# House Allocation with Existing Tenants: An Equivalence*

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**Abstract**

In this paper we analyze two house allocation mechanisms each of which is designed to eliminate inefficiencies in real-life house allocation problems where there are both existing tenants and newcomers. The first mechanism chooses the unique core allocation of a "sister" exchange economy which is constructed by assigning each existing tenant her current house and randomly assigning each newcomer a vacant house. The second mechanism -top trading cycles mechanism- first chooses an ordering from a given distribution and next determines the final outcome as follows: Assign first agent her top choice, next agent her top choice among remaining houses and so on, until someone demands house of an existing tenant who is still in the line. At that point modify the queue by inserting her at the top and proceed. Similarly, insert any existing tenant who is not already served at the top of the queue once her house is demanded. Whenever a loop of existing tenants forms, assign each of them the house she demands and proceed. Our main result is that the core based mechanism is equivalent to an extreme case of the top trading cycles mechanism which orders newcomers before the existing tenants.

**Keywords:** Core, House Allocation, Housing Lottery, Indivisible Goods, Matching.

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# 1 Introduction

Motivated by real-life on-campus housing practices Abdulkadiroğlu and Sönmez (1999) introduce *house allocation problems with existing tenants*: A set of houses should be allocated to a set of agents by a centralized clearing house. Some of the agents are existing tenants each of whom already occupies a house and the rest of the agents are newcomers. In addition to occupied houses, there are vacant houses. Existing tenants are not only entitled to keep their current houses but also apply for other houses.

The mechanism known as *random serial-dictatorship with squatting rights* is used in most real-life applications of these problems.[1] This mechanism works as follows:

(a) Each existing tenant decides whether she will enter the housing lottery or keep her current house. Those who prefer keeping their houses are assigned their houses. All other houses become available for allocation.

(b) An ordering of agents in the lottery is randomly chosen from a given distribution of orderings. This distribution may be uniform or it may favor some groups.

(c) Once the agents are ordered, available houses are allocated using the induced *serial dictatorship*: The first agent receives her top choice, the next agent receives her top choice among the remaining houses and so on.

While this mechanism is very popular in real-life applications, it suffers from a major deficiency. Since it does not guarantee each existing tenant a house that is as good as her own, some existing tenants may choose to keep their houses even though they wish to move, and this may result in loss of potentially large gains from trade. Hence this popular mechanism is neither individually rational nor Pareto efficient.[2] One can fix this deficiency via two alternative approaches:

1. The first approach is based on the key mechanism for an important special case of our model. Consider the case where there are only existing tenants and occupied houses. This special case is known as *housing markets* (Shapley and Scarf, 1974). For each housing market there is a unique core allocation which also coincides with the unique competitive allocation (Roth and Postlewaite, 1977). Core, as a mechanism, is strategy-proof (Roth, 1982) and it is the only mechanism that is Pareto efficient, individually rational and strategy-proof (Ma, 1994). Based on these results, core (or equivalently the competitive mechanism) is considered the key mechanism for housing markets and hence it is natural to consider the following mechanism for house allocation problems with existing tenants:

   (a) First construct an initial allocation by (i) assigning each existing tenant her own house and (ii) randomly assigning the vacant houses to newcomers with uniform distribution, and

   (b) next choose the core of the induced housing market to determine the final outcome.

[1] Some examples include undergraduate housing at Carnegie-Mellon, Duke, Michigan, Northwestern and Pennsylvania.

[2] See Chen and Sönmez (2002, 2004) for experimental evidence of this inefficiency.

This mechanism is individually rational, Pareto efficient and strategy-proof.

2. The second approach is a direct one. First choose an ordering of agents from a given distribution of orderings and next determine the final outcome using the following "you request my house-I get your turn (YRMH-IGYT)" algorithm: Assign first agent her top choice, second agent her top choice among the remaining houses and so on, until someone demands house of an existing tenant. If at that point the existing tenant is already served then do not disturb the procedure. Otherwise modify the remainder of the queue by inserting her at the top and proceed. Similarly, insert any existing tenant who is not already served at the top of the queue once her house is demanded. If at any point a loop forms, it is formed by existing tenants and in such cases remove all agents in the loop by assigning them the houses they demand and proceed.

   The key innovation in this mechanism is that an existing tenant whose current house is requested is upgraded to the top of the queue before her house is assigned. As a result it is individually rational as it assures every existing tenant a house that is at least as good as her own. In addition it is also Pareto efficient and strategy-proof. YRMH-IGYT algorithm reduces Gale's top trading cycles algorithm for the special case of housing markets and following Abdulkadiroğlu and Sönmez (1999) we refer above mechanism as the *top trading cycles* mechanism.

In this paper we show that there is an important relation between the two mechanisms described above: *The core based mechanism is equivalent to an extreme case of the top-trading cycles mechanism where newcomers are randomly ordered first and existing tenants are randomly ordered next.* This result illustrates that there is a hidden bias in the core based mechanism. Recall that in that mechanism an initial allocation is constructed by assigning each existing tenant her current house and randomly assigning vacant houses to newcomers. This might be interpreted as granting property rights of vacant houses to newcomers. Therefore existing tenants who also have claims on vacant houses give up these claims under the core based mechanism. In that sense the bias in the core based mechanism is quite intuitive.

Our main result has an important corollary for the special case of *house allocation problems* (without existing tenants): The popular real-life mechanism *random serial dictatorship* is equivalent to *core from random endowments*. (Here random serial dictatorship randomly orders the agents and assigns the first agent her top choice, the next agent her top choice among remaining houses and so on whereas core from random endowments randomly chooses an initial allocation and chooses the core of the induced housing market.) This equivalence result is originally shown by Abdulkadiroğlu and Sönmez (1998) and it provides important support for both mechanisms since the two key mechanisms for house allocation problems are equivalent. The policy implication of our paper is quite different than that of Abdulkadiroğlu and Sönmez (1998). While core from random endowments is a key mechanism for house allocation problems, its extension to house allocation problems with existing tenants is extremely biased in favor of newcomers. In most real-life applications the priority is intended for existing tenants and our result shows that the core based approach is not the best choice in such cases.[3] Encouraged

[3] One could argue that the setup itself favors existing tenants since they each have a current house that they could keep assuring a lower bound on their welfare and therefore it is only fair that the chosen mechanism favors the newcomers for the vacant houses. In our view this normative issue shall be resolved by the central planner.

by Abdulkadiroğlu and Sönmez (1998), one may be tempted to use the core based mechanism for house allocation problems with existing tenants. Our paper shows that this approach may produce an undesired bias which can be avoided via the top trading cycles mechanism.

The rest of the paper is organized as follows: In Section 2 we formally introduce the model as well as the special case of housing markets and Gale's top trading cycles algorithm. In Section 3 we introduce the two mechanisms studied in the paper and analyze the dynamics of the YRMH-IGYT algorithm. In Section 4 we present our equivalence result and its corollary in the context of house allocation problems. Finally in Section 5 we conclude.

# 2 House Allocation with Existing Tenants

A set of houses (or other indivisible goods) should be allocated to a set of agents by a centralized clearing-house. Some of these agents are existing tenants each of whom already occupies a house, the rest of the agents are newcomers and there are houses which are vacant. Existing tenants are not only entitled to keep their current houses but also to apply for other houses if they wish. The main real-life application we have in mind is on-campus house allocation.

Formally, a **house allocation problem with existing tenants** (Abdulkadiroğlu and Sönmez, 1999) is a five-tuple $\langle A_E, A_N, H_O, H_V, P\rangle$ where $A_E = \{a_1, a_2, \ldots, a_n\}$ is a finite set of existing tenants, $A_N = \{a_{n+1}, \ldots, a_{n+m}\}$ is a finite set of newcomers, $H_O = \{h_a\}_{a\in A_E}$ is a finite set of occupied houses, $H_V$ is a finite set of vacant houses, and $P = (P_a)_{a\in A_E\cup A_N}$ is a list of strict preference relations. Let $A = A_E \cup A_N$ denote the set of all agents and $H = H_O \cup H_V$ denote the set of all houses. We assume that $|H| = |A| = n + m$ and thus $|H_V| = |A_N| = m$. Each agent $a \in A$ has a strict preference relation $P_a$ on the set of houses $H$. Let $R_a$ denote the "at-least-as-good-as" relation associated with $P_a$. Preferences are fixed throughout the paper.

A **matching** $\mu$ is an assignment of houses to agents such that each agent is assigned one house and each house is assigned to a different agent. Formally speaking a matching is a one-to-one mapping $\mu : A \to H$. For all $a \in A$, we refer $\mu(a)$ as the assignment of agent $a$ under $\mu$. Let $\mathcal{M}$ be the set of all matchings. Note that $|\mathcal{M}| = (n + m)!$.

A **lottery** is a probability distribution over all matchings. Let $\triangle\mathcal{M}$ denote the set of all lotteries. In order to simplify the exposition we abuse the notation and let $\mu$ also denote the lottery that assigns probability 1 to matching $\mu$.

## 2.1 Housing Markets

The class of **housing markets** (Shapley and Scarf, 1974) is an important subclass of our model where there are only existing tenants and occupied houses.[4] Formally a housing market is a four-tuple $\langle A, H, P, \mu\rangle$ where $A$ is a finite set of agents, $H$ is a finite set of houses, $P$ is a list of strict preference relations, and $\mu$ is a matching which specifies the initial allocation. Throughout the paper we fix $A$, $H$ and $P$ so that each matching $\mu$ defines a housing market.

Given a housing market $\mu$, the coalition $T \subseteq A$ **blocks** a matching $\eta \in \mathcal{M}$ if there exists a matching $\nu \in \mathcal{M}$ such that (i) $\nu(a) \in \{h \in H : h = \mu(a') \text{ for some } a' \in T\}$ for all $a \in T$, (ii) $\nu(a)R_a\eta(a)$ for all $a \in T$, and (iii) $\nu(a)P_a\eta(a)$ for some $a \in T$. A matching $\eta$ is in the **core** of a housing market $\mu$ if it is not blocked by any coalition.

[4]See Moulin (1995) for an extensive analysis of housing markets.

The core plays the key role for housing markets. Roth and Postlewaite (1977) show that there is a unique matching in the core of each housing market which also coincides with the unique competitive allocation. The core as a mechanism is strategy-proof (Roth, 1982) and it is the only mechanism that is Pareto efficient, individually rational and strategy-proof (Ma, 1994).

## 2.2 Gale's Top Trading Cycles Algorithm

**Gale's top trading cycles algorithm** (GTTCA) is an iterative algorithm which is used to find the unique core allocation of a housing market. This algorithm is one of the two key algorithms in this paper and it is defined as follows:

**Round 1:** Each agent points to the agent who owns her most preferred house. Since the number of agents is finite, there is at least one cycle (a **cycle** is either a singleton $(a_1)$ who points to herself or an ordered list $(a_1, \ldots, a_k)$ of agents where $a_1$ points to $a_k$, $a_k$ points to $a_{k-1}, \ldots, a_2$ points to $a_1$). In each cycle corresponding trades are performed and all agents in a cycle are removed together with their assignments. (Note that all of them are assigned their most preferred houses.) If there are remaining agents then we proceed with the next round.

In general,

**Round t:** Each remaining agent points to the agent who owns her most preferred house among those remaining in the market. In each cycle corresponding trades are performed and all agents in a cycle are removed together with their assignments. If there are remaining agents then we proceed with the next round.

By the finiteness of agents, at least one cycle forms at each round so that the algorithm terminates in at most $|A|$ rounds.

# 3 Matching and Lottery Mechanisms

A **matching mechanism** is a systematic procedure to select a matching for each house allocation problem with existing tenants. Similarly a **lottery mechanism** is a systematic procedure to select a lottery for each problem.

## 3.1 Core Based Mechanisms

Let $\mathcal{M}^* = \{\mu \in \mathcal{M} : \mu(a) = h_a \text{ for all } a \in A_E\}$ be the set of matchings which assign each existing tenant her current house. Note that $|\mathcal{M}^*| = m!$. For given $A$, $H$ and for each $\mu \in \mathcal{M}^*$ define mechanism $\varphi^\mu$ as follows: For any preference profile mechanism $\varphi^\mu$ interprets $\mu$ as the initial allocation and chooses the core of the induced housing market. Since the preferences are fixed throughout the paper, we denote the outcome of mechanism $\varphi^\mu$ also with $\varphi^\mu$ dropping the argument in $\varphi^\mu(P)$.

Since core is the key mechanism for housing markets, it is natural to consider the following lottery mechanism for house allocation problems with existing tenants:

1. For each problem, first construct an initial endowment by (i) assigning each existing tenant her current house and (ii) randomly assigning vacant house to newcomers with uniform distribution, and

2. next choose the core of the induced housing market as the final outcome.

Let us refer this mechanism as mechanism $\Phi$. Formally,

$$\Phi = \sum_{\mu \in \mathcal{M}^*} \frac{1}{m!} \varphi^{\mu}$$

## 3.2 Mechanisms Through a Direct Approach

Let $f : \{1, \ldots, n+m\} \to A$ be a bijection and $\mathcal{F}$ be the class of all such bijections. We refer each such bijection as an **ordering** of agents and denote it as the ordered list $(f(1), f(2), \ldots, f(n+m))$. For any ordering $f \in \mathcal{F}$, its inverse $f^{-1}(.)$ is defined as $f^{-1}(a) = i$ if and only if $f(i) = a$. For each $A^* \subset A$, a bijection $f : \{1, \ldots, |A^*|\} \to A^*$ is referred as a **sub-order**. Here agent $f(1)$ is the **head** and agent $f(|A^*|)$ is the **tail** of the sub-order $f$.

For a given ordering $f \in \mathcal{F}$ consider the following "**you request my house - I get your turn** (YRMH-IGYT)" algorithm (Abdulkadiroğlu and Sönmez, 1999): For any given ordering $f$, assign the first agent her top choice, the second agent her top choice among the remaining houses, and so on, until an agent $a$ demands house $h_{a'}$ of an existing tenant $a'$. If at that point existing tenant $a'$ is already served then do not disturb the procedure. Otherwise, modify the queue by inserting existing tenant $a'$ to the top so that existing tenant $a'$ is at the top of the line, agent $a$ is second in the line and the rest of the line is uninterrupted. Next it is the turn of existing tenant $a'$ and there are three possibilities:

1. Existing tenant $a'$ demands her own house $h_{a'}$: In this case existing tenant $a'$ is assigned her own house $h_{a'}$; next, once again, it is the turn of agent $a$ and she demands her top choice among the remaining houses and the procedure continues in a similar way.

2. Existing tenant $a'$ demands an available house $h$ that is either vacant or that used to be the house of an existing tenant who is already assigned another house: In this case existing tenant $a'$ is assigned the available house $h$, agent $a$ is assigned house $h_{a'}$, and the procedure continues with the next agent in line.

3. Existing tenant $a'$ demands house $h_{a''}$ of another existing tenant $a''$ who is still in the line: In this case modify the queue by inserting existing tenant $a''$ at the top so that existing tenant $a''$ is at the top of the line, existing tenant $a'$ is second in the line, agent $a$ is third in the line and the rest of the line is uninterrupted. Next it is the turn of existing tenant $a''$ and the procedure continues in a similar way.

As we proceed, existing tenants may form loop-orders. (A **loop-order** is either a singleton $(a_1)$ who demands her own house or an ordered list $(a_1, \ldots, a_k)$ of existing tenants where agent $a_1$ demands the house of agent $a_k$, agent $a_k$ demands the house of agent $a_{k-1}$, …, agent $a_2$ demands the house of agent $a_1$.) In such cases, remove all agents in the loop-order by assigning them the houses they demand and proceed.

For any ordering $f \in \mathcal{F}$, let $\psi^f$ denote the induced matching mechanism through YRMH-IGYT algorithm. Following Abdulkadiroğlu and Sönmez (1999), we refer this mechanism as the **top trading cycles mechanism**. Since the preferences are fixed, we denote the outcome of YRMH-IGYT algorithm also with $\psi^f$ dropping the argument in $\psi^f(P)$. In this paper we are particularly interested in orderings which place existing tenants at the end of the line giving priority to newcomers. Define $\widetilde{\mathcal{F}} = \{f \in \mathcal{F} : f^{-1}(a) < f^{-1}(a') \text{ for all } a \in A_N \text{ and } a' \in A_E\}$. Note that $|\widetilde{\mathcal{F}}| = n!m!$. Define mechanism $\Psi$ as

$$\Psi = \sum_{f \in \widetilde{\mathcal{F}}} \frac{1}{m!n!} \psi^f$$

That is, an ordering $f$ among those which give priority to newcomers is randomly chosen with uniform distribution and next the outcome is obtained using YRMH-IGYT algorithm.

## 3.3 Dynamics of YRMH-IGYT Algorithm

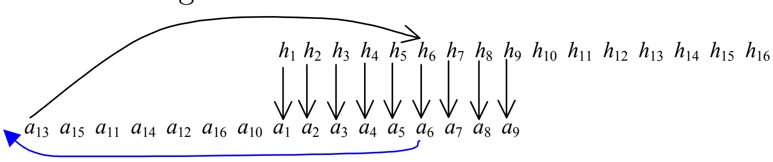
Since YRMH-IGYT algorithm is key to this paper, it is crucial to understand how it works. For a given ordering $f$, the **serial-dictatorship** induced by $f$ allocates the houses as follows: The first agent receives her top choice, the next agent receives her top choice among the remaining houses and so on. For a given ordering $f \in \mathcal{F}$, construct the **effective-order** $e_f \in \mathcal{F}$ as follows: Run YRMH-IGYT algorithm and order agents in the same order their assignments are finalized. When there is a loop-order, order these agents as in the loop-order.

We illustrate the construction of $e_f$ with the following example. Later on we use the same example to illustrate other constructions that are crucial to this paper. Example 1 is rather involved in order to capture every key aspect of these constructions.

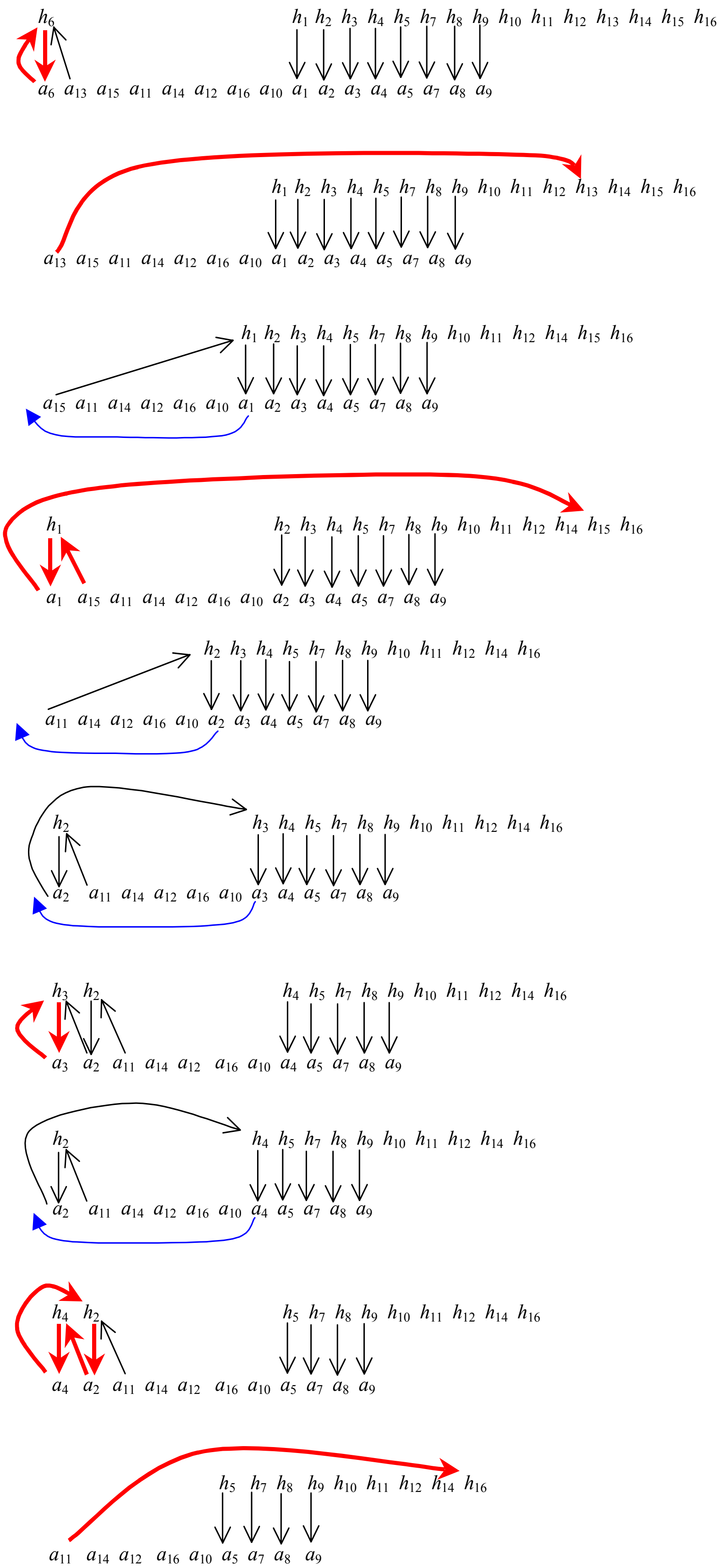
**Example 1:** Let $A_E = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$ be the set of existing tenants, $A_N = \{a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}\}$ be the set of newcomers, $H_0 = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\}$ be the set of occupied houses, and $H_V = \{h_{10}, h_{11}, h_{12}, h_{13}, h_{14}, h_{15}, h_{16}\}$ be the set of vacant houses. (Here $h_i$ is the current house of existing tenant $a_i$ for $i \leq 9$.) Let the preference profile $P$ be given as:[5]

| $A_E$ | | | | | | | | | $A_N$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $a_{16}$ |
| $h_{15}$ | $h_3$ | $h_1$ | $h_2$ | $h_9$ | $h_6$ | $h_6$ | $h_6$ | $h_{11}$ | $h_7$ | $h_2$ | $h_4$ | $h_6$ | $h_8$ | $h_1$ | $h_5$ |
| $\vdots$ | $h_4$ | $h_3$ | $\vdots$ | $\vdots$ | $\vdots$ | $h_7$ | $h_{12}$ | $\vdots$ | $h_3$ | $h_4$ | $h_{14}$ | $h_{13}$ | $\vdots$ | $\vdots$ | $\vdots$ |
| | $\vdots$ | $\vdots$ | | | | $\vdots$ | $\vdots$ | | $h_{12}$ | $h_{16}$ | $\vdots$ | $\vdots$ | | | |
| | | | | | | | | | $h_{10}$ | $\vdots$ | | | | | |
| | | | | | | | | | $\vdots$ | | | | | | |

Let $f = (a_{13}, a_{15}, a_{11}, a_{14}, a_{12}, a_{16}, a_{10}, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ be the ordering of the agents. The following series of figures illustrates the dynamics of the YRMH-IGYT algorithm. When an agent's assignment under $\psi^f$ is finalized, that is indicated with bold arrows and reported at the right end of the figure. The effective-order $e_f$ orders the agents in the same order as their assignments are finalized.




[5] After the best few houses the rest of the preferences are arbitrary for each agent.




$\psi^f(a_6) = h_6$

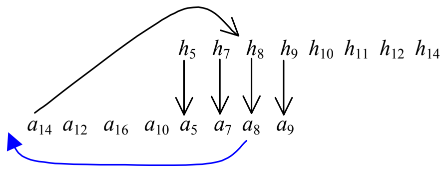
$\psi^f(a_{13}) = h_{13}$

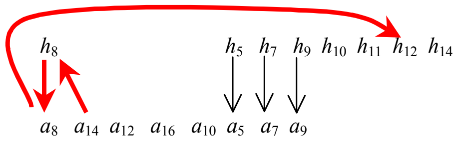
$\psi^f(a_1) = h_{15}$
$\psi^f(a_{15}) = h_1$

$\psi^f(a_3) = h_3$

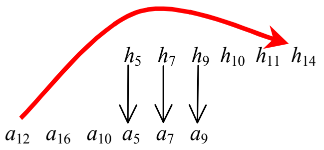
$\psi^f(a_4) = h_2$
$\psi^f(a_2) = h_4$

$\psi^f(a_{11}) = h_{16}$

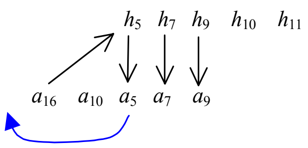


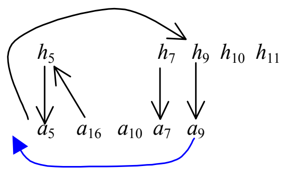



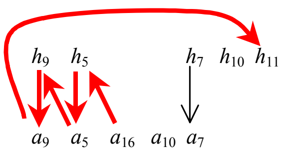

$\psi^f(a_8) = h_{12}$
$\psi^f(a_{14}) = h_8$



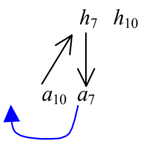

$\psi^f(a_{12}) = h_{14}$

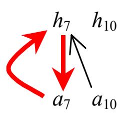



$\psi^f(a_9) = h_{11}$
$\psi^f(a_5) = h_9$
$\psi^f(a_{16}) = h_5$




$\psi^f(a_7) = h_7$


$\psi^f(a_{10}) = h_{10}$

In this example agents' assignments are finalized in the following order:

$$e_f = (a_6, a_{13}, a_1, a_{15}, a_3, a_4, a_2, a_{11}, a_8, a_{14}, a_{12}, a_9, a_5, a_{16}, a_7, a_{10})$$

The outcome of the algorithm is

$$\psi^f = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ h_{15} & h_4 & h_3 & h_2 & h_9 & h_6 & h_7 & h_{12} & h_{11} & h_{10} & h_{16} & h_{14} & h_{13} & h_8 & h_1 & h_5 \end{pmatrix}$$

Recall that only existing tenants are inserted to the top of the line in the YRMH-IGYT algorithm. Therefore the relative order of newcomers in an ordering $f$ and its effective-order $e_f$ are the same.

**Observation 1:** For all $f \in \mathcal{F}$ and $a, a' \in A_N$ we have $f^{-1}(a) < f^{-1}(a') \Longleftrightarrow e_f^{-1}(a) < e_f^{-1}(a')$

Next consider an ordering $f \in \widetilde{\mathcal{F}}$. Here agents $f(1), \ldots, f(m)$ are newcomers. Since the relative order of newcomers are identical in $f$ and $e_f$, the effective-order $e_f$ will order agents as follows: Some existing tenants (possibly none) are followed by $f(1)$, followed by some existing tenants (possibly none), followed by $f(2)$, …, followed by $f(m)$, followed by some existing tenants (possibly none).

Consider newcomer $f(1)$ who is at the top of ordering $f$. If she is not at the top of effective-order $e_f$ that means she requested the current house of an existing tenant who might have requested the current house of another existing tenant and so on. Insertion of existing tenants will stop once any of these existing tenants (or the newcomer $f(1)$ herself) requests a vacant house. Therefore one and only one agent among newcomer $f(1)$ and her predecessors in $e_f$ will be assigned a vacant house. Similarly for any $k \leq m$, $k$ agents will be assigned vacant houses among newcomer $f(k)$ and her predecessors in $e_f$. Hence we have the following observation:

**Observation 2:** Let $f \in \widetilde{\mathcal{F}}$ and consider the matching $\psi^f$. There is one and only one agent between $e_f(1)$ and $f(1)$ in effective-order $e_f$ who is assigned a vacant house. Similarly for each $k \leq m$, there is one and only one agent between the immediate successor of $f(k-1)$ and $f(k)$ in $e_f$ who is assigned a vacant house.

For each $f \in \widetilde{\mathcal{F}}$, YRMH-IGYT algorithm assigns houses in one of two possible ways:

1. There is a sub-order $(a_1, \ldots, a_k)$ of agents where

   (a) $a_k$ is a newcomer, $a_1, \ldots, a_{k-1}$ are existing tenants and

   (b) $a_1$ receives a vacant house, $a_2$ receives $a_1$'s house, … , $a_k$ receives $a_{k-1}$'s house.

   We call each such sub-order a **serial-order** ($S$).

2. There is a sub-order $(a_1, \ldots, a_k)$ of existing tenants where $a_1$ receives $a_k$'s house, $a_k$ receives $a_{k-1}$'s house, …, $a_2$ receives $a_1$'s house. Recall that we call each such sub-order a **loop-order** ($L$).

Therefore effective-order $e_f$ is a sequence $L, \ldots, L, S_1, L, \ldots, L, S_2, \ldots, L, S_m, L, \ldots, L$ of serial-orders and loop-orders where the tail of serial-order $S_i$ is newcomer $f(i)$ for $i \leq m$.

**Example 1 continued:** Following the dynamics of YRMH-IGYT algorithm in Example 1, effective-order $e_f$ is the following sequence of loop-orders and serial-orders.

$$\underbrace{(a_6)}_{L_1}, \underbrace{(a_{13})}_{S_1}, \underbrace{(a_1, a_{15})}_{S_2}, \underbrace{(a_3)}_{L_2}, \underbrace{(a_4, a_2)}_{L_3}, \underbrace{(a_{11})}_{S_3}, \underbrace{(a_8, a_{14})}_{S_4}, \underbrace{(a_{12})}_{S_5}, \underbrace{(a_9, a_5, a_{16})}_{S_6}, \underbrace{(a_7)}_{L_4}, \underbrace{(a_{10})}_{S_7}$$

### 3.4 A Simplification of Mechanism $\Psi$

Define $\mathcal{F}^* = \{f \in \widetilde{\mathcal{F}} : a < a' \Rightarrow f^{-1}(a) < f^{-1}(a') \text{ for all } a, a' \in A_E\}$. That is, orderings in $\mathcal{F}^*$ not only order newcomers before existing tenants but also order existing tenants based on their index. Note that $|\mathcal{F}^*| = m!$. The following lemma states that the outcome of YRMH-IGYT algorithm is identical for two orderings in $\widetilde{\mathcal{F}}$ as long as newcomers are ordered in the same way under both orderings.

**Lemma 1:** Let $f, g \in \widetilde{\mathcal{F}}$ be such that $f(i) = g(i)$ for all $i \leq m$. Then $\psi^f = \psi^g$.

*Proof*: Let $f, g \in \widetilde{\mathcal{F}}$ be such that $f(i) = g(i)$ for all $i \leq m$. Since $f(i) = g(i)$ for all $i \leq m$, YRMH-IGYT algorithm works identical for both orderings until newcomer $f(m)$ (i.e. the last newcomer) is assigned a house. Therefore for each $i \leq m$, agent $f(i)$ is assigned the same house under $\psi^f$ and $\psi^g$. Next consider the rest of the agents each of whom is an existing tenant. YRMH-IGYT algorithm is equivalent to GTTCA when there are only existing tenants (Abdulkadiroğlu and Sönmez, 1999) and therefore each of the remaining agents receive the unique core assignment of the remaining market under either ordering. Hence $\psi^f = \psi^g$. $\diamondsuit$

Using Lemma 1 we can obtain the following simpler expression for mechanism $\Psi$:

$$\Psi = \sum_{f \in \mathcal{F}^*} \frac{1}{m!} \psi^f$$

That is, first randomly order the newcomers with uniform distribution, next order the existing tenants based on their index and finally obtain the outcome using YRMH-IGYT algorithm.

# 4 Main Result

Our main contribution is that the two lottery mechanisms $\Phi$ and $\Psi$ are equivalent. Recall that both mechanisms select a uniform lottery over $m!$ matchings for each problem. Here is our proof strategy: For each ordering $f \in \mathcal{F}^*$ we construct a matching $\eta(f) \in \mathcal{M}^*$ such that $\psi^f = \varphi^{\eta(f)}$. Next we show that mapping $\eta : \mathcal{F}^* \to \mathcal{M}^*$ is a bijection by constructing its inverse mapping. Therefore mapping $\eta$ is such that $f \neq g \Leftrightarrow \eta(f) \neq \eta(g)$ for all $f, g \in \mathcal{F}^*$ and this in turn implies that $\Phi = \Psi$.

### 4.1 Construction of Mapping $\eta$

Construction of mapping $\eta : \mathcal{F}^* \to \mathcal{M}^*$ is quite involved and it requires additional notation. The key challange in this construction is finding a mapping which is a bijection (i.e. one-to-one and onto). Otherwise it would be a straightforward task to construct a mapping $\nu : \mathcal{F}^* \to \mathcal{M}^*$ such that $\psi^f = \varphi^{\nu(f)}$ for each $f \in \mathcal{F}^*$. For example one such mapping $\nu$ can be constructed by simply

1. finding the effective order, loop-orders, serial-orders, and

2. assigning each agent at the tail of a serial-order (who is by definition a newcomer) the vacant house allocated in the serial-order.

When we run GTTCA with this initial allocation, each of the loop-orders and the serial-orders obtained in the YRMH-IGYT algorithm will form as a cycle, and hence the same outcome will be obtained by the two algorithms. However the mapping $\nu$ is not one-to-one and thus two distinct orderings $f, g$ may yield the same initial allocation $\nu(f) = \nu(g)$. We illustrate this point with our running example.

**Example 1 continued:** Recall that for ordering

$$f = (a_{13}, a_{15}, a_{11}, a_{14}, a_{12}, a_{16}, a_{10}, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9),$$

the effective-order $e_f$ is the following sequence of loop-orders and serial-orders:

$$\underbrace{(a_6)}_{L_1}, \underbrace{(a_{13})}_{S_1}, \underbrace{(a_1, a_{15})}_{S_2}, \underbrace{(a_3)}_{L_2}, \underbrace{(a_4, a_2)}_{L_3}, \underbrace{(a_{11})}_{S_3}, \underbrace{(a_8, a_{14})}_{S_4}, \underbrace{(a_{12})}_{S_5}, \underbrace{(a_9, a_5, a_{16})}_{S_6}, \underbrace{(a_7)}_{L_4}, \underbrace{(a_{10})}_{S_7}$$

Next consider the ordering

$$g = (a_{15}, a_{13}, a_{11}, a_{14}, a_{12}, a_{16}, a_{10}, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$$

which differs from ordering $f$ in only the order of agents $a_{13}$ and $a_{15}$. In this case the effective order $e_g$ is the following sequence of loop-orders and serial-orders

$$\underbrace{(a_1, a_{15})}_{S_2}, \underbrace{(a_6)}_{L_1}, \underbrace{(a_{13})}_{S_1}, \underbrace{(a_3)}_{L_2}, \underbrace{(a_4, a_2)}_{L_3}, \underbrace{(a_{11})}_{S_3}, \underbrace{(a_8, a_{14})}_{S_4}, \underbrace{(a_{12})}_{S_5}, \underbrace{(a_9, a_5, a_{16})}_{S_6}, \underbrace{(a_7)}_{L_4}, \underbrace{(a_{10})}_{S_7}$$

which consists of the same loop-orders and same serial-orders as effective order $e_f$, although in a different sequence. Since the serial-orders are the same for the two effective orders, the above mentioned mapping $\nu$ yields

$$\nu(f) = \nu(g) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{16} & h_{14} & h_{13} & h_{12} & h_{15} & h_{11} \end{pmatrix}.$$

We next proceed with additional notation need for the construction of mappning $\eta$. Recall that for each $f \in \mathcal{F}^*$ effective-order $e_f$ is a sequence of serial-orders and loop-orders $L, \ldots, L, S_1, L, \ldots, L, S_2, \ldots, L, S_m, L, \ldots, L$. Moreover newcomer $f(1)$ is the tail of serial-order $S_1$, newcomer $f(2)$ is the tail of serial-order $S_2$, ..., newcomer $f(m)$ is the tail of serial-order $S_m$. We partition serial-orders and loop-orders of effective-order $e_f$ as follows:

**Step 1:** Starting with agent $e_f(1)$ clear each agent in order until it is the turn of an agent $a$ for whom her assignment $\psi^f(a)$ is worse than a house previously assigned to an agent in a serial-order. Terminate first step right after the *last serial-order before* agent $a$. Next proceed to step 2. If such an agent does not exist then $e_f$ consists of a single step.

In general,

**Step t:** Starting with the next agent clear agents one at a time until it is the turn of an agent $a$ for whom her assignment $\psi^f(a)$ is worse than a house previously assigned to an agent in a serial-order of current step $t$. Terminate step $t$ after the *last serial-order before* agent $a$. Next proceed to step $t + 1$. If such an agent does not exist then $e_f$ consists of $t$ steps.

Let $e_f$ consist of $T$ steps. For each $t \leq T$, let $\mathcal{S}^t$ denote the set of serial-orders of $e_f$ at step $t$, $\mathcal{L}^t$ denote the set of loop-orders of $e_f$ at step $t$ and $A_N^t$ denote the set of newcomers at step

$t$. For each $t \le T$, let $A_{\mathcal{S}^t}$ denote the set of agents in serial-orders of $\mathcal{S}^t$ and $A_{\mathcal{L}^t}$ denote the set of agents in loop-orders of $\mathcal{L}^t$. For each $t \le T$, let $\psi^f(\mathcal{S}^t)$ denote the set of houses assigned to agents in $A_{\mathcal{S}^t}$ and $\psi^f(\mathcal{L}^t)$ denote the set of houses assigned to agents in $A_{\mathcal{L}^t}$. For any loop-order $L$ let $\psi^f(L)$ denote the set of houses assigned to members of $L$ and for any serial-order $S$ let $\psi^f(S)$ denote the set of houses assigned to members of $S$.

Now we iteratively construct sets $G^1, \ldots, G^T$ of houses as follows: First consider houses which are assigned to agents in serial-orders. For any $t \le T$, include in $G^t$ all houses in $\psi^f(\mathcal{S}^t)$. We may include additional houses to $G^1, \ldots, G^{T-1}$ as explained below.

Next consider houses which are assigned to agents in loop-orders. We skip loop-orders in $\mathcal{L}^1$. Start with the first loop-order $L$ in $\mathcal{L}^2$. If any agent $a \in L$ prefers any of the current houses in $G^1$ to her own assignment $\psi^f(a)$ then enlarge $G^1$ by including houses in $\psi^f(L)$. If no such agent exists, do not change $G^1$ at this point. Similarly consider each loop-order one at a time in order. For any loop-order first determine which step of $e_f$ it belongs. Suppose it is the turn of loop-order $L \in \mathcal{L}^t$. If any agent $a$ in loop-order $L$ prefers any of the current houses in $G^{t-1}$ to her own assignment $\psi^f(a)$ then enlarge $G^{t-1}$ by including houses in $\psi^f(L)$. If no such agent exists then check whether any agent $a$ in loop-order $L$ prefers any of the current houses in $G^{t-2}$ to her own assignment $\psi^f(a)$. If so then enlarge $G^{t-2}$ by including houses in $\psi^f(L)$. If no such agent exists then check whether .... If no such agent exists then check whether any agent $a$ in loop-order $L$ prefers any of the current houses in $G^1$ to her own assignment $\psi^f(a)$. If so then enlarge $G^1$ by including houses in $\psi^f(L)$. If no such agent exists then do not change any of $G^{t-1}, \ldots, G^1$ at this point and proceed with the next loop-order.[6]

**Remark 1**: Consider any $t \in \{1, \ldots, T\}$. Pick any loop-order $L$ in $e_f$. We have $\psi^f(L) \subseteq G^t$ if and only if (i) there exists an agent $a$ in loop-order $L$ and a house $h \in G^t$ such that $hP_a\psi^f(a)$, and (ii) for any agent $a'$ in loop-order $L$, for any $r \in \{t+1, \ldots, T\}$, and for any $h' \in G^r$ we have $\psi^f(a')P_{a'}h'$.

For $t > 1$, let $\mathcal{S}^{*t} \subseteq \mathcal{S}^t$ be the set of serial-orders at step $t$ where at least one member of the serial-order prefers a house in $G^{t-1}$ to her assignment under $\psi^f$. That is

$$\mathcal{S}^{*t} = \{S \in \mathcal{S}^t : hP_a\psi^f(a) \text{ for some house } h \in G^{t-1} \text{ and some agent } a \text{ in serial-order } S\}$$

For each $t > 1$, $S^{*t}$ is non-empty by construction of step $t$ of $e_f$ together with construction of $G^{t-1}$. Finally for $t > 1$, let $A_N^{*t}$ be the set of newcomers each of whom is the tail of a serial-order in $\mathcal{S}^{*t}$.[7]

We are ready to construct mapping $\eta : \mathcal{F}^* \to \mathcal{M}^*$. For each $f \in \mathcal{F}^*$:

[6] Construction of sets $G^1, \ldots, G^T$ makes it possible to "link" the newcomers in the same step so that their assignments are finalized simultaneously under the GTTCA. Moreover the construction assures that newcomers in Step $t$ leave GTTCA before newcomers in Step $s$ for $t < s$. Therefore it will be possible to recover the relative ordering of two newcomers in the original ordering used by the YRMH-IGYT algorithm, provided that the two newcomers belong to serial-orders of different steps.

[7] As we have already indicated the construction of steps of $e_f$ and sets $G^1, \ldots, G^T$ make it possible to recover the relative ordering of two newcomers in different steps. Depending on which serial-orders join to form cycles in the GTTCA, recovering the relative ordering of some of the newcomers in the same cycle (who are necessarily in the same step) will also be possible. However this will not uniquely determine

1. the relative ordering of newcomers in Step 1, or
2. the relative ranking of newcomers in $A_N^{*t}$ for $t > 1$.

In the construction of matching $\eta(f)$, the indices of these newcomers will be utilized for this purpose.

1. Find effective-order $e_f$. Find loop-orders, serial-orders and steps of $e_f$ as well as sets $\{G^t\}$, $\{\mathcal{S}^{*t}\}$ and $\{A_N^{*t}\}$.

2. For each existing tenant $a \in A_E$, let $\eta(f)(a) = h_a$. That is, each existing tenant is assigned her current house under $\eta(f)$.

3. Next we handle newcomers in step 1 of $e_f$. The number of vacant houses assigned at step 1 is equal to the number of newcomers at step 1. Let $G_V^1$ be the set of vacant houses assigned at step 1 of $e_f$. Assign the first newcomer in $e_f$ the smallest indexed house in $G_V^1$, the second newcomer in $e_f$ the second smallest indexed house in $G_V^1$, ..., the last newcomer in step 1 of $e_f$ the biggest indexed house in $G_V^1$ under matching $\eta(f)$.

4. Finally we handle newcomers in step $t$ of $e_f$ for $t > 1$.

   Recall that (i) each newcomer at step $t$ is the tail of a serial-order and (ii) in each serial-order only the head agent is assigned a vacant house. Newcomers in $A_N^{*t}$ will be treated differently than newcomers in $A_N^t \setminus A_N^{*t}$.

   (a) Newcomers in $A_N^{*t}$: Recall that $A_N^{*t}$ is the set of newcomers each of whom is the tail of a serial-order in $\mathcal{S}^{*t}$. For each serial-order $S \in \mathcal{S}^{*t}$, find the vacant house that is assigned in the *next* serial-order of $e_f$ unless $S$ is the last serial-order of step $t$. If $S \in \mathcal{S}^{*t}$ is the last serial-order of step $t$ then find the vacant house that is assigned in the *first* serial-order of step $t$. Let $G_V^{*t}$ be the resulting set of vacant houses. Order newcomers in $A_N^{*t}$ based on their order in $e_f$. Under matching $\eta(f)$ the first newcomer in $A_N^{*t}$ receives the smallest indexed house in $G_V^{*t}$, the second newcomer in $A_N^{*t}$ receives the second smallest indexed house in $G_V^{*t}$ and so on.

   (b) Newcomers in $A_N^t \setminus A_N^{*t}$: Under $\eta(f)$ each such newcomer who is not the last newcomer of step $t$ receives the vacant house that is assigned in the next serial-order. If the newcomer is the last newcomer of step $t$ then she receives the vacant house that is assigned in the first serial-order of step $t$.

As we already emphasized, the set of newcomers $A_N^{*t}$ plays a key role in construction of matching $\eta(f)$. Under $\eta(f)$ each agent in a serial-order $S \in \mathcal{S}^t$ is assigned a house $h \in \psi^f(\mathcal{S}^t)$. Moreover when GTTCA is executed for housing market $\eta(f)$, (i) serial-orders in $\mathcal{S}^t$ will form one or more cycles among themselves and (ii) each of these cycles will contain at least one newcomer in $A_N^{*t}$. Here the second point ensures that each serial-order in $\mathcal{S}^t$ becomes part of a cycle and leaves the market after at least one of the serial orders in $\mathcal{S}^{t-1}$. In Section 4.2 we execute GTTCA in such a way that cycles that include newcomers are removed from the market simultaneously. That assures that agents in $A_{\mathcal{S}^{t-1}}$ leave the market before agents in $A_{\mathcal{S}^t}$ for any $t > 1$. This point is key for construction of inverse mapping $g$ in Section 4.2.

**Remark 2:** Pick any $t > 1$. Execute GTTCA for housing market $\eta(f)$ as explained in Section 4.2. No agent in $A_{\mathcal{S}^t}$ leaves the market before each agent in $A_{\mathcal{S}^{t-1}}$ does.

Next we illustrate partition of $e_f$ into its steps, construction of sets $G^1, \ldots, G^T$, $\mathcal{S}^{*1}, \ldots, \mathcal{S}^{*T}$, $A_N^{*1}, \ldots, A_N^{*T}$ and construction of mapping $\eta$ with our running example.

**Example 1 continued:** In order to construct $\eta(f)$ we first partition $e_f$ into its steps. Clear each agent one at a time following the order in $e_f$. We can skip the agents until the end of the first serial-order $S_1$. Consider agent $a_1$. We have $\psi^f(a_1) = h_{15}$ which is the top choice of agent $a_1$. So skip to agent $a_{15}$. We have $\psi^f(a_{15}) = h_1$ which is the top choice of agent $a_{15}$. So skip to agent $a_3$. We have

$$\underbrace{\psi^f(a_{15})}_{=h_1} P_{a_3} \underbrace{\psi^f(a_3)}_{=h_3}$$

and moreover agent $a_{15}$ is a member of serial-order $S_2$. Therefore Step 1 ends right after the last serial-order before agent $a_3$, namely serial-order $S_2 = (a_1, a_{15})$.

We can skip the agents until the end of the first serial-order of Step 2. Consider agent $a_8$. We have $\psi^f(a_8) = h_{12}$ and for agent $a_8$ only house $h_6$ is better. However house $h_6$ is assigned to agent $a_6$ who is a member of a loop-order. So skip to agent $a_{14}$. We have $\psi^f(a_{14}) = h_8$ which is the top choice of agent $a_{14}$. So skip to agent $a_{12}$. We have $\psi^f(a_{12}) = h_{14}$ and for agent $a_{12}$ only house $h_4$ is better. However house $h_4$ is assigned to agent $a_4$ who is a member of a loop-order. So skip to agent $a_9$. We have $\psi^f(a_9) = h_{11}$ which is the top choice of agent $a_9$. So skip to agent $a_5$. We have $\psi^f(a_5) = h_9$ which is the top choice of agent $a_5$. So skip to agent $a_{16}$. We have $\psi^f(a_{16}) = h_5$ which is the top choice of agent $a_{16}$. So skip to agent $a_7$. We have $\psi^f(a_7) = h_7$ and for agent $a_7$ only house $h_6$ is better. However house $h_6$ is assigned to agent $a_6$ who is a member of a loop-order. So skip to agent $a_{10}$. We have

$$\underbrace{\psi^f(a_8)}_{=h_{12}} P_{a_{10}} \underbrace{\psi^f(a_{10})}_{=h_{10}}$$

and moreover agent $a_8$ is a member of serial-order $S_4$ which is in Step 2. Therefore Step 2 ends right after the last serial-order before agent $a_{10}$, namely the serial-order $S_6 = (a_9, a_5, a_{16})$.

Agent $a_{10}$ was the last agent in $e_f$ so effective-order $e_f$ has 3 steps:

$$\underbrace{\underbrace{(a_6)}_{L_1}, \underbrace{(a_{13})}_{S_1}, \underbrace{(a_1, a_{15})}_{S_2}}_{\text{Step 1}} \Bigg| \underbrace{\underbrace{(a_3)}_{L_2}, \underbrace{(a_4, a_2)}_{L_3}, \underbrace{(a_{11})}_{S_3}, \underbrace{(a_8, a_{14})}_{S_4}, \underbrace{(a_{12})}_{S_5}, \underbrace{(a_9, a_5, a_{16})}_{S_6}}_{\text{Step 2}} \Bigg| \underbrace{\underbrace{(a_7)}_{L_4}, \underbrace{(a_{10})}_{S_7}}_{\text{Step 3}}$$

Therefore $A_N^1 = \{a_{13}, a_{15}\}$, $A_N^2 = \{a_{11}, a_{14}, a_{12}, a_{16}\}$ and $A_N^3 = \{a_{10}\}$.

Next, we construct sets of houses $G^1$, $G^2$ and $G^3$:

First consider houses that are assigned to members of serial-orders. Serial-orders of Step 1 are $S_1$, $S_2$ and houses assigned in these serial-orders are $h_{13}$, $h_{15}$, $h_1$. Therefore $\{h_{13}, h_{15}, h_1\} \subseteq G^1$. Serial-orders of Step 2 are $S_3$, $S_4$, $S_5$, $S_6$ and houses assigned in these serial-orders are $h_{16}$, $h_{12}$, $h_8$, $h_{14}$, $h_{11}$, $h_9$, $h_5$. Therefore $\{h_{16}, h_{12}, h_8, h_{14}, h_{11}, h_9, h_5\} \subseteq G^2$. The only serial-order of Step 3 is $S_7$ and the only house that is assigned in that serial-order is $h_{10}$. Therefore $\{h_{10}\} \subseteq G^3$.

Next consider houses which are assigned to agents in loop-orders. Skip loop-order $L_1$ which is in Step 1. Consider loop-order $L_2 = (a_3)$ which is in Step 2. We have

$$h_1 P_{a_3} \underbrace{\psi^f(a_3)}_{=h_3} \text{ and } h_1 \in G^1$$

Hence we include house $h_3$ to $G^1$ since it is the only house assigned in loop-order $L_2$. Thus $\{h_3\} \subseteq G^1$. Next consider loop-order $L_3 = (a_4, a_2)$ which is also in Step 2. We have

$$h_3 P_{a_2} \underbrace{\psi^f(a_2)}_{=h_4} \text{ and } h_3 \in G^1$$

Hence we include houses $h_4, h_2$ to $G^1$ since they are the houses assigned in loop-order $L_3$. Thus $\{h_4, h_2\} \subseteq G^1$. Finally consider loop-order $L_4 = (a_7)$ which is in step 3. There is no house $h \in G^2$ such that

$$hP_{a_7} \underbrace{\psi^f(a_7)}_{=h_7}$$

so $h_7 \notin G^2$. There is no house $h \in G^1$ such that

$$hP_{a_7} \underbrace{\psi^f(a_7)}_{=h_7}$$

so $h_7 \notin G^1$ either. $L_4$ is the last loop-order so $G^1$, $G^2$ and $G^3$ are finalized as:

$$G^1 = \{h_{13}, h_{15}, h_1, h_3, h_2, h_4\}, \quad G^2 = \{h_{16}, h_{12}, h_8, h_{14}, h_{11}, h_9, h_5\}, \quad G^3 = \{h_{10}\}$$

Next we construct sets $\mathcal{S}^{*2}$ and $\mathcal{S}^{*3}$ of serial-orders and sets $A_N^{*2}$, $A_N^{*3}$ of newcomers. We have $\mathcal{S}^{*2} = \{S_3, S_5\}$ since

- $a_{11}$ is a member of $S_3$, $h_2 \in G^1$, $h_2 P_{a_{11}} \underbrace{\psi^f(a_{11})}_{=h_4}$,
- $a_{12}$ is a member of $S_5$, $h_4 \in G^1$, $h_4 P_{a_{12}} \underbrace{\psi^f(a_{12})}_{=h_2}$,
- and no member of $S_4$ or $S_6$ prefer any house in $G^1$ to their assignment under $\psi^f$.

We have $\mathcal{S}^{*3} = \{S_7\}$ since

- $a_{10}$ is a member of $S_7$, $h_{12} \in G^2$, $h_{12} P_{a_{10}} \underbrace{\psi^f(a_{10})}_{=h_{10}}$.

Therefore $A_N^{*2}$ consists of the tails of serial-orders $S_3$, $S_5$ and $A_N^{*3}$ consists of the tail of serial-order $S_7$. Hence $A_N^{*2} = \{a_{11}, a_{12}\}$ and $A_N^{*3} = \{a_{10}\}$.

We are ready to construct matching $\eta(f)$:

1. For each existing tenant $a_j \in A_E$ we have $\eta(f)(a_j) = h_j$. That is $\eta(f)(a_j) = h_j$ for $j \leq 9$.
2. Next consider the newcomers:
   (a) Newcomers in Step 1 of $e_f$:
   $G_V^1 = \{h_{13}, h_{15}\}$ is the set of vacant houses assigned at Step 1 and $A_N^1 = \{a_{13}, a_{15}\}$. Since $a_{13}$ is ordered before $a_{15}$ in $e_f$, newcomer $a_{13}$ is assigned the smaller indexed house in $G_V^1$ and $a_{15}$ is assigned the bigger indexed house in $G_V^1$ under $\eta(f)$. Therefore $\eta(f)(a_{13}) = h_{13}$ and $\eta(f)(a_{15}) = h_{15}$.
   (b) Newcomers in Step 2 of $e_f$:
      i. Newcomers in $A_N^{*2} = \{a_{11}, a_{12}\}$ :
      The two serial-orders in $\mathcal{S}^{*2}$ are $S_3$ and $S_5$. We have $G_V^{*2} = \{h_{11}, h_{12}\}$ because

- $h_{12}$ is the vacant house that is assigned in $S_4$ which is the next serial-order after $S_3$ and
- $h_{11}$ is the vacant house that is assigned in $S_6$ which is the next serial-order after $S_5$.

Since $a_{11}$ is ordered before $a_{12}$ in $e_f$, newcomer $a_{11}$ is assigned the smaller indexed house in $G_V^{*2}$ and $a_{12}$ is assigned the bigger indexed house in $G_V^{*2}$ under $\eta(f)$. Therefore $\eta(f)(a_{11}) = h_{11}$ and $\eta(f)(a_{12}) = h_{12}$.

ii. Newcomers in $A_N^2 \backslash A_N^{*2} = \{a_{14}, a_{16}\}$:
Since newcomer $a_{14}$ is a member of serial-order $S_4$, her assignment under $\eta(f)$ is the vacant house that is assigned in the next serial-order $S_5$. Therefore $\eta(f)(a_{14}) = h_{14}$. Newcomer $a_{16}$ is a member of serial-order $S_6$ which is the last serial-order of Step 2. Therefore her assignment under $\eta(f)$ is the vacant house that is assigned in the first serial-order $S_3$ of Step 2. Hence $\eta(f)(a_{16}) = h_{16}$.

(c) Newcomers in Step 3 of $e_f$:

i. Newcomers in $A_N^{*3} = \{a_{10}\}$:
The only vacant house that is assigned in Step 3 is $h_{10}$. Therefore $G_V^{*3} = \{h_{10}\}$ and $\eta(f)(a_{10}) = h_{10}$.

There are no remaining agents and therefore

$$\eta(f) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \end{pmatrix}$$

We next show that for any $f \in \mathcal{F}^*$, the outcome of the YRMH-IGYT algorithm is the same as the core of the housing market induced by initial allocation $\eta(f)$.

**Lemma 2:** For any $f \in \mathcal{F}^*$ we have $\varphi^{\eta(f)} = \psi^f$.

*Proof*: Let $f \in \mathcal{F}^*$. By definition $\varphi^{\eta(f)}$ is the core allocation of housing market $\eta(f)$.

We first show that $\varphi^{\eta(f)}(a) = \psi^f(a)$ for each agent $a$ in the first step of $e_f$. Once agents in step 1 are handled, iteration of the same logic implies the desired conclusion. Let step 1 of $e_f$ be in the following structure: $L_1^1, L_2^1, \ldots, L_{\ell_1}^1, S_1^1, L_{\ell_1+1}^1, \ldots, L_{\ell_2}^1, S_2^1, \ldots, L_{\ell_k}^1, S_k^1$. So there are $\ell_k$ loop-orders and $k$ serial-orders in step 1 of $e_f$. Recall that (i) all agents in a loop-order are existing tenants, (ii) every existing tenant $a \in A_E$ is assigned her current house $h_a$ under $\eta(f)$ and (iii) each agent in a loop-order receives her top choice among the houses those are assigned to members of her loop-order. Consider agents in loop-order $L_1^1$. By definition of a loop-order each member of $L_1^1$ is assigned the current house of a member of loop-order $L_1^1$ under matching $\psi^f$. Since $e_f$ is a serial-dictatorship and since $L_1^1$ is a loop-order, each agent in $L_1^1$ receives her top choice among all houses under $\psi^f$. Moreover by construction each agent in $L_1^1$ is assigned her current house under matching $\eta(f)$. Therefore each member of $L_1^1$ should be assigned her top choice under matching $\varphi^{\eta(f)}$ or otherwise members of $L_1^1$ will block $\varphi^{\eta(f)}$ contradicting $\varphi^{\eta(f)}$ is the core allocation for housing market $\eta(f)$. Hence $\varphi^{\eta(f)}(a) = \psi^f(a)$ for each agent $a$ in loop-order $L_1^1$. Fix the assignments of these agents under $\varphi^{\eta(f)}$.

Next consider agents in loop-order $L_2^1$. By definition of a loop-order each member of $L_2^1$ is assigned the current house of a member of loop-order $L_2^1$ under matching $\psi^f$. Since $e_f$ is a serial-dictatorship and since $L^1_2$ is a loop-order, each agent in $L^1_2$ receives her top choice among all remaining houses (i.e. houses in $H \setminus \psi^f(L^1_1)$) under $\psi^f$. By construction each agent in $L^1_2$ is assigned her current house under $\eta(f)$ and therefore under matching $\varphi^{\eta(f)}$ each member of $L^1_2$ should be assigned her top choice among all remaining houses or otherwise members of $L^1_2$ will block $\varphi^{\eta(f)}$. Hence $\varphi^{\eta(f)}(a) = \psi^f(a)$ for each agent $a$ in loop-order $L^1_2$ as well. Proceeding in a similar way we shall have $\varphi^{\eta(f)}(a) = \psi^f(a)$ for any agent $a$ who is a member of a loop-order preceding serial-order $S^1_1$. Fix assignments of these agents under $\varphi^{\eta(f)}$ as well.

For the moment skip serial-order $S^1_1$ and proceed with the next loop-order $L^1_{\ell_1+1}$. By definition of a loop-order each member of $L^1_{\ell_1+1}$ is assigned the current house of a member of loop-order $L^1_{\ell_1+1}$ under matching $\psi^f$. By construction of the steps of effective-order $e_f$, each agent in $L^1_{\ell_1+1}$ prefers her assignment under $\psi^f$ to each house in $\psi^f(S^1_1)$. This together with $e_f$ being a serial-dictatorship and $L^1_{\ell_1+1}$ being a loop-order imply that each agent in $L^1_{\ell_1+1}$ receives her top choice among all houses in $H \setminus \cup_{r=1}^{\ell_1} \psi^f(L^1_r)$ (i.e. all houses whose recipients are not fixed so far under $\varphi^{\eta(f)}$) under matching $\psi^f$. By construction each agent in $L^1_{\ell_1+1}$ is assigned her current house under $\eta(f)$ and therefore under $\varphi^{\eta(f)}$ each member of $L^1_{\ell_1+1}$ should be assigned her top choice among all houses in $H \setminus \cup_{r=1}^{\ell_1} \psi^f(L^1_r)$ or otherwise members of $L^1_{\ell_1+1}$ will block $\varphi^{\eta(f)}$. Hence $\varphi^{\eta(f)}(a) = \psi^f(a)$ for each agent $a$ in loop-order $L^1_{\ell_1+1}$ as well. Fix the assignments of these agents as well under $\varphi^{\eta(f)}$. Note that we are able to use the same argument as before once we observe that any member of a loop-order at step 1 prefers her assignment under $\psi^f$ to any house that is assigned to members of serial-orders at step 1 even if the serial-order precedes the loop-order agent belongs. Proceeding in a similar way we shall have $\varphi^{\eta(f)}(a) = \psi^f(a)$ for any agent $a$ who is a member of a loop-order at step 1. Fix the assignments of these agents under $\varphi^{\eta(f)}$ as well.

Next consider *all* agents in serial-orders $S^1_1, \ldots, S^1_k$ at step 1. By definition of a serial-order, under matching $\psi^f$ each agent in $A_{\mathcal{S}^1}$ is assigned either the current house of an existing tenant in $A_{\mathcal{S}^1}$ or a vacant house (that is clearly assigned at step 1). By construction of the steps of $e_f$, each agent in $A_{\mathcal{S}^1}$ prefers her assignment under $\psi^f$ to each of the houses in $\psi^f(\mathcal{S}^1)$. This, together with $e_f$ being a serial-dictatorship imply that each agent in $A_{\mathcal{S}^1}$ receives her top choice among all houses in $H \setminus \psi^f(\mathcal{L}^1)$ (i.e. all houses whose recipients are not fixed so far under $\varphi^{\eta(f)}$ under matching $\psi^f$). By construction houses in $\psi^f(\mathcal{S}^1)$ are given to members of $A_{\mathcal{S}^1}$ under matching $\eta(f)$. In particular vacant houses are given to newcomers at step 1 and occupied houses are given to their current owners each of whom is also in $A_{\mathcal{S}^1}$. Therefore each agent in $A_{\mathcal{S}^1}$ should receive her top choice in $H \setminus \psi^f(\mathcal{L}^1)$ under $\varphi^{\eta(f)}$ or otherwise members of $A_{\mathcal{S}^1}$ will block. Hence $\varphi^{\eta(f)}(a) = \psi^f(a)$ for any agent $a \in A_{\mathcal{S}^1}$ . Fix the assignments of these agents under $\varphi^{\eta(f)}$ as well.

Once $\varphi^{\eta(f)}(a)$ is fixed for any agent $a$ in step 1, we can iterate the same arguments (by handling the loop-orders first and all the serial-orders next) for agents in step 2 of $e_f$ and so on. This completes the proof of Lemma 2. $\diamondsuit$

**Corollary 1:** Let $f \in \mathcal{F}^*$ and let $L$ be any loop-order of $e_f$. Loop order $L$ forms a cycle via GTTCA for housing market $\eta(f)$.

*Proof*: Directly follows from the proof of Lemma 2. $\diamondsuit$

## 4.2 Construction of Inverse Mapping $g$

Let $\mu \in \mathcal{M}^*$. Execute GTTCA. Note that it does not matter in what order cycles are removed from the market. That is because any cycle remains a cycle as long as its members are in the market. We first iteratively construct sets $H^1, \ldots, H^U$ of houses and sets $\mathcal{C}^1, \ldots, \mathcal{C}^U$ of cycles as follows:

1. Remove any cycle that exclusively consists of existing tenants. In the process new cycles may form. Remove any new cycle that exclusively consists of existing tenants as well. Proceed until each remaining cycle contains at least one newcomer.

2. (a) Remove all remaining cycles (each of which contains at least one newcomer) *simultaneously*. Let $\mathcal{C}^1$ be the set of these cycles. Order the agents in these cycles so that the last agent in each cycle is a newcomer.[8] Start constructing $H^1$ by including all houses that are assigned to members of cycles in $\mathcal{C}^1$. At this point new cycles that include newcomers may form. Do not remove them yet.

   (b) Remove any newly formed cycle that exclusively consists of existing tenants. Include in $H^1$ all houses that are assigned to members of these cycles. Proceed until each remaining cycle contains at least one newcomer.

   In general,

$t$. (a) Remove all remaining cycles (each of which contains at least one newcomer) *simultaneously*. Let $\mathcal{C}^{t-1}$ be the set of these cycles. Order agents in these cycles so that the last agent in each cycle is a newcomer. Start constructing $H^{t-1}$ by including all houses that are assigned to members of cycles in $\mathcal{C}^{t-1}$. At this point new cycles that include newcomers may form. Do not remove them yet.

   (b) Remove any newly formed cycle that exclusively consists of existing tenants. Include in $H^{t-1}$ all houses that are assigned to members of these cycles. Proceed until each remaining cycle contains at least one newcomer.

The process ends when no agent remains. For each $t \leq U$, let $A_{\mathcal{C}^t}$ denote the set of agents in cycles of $\mathcal{C}^t$ and let $\varphi^\mu(\mathcal{C}^t)$ denote the set of houses assigned to members of $A_{\mathcal{C}^t}$. For any cycle $C$ let $\varphi^\mu(C)$ denote the set of houses assigned to members of $C$.

Next construct sets $\widetilde{A}_N^2, \ldots, \widetilde{A}_N^U$ of newcomers as follows: For any $t > 1$, consider $C \in \mathcal{C}^t$. By construction there is at least one newcomer in cycle $C$. A cycle with $k$ newcomers can be divided into $k$ serial-orders where each serial-order starts with an agent who receives a vacant house under $\varphi^\mu$ and ends with a newcomer. Let newcomer $a$ be a member of cycle $C \in \mathcal{C}^t$. In order to determine whether newcomer $a$ belongs $\widetilde{A}_N^t$ (i) divide cycle $C$ into its serial orders, (ii) find the serial-order $S$ newcomer $a$ belongs to and (iii) check whether there exists any agent $a'$ in serial-order $S$ such that $hP_{a'}\varphi^\mu(a')$ for some $h \in H^{t-1}$. We have $a \in \widetilde{A}_N^t$ if and only if such an agent $a'$ exists.

For any $t > 1$, each cycle $C \in \mathcal{C}^t$ hosts a newcomer who is also a member of $\widetilde{A}_N^t$. Otherwise cycle $C$ could have been removed before. For notational convenience re-organize each cycle $C \in \mathcal{C}^t$ for any $t > 1$ so that the last member of $C$ belongs to $\widetilde{A}_N^t$.

[8] Note that cycle $(a_1, a_2, a_3, \ldots, a_k)$ is the same cycle with each of the following cycles: $(a_2, a_3, \ldots, a_k, a_1)$, $(a_3, a_4, \ldots, a_k, a_1, a_2)$, …, $(a_k, a_1, a_2, \ldots, a_{k-1})$.

We are now ready to construct inverse mapping $g : \mathcal{M}^* \to \mathcal{F}^*$. For any $\mu \in \mathcal{M}^*$:

1. Construct sets $\{\mathcal{C}^t\}$, $\{H^t\}$, $\{\widetilde{A}^t_N\}$. Make sure that for $t > 1$ cycles are re-organized so that the last agent in each cycle $C \in \mathcal{C}^t$ is a member of $\widetilde{A}^t_N$.
2. For any $t$, order the newcomers in $A_{\mathcal{C}^t}$ before the newcomers in $A_{\mathcal{C}^{t+1}}$.
3. Order the newcomers in $A_{\mathcal{C}^1}$ based on the index of their endowment in $\mu$ starting with the agent who has the house with the smallest index.
4. For any step $t > 1$ order the newcomers in $A_{\mathcal{C}^t}$ as follows:
   (a) First order the newcomers in $\widetilde{A}^t_N$ based on the index of their endowment in $\mu$ starting with the agent who has the house with the smallest index. Let $\widetilde{A}^t_N = \{\tilde{a}^t_1, \ldots, \tilde{a}^t_\ell\}$ and without loss of generality suppose house $\mu(\tilde{a}^t_1)$ has the smallest index, house $\mu(\tilde{a}^t_2)$ has the second smallest index and so on so forth. Therefore we order agents in $\widetilde{A}^t_N$ as $(\tilde{a}^t_1, \ldots, \tilde{a}^t_\ell)$ among themselves.
   (b) In order to complete the sub-order we will insert the remaining newcomers in $A_{\mathcal{C}^t}$ between agents in $\widetilde{A}^t_N$. The treatment for agent $\tilde{a}^t_1$ will be slightly different so start with newcomer $\tilde{a}^t_2$. Find the cycle newcomer $\tilde{a}^t_2$ belongs. Find the closest newcomer $a'$ who precedes newcomer $\tilde{a}^t_2$ in the cycle. If $a' \in \widetilde{A}^t_N$ then she is already handled and skip to agent $\tilde{a}^t_3$. If $a' \notin \widetilde{A}^t_N$ then order her right in front of $\tilde{a}^t_2$ and find the closest newcomer $a''$ who precedes newcomer $a'$ in the cycle. If $a'' \in \widetilde{A}^t_N$ then she is already handled and skip to agent $\tilde{a}^t_3$. If $a'' \notin \widetilde{A}^t_N$ then order her right in front of newcomer $a'$ and proceed in a similar way until encountering a newcomer $\tilde{a} \in \widetilde{A}^t_N$. Newcomer $\tilde{a}$ is already handled so skip to agent $\tilde{a}^t_3$.
   Repeat this procedure for each of the agents $\tilde{a}^t_3, \ldots, \tilde{a}^t_\ell$.
   Finally consider newcomer $\tilde{a}^t_1$ and find the cycle she belongs. Find the closest newcomer $a'$ who precedes newcomer $\tilde{a}^t_1$ in the cycle. If $a' \in \widetilde{A}^t_N$ then she is already handled and terminate the procedure. If $a' \notin \widetilde{A}^t_N$ then order her at the very end of the sub-order (that orders newcomers in $A_{\mathcal{C}^t}$) and find the closest newcomer $a''$ who precedes newcomer $a'$ in the cycle. If $a'' \in \widetilde{A}^t_N$ then she is already handled and terminate the procedure. If $a' \notin \widetilde{A}^t_N$ then order her right in front of newcomer $a'$ and proceed in a similar way until encountering a newcomer $\tilde{a} \in \widetilde{A}^t_N$. Newcomer $\tilde{a}$ is already handled so terminate the procedure.

   This orders newcomers in $A_{\mathcal{C}^t}$ among themselves in a unique way.
5. Order the existing tenants after newcomers based on their index starting with the existing tenant with the smallest index.[9]

[9] Abdulkadiroğlu and Sönmez (1998) study a special case of our model where there are no existing tenants. For this case construction of mapping $g$ simplifies as follows:

(a) When $A_E = \emptyset$, construction of sets of houses $H^1, \ldots, H^U$ simplifies considerably: Here $H^t$ is simply the set of houses that can be removed in Round $t$ of GTTCA.

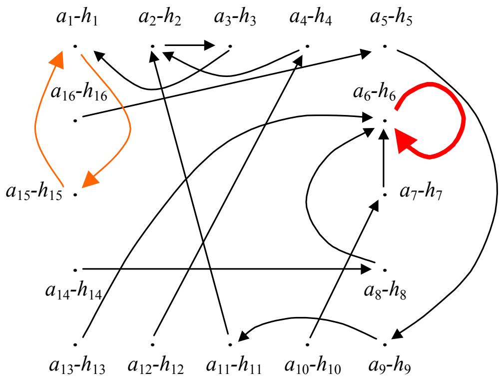
(b) When $A_E = \emptyset$, construction of sets of agents $\widetilde{A}^2_N, \ldots, \widetilde{A}^U_N$ also simplifies considerably: $\widetilde{A}^t_N$ simply consists

Next we illustrate construction of sets $H^1, \ldots, H^U$, $\mathcal{C}^1, \ldots, \mathcal{C}^U$, $\widetilde{A}_N^2, \ldots, \widetilde{A}_N^U$ as well as construction of mapping $g$ with our running example.

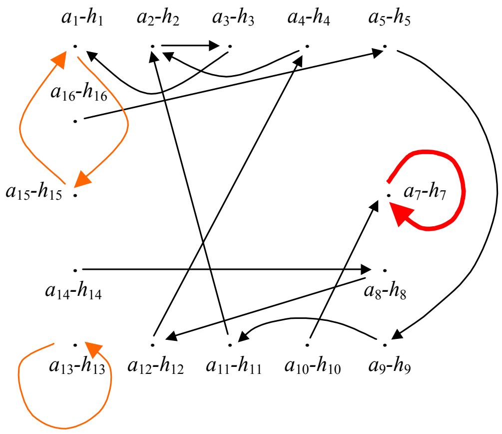
**Example 1 continued:** Let

$$\mu = \eta(f) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \end{pmatrix}$$

We first construct sets of houses $\{H^t\}$ and sets of cycles $\{\mathcal{C}^t\}$ as we execute GTTCA. For each market, cycles that shall be immediately removed is indicated with bold arrows and cycles that shall remain for the remaining market is indicated with light arrows.



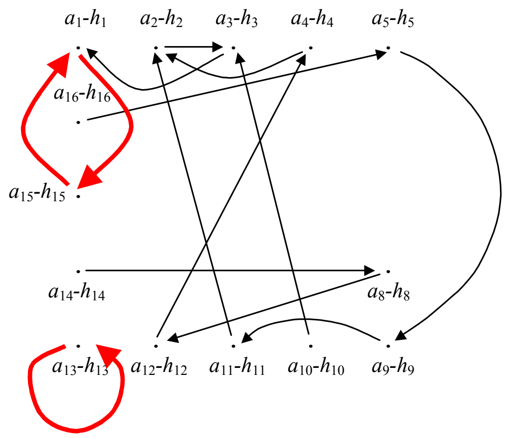

There are two cycles $(a_1, a_{15})$ and $(a_6)$ for the initial market. Among the two, the former hosts newcomer $a_{15}$ so remove only cycle $(a_6)$ and set $\varphi^\mu(a_6) = h_6$.



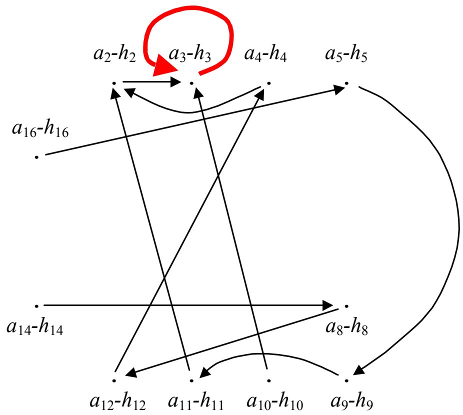

In the remaining market there are three cycles $(a_1, a_{15})$, $(a_{13})$ and $(a_7)$. Among the three the first one hosts newcomer $a_{15}$ and second one hosts newcomer $a_{13}$ so remove only cycle $(a_7)$ and set $\varphi^\mu(a_7) = h_7$.

of agents each of whom prefers some house in $H^{t-1}$ to her assignment under $\varphi^\mu$. This is because each serial-order is a singleton for this case.

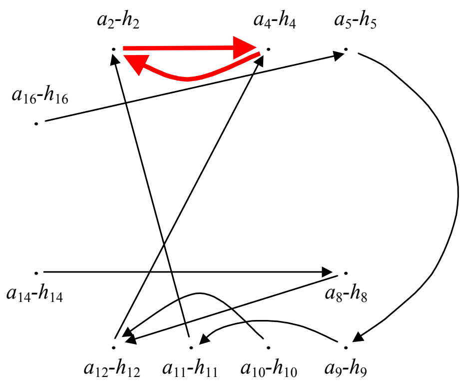
Mapping $g$ reduces to an analogous mapping in Abdulkadiroğlu and Sönmez (1998) under these simplifications.



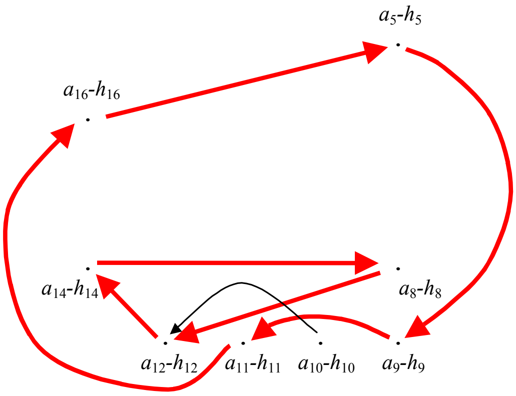

No new cycle forms in the remaining market and each of the two present cycles hosts a newcomer. Remove both cycles $(a_1, a_{15})$ and $(a_{13})$ from the market simultaneously and let $\mathcal{C}^1$ be the set of these cycles. Set $\varphi^\mu(a_1) = h_{15}$, $\varphi^\mu(a_{15}) = h_1$, $\varphi^\mu(a_{13}) = h_{13}$ and include houses $h_{15}$, $h_1$, $h_{13}$ in $H^1$. At this point we have $\mathcal{C}^1 = \{(a_1, a_{15}), (a_{13})\}$ and $\{h_{15}, h_1, h_{13}\} \subseteq H^1$. Note that while set $\mathcal{C}^1$ is already determined, set $H^1$ may grow as we proceed.



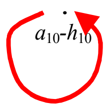

In the remaining market there is only one cycle $(a_3)$ and it does not host a newcomer. Remove it from the market, set $\varphi^\mu(a_3) = h_3$ and include house $h_3$ in $H^1$. At this point we have $\{h_{15}, h_1, h_{13}, h_3\} \subseteq H^1$.




In the remaining market there is only one cycle $(a_2, a_4)$ which exclusively consists of existing tenants. Remove it from the market, set $\varphi^\mu(a_2) = h_4$, $\varphi^\mu(a_4) = h_2$ and include houses $h_4$, $h_2$ in $H^1$. At this point we have $\{h_{15}, h_1, h_{13}, h_3, h_4, h_2\} \subseteq H^1$.




In the remaining market there are two cycles $(a_{11}, a_9, a_5, a_{16})$, $(a_{12}, a_8, a_{14})$ each of which includes a newcomer. Remove them from the market simultaneously and let $\mathcal{C}^2$ be the set of these cycles. Set $\varphi^\mu(a_9) = h_{11}$, $\varphi^\mu(a_5) = h_9$, $\varphi^\mu(a_{16}) = h_5$, $\varphi^\mu(a_{11}) = h_{16}$, $\varphi^\mu(a_8) = h_{12}$, $\varphi^\mu(a_{14}) = h_8$, $\varphi^\mu(a_{12}) = h_{14}$, and include houses $h_{16}$, $h_{11}$, $h_9$, $h_5$, $h_{12}$, $h_8$, $h_{14}$ in $H^2$. At this point we have $\mathcal{C}^2 = \{(a_{11}, a_9, a_5, a_{16}), (a_{12}, a_8, a_{14})\}$ and $\{h_{16}, h_{11}, h_9, h_5, h_{12}, h_8, h_{14}\} \subseteq H^2$.




In the remaining market there is only one cycle $(a_{10})$ which hosts newcomer $a_{10}$. Remove it from the market and let $\mathcal{C}^3 = \{(a_{10})\}$. Set $\varphi^\mu(a_{10}) = h_{10}$ and include house $h_{10}$ in $H^3$. Since there are no remaining agents we have

$$\mathcal{C}^1 = \{(a_1, a_{15}), (a_{13})\}, \quad \mathcal{C}^2 = \{(a_{11}, a_9, a_5, a_{16}), (a_{12}, a_8, a_{14})\}, \quad \mathcal{C}^3 = \{(a_{10})\},$$
$$H^1 = \{h_{15}, h_1, h_{13}, h_3, h_4, h_2\}, \quad H^2 = \{h_{16}, h_{11}, h_9, h_5, h_{12}, h_8, h_{14}\} \quad H^3 = \{h_{10}\}.$$

Next we find $\widetilde{A}^2_N$ and $\widetilde{A}^3_N$. First consider cycle $(a_{11}, a_9, a_5, a_{16}) \in \mathcal{C}^2$ which can be divided into two serial-orders $a_{11}$ and $(a_9, a_5, a_{16})$. Agent $a_{11}$, a member of serial-order $(a_{11})$, prefers $h_2 \in H^1$ to her assignment $\varphi^\mu(a_{11}) = h_{16}$. Therefore the tail of this serial order, namely newcomer $a_{11}$, is a member of $\widetilde{A}^2_N$. No-one in serial-order $(a_9, a_5, a_{16})$ prefers a house in $H^1$ to her own assignment under $\varphi^\mu$. Therefore the tail of this serial-order, namely newcomer $a_{16}$, is not a member of $\widetilde{A}^2_N$.

Next consider cycle $(a_{12}, a_8, a_{14}) \in \mathcal{C}^2$ which can be divided into two serial-orders $(a_{12})$ and $(a_8, a_{14})$. Agent $a_{12}$, a member of serial-order $(a_{12})$, prefers $h_4 \in H^1$ to her assignment $\varphi^\mu(a_{12}) = h_{14}$. Therefore the tail of this serial order, namely newcomer $a_{12}$, is a member of $\widetilde{A}^2_N$. No-one in serial-order $(a_8, a_{14})$ prefers a house in $H^1$ to her own assignment under $\varphi^\mu$. Therefore the tail of this serial-order, namely newcomer $a_{14}$, is not a member of $\widetilde{A}^2_N$.

Finally consider cycle $(a_{10}) \in \mathcal{C}^3$. Agent $a_{10}$, a member of serial-order $(a_{10})$, prefers $h_{12} \in H^2$ to her assignment $\varphi^\mu(a_{10}) = h_{10}$. Therefore the tail of this serial order, namely newcomer $a_{10}$, is a member of $\widetilde{A}^3_N$. Hence $\widetilde{A}^2 = \{a_{11}, a_{12}\}$ and $\widetilde{A}^3 = \{a_{10}\}$.

We are now ready to construct ordering $g(\mu)$.

1. We first reorganize cycle $(a_{11}, a_9, a_5, a_{16})$ as $(a_9, a_5, a_{16}, a_{11})$ and cycle $(a_{12}, a_8, a_{14})$ as $(a_8, a_{14}, a_{12})$ so that the last agent in both cycles is a member of $\widetilde{A}^2_N$.

2. Newcomers in $A_{\mathcal{C}^1}$ (i.e. agents $a_{15}, a_{13}$) are ordered before newcomers in $A_{\mathcal{C}^2}$ (i.e. agents $a_{11}, a_{16}, a_{14}, a_{12}$) who are ordered before the newcomer in $A_{\mathcal{C}^3}$ (i.e agent $a_{10}$).

3. Under $\mu$, newcomer $a_{13}$ has the smaller indexed house $h_{13}$ and newcomer $a_{15}$ has the bigger indexed house $h_{15}$. Therefore newcomers in $A_{\mathcal{C}^1}$ are ordered as $(a_{13}, a_{15})$.

4. Among newcomers in $A_{\mathcal{C}^2}$, first consider agents $a_{11}$, $a_{12}$ who are members of $\widetilde{A}^2_N$. Under $\mu$, newcomer $a_{11}$ has the smaller indexed house $h_{11}$ and newcomer $a_{12}$ has the bigger indexed house $h_{12}$. Therefore agents in $\widetilde{A}^2_N$ are ordered as $(a_{11}, a_{12})$. Newcomer $a_{12}$ belongs to cycle $(a_8, a_{14}, a_{12})$. The closest newcomer that precedes $a_{12}$ is newcomer $a_{14}$. Since $a_{14} \notin \widetilde{A}^2_N$, order her right in front of newcomer $a_{12}$. So far agents $a_{11}$, $a_{12}$, $a_{14}$ are ordered as $(a_{11}, a_{14}, a_{12})$. There is no other newcomer in cycle $(a_8, a_{14}, a_{12})$ so skip to newcomer $a_{11}$ who belongs to cycle $(a_9, a_5, a_{16}, a_{11})$. The closest newcomer that precedes $a_{11}$ is newcomer $a_{16}$. Since $a_{16} \notin \widetilde{A}^2_N$, order her at the end of the sub-order that orders newcomers in $A_{\mathcal{C}^2}$. That takes care of newcomers in $A_{\mathcal{C}^2}$ and they are ordered as $(a_{11}, a_{14}, a_{12}, a_{16})$.

   Since $a_{10}$ is the only newcomer in $A_{\mathcal{C}^3}$, she is ordered last among the newcomers.

5. Finally we order existing tenants after the newcomers based on their index.

Therefore we have $g(\mu) = (a_{13}, a_{15}, a_{11}, a_{14}, a_{12}, a_{16}, a_{10}, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$. Note that, since $\mu = \eta(f)$ we have $g(\mu) = f$.

## 4.3 g is inverse mapping of $\eta$

We are going to show that $g(\eta(f)) = f$ via three lemmata. Let $f \in \mathcal{F}^*$, and construct $e_f$, $\{\mathcal{S}^t\}$, $\{\mathcal{L}^t\}$,$\{G^t\}$, $\{A^{*t}_N\}$ and $\eta(f)$. For matching $\eta(f)$, construct $\{\mathcal{C}^t\}$,$\{H^t\}$, $\{\widetilde{A}^t_N\}$ and $g(\eta(f))$.

**Lemma 3:** For any $t$, each serial-order $S \in \mathcal{S}^t$ is part of a cycle $C \in \mathcal{C}^t$. Conversely for any $t$, any cycle $C \in \mathcal{C}^t$ can be reorganized as $C = (S_1, \ldots, S_k)$ such that $S_i \in \mathcal{S}^t$ for all $i \in \{1, \ldots, k\}$.
*Proof*: We are going to prove the lemma iteratively for each $t$. Fix $t$. Some of the loop-orders in step $t$ of e$_f$ may have already formed cycles and left the market via GTTCA before agents in serial-orders of $\mathcal{S}^{t-1}$. The remaining ones will form cycles by Corollary 1 and leave the market after agents in serial-orders of $\mathcal{S}^{t-1}$ form one or more cycles and leave the market. By Remark 2 serial-orders in $\mathcal{S}^t$ will not leave the market via GTTCA before serial-orders in $\mathcal{S}^{t-1}$. Once all loop-orders in step t of $e_f$ leave the market, serial-orders in $\mathcal{S}^t$ will form one or more cycles among themselves and leave the market via GTTCA. That is because (i) by construction of step $t$ of $e_f$, for any $a \in A_{\mathcal{S}^t}$ and any house $h \in H \Big\backslash \Big( \Big( \underset{r<t}{\cup} \psi^f(\mathcal{S}^r) \Big) \cup \psi^f(\mathcal{L}^t) \Big)$ we have $\psi^f(a) P_a h$, and (ii) by construction of $\eta(f)$, for any $a \in A_{\mathcal{S}^t}$ we have $\eta(f)(a) {=} \psi^f(a')$ for some $a' \in A_{\mathcal{S}^t}$. Moreover, by Remark 2 no agent in $\underset{r>t}{\cup} A_{\mathcal{S}^r}$ leaves the market before each agent in $A_{\mathcal{S}^t}$ does. These together with construction of $\mathcal{C}^t$ complete the proof of Lemma 3. $\diamondsuit$

**Corollary 2:** Execute GTTCA for housing market $\eta(f)$. Any cycle that exclusively consists of existing tenants is a loop-order in $e_f$.

*Proof*: Each loop-order in $e_f$ forms a cycle via GTTCA by Corollary 1. Each remaining agent is a member of a serial-order of $e_f$ and by Lemma 3 she leaves the market together with all members of her serial-order (which includes a newcomer) as a part of a cycle via GTTCA. This implies the desired conclusion. $\diamondsuit$

**Lemma 4 :** $H^t = G^t$ for all $t$.

*Proof*: For each $t$ we have $\psi^f(\mathcal{S}^t) \subseteq G^t$ by construction of $G^t$. Similarly for each $t$ we have $\varphi^{\eta(f)}(\mathcal{C}^t) \subseteq H^t$ by construction of $H^t$. Moreover by Lemma 2 we have $\varphi^{\eta(f)} = \psi^f$ and by Lemma 3 we have $A_{\mathcal{C}^t} = A_{\mathcal{S}^t}$ for all $t$. Therefore $\varphi^{\eta(f)}(\mathcal{C}^t) = \psi^f(\mathcal{S}^t)$ for all t.

**Claim 1:** $G^t \subseteq H^t$ for all $t$.

*Proof of Claim 1*: Fix $t$. We will show that $G^t \backslash \psi^f(\mathcal{S}^t) \subseteq H^t$. By construction of $G^t$, set $G^t \backslash \psi^f(\mathcal{S}^t)$ consists of houses allocated in loop-orders of $e_f$ each of which belongs to a step $s > t$. Consider each of these loop-orders following their order in $e_f$. Let $L$ be the first loop-order such that $\psi^f(L) \subseteq G^t \backslash \psi^f(\mathcal{S}^t)$. By Corollary 1, $L$ will form a cycle via GTTCA and leave the market. By Remark 1(i), there exists a house $h \in G^t$ and an agent $a$ in $L$ such that $hP_a\psi^f(a)$. Since $L$ is the first loop-order with $\psi^f(L) \subseteq G^t \backslash \psi^f(\mathcal{S}^t)$, by construction of $G^t$ we have $h \in \psi^f(\mathcal{S}^t)$. Therefore since we have $A_{\mathcal{C}^t} = A_{\mathcal{S}^t}$ as well as $\varphi^{\eta(f)} = \psi^f$ and since all cycles in $\mathcal{C}^t$ leave the market simultaneously via GTTCA, loop-order $L$ forms a cycle via GTTCA and leaves the market *after* each of the cycles in $\mathcal{C}^t$. Moreover by Remark 1(ii), any agent $a''$ in $L$ prefers $\psi^f(a'')$ to any house in $G^s$ for any $s > t$. This together with $A_{\mathcal{C}^s} = A_{\mathcal{S}^s}$ for all $s$ and $\varphi^{\eta(f)} = \psi^f$ imply that $L$ forms a cycle via GTTCA and leaves the market *before* any cycle in $\mathcal{C}^s$ for any $s > t$. Therefore by construction of $H^t$ we have $\psi^f(L) \subseteq H^t$. Next let $L'$ be the second loop-order in $e_f$ such that $\psi^f(L') \subseteq G^t \backslash \psi^f(\mathcal{S}^t)$. By Remark 1(i), there exists a house $h' \in G^t$ and an agent $a'$ in $L'$ such that $h'P_{a'}\psi^f(a')$. By construction of $G^t$ we have $h \in \psi^f(\mathcal{S}^t) \cup \psi^f(L)$. Therefore since we have $A_{\mathcal{C}^t} = A_{\mathcal{S}^t}$ as well as $\varphi^{\eta(f)} = \psi^f$ and since $L$ leaves the market after all cycles in $\mathcal{C}^t$ all of which leave the market simultaneously, loop-order $L'$ forms a cycle via GTTCA and leaves the market *after* each of the cycles in $\mathcal{C}^t$. Moreover by Remark 1(ii), any agent $a''$ in $L'$ prefers $\psi^f(a'')$ to any house in $G^s$ for any $s > t$. This together with $A_{\mathcal{C}^s} = A_{\mathcal{S}^s}$ for all $s$ and $\varphi^{\eta(f)} = \psi^f$ imply that $L'$ forms a cycle via GTTCA and leaves the market before any cycle in $\mathcal{C}^s$ for any $s > t$. Therefore by construction of $H^t$ we have $\psi^f(L') \subseteq H^t$. Following in a similar way we obtain $G^t \backslash \psi^f(\mathcal{S}^t) \subseteq H^t$. Moreover since $\psi^f(\mathcal{S}^t) = \varphi^{\eta(f)}(\mathcal{C}^t) \subseteq H^t$ we have $G^t \subseteq H^t$ completing the proof of Claim 1.

**Claim 2:** $H^t \subseteq G^t$ for all $t$.

*Proof of Claim 2* : Fix $t$. We will show that $H^t \backslash \varphi^{\eta(f)}(\mathcal{C}^t) \subseteq G^t$. By construction of $H^t$, the set $H^t \backslash \varphi^{\eta(f)}(\mathcal{C}^t)$ consists of the houses allocated to existing tenants who form cycles via GTTCA and leave the market after the cycles in $\mathcal{C}^t$ and before the cycles in $\mathcal{C}^{t+1}$. Consider each of these cycles one at a time following the order they leave the market via GTTCA (and arbitrarily order the cycles which form simultaneously). Let $C$ be the first such cycle. By Corollary 2, $C$ is a loop-order in $e_f$. Since $C$ forms and leaves the market only after the cycles in $\mathcal{C}^t$, there exists an agent $a$ in $C$ and a house $h \in \varphi^{\eta(f)}(\mathcal{C}^t) = \psi^f(\mathcal{S}^t) \subseteq G^t$ such that $hP_a\varphi^{\eta(f)}(a)$. Moreover since $C$ forms and leaves the market before the cycles in $\mathcal{C}^{t+1}$, we have $\varphi^{\eta(f)}(a'')P_{a''}h''$ for any agent $a''$ in $C$ and for any house $h'' \in H^s$ for any $s > t$. Therefore since $G^s \subseteq H^s$ for all $s$ by Claim 1, we have $\varphi^{\eta(f)}(a'')P_{a''}h''$ for any agent $a''$ in $C$ and any house $h'' \in G^s$ for any $s > t$. Hence by Remark 1 we have $\varphi^{\eta(f)}(C) \subseteq G^t$. Next consider the second such cycle $C'$. By Corollary 2, $C'$ is a loop-order in $e_f$. Since $C'$ forms and leaves the market only after the cycles in $\mathcal{C}^t$ and not before cycle $C$, there exists an agent $a'$ in $C'$ and a house $h' \in \varphi^{\eta(f)}(\mathcal{C}^t) \cup \varphi^{\eta(f)}(C) = \psi^f(\mathcal{S}^t) \cup \varphi^{\eta(f)}(C) \subseteq G^t$ such that $h'P_{a'}\varphi^{\eta(f)}(a')$. Since $C'$ forms and leaves the market before the cycles in $\mathcal{C}^{t+1}$, we have $\varphi^{\eta(f)}(a'')P_{a''}h''$ for any agent $a''$ in $C'$ and for any house $h'' \in H^s \supseteq G^s$ for any $s > t$. Hence by Remark 1 we have $\varphi^{\eta(f)}(C') \subseteq G^t$. Following in a similar way, we obtain $H^t \backslash \varphi^{\eta(f)}(\mathcal{C}^t) \subseteq G^t$. Moreover since $\varphi^{\eta(f)}(\mathcal{C}^t) = \psi^f(\mathcal{S}^t) \subseteq G^t$ we have $H^t \subseteq G^t$ completing the proofs of Claim 2 and Lemma 4. $\diamondsuit$

Define $H_V^1 = H^1 \cap H_V$. For each $t > 1$ define $\widetilde{H}_V^t = \{h \in H^t \cap H_V : h = \eta(f)(a)$ for some newcomer $a \in \widetilde{A}_N^t\}$

**Corollary 3:** $\widetilde{A}_N^t = A_N^{*t}$ and $\widetilde{H}_V^t$=$G_V^{*t}$ for any $t > 1$ .

*Proof*: Immediately follows from Lemma 2, Lemma 3, Lemma 4 and construction of sets $\widetilde{A}_N^t$, $A_N^{*t}$,$\widetilde{H}_V^t$,$G_V^{*t}$ for any $t$. $\diamondsuit$

**Lemma 5:** $g(\eta(f)) = f$.

*Proof*: We will show that each agent's order is the same under $f$ and $g(\eta(f))$. We proceed by induction.

1. Let $a \in A_N^1 = A_{\mathcal{S}^1} \cap A_N = A_{\mathcal{C}^1} \cap A_N$ be a newcomer with $f(i) = a$ for some order $i$. By construction of $\eta(f)$, house $\eta(f)(a)$ is the $i$'th smallest indexed house in $G_V^1$. Moreover $H_V^1 = G_V^1$ by Lemma 4. Therefore $\eta(f)(a)$ is also the $i$'th smallest indexed house in $H_V^1$. Since mapping $g$ orders newcomers in $A_{\mathcal{C}^1}$ before other newcomers and since this order is based on the index of their endowments under $\eta(f)$, agent $a$ should be ordered $i$'th by ordering $g(\eta(f))$ as well.

2. Assume that for each step $r \in \{2, ..., t-1\}$ of $e_f$ and for any agent $a \in A_N^r$, agent $a$'s order is the same under $f$ and $g(\eta(f))$.

   We will show that for each $a \in A_N^t = A_{\mathcal{S}^t} \cap A_N = A_{\mathcal{C}^t} \cap A_N$, agent $a$'s order should be same under $f$ and $g(\eta(f))$.

   Recall that $A_N^{*t} = \widetilde{A}_N^t$ by Corollary 3. First we show that agents in $A_N^{*t}$ are ordered the same among themselves under $f$ and $g(\eta(f))$. Fix $a^* \in A_N^{*t}$ and let ordering $f$ order her $i$'th among agents in $A_N^{*t}$. By construction of $\eta(f)$, house $\eta(f)(a^*)$ is the $i$'th smallest indexed house in $G_V^{*t}$. Moreover by Corollary 3 we have $\widetilde{H}_V^t = G_V^{*t}$. Therefore house $\eta(f)(a^*)$ is the $i$'th smallest indexed house in $\widetilde{H}_V^t$. Since mapping $g$ orders newcomers in $\widetilde{A}_N^t$ based on the index of their endowments under $\eta(f)$, agent $a^*$ should be ordered $i$'th among agents in $\widetilde{A}_N^t = A_N^{*t}$ by ordering $g(\eta(f))$ as well. Next we show that the remaining newcomers in $A_N^t$ are ordered the same under $f$ and $g(\eta(f))$. We have two cases to consider:

   *Case 1*: $i > 1$. Let newcomer $a^{**} \in A_N^{*t}$ be such that $a^{**}$ is ordered $(i-1)$'th among newcomers in $A_N^{*t}$ under $f$ as well as $g(\eta(f))$. Consider the newcomers who are ordered between newcomers $a^{**}$ and $a^*$ under $f$. First consider newcomer $a \in A_N^t \backslash A_N^{*t}$ who is ordered right before $a^*$ under $f$. Let $S^*$ be the serial-order newcomer $a^*$ belongs in $e_f$. By construction of $\eta(f)$ house $\eta(f)(a)$ is the vacant house allocated in $S^*$ under $\psi^f$ and by Lemma 3 serial-order $S^*$ is a part of a cycle $C \in \mathcal{C}^t$. These together with $\varphi^{\eta(f)} = \psi^f$ imply that newcomer $a$ also belongs to cycle $C$ and she is ordered right before $S^*$ in cycle $C$. Since $a \in A_N^t \backslash A_N^{*t} = A_N^t \backslash \widetilde{A}_N^t$, she is also ordered right before newcomer $a^*$ under $g(\eta(f))$ by construction of mapping $g$. Next consider newcomer $a' \in A_N^t \backslash A_N^{*t}$ who is ordered right before newcomer $a$ under $f$. Let $S$ be the serial-order newcomer $a$ belongs in $e_f$. By construction of $\eta(f)$ house $\eta(f)(a')$ is the vacant house allocated in $S$ under

$\psi^f$. Since newcomer $a$ belongs to cycle $C$, serial-order $S$ is part of cycle $C$ by Lemma 3. These together with $\varphi^{\eta(f)} = \psi^f$ imply that newcomer $a'$ also belongs to cycle $C$ and she is ordered right before $S$ in cycle $C$. Since $a' \in A_N^t \backslash A_N^{*t} = A_N^t \backslash \widetilde{A}_N^t$, she is ordered right before newcomer $a$ under $g(\eta(f))$. Following in a similar way, we show that newcomers between $a^{**}$ and $a^*$ under $f$ are ordered the same among themselves under $f$ and $g(\eta(f))$.

*Case 2*: $i = 1$. By construction of $A_N^{*t}$, newcomer $a^*$ is ordered first among agents in $A_N^t$ under $f$. Let $S^*$ be the serial order newcomer $a^*$ belongs in $e_f$. Let newcomer $a^{**} \in A_N^{*t}$ be the agent who is ordered last among newcomers in $A_N^{*t}$ under $f$ and $g(\eta(f))$. Consider newcomers in $A_N^t$ who are ordered after newcomer $a^{**}$ under $f$. Let newcomer $a \in A_N^t \backslash A_N^{*t}$ be the last agent under $f$ among newcomers in $A_N^t$. By construction of $\eta(f)$ house $\eta(f)(a)$ is the vacant house allocated in $S^*$ under $\psi^f$ and by Lemma 3 serial-order $S^*$ is a part of a cycle $C \in \mathcal{C}^t$. These together with $\varphi^{\eta(f)} = \psi^f$ imply that newcomer $a$ also belongs to cycle $C$ and she is ordered right before $S^*$ in cycle $C$. Since $a \in A_N^t \backslash A_N^{*t} = A_N^t \backslash \widetilde{A}_N^t$, she is also ordered last among newcomers in $A_N^t$ under $g(\eta(f))$ by construction of mapping $g$. Next consider newcomer $a' \in A_N^t \backslash A_N^{*t}$ who is ordered right before newcomer $a$ under $f$. Let $S$ be the serial-order newcomer $a$ belongs in $e_f$. By construction of $\eta(f)$ house $\eta(f)(a')$ is the vacant house allocated in $S$ under $\psi^f$. Since newcomer $a$ belongs to cycle $C$, serial-order $S$ is part of cycle $C$ by Lemma 3. These together with $\varphi^{\eta(f)} = \psi^f$ imply that newcomer $a'$ also belongs to cycle $C$ and she is ordered right before $S$ in cycle $C$. Since $a' \in A_N^t \backslash A_N^{*t} = A_N^t \backslash \widetilde{A}_N^t$, she is ordered right before newcomer $a$ under $g(\eta(f))$ as well. Following in a similar way we show that newcomers after $a^{**}$ in step t of $e_f$ are ordered the same among themselves under $f$ and $g(\eta(f))$.

This covers all newcomers in $A_N^t$ and shows that they are ordered the same under $f$ and $g(\eta(f))$.

This shows that newcomers are ordered the same under $f$ and $g(\eta(f))$. Finally existing tenants are ordered after the newcomers based on their index under both $f$ and $g(\eta(f))$. This concludes the proof of Lemma 5. $\diamondsuit$

## 4.4 Proof of the Main Result

We are now ready to prove our main result.

**Theorem 1:** Lottery mechanisms $\Phi$ and $\Psi$ are equivalent.

*Proof*: We have

$$\Phi = \sum_{\mu \in \mathcal{M}^*} \frac{1}{m!} \varphi^\mu \qquad \text{and} \qquad \Psi = \sum_{f \in \mathcal{F}^*} \frac{1}{m!} \psi^f$$

Both mechanisms select a uniform lottery over $m!$ matchings for each problem. Fix a problem. For each ordering $f \in \mathcal{F}^*$ construct matching $\eta(f) \in \mathcal{M}^*$. By Lemma 2 we have $\psi^f = \varphi^{\eta(f)}$ and by Lemma 5 mapping $\eta$ is invertible. Hence $\Phi = \Psi$. $\diamondsuit$

### 4.5 Implications for the House Allocation Problems

A **house allocation problem** (Hylland and Zeckhauser, 1977) is a special case of our model where there are only newcomers and vacant houses.[10] A popular real-life mechanism in this context is **random serial-dictatorship:** Randomly order the agents and assign the first agent her top choice, the second agent her top choice among the remaining houses and so on. Another natural mechanism is **core from random endowments:** Randomly allocate the houses to agents, interpret it as an initial endowment, and choose the core (or equivalently competitive allocation) of the induced housing market. Abdulkadiroğlu and Sönmez (1998) show that the two mechanisms are equivalent and we obtain their result as an immediate corollary to Theorem 1.

**Corollary 4:** The random serial dictatorship is equivalent to core from random endowments for house allocation problems.

*Proof*: YRMH-IGYT algorithm reduces to a serial-dictatorship when there are no existing tenants. This together with Theorem 1 imply the desired result. $\diamondsuit$

## 5 Conclusion

In this paper we show that there is an important relation between two intuitive house allocation mechanisms which are designed to avoid inefficiencies in those situations where there are existing tenants and newcomers. Since the core (or equivalently the competitive mechanism) is the undisputed mechanism in the context of housing markets, it is tempting to extend this mechanism via constructing an initial allocation by assigning existing tenants their current houses and randomly assigning vacant houses to newcomers. However this extended mechanism grants initial property rights of vacant houses to newcomers and therefore its equivalence to "newcomer favoring" top trading cycles algorithm is quite intuitive. We believe our result provides additional support for the top trading cycles mechanism by showing that its main competitor is a very biased special case.

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