates that an economy can grow endogenously without the growth of population contrary to non-scale models developed initially by Jones (1995b).

2. Asymptotically diminishing scale effects

The model in the paper has a feature of asymptotically diminishing scale effects. Before examining this feature, scale effects are defined as follows:

Definition: Scale effects are defined as $S(L_t) = \varepsilon\frac{\dot{c}_t}{c_t} + \theta$.

Hence, scale effects are defined here as the population related part of $\frac{\dot{c}_t}{c_t}$. The scale effect in the above case is thereby $S(L_t) = \dfrac{mL_t(1-\alpha)\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{mL_t(1-\alpha)+\alpha}$.

Proposition 2: The scale effect $S(L_t)=\dfrac{mL_t(1-\alpha)\left[\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t\right]-\alpha n_t}{mL_t(1-\alpha)+\alpha}$ has an upper bound if $\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t>0$.

Proof: $\lim_{L_t\to\infty} S(L_t)=\lim_{L_t\to\infty}\dfrac{mL_t(1-\alpha)\left[\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t\right]-\alpha n_t}{mL_t(1-\alpha)+\alpha}=\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t=\text{constant}$, and

if $\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t>0$ then $\dfrac{dS(L_t)}{dL_t}>0$ and $\dfrac{d^2S(L_t)}{dL_t^{\,2}}<0$.

Q.E.D.

Existence of the upper bound has an important meaning that can be understood by the following example.

Example: Assume for example that $\alpha=0.6$, $n_t=0$, $\theta=0.05$, $\varepsilon=1.5$, $v=3000$ and $m=0.03$, then the degrees of scale effects for various cases of population are shown in the table.

The example clearly shows that scale effects are economically important if the size of population is very small, i.e. the number of firms is very small, while if the size of population is sufficiently large, i.e. the number of firms is sufficiently large, scale effects are economically unimportant. In the early days of human history, scale effects may have played a crucial role and actually early civilizations developed in the areas where the size of population was relatively large. However, in modern day industrialized economies, it may not be necessary to treat scale effects as an important factor, because the number of firms in these economies seems sufficiently large.

This example implies that if it is assumed that the number of firms is irrelevant to the number of population, the familiar scale effects emerge.

Remark: If M_t is constant such as $M_t = M_0$ for any *t*, and thus if *m* is time-variable, then the usual scale effects emerge in the model.

Proof: If M_t is constant such as $M_t = M_0$ for any *t*, and thus if *m* is time-variable, then

$$\frac{\dot{c}_t}{c_t} = \frac{\dfrac{mL_t(1-\alpha)\left[\left(\dfrac{\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{mL_t(1-\alpha)+\alpha} - \theta}{\varepsilon} = \frac{\dfrac{M_0{}^{\rho}(1-\alpha)\left[L_t^{\alpha}\left(\dfrac{\alpha}{M_0{}^{\rho}v}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{M_0{}^{\rho}(1-\alpha)+\alpha} - \theta}{\varepsilon}.$$

Q.E.D.

IV. DECENTRALIZED ECONOMIES

1. The model of decentralized economies

In the decentralized economy examined below, a patent system is introduced to enhance an incentive to inventions. Assume that by inventing a new technology, a firm that invented the technology can enjoy some degree of monopoly for a predetermined period and after that period the monopoly ends and every firm can use this new technology freely. The monopoly will prevent other firms from using this new technology for their production to some degree. Then with the patent system, available knowledge for firms will be older compared to that without patent system. Considering the above features of the patent system, the following assumption is introduced.

Assumption:

(A5) The production function in decentralized economies is $y_t = A_{t-\delta}^{\alpha} f(k_t)$, where δ is the length of patented period, while the accumulation of capital follows $\dot{k}_t = y_t - c_t - \frac{v\dot{A}_{t-\delta}}{L_t} - n_t k_t$.

(A6) The returns on investing in A_t for a firm that invests in A_t in decentralized economies is $\frac{1}{\gamma m v}\frac{\partial y_t}{\partial A_{t-\delta}}$, where $\gamma(0 < \gamma \le 1)$ is the function of the period of patent $\delta(> 0)$ such that $\frac{d\gamma}{d\delta} < 0$.

$\gamma(0 < \gamma \le 1)$ can be interpreted as the degree of monopoly. Larger γ means that a firm that invests in A_t can not fully exploit profit because the period of monopoly is shorter.

Because $\frac{1}{\gamma m v}\frac{\partial y_t}{\partial A_{t-\delta}} = \frac{\partial y_t}{\partial k_t} \Leftrightarrow \frac{\alpha}{\gamma m v} A_{t-\delta}^{\alpha-1} f(k_t) = A_{t-\delta}^{\alpha} f'(k_t)$, thereby $A_{t-\delta} = \frac{\alpha}{\gamma m v}\frac{f(k_t)}{f'(k_t)}$ and thus $y_t = A_{t-\delta}^{\alpha} f = \left(\frac{\alpha}{\gamma m v}\right)^{\alpha} \frac{f^{1+\alpha}}{f'^{\alpha}}$ and $\dot{A}_{t-\delta} = \frac{\alpha}{\gamma m v}\dot{k}_t\left(1 - \frac{f f''}{f'^2}\right)$. Thereby, $\dot{k}_t = y_t - c_t - \frac{v\dot{A}_{t-\delta}}{L_t} - n_t k_t = \left(\frac{\alpha}{\gamma m v}\right)^{\alpha} \frac{f^{1+\alpha}}{f'^{\alpha}} - c_t - \frac{\alpha}{\gamma m L_t}\dot{k}_t\left(1 - \frac{f f''}{f'^2}\right) - n_t k_t$, and thus

$$\dot{k}_t = \frac{\left(\frac{\alpha}{\gamma m v}\right)^{\alpha} \frac{f^{1+\alpha}}{f'^{\alpha}} - c_t - n_t k_t}{1 + \frac{\alpha}{\gamma m L_t}\left(1 - \frac{f f''}{f'^2}\right)}.$$

2. Growth rates

The optimization problem of a representative household is

Max $E_0 \int_0^{\infty} u(c_t) \exp(-\theta t)\, dt$

subject to

$$\dot{k}_t = \frac{\left(\frac{\alpha}{\gamma m v}\right)^{\alpha} \frac{f^{1+\alpha}}{f'^{\alpha}} - c_t - n_t k_t}{1 + \frac{\alpha}{\gamma m L_t}\left(1 - \frac{f f''}{f'^2}\right)}.$$

Let Hamiltonian H be

$$H = u(c_t)\exp(-\theta t) + \lambda_t \left[\frac{\left(\frac{\alpha}{\gamma m v}\right)^{\alpha} \frac{f^{1+\alpha}}{f'^{\alpha}} - c_t - n_t k_t}{1 + \frac{\alpha}{\gamma m L_t}\left(1 - \frac{f f''}{f'^2}\right)} \right]$$

where λ_t is a costate variable. The optimality conditions for the problem are

$$\frac{\partial u(c_t)}{\partial c_t}\exp(-\theta t) = \frac{\lambda_t}{1 + \frac{\alpha}{\gamma m L_t}\left(1 - \frac{f f''}{f'^2}\right)}, \tag{8}$$

$$\dot{\lambda}_t = -\frac{\partial H}{\partial k_t}, \tag{9}$$

$$\dot{k}_t = \frac{\left(\frac{\alpha}{\gamma m v}\right)^{\alpha} \frac{f^{1+\alpha}}{f'^{\alpha}} - c_t - n_t k_t}{1 + \frac{\alpha}{\gamma m L_t}\left(1 - \frac{f f''}{f'^2}\right)}, \tag{10}$$

$$\lim_{t \to \infty} \lambda_t k_t = 0. \tag{11}$$

As section III, to begin with, three lemmas are proved to show that the growth rates of output, knowledge/idea, consumption and capital converge at the same rate.

Lemma 4: The growth rate of consumption is $\frac{\dot{c}_t}{c_t} = \dfrac{\dfrac{\gamma m L_t (1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{\gamma m L_t (1-\alpha) + \alpha} - \theta}{\varepsilon}$.

Proof: Because the production function $y_t = A_t^\alpha f(k_t)$ is such that $y_t = A_t^\alpha k_t^{1-\alpha}$ and thus $Y_t = K_t^{1-\alpha}(A_t L_t)^\alpha$, then $\frac{f f''}{f'^2} = -\frac{\alpha}{1-\alpha}$. Hence,

$$H = u(c_t)\exp(-\theta t) + \lambda_t \left[\frac{\left(\frac{\alpha}{\gamma m v}\right)^\alpha \frac{f^{1+\alpha}}{f'^\alpha} - c_t - n_t k_t}{1 + \frac{\alpha}{\gamma m L_t}\left(1 - \frac{f f''}{f_t'}\right)} \right]$$

$$= u(c_t)\exp(-\theta t) + \lambda_t \frac{\gamma m L_t (1-\alpha)}{\gamma m L_t (1-\alpha) + \alpha} \left[\left(\frac{\alpha}{\gamma m v}\right)^\alpha (1-\alpha)^{-\alpha} k_t - c_t - n_t k_t \right].$$

Then condition (9) is

$$\dot{\lambda}_t = -\lambda_t \frac{\gamma m L_t (1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^\alpha (1-\alpha)^{-\alpha} - n\right]}{\gamma m L_t (1-\alpha) + \alpha} \tag{12}$$

and condition (10) is

$$\dot{k}_t = \frac{\gamma m L_t (1-\alpha)}{\gamma m L_t (1-\alpha) + \alpha} \left[\left(\frac{\alpha}{\gamma m v}\right)^\alpha (1-\alpha)^{-\alpha} k_t - c_t - n_t k_t \right]. \tag{13}$$

Hence, by condition (8) and equation (12),

$$\frac{\dot{c}_t}{c_t} = \frac{\frac{\gamma m L_t (1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^\alpha (1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{\gamma m L_t (1-\alpha) + \alpha} - \theta}{\left(-\frac{c_t u''}{u'}\right)} = \frac{\frac{\gamma m L_t (1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^\alpha (1-\alpha)^{-\alpha} - n_t\right] - \alpha n_t}{\gamma m L_t (1-\alpha) + \alpha} - \theta}{\varepsilon}.$$

Therefore the growth rate of consumption is constant if the utility function is a CRRA type.

Q.E.D.

Lemma 5: The transversality condition (11) $\lim_{t\to\infty} \lambda_t k_t = 0$ is not satisfied if and only if $\frac{c_t}{k_t} = 0$.

Proof: By equation (13), $\dot{k}_t = \frac{\gamma m L_t(1-\alpha)}{\gamma m L_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}k_t - c_t - n_t k_t\right]$, and thus

$\frac{\dot{k}_t}{k_t} = \frac{\gamma m L_t(1-\alpha)}{\gamma m L_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t - \frac{c_t}{k_t}\right]$. If $\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t > \frac{c_t}{k_t}$ then $\frac{\dot{k}_t}{k_t} > 0$.

On the other hand, by equation (12), $\dot{\lambda}_t = -\lambda_t \frac{\gamma m L_t(1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha} - n\right]}{\gamma m L_t(1-\alpha)+\alpha}$ and thus

$$\frac{\dot{\lambda}_t}{\lambda_t} = -\frac{\gamma m L_t(1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha} - n\right]}{\gamma m L_t(1-\alpha)+\alpha}.$$

Here, $\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{k}_t}{k_t} = -\frac{\gamma m L_t(1-\alpha)\left[\left(\frac{\alpha}{v}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right]}{\gamma m L_t(1-\alpha)+\alpha} + \frac{\gamma m L_t(1-\alpha)\left[\left(\frac{\alpha}{v}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t - \frac{c_t}{k_t}\right]}{\gamma m L_t(1-\alpha)+\alpha}$

$= -\frac{\gamma m L_t(1-\alpha)}{\left[\gamma m L_t(1-\alpha)+\alpha\right]}\frac{c_t}{k_t}$. Thereby if $\frac{c_t}{k_t} > 0$, then $\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{k}_t}{k_t} < 0$. Hence, the transversality condition (11) $\lim_{t\to\infty}\lambda_t k_t = 0$ is not satisfied if and only if $\frac{c_t}{k_t} = 0$ (Because $c_t \geq 0$ and $k_t \geq 0$).

Q.E.D.

Lemma 6: If and only if $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$, all the conditions are satisfied.

Proof:

(Step 1) Since the growth rate of population n_t is constant by assumption (A3), L_t increases exponentially, and thus for any n_t $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \text{constant}$ because for $n_t > 0$

$$\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dfrac{\gamma m L_t(1-\alpha)\left[\left(\dfrac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t\right]-\alpha n_t}{\gamma m L_t(1-\alpha)+\alpha}-\theta}{\varepsilon}=\frac{\left(\dfrac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t-\theta}{\varepsilon}=\text{constant}$$, and

for $n_t=0$

$$\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dfrac{\gamma m L_t(1-\alpha)\left[\left(\dfrac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t\right]-\alpha n}{\gamma m L_t(1-\alpha)+\alpha}-\theta}{\varepsilon}=\frac{\dfrac{\gamma m L_t\left(\dfrac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{1-\alpha}}{\gamma m L_t(1-\alpha)+\alpha}-\theta}{\varepsilon}=\text{constant}.$$

On the other hand, for $n_t>0$

$$\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}=\lim_{t\to\infty}\frac{\gamma m L_t(1-\alpha)}{\gamma m L_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t-\frac{c_t}{k_t}\right]=\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t-\lim_{t\to\infty}\frac{c_t}{k_t}$$, and for $n_t=0$

$$\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}=\lim_{t\to\infty}\frac{\gamma m L_t(1-\alpha)}{\gamma m L_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t-\frac{c_t}{k_t}\right]=\frac{\gamma m L_t(1-\alpha)}{\gamma m L_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-\lim_{t\to\infty}\frac{c_t}{k_t}\right].$$

(Step 2) If $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}>\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$, then $\frac{c_t}{k_t}$ diminishes as time passes, then $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$ increases. Hence, eventually $\frac{c_t}{k_t}$ diminishes to zero. Therefore, by lemma 5, the transversality condition (11) is not satisfied.

If $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}<\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$, then $\frac{c_t}{k_t}$ increases as time passes, then $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$ diminishes and eventually becomes negative. Hence, k_t decreases and eventually violates equation (13) since $k_t\geq 0$ and thus k_t can not be negative.

However, if $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$, then $\frac{c_t}{k_t}$ is constant and thus $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$ and $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$ continue to be constant and identical.

Q.E.D.

Unquestionably rational consumers will set the initial consumption that leads to a growth path satisfying all the conditions, i.e. consumers will chose intentionally the “steady state growth path.”

Assumption:

(A7) Given the initial A_0 and k_0, consumers set the initial consumption so as to achieve a growth path that satisfies all the conditions, i.e. a growth path of $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$, while firms adjust k_t so as to achieve $\frac{\partial Y_t}{\partial K_t}=\frac{1}{\gamma M_t^{\rho}}\frac{\partial Y_t}{\partial (vA_t)}$.[11]

By lemma 4, lemma 5 and lemma 6, it is proved that the growth rates of output, knowledge/idea, consumption and capital converge at the same rate, if consumers behave according to the above assumption.

Proposition 3: $\lim_{t\to\infty}\frac{\dot{y}_t}{y_t}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$.

Proof:

(Step 1) As for y_t, because $y_t = A_{t-\delta}^{\alpha} k_t^{1-\alpha}$,

$$\dot{y}_t=\left(\frac{A_{t-\delta}}{k_t}\right)^{\alpha}\left[(1-\alpha)\dot{k}_t+\alpha\frac{k_t}{A_{t-\delta}}\dot{A}_{t-\delta}\right]. \tag{14}$$

[11] Because $A_t=\frac{\alpha}{\gamma mv(1-\alpha)}k_t$, it is assumed that $A_0=\frac{\alpha}{\gamma mv(1-\alpha)}k_0$.

Since, $\dot{A}_{t-\delta} = \frac{\alpha}{\gamma m v}\dot{k}_t\left(1-\frac{f f''}{f'^2}\right) = \frac{\alpha}{\gamma m v(1-\alpha)}\dot{k}_t$, then $\dot{y}_t = \dot{k}_t\left(\frac{A_{t-\delta}}{k_t}\right)^{\alpha}\left[(1-\alpha)+\frac{\alpha^2}{\gamma m v(1-\alpha)}\frac{k_t}{A_{t-\delta}}\right]$,

and thus $\frac{\dot{y}_t}{y_t} = \frac{\dot{k}_t}{k_t}\left[(1-\alpha)+\frac{\alpha^2}{\gamma m v\,(1-\alpha)}\frac{k_t}{A_{t-\delta}}\right]$. Because $A_{t-\delta} = \frac{\alpha}{\gamma m v\,(1-\alpha)}k_t$,

$\frac{\dot{y}_t}{y_t} = \frac{\dot{k}_t}{k_t}[(1-\alpha)+\alpha] = \frac{\dot{k}_t}{k_t}$. Hence $\lim_{t\to\infty}\frac{\dot{y}_t}{y_t} = \lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$.

(Step 2) As for A_t, by equation (14) and $\dot{A}_{t-\delta} = \frac{\alpha}{\gamma m v\,(1-\alpha)}\dot{k}_t$,

$\dot{y}_t = \dot{A}_{t-\delta}\left(\frac{A_{t-\delta}}{k_t}\right)^{\alpha}\left[\frac{\gamma m v\,(1-\alpha)^2}{\alpha}+\alpha\frac{k_t}{A_{t-\alpha}}\right]$, and thus $\frac{\dot{y}_t}{y_t} = \frac{\dot{A}_{t-\delta}}{k_t}\frac{\gamma m v\,(1-\alpha)^2}{\alpha}+\alpha\frac{\dot{A}_{t-\delta}}{A_{t-\delta}}$.

Because $\dot{A}_{t-\delta} = \frac{\alpha}{\gamma m v(1-\alpha)}\dot{k}_t$, then $\frac{\dot{y}_t}{y_t} = (1-\alpha)\frac{\dot{k}_t}{k_t}+\alpha\frac{\dot{A}_{t-\delta}}{A_{t-\delta}}$. Hence, $\frac{\dot{y}_t}{y_t} = \frac{\dot{k}_t}{k_t} = (1-\alpha)\frac{\dot{k}_t}{k_t}+\alpha\frac{\dot{A}_{t-\delta}}{A_{t-\delta}}$

and thus $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} = \lim_{t\to\infty}\frac{\dot{A}_{t-\delta}}{A_{t-\delta}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \text{constant}$. Therefore $\lim_{t\to\infty}\frac{\dot{y}_t}{y_t} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$.

Q.E.D.

Hence, if consumers set the initial consumption so as to achieve $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$, the growth rates of y_t, c_t, k_t and A_t asymptotically converge at the same rate.

Like section III, in the special case such that $n_t = 0$, the growth rates are equal at any time. That is, an economy can grow without population growth.

Corollary 2: If $n_t = 0$, then $\frac{\dot{y}_t}{y_t} = \frac{\dot{A}_t}{A_t} = \frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t} = \text{constant}$.

Proof: $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\frac{\gamma m L_t(1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t\right]-\alpha n_t}{\gamma m L_t(1-\alpha)+\alpha}-\theta}{\varepsilon}=\frac{\frac{\gamma m L_t\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{1-\alpha}}{\gamma m L_t(1-\alpha)+\alpha}-\theta}{\varepsilon}$

$=\frac{\dot{c}_t}{c_t}=\text{constant}$, and

$$\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}=\lim_{t\to\infty}\frac{\gamma m L_t(1-\alpha)}{\gamma m L_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t-\frac{c_t}{k_t}\right]=\frac{\gamma m L_t(1-\alpha)}{\gamma m L_t(1-\alpha)+\alpha}\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-\lim_{t\to\infty}\frac{c_t}{k_t}\right].$$

Therefore, if and only if $\frac{\dot{c}_t}{c_t}=\frac{\dot{k}_t}{k_t}$, $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$. Hence, by the proof of proposition 3,

$\frac{\dot{y}_t}{y_t}=\frac{\dot{A}_t}{A_t}=\frac{\dot{c}_t}{c_t}=\frac{\dot{k}_t}{k_t}=\text{constant}\cdot$

Q.E.D.

3. Asymptotically diminishing scale effect

Scale effects asymptotically diminish also in the decentralized economy as the following proposition 4 shows.

Proposition 4: The scale effect $S(L_t)=\frac{\gamma m L_t(1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t\right]-\alpha n_t}{\gamma m L_t(1-\alpha)+\alpha}$ has also an upper bound in decentralized economies if $\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t>0$.

Proof: $\lim_{L_t\to\infty}S(L_t)=\lim_{L_t\to\infty}\frac{\gamma m L_t(1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t\right]-\alpha n_t}{\gamma m L_t(1-\alpha)+\alpha}=\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t=\text{constant}$, and if $\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t>0$ then $\frac{dS(L_t)}{dL_t}>0$ and $\frac{d^2S(L_t)}{dL_t{}^2}<0$.

Q.E.D.

4. The optimal patent period

For sufficiently large $mL_t=M_t{}^{\rho}$, the growth rate of consumption is higher if γ is smaller, i.e. the period of patent δ is longer, because $\lim_{L_t\to\infty}\frac{\dot{c}_t}{c_t}=$

$$\lim_{L_t\to\infty}\frac{\frac{\gamma m L_t(1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t\right]-\alpha n_t}{\gamma m L_t(1-\alpha)+\alpha}-\theta}{\varepsilon}=\frac{\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t-\theta}{\varepsilon}.$$

However, if γ becomes very small, i.e. a very long period of patent, and thus if γm and $\gamma m L_t$ approach zero, the growth rate of consumption, in reverse, becomes lower as γ becomes smaller because

$$\lim_{\gamma\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{\gamma\to\infty}\frac{\frac{\gamma m L_t(1-\alpha)\left[\left(\frac{\alpha}{\gamma m v}\right)^{\alpha}(1-\alpha)^{-\alpha}-n_t\right]-\alpha n_t}{\gamma m L_t(1-\alpha)+\alpha}-\theta}{\varepsilon}=-\frac{\theta}{\varepsilon}.$$

This is due to slow capital accumulation. As the benefit of patent increases, firms are tempted to invest more in A_t than in K_t. On the other hand, the level of production $y_t=A_{t-\delta}^{\alpha}f(k_t)$ is smaller in every period if δ is longer. From this point of view, the shorter period of patent is more favorable for the higher level of consumption.

Combining the above arguments, therefore, the optimal patent period δ^* is given by δ that satisfies $E_0\frac{d\int_0^{\infty}u(c_t^*)\exp(-\theta t)dt}{d\delta}=0$ where c_t^* is the consumption in the case of the optimal patent period δ^*.

V. CONCLUDING REMARKS

Scale effects have been the central issue in the field of endogenous growth models over the last decade. The early endogenous growth models, e.g. Romer (1986, 1987) or Lucas (1988), had the nature of scale effects. Jones (1995b) presents a different type of endogenous growth model that can eliminate scale effects, but the growth rate of population, instead, plays a crucial role in this model. Models that are developed by Young (1998), Peretto (1998), Aghion and Howitt (1998), and Dinopoulos and Thompson (1998) eliminate the influence of population growth as well as scale effects, but Jones (1999) shows that it crucially depends on a very special assumption such that $\varphi_5 = 1$. Peretto and Smulders (2002) take the fourth approach. They assume that $A_t L_t$, instead of A_t, and K_t are positively linked and $\lim_{L_t \to \infty} A_t L_t = \varphi_6 K_t$ where φ_6 is a constant. Hence $\frac{A_t}{k_t} = \frac{A_t L_t}{K_t}$ and $\lim_{L_t \to \infty} \frac{A_t}{k_t} = \lim_{L_t \to \infty} \frac{A_t L_t}{K_t} = \varphi_6$, and thus asymptotically scale effects vanish.

The basic strategy of the model in the paper is this fourth approach. However, the model in the paper is fundamentally different from the model in Peretto and Smulders (2002) with regard to the mechanism how the relation such that $\lim_{L_t \to \infty} A_t L_t = \varphi_6 K_t$ emerges. The novelty of the paper is that it uncovers a completely different and more natural mechanism that generates a fourth type endogenous growth model that has the feature of $\lim_{L_t \to \infty} A_t L_t = \varphi_6 K_t$ and asymptotically diminishing scale effects. The model in the paper, contrary to the model in Peretto and Smulders (2002), assumes that uncompensated knowledge spillovers become more active when the number of firms increases as the theories of knowledge spillovers predict. This reverse of assumption does not make scale effects much worse, if we consider substitution between accumulations of capital and knowledge. An intuitive explanation behind this result is that a firm will invest more in K_t than in A_t if firms that invest in A_t is less compensated due to more active uncompensated knowledge spillovers.

According to the theories of knowledge spillovers, both the theory of MAR and the theory of Jacobs, knowledge spillovers are more active if the number of firms is larger. The paper incorporates this feature in the model in which substitution between investing in K_t and in A_t can be tractable. The model that has these features shows that if the number of firms increases and thus knowledge spillovers becomes more active, each firm tends to invest more in K_t rather than in A_t while the economy wide returns on investing in A_t increases, hence the relation such that $\lim_{L_t \to \infty} A_t L_t = \varphi_6 K_t$ emerges. This is the key mechanism of the model that results in asymptotically diminishing scale effects. As a result, the model can eliminate both scale effects and the influence of population growth.

Asymptotically diminishing scale effects indicate that if the number of firms is very small, scale effects have significant influence on growth rates, however, if the number of firms becomes sufficiently large, scale effects vanish. This result suggests that in the early history of civilizations, scale effects may have been a crucial factor for economic growth, but in modern day industrialized economies, scale effects may not have to be seen as an important factor for economic growth. Hence, although the model in the paper does not escape from scale effects completely, it escapes from scale effects virtually without influence of population growth.

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Table:

L_t	$\frac{S(L_t)}{S(10{,}000)}$	$\lim_{t\to\infty}\frac{\dot{y}_t}{y_t} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$
10,000	1.00	0.0061
100,000	1.39	0.021
1,000,000	1.44	0.024
10,000,000	1.45	0.024
100,000,000	1.45	0.024

Note: $\alpha = 0.6$, $n_t = 0$, $\theta = 0.05$, $\varepsilon = 1.5$, $v = 3000$ and $m = 0.03$

Appendix

If technological progress is not Harrod neutral such that $y_t = vA_t^{\alpha} k_t^{1-\beta}$ for $\alpha \neq \beta$, it is not possible for a central planner to set the initial consumption level that satisfies the growth path $\lim_{t\to\infty} \frac{\dot{k}_t}{k_t} = \lim_{t\to\infty} \frac{\dot{c}_t}{c_t} \neq 0$.

Proof:

(Step 1) If the production function $y_t = A_t^{\alpha} f(k_t)$ is such that $y_t = A_t^{\alpha} k_t^{1-\beta}$ and thus $Y_t = A_t^{\alpha} K_t^{1-\beta} L_t^{\beta}$, then $\frac{f f''}{f'^2} = -\frac{\beta}{1-\beta}$ since $f = k_t^{1-\beta}$. Hence,

$$H = u(c_t)\exp(-\theta t) + \lambda_t \left[\frac{\left(\frac{\alpha}{mv}\right)^{\alpha} \frac{f^{1+\alpha}}{f'^{\alpha}} - c_t - n_t k_t}{1 + \frac{\alpha}{mL_t}\left(1 - \frac{f f''}{f'^2}\right)} \right]$$

$$= u(c_t)\exp(-\theta t) + \lambda_t \frac{mL_t(1-\beta)}{mL_t(1-\beta)+\alpha} \left[\left(\frac{\alpha}{mv}\right)^{\alpha} (1-\beta)^{-\alpha} k_t^{1+\alpha-\beta} - c_t - n_t k_t \right].$$

Then condition (2) is

$$\dot{\lambda}_t = -\lambda_t \frac{mL_t(1-\beta)}{mL_t(1-\beta)+\alpha} \left[\left(\frac{\alpha}{mv}\right)^{\alpha} (1-\beta)^{-\alpha} k_t^{\alpha-\beta} - c_t - n_t \right] \tag{a1}$$

and condition (3) is

$$\dot{k}_t = \frac{mL_t(1-\beta)}{mL_t(1-\beta)+\alpha} \left[\left(\frac{\alpha}{mv}\right)^{\alpha} (1-\beta)^{-\alpha} k_t^{1+\alpha-\beta} - c_t - n_t k_t \right]. \tag{a2}$$

Combining condition (1) and equation (a1),

$$\frac{\dot{c}_t}{c_t} = \frac{\dfrac{mL_t(1-\beta)\left[\left(\frac{\alpha}{mv}\right)^{\alpha} (1-\beta)^{-\alpha} k_t^{\alpha-\beta} - n_t\right] - \alpha n_t}{mL_t(1-\beta)+\alpha} - \theta}{\varepsilon}. \tag{a3}$$

On the other hand, by equation (a2),

$$\dot{k}_t = \frac{mL_t(1-\beta)}{mL_t(1-\beta)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\beta)^{-\alpha}k_t^{1+\alpha-\beta} - c_t - n_t k_t\right] \text{, and thus}$$

$$\frac{\dot{k}_t}{k_t} = \frac{mL_t(1-\beta)}{mL_t(1-\beta)+\alpha}\left[\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\beta)^{-\alpha}k_t^{\alpha-\beta} - n_t - \frac{c_t}{k_t}\right]. \tag{a4}$$

(Step 2) Here assume that $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} = \lim_{t\to\infty}\frac{\dot{c}_t}{c_t} \neq 0$. Then, by equations (a3) and (a4),

$$\lim_{t\to\infty} k_t^{\alpha-\beta} = \lim_{t\to\infty}\frac{\frac{\theta}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon}\frac{c_t}{k_t} + n_t}{\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\beta)^{-\alpha}} = \frac{\frac{\theta}{1-\varepsilon} + n_t}{\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\beta)^{-\alpha}} - \frac{\varepsilon(1-\beta)^{\alpha}}{(1-\varepsilon)\left(\frac{\alpha}{mv}\right)^{\alpha}}\lim_{t\to\infty}\frac{c_t}{k_t} \quad \text{for } n_t > 0 \text{, and}$$

$$\lim_{t\to\infty} k_t^{\alpha-\beta} = \lim_{t\to\infty}\frac{\theta - \varepsilon\left[\frac{mL_t(1-\beta)}{mL_t(1-\beta)+\alpha}\right]\frac{c_t}{k_t}}{(1-\varepsilon)\left[\frac{mL_t(1-\beta)}{mL_t(1-\beta)+\alpha}\right]\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\beta)^{-\alpha}}$$

$$= \frac{\theta}{(1-\varepsilon)\left[\frac{mL_t(1-\beta)}{mL_t(1-\beta)+\alpha}\right]\left(\frac{\alpha}{mv}\right)^{\alpha}(1-\beta)^{-\alpha}} - \frac{\varepsilon(1-\beta)^{\alpha}}{(1-\varepsilon)\left(\frac{\alpha}{mv}\right)^{\alpha}}\lim_{t\to\infty}\frac{c_t}{k_t} \quad \text{for } n_t = 0.$$

(Step 3) Because $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} = \lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$, then $\lim_{t\to\infty}\frac{c_t}{k_t} = \text{constant}$, and thus $\lim_{t\to\infty} k_t^{\alpha-\beta} = \text{constant}$. Thereby $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} = 0$. This result contradicts the assumption $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} = \lim_{t\to\infty}\frac{\dot{c}_t}{c_t} \neq 0$. Therefore, it is impossible to achieve $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} = \lim_{t\to\infty}\frac{\dot{c}_t}{c_t} \neq 0$.

Q.E.D.