Markov Equilibrium in Models of Dynamic Endogenous Political Institutions

Roger Lagunoff*

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Abstract

This paper examines existence of Markov equilibria in the class of *dynamic political games (DPGs)*. DPGs are dynamic games in which political institutions are endogenously determined each period. The process of change is both *recursive* and *instrumental*: the rules for political aggregation at date $t+1$ are decided by the rules at date t, and the resulting institutional choices do not affect payoffs or technology directly.

Equilibrium existence in dynamic political games requires a resolution to a "political fixed point problem" in which a current political rule (e.g., majority voting) admits a solution only if all feasible political rules in the future admit solutions in all states. If the class of political rules is dynamically consistent, then DPGs are shown to admit political fixed points. This result is used to prove two equilibrium existence theorems, one of which implies that equilibrium strategies, public and private, are smooth functions of the economic state. We discuss practical applications that require existence of smooth equilibria.

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*Department of Economics, Georgetown University, Washington DC 20057 USA. 202 687-1510. lagunofr@georgetown.edu. http://www.georgetown.edu/lagunoff/lagunoff.htm . This project began while the author visited Johns Hopkins University and the W. Allen Wallis Institute of Political Economy at the University of Rochester. I am grateful for their hospitality. I also thank Luca Anderlini, Jimmy Chan, William Jack, Jim Jordan, Mark Stegeman, and seminar participants at Berkeley, Cal Tech, Georgetown, Penn State, VPI, and the Latin American and North American Meetings of the Econometric Society for their helpful comments.

1 Introduction

This paper introduces, and examines equilibrium existence in, a class of *dynamic political games*. Dynamic political games (or DPGs) are infinite horizon, stochastic games in which political institutions are determined endogenously.

Dynamic political games have two defining features. First, the process of institutional change is recursive: the rules for choosing public decisions in period $t+1$ are, themselves, objects of choice in period t. Second, the process is instrumental: the rules do not affect payoffs or technology directly. DPGs distinguish between "economic" and "political" states that are jointly determined each period. The economic states comprise substantive parameters that affect preferences and technology directly. Whereas, political states are procedural parameters that characterize the explicit *political rule*, such as the majority voting rule, for determining public sector decisions. Decision makers in a DPG can modify existing institutions, not because the details of political procedures enter into the utility functions, but rather because institutional changes modify future economic states.

DPGs admit a broad array of institutional changes. We give examples of these that include changes in the voting rule (majority vs supermajority rules), changes in voting rights (e.g. larger vs smaller voting franchise), and changes in the scope of the public sector (e.g., expansions vs contractions of regulatory authority).

Because the pace of institutional change is captured succinctly by the evolution of political states, DPGs are natural objects in which to study stability and reform of political institutions. These issues are taken up in a companion paper, Lagunoff (2004b).[1]

The present paper, meanwhile, establishes existence of a natural adaptation of Markov Perfect equilibrium to dynamic political games. According to this adaptation, a Markov equilibrium is a collection of private and public sector Markov strategies such that (a) the strategy of each individual in the private sector is optimal for that individual in every state, and (b) decisions in the public sector are consistent with the prevailing political rule in every state. Our restriction to Markov equilibria follows a general rule of thumb: changes in political institutions often occur on an economy-wide scale; in an economy-wide context, the participants are less apt to coordinate on non payoff-relevant history than they would if they were in a small group.

A number of interesting issues arise in DPGs that make equilibrium existence more than a standard technical exercise. Any recursive political model incurs the following "political fixed point problem." Unlike in standard social choice problems, political rules here operate on endogenous objects. A profile of payoffs in this model is a profile of average discounted

[1] That paper also contains a broader motivation, citing a number historical examples in which political institutions are modified.

vectors of the form $(1-\delta)u_t + \delta V_{t+1}$ where u_t is the current stage payoff vector, δ is the discount factor, and V_{t+1} is the vector of continuations. The endogeneity arises because the continuation V_{t+1} is the result of the application of a political rule in period $t+1$. In turn, this period $t+1$ political rule is defined on payoff profiles that depend on continuation V_{t+2} that, itself, results from the application of a political rule in period $t+2$, and so on... To ensure that these objects are well defined, the implied map from future (state-contingent) strategies to current ones must have a "political fixed point."

Existence of political fixed points, however, is not automatic. If, for example, the current political rule is a simple majority rule, then the feasible set of outcomes under majority rule is typically given by the set of Condorcet Winners — the choices which cannot be defeated by any alternative choice in a majority vote. In this case, however, voting cycles can arise, and voting cycles present well known problems for existence. While this problem arises in all recursive models of political aggregation, it is especially troublesome in DPGs because public decisions here are inherently multi-dimensional: both the current policy and the future political rule are chosen each period. Consequently, standard "fixes" such as single peakedness do not work.

Our results emphasize the importance of dynamic consistency for resolving this problem. Roughly, a political rule is *dynamically consistent* if it is rationalized by a time separable social welfare function. We show that if rules are dynamically consistent then under standard technical conditions (continuity and compactness), DPGs admit political fixed points. This is proved by showing that the associated "Bellman's map" has a fixed point in the space of continuation value functions. An important special case is the class of all voting rules. If stage game payoffs admit an affine representation then each voting rule is partially rationalized by the preferences of the same pivotal voter in all states. This affine preference representation is similar to (and somewhat more restrictive than) the class of Intermediate Preferences introduced by Grandmont (1978) to prove a multi-dimensional Median Voter Theorem. Since the preferences of the pivotal voter are dynamically consistent, political fixed points exist under voting rules and affine stage payoffs.

We use these results to establish two equilibrium existence theorems. The first asserts that (mixed) Markov Perfect equilibria exist when state and action spaces are finite. The second, and main, theorem establishes sufficient conditions under which equilibria exist that are smooth functions of the economic state. Both results make critical use of dynamic consistency. The intuition, roughly, is that because dynamically consistent rules are rationalized by time separable welfare functions, each such rule can be treated as a "player" in the game. The "trick" is to find the right mapping that transforms the DPG into a standard stochastic game with private decisions.

The second result also adapts arguments from an elegant result of Horst (2003) who proves the existence of almost-everywhere differentiable Markov equilibria in standard, stochastic games. Horst's result applies to many economically interesting dynamic environments with unbounded state space such as environments with capital accumulation. The key assumption in his result is a "moderate social influence" condition whereby an individual's own actions have a relatively greater effect on his marginal dynamic payoff than those of all other individuals combined. The moderate social influence condition is shown to bound uniformly the marginal best replies via the Implicit Function Theorem. It serves a similar purpose in the present result.

Smooth Markov equilibria are useful for practical applications. For example, they provide one clear justification for Euler equation characterizations in dynamic politico-economic models. Examples include Klein, Krusell, and Rios-Rull (2002) and Jack and Lagunoff (2003). In particular, computational techniques developed by Klein, Krusell, and Rios-Rull (2002) characterize the optimal policy functions resulting from median voter and other political rules. They compute these functions from "generalized Euler equations" that are higher order differential equations of future policy functions. These Euler equations are well-defined only if there exist Markov equilibria that admit higher order derivatives of the economic state.

This paper and its companion follow a growing literature on dynamic, endogenous institutional change. The list includes Roberts (1998, 1999), Justman and Gradstein (1999), Acemoglu and Robinson (2000, 2001), Lagunoff (2001), Barbera, Maschler, and Shalev (2001), Messner and Polborn (2002), Jack and Lagunoff (2003), Gradstein (2003), and Greif and Laitin (2004). These are dynamic/repeated game models in which some attribute of an aggregation rule may be chosen each period. The main difference between the present paper and most of the literature is that the choice of rule here is both recursive and instrumental.[2]

The paper is organized as follows. In Section 2, we present a less formal version of the model in order to highlight some issues and problems of in our framework. The general model is described in Section 3. There, we introduce the class of *political rules*. Political rules are natural objects of choice in a recursive model of endogenous institutions. A *dynamic political game* combines both the public and private sectors of the environment. An "equilibrium" combines standard Markov Perfection in private decisions with a political fixed point requirement for public decisions. Section 4 examines the political fixed point problem and describes the existence results more generally. Section 5 offer concluding remarks. Details of some of the proofs are in the Appendix in Section 6.

[2]Jack and Lagunoff (2003) is an exception. See the companion paper, Lagunoff (2004b), for a more detailed discussion of the literature.

2 Political Fixed Points: An Example

We begin with a simplified, "stripped down" version of the general model. Consider initially an equilibrium model of period-by-period majority voting with infinitely lived agents. For now, the institutional environment is fixed.

At each date $t = 1, 2, \ldots$, a set $I = \{1, \ldots, n\}$ of individuals must vote to decide a policy p_t at date t. In the general model, p_t is an abstract representation of any vector of public sector decisions. Let P denote the set of all feasible policies. For example, one could have $P = [0, 1]$ as the set of all flat tax rates. The current state is ω_t drawn from a set Ω. In the flat tax example, ω_t could describe the distribution of earnings across individuals. The state enters directly into each individual's stage payoff, and it also enters the transition function that determines the future state. For now, omit (notationally) private sector behavior such as individuals' savings decisions.[3]

A *policy strategy* ψ is a Markov strategy specifying the policy $p_t = \psi(\omega_t)$ as a function of state ω_t at date t. Given state ω_t, individual i's ($i = 1, \ldots, n$) average discounted dynamic payoff over policies is expressed in a recursive form by

$$U_i(\omega_t;\ \psi)(p_t) \ \equiv\ (1-\delta)u_i(\omega_t, p_t) + \delta \int V_i(\omega_{t+1}; \psi) dq(\omega_{t+1} |\ \omega_t, p_t) \tag{1}$$

where δ is the discount factor, u_i is the stage game payoff received in each period, q denotes the stochastic transition function mapping current states and policies into probability distributions over future states, and V_i is i's continuation payoff given policy rule ψ.[4] If p_t is chosen each period by a simple majority vote, then pairwise comparisons of policy are evaluated by each individual using his recursive payoff function $U_i(\omega_t; \psi)(\cdot)$ in (1). By construction, $U_i(\omega_t;\ \psi)(\psi(\omega_t)) = V_i(\omega_t; \psi)$. In other words, the continuation payoff is the recursive payoff function evaluated the policy strategy.

The profile of payoff functions is $U(\omega_t; \psi) \ = \ (U_i(\omega_t; \psi))_{i=1}^n$. For each such profile, the outcome of majority voting is typically represented by the set of *Condorcet Winners* — outcomes that survive all pairwise comparisons in a majority vote. Denote this set by $C(U(\omega_t; \psi))$.[5] For the policy strategy ψ to be consistent with $C(U(\omega_t; \psi))$, it must satisfy the "political fixed point" problem

$$\psi(\omega_t) \in C(U(\omega_t; \psi)\ ),\ \forall \omega_t \tag{2}$$

[3]Private sector behavior will be introduced in the next Section.

[4]The transition q is assumed to satisfy the standard measurability assumptions.

[5]Formally, $p \in C(U(\omega_t; \psi))$ if for all $\hat{p} \neq p$,

$$|\{i:\ U_i(\omega_t; \psi)(\hat{p}) > U_i(\omega_t; \psi)(p)\}| \leq n/2.$$

Clearly, (2) defines a fixed point. Condition (2) is also reminiscent of an implementation property, although it differs from implementation in the critical sense the preference domain is endogenously determined by the fixed point property. For this reason, it makes no sense to speak of "truth-telling" constraints in a fixed domain of preference characteristics.

In the current period t, a Condorcet Winner exists if voting is cycle proof. But since voting takes place each period, the continuation payoff V_i already encodes future voting outcomes, and so the voting rule at date t is cycle-proof only if Condorcet Winners exist in all future dates.

Though couched in different terminology, this political fixed point problem has recent antecedents in the politico-economic literature, beginning with the pioneering work of Krusell, Quadrini, and Ríos-Rull (1997).[6] They specify a general "political aggregator function" analogous to our Condorcet operator. One key insight of theirs is that any social choice function, Condorcet or otherwise, must operate on the profile of *endogenous*, recursive policy preferences. In the present context, this profile is $U(\omega_t; \psi)$.

The political fixed problem is resolved in Krusell, et. al., and in most of the politico-economic literature that followed, by assuming either that there are only two types of agents, or that p_t is uni-dimensional and stage game preferences have a special form to generate single peaked recursive preferences.[7]

Unfortunately, single peakedness will not suffice if both the policy and the political institution itself are part of the decision problem. We now introduce a parameter θ_t that determines the political institution. In this Section, suppose that $\theta_t \in \{\theta^1, \theta^2\}$ where $\theta_t = \theta^1$ means that the policy is determined by a majority vote, and $\theta_t = \theta^2$ means that the policy is imposed by a "dictator," whom we assume to be individual $i = 1$. The "political state" θ_t is then distinct from the payoff-relevant "economic" state ω_t. The political state summarizes the political process by which public decisions are made. In particular, the political state θ_t determines the political rule, in this case either majority rule or dictatorship, for choosing both the policy p_t and the subsequent political state, θ_{t+1}.

Again, omit the private sector. Now, the public sector decision includes both the choice of current policy *and* the choice of institution for the following period. Keeping notation the same whenever possible, the policy strategy ψ determines policy $p_t = \psi(\omega_t, \theta_t)$ given the current economic and political state. An *institutional strategy* μ is a mapping that determines the future political state $\theta_{t+1} = \mu(\omega_t, \theta_t)$. The institutional strategy describes a recursive process of institutional change. The rule in period t produces the new rule for period $t + 1$. The *public sector decision* each period is a pair (p_t, θ_{t+1}).

[6] Their work builds on an older literature on time consistent policy. See Persson and Tabellini (2001) and Hassler, et. al. (2003) for references.

[7] A few exceptions to these two standards sets of conditions are mentioned below.

To save on notation, let $s_t = (\omega_t, \theta_t)$ be the composite state. An individual's payoff now is

$$U_i(s_t;\ \psi, \mu)(p_t, \theta_{t+1}) \equiv (1-\delta)u_i(\omega_t, p_t) + \delta \int V_i(s_{t+1};\ \psi, \mu) dq(\omega_{t+1} |\ \omega_t, p_t) \tag{3}$$

If $s_t = (\omega_t, \theta^1)$, then the set $C(U(s_t; \psi, \mu), s_t)$, now indexed by state s_t, describes the set of Condorcet Winning public decisions as before. However, if $s_t = (\omega_t, \theta^2)$, then $C(U(s_t; \psi, \mu), s_t)$ describes the public sector decisions that maximize the dictator's payoff function $U_1(s_t; \psi, \mu)(\cdot)$. The political fixed point problem is restated as

$$(\psi(s_t), \mu(s_t)) \in C(U(s_t;\ \psi, \mu), s_t),\ \forall s_t \tag{4}$$

The fixed point problems (2) and (4) are distinct in several respects. In (2), the map admits a fixed point in the space of policy strategies. Since the institution — majority voting — was fixed, the recursive payoff profiles were required to admit Condorcet Winners for each economic state ω. This is a nontrivial problem by itself. However, the mapping in (4) varies by institutional state, θ_t, as well as by economic state ω_t, and so we further require that (4) admits solutions for all political aggregation rules in some feasible set.

This fixed point problem in institutions is reminiscent of "self-selected rules" in the static social choice models of Koray (2000) and Barbera and Jackson (2000), and also the infinite regress model of choice of rules by Lagunoff (1992). These all posit social orderings on the rules themselves based on the outcomes that these rules prescribe. The present model differs from these in that institutional choice occurs in real time. This makes possible an analysis of explicit dynamics of change. Additionally, the present model is more concrete, distinguishing between public and private sector decisions (in the next Section). Once a private sector is introduced, then (4) must hold for individuals' private decision rules that best respond to public sector decisions and to each other in all states.

Notice, finally, that the decision problem is necessarily multi-dimensional. Hence, the simplest Median Voter Theorems are not useful for solving (4). A few papers examine dynamic models of voting that specifically allow for multi-dimensional choice spaces (though keeping the voting mechanism fixed). These include Bernheim and Nataraj (2002), Kalandrakis (2002), and Banks and Duggan (2003). Here, the political fixed point problem is compounded by fact that different institutions each have possibly distinct requirements for achieving recursive consistency. The sequel addresses this problem in fuller generality.

3 The General Model

In general, the feasible set of political institutions can easily include more than merely "pure dictatorship" and "pure democracy". Here we introduce the general model. Our specification includes private (individual) decisions as well as public ones, and a fairly arbitrary space of political institutions. Let Θ denote an index set of feasible institutions. Let $S = \Omega \times \Theta$ denote the composite state space. In what follows, we will find it useful to adopt the standard convention and drop time subscripts, using instead the notation of primes, e.g., θ' to denote subsequent period's variables, θ_{t+1}, and so on. Let $|\cdot|$ denote the cardinality of a set.

3.1 Political Rules

For expositional reasons, we first describe the political institutions that aggregate arbitrary profiles of payoff functions without reference to the dynamic game. Let v_i denote an arbitrary function expressing the payoff $v_i(p, \theta')$ over current policy p and next period's political state, θ'. One example is the dynamic model of the previous section whereby $v_i(p, \theta') = U_i(s;\ \psi, \mu)(p, \theta')$. Let $\mathcal{V}$ denote the set of all profiles, $v = (v_1, \ldots, v_n)$, of such payoff functions.

A *class of political rules* corresponds to a state-contingent social choice correspondence,

$$C : \mathcal{V} \times S \rightarrow\rightarrow P \times \Theta,$$

that associates to each state s and each payoff profile v, a set $C(v, s)$ of public decisions. If $(p, \theta') \in C(v, s)$, then (p, θ') is a feasible public decision under C. Each particular political rule in the class C is given by $C(\cdot, s)$. Examples of political rules include:

(i) *A Time Consistent Planner* The standard benchmark model for dynamic optimal (Ramsey) policy formation is of a time consistent government or social planner. The set Θ is a singleton set $\{\theta\}$ and $\theta \in \mathbb{R}^n_+$ is a vector that determines the welfare weights in a planner's objective. For each $s = (\omega, \theta) \in S$, let $(p, \theta') \in C(v, s)$ if $\theta' = \theta$ and p maximizes $\sum_i \theta_i\ v_i(p, \theta)$.

(ii) *Supermajority rules.* The political state identifies the fraction, $\theta \geq 1/2$ of individuals required to pass a public decision. A super-majority voting rule therefore determines which supermajority rule is used in the future: formally $\Theta \subset (.5, 1]$ and for each $s = (\omega, \theta)$, let $(p, \theta') \in C(v, s)$ if for all $(\hat{p}, \hat{\theta}')$

$$|\{i \in I :\ v_i(\hat{p}, \hat{\theta}') > v_i(p, \theta')\}|\ \leq \theta n$$

(iii) *Voting rights.* The political state θ identifies the subset of individuals who currently possess the right to vote (the voting franchise). The chosen public decision is the one that is majority preferred within this restricted group. Each restricted voting franchise today uses a majority vote to determine what group of individuals has the right to vote tomorrow: formally, $\Theta \supseteq 2^I$, and let $(p, \theta') \in C(v, s)$ if for all $(\hat{p}, \hat{\theta}')$,

$$|\{i \in \theta :\ v_i(\hat{p}, \hat{\theta}') > v_i(p, \theta')\}|\ \leq\ \frac{1}{2}|\theta|$$

(iv) *The scope of public sector decisions.* The political state identifies the domain of public decisions. Let $\theta \subset P$ so that θ denotes the set of feasible policies. Let $(p, \theta') \in C(v, s)$ if $p \in \theta$ and for all $(\hat{p}, \hat{\theta}')$ satisfying $\hat{p} \in \theta$,

$$|\{i \in I : \ v_i(\hat{p}, \hat{\theta}') > v_i(p, \theta')\}| \ \leq \frac{1}{2}n$$

Although these are all examples in which the procedure for changing policy is the same as that which changes institutions, this need not be the case. Consider the following modification of (ii). The political state is $\theta = (\theta_a, \theta_b)$, whereby θ_a is the supermajority required to determine policy, while θ_b is the supermajority required to determine the subsequent rule. Then $(p, \theta') \in C(v, s)$ if, for all $(\hat{p}, \hat{\theta}')$, EITHER

$$|\{i \in I : \ v_i(\hat{p}, \hat{\theta}') > v_i(p, \hat{\theta}')\}| \ \leq \theta_a n \ \text{ OR } \ |\{i \in I : \ v_i(\hat{p}, \hat{\theta}') > v_i(\hat{p}, \theta')\}| \ \leq \theta_b n$$

Hence, the framework allows for provisions, such as those in the U.S. Constitution, that distinguish rules for changing policy from rules for changing rules.

Call a class of rules C *single valued* if for each $v \in \mathcal{V}$ and for each s, there exists a (p, θ') pair such that $C(v, s) = \{(p, \theta')\}$. In many cases, the political rules of interest are those that can rationalized by some social welfare criterion. Formally, a class of rules C is *(partially) rationalized by* a social welfare function $F : \mathbb{R}^n \times S \to \mathbb{R}$ if F is weakly increasing in each dimension of $\mathbb{R}^n$ and if

$$C(v, s) = (\supseteq) \arg\max_{p, \theta'} F(v(p, \theta'), s)$$

When $C(\cdot)$ describes a voting rule in every state (e.g., Examples (ii) and (iii) above), then there are two well known conditions under which the public decisions under C are partially rationalized by the preferences of at least one pivotal voter. The first is the standard restriction to single peaked preferences on a one dimensional decisions space.[8] The second is the order restriction property of Rothstein (1990). Similar results can be found in the application of single crossing properties by Roberts (1977) and by Gans and Smart (1996).

3.2 Dynamic Political Games

Recall that $I = \{1, \ldots, n\}$ is the set of individuals in this society. P is the set of feasible policies in each period, and $S = \Omega \times \Theta$ is the composite state space. Let $s_0 = (\omega_0, \theta_0)$ denote the initial state. We now introduce private sector decisions. Let e_{it} denote i's private decision at date t, chosen from a feasible set E. A profile of private decisions is $e_t = (e_{1t}, \ldots, e_{nt})$. These decisions may capture any number of activities, including labor effort, savings, or investment activities. They may also include "non-economic" activities such as religious worship or one's participation in a violent revolt.

[8]See Arrow (1951) and Black (1958).

To express the dependence of payoffs and technology in private sector decisions, let $u_i(\omega_t, e_t, p_t)$ denote i's stage payoff and let $q(B|\ \omega_t,\ e_t,\ p_t)$ denote the probability that ω_{t+1} belongs to the (Borel measurable) subset $B \subseteq \Omega$, given the economic state ω_t, the private decision profile e_t, and the policy p_t. Each individual's dynamic objective is to maximize average discounted payoff,

$$E\left[\sum_{t=0}^{\infty} \delta^t (1-\delta)\ u_i(\omega_t,\ e_t,\ p_t\ )\right]. \tag{5}$$

Define a *Dynamic Political Game (DPG)* to be the collection

$$G \equiv \langle (u_i)_{i\in I}, q, E, P, \Omega,\ \Theta, C,\ s_0 \rangle$$

The class of dynamic political games (DPGs) constitutes a broad set of problems in which institutional changes occur endogenously and incrementally. The specification of a DPG includes an "economic structure" in form of $(u_i)_{i\in I}, q, E, P$ and Ω. This part is found in any standard stochastic game. The addition is the "political structure" given by Θ and C. The class of political rules is described by operator C. The operator C is defined on the profiles of dynamic recursive payoffs in each state. In turn, these dynamic payoffs depend on future strategies (see, for example, Equation (3) in the previous Section) which are defined explicitly in Section 3.3. The initial state s_0 completes the specification.

For tractability, we restrict attention to dynamic political games that satisfy one of the following two exclusive sets of assumptions.

(A1) Θ is a finite set. P and E are compact, convex subsets of Euclidian spaces, and Ω is a convex subset of a Euclidian space; the payoff function u_i for each i is continuous and uniformly bounded above by some $K > 0$; for each (ω, e, p), $q(\cdot|\ \omega, e, p)$ admits a norm continuous, conditional density $f(\cdot|\ \omega, e, p)$ with respect to a probability measure η.

(A1') Θ, E, P and Ω are all finite sets.

Unless otherwise stated, all the subsequent results assume that either (A1) or (A1') holds.

The continuity and boundedness assumptions in (A1) are fairly standard in existence Theorems of Markov equilibria in stochastic games.[9] However, they are not harmless. In particular, they imply that dynamic payoffs are bounded by K, and, more importantly, that transitions cannot be deterministic. Dutta and Sundaram (1994) contains a cogent discussion of the role of these assumptions in existence results.

[9]See, for example, Mertens and Parthasarathy (1987), Dutta and Sundaram (1994), Amir (1996), Curtat (1996), and Horst (2003).

3.3 Strategies and Equilibrium

To make the theory tractable, we restrict attention to Markov strategies. Such strategies only encode the payoff-relevant states of the game. Consequently, individuals are not required to coordinate on the history of past play.

Recall that $\psi : S \to P$ and $\mu : S \to \Theta$ describe the policy and institutional strategies, respectively. Together they constitute the *public sector strategies*. A *private sector strategy* for individual i is a function $\sigma_i : S \to E_i$ that prescribes private action $e_{it} = \sigma_i(s_t)$ in state s_t. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$. The strategy profile is therefore summarized by the triple

$$\pi \equiv (\underbrace{\sigma,}_{\text{private sector profile}} \underbrace{\psi,}_{\text{policy strategy}} \underbrace{\mu}_{\text{institutional strategy}})$$

An individual deviation from π is denoted by, for example, $\pi\backslash\sigma_i$. For any $s_t = (\omega_t, \theta_t)$ the payoff to citizen i in profile π at date t is defined recursively by:

$$V_i(s_t;\ \pi) = (1-\delta)u_i(\omega_t, \sigma(s_t), \psi(s_t)\ ) + \delta \int V_i(\omega_{t+1}, \mu(s_t); \pi) dq(\omega_{t+1}|\ \omega_t, \sigma(s_t), \psi(s_t)\ ) \quad (6)$$

The function V depends on and varies with arbitrary Markov strategy profiles $\pi = (\sigma, \psi, \mu)$. Along an equilibrium path (defined below), the function V_i defines a Bellman's equation for citizen i. Given any strategy π, and any state s_t at date t, an individual's public payoff function $U_i(s_t, \pi) : P \times \Theta \to \mathbb{R}$ is defined by

$$U_i(s_t, \pi)(p_t, \theta_{t+1}) \equiv (1-\delta)u_i(\omega_t, \sigma(s_t), p_t) + \delta \int V_i(\omega_{t+1}, \theta_{t+1}; \pi) dq(\omega_{t+1}|\ \omega_t, \sigma(s_t), p_t\ ) \quad (7)$$

Let $U(s_t, \pi) = (U_i(s_t, \pi)\ )_{i\in I}$. We now drop time subscripts and define an equilibrium for any dynamic political game.

Definition 1 An *Equilibrium* of a dynamic political game, G, is a profile $\pi = (\sigma, \psi, \mu)$ of Markov strategies such that for all states $s = (\omega, \theta)$,

(a) *Private decision rationality*: For each citizen i, and each private decision rule, $\hat{\sigma}_i$,

$$V_i(s;\ \pi) \geq V_i(s;\ \pi\backslash\hat{\sigma}_i\ ) \quad (8)$$

(b) *Existence of political fixed points*: The public decision pair $(\psi(s), \mu(s)\ )$ satisfies

$$(\psi(s), \mu(s)\ ) \in C\,(U(s, \pi),\ s\ ) \quad (9)$$

Part (a) is the standard Markov Perfection property of a stochastic game. Private sector actions must be individually optimal in each state. Part (b) asserts the existence of political fixed points. Public sector decisions are required to be consistent with political rules in the class C. In keeping with the standard definition of a stochastic game, both types of decisions are simultaneous. Therefore, an equilibrium of a DPG requires both Markov Perfection from individuals' private sector choices and recursive consistency of public sector choices with a political rule.

Since this equilibrium concept departs from Nash equilibrium, it deserves further comment. One could make a reasonable case that, instead of modelling political rules in reduced form as social choice rules (e.g., Condorcet Winners to represent majority voting), all political rules should be explicitly modelled as a part of a noncooperative game. The argument goes that endogenous institutions can arise as Nash equilibrium outcomes of a standard stochastic game. The issue then becomes: which game? While there are agreed upon canonical social choice representations of voting, there are fewer in non-cooperative games.[10] Hence, the trade-off is one of "explicitness" versus "representativeness." The approach taken here favors the latter.[11]

4 Equilibrium Existence and Political Fixed Points

There are two main problems in establishing equilibrium existence. First, there is the "standard" existence problem in all stochastic games. This problem amounts to finding a solution to Part (a) in the equilibrium definition. For even if there were no public decisions (or if C was constant in all states), known existence results generally employ restrictive conditions on feasible choice sets, preferences, and transition technology. Second, one must address the political fixed point problem as expressed in Part (b) of the equilibrium definition. The problem of finding political fixed points was outlined in Section 2. Both problems must be solved in order for a DPG to admit equilibria. However, while the standard existence problem applies to all stochastic games, the political fixed problem is specific to dynamic political games. We address the specific problem first.

4.1 Dynamically Consistent Rules

Recall that most of the common political rules of interest are those that are rationalized or partially rationalized by social welfare functions. Among these, the most obvious ones satisfy

[10] Arguably, the closest to a canonical model among non-cooperative games is the endogenous candidate model of Besley and Coate (1997) and Osborne and Slivinsky (1996). Even in their model, however, there are reasonable, alternative specifications of how candidates could emerge.

[11] It should also be noted that existence Theorems in this paper apply to purely noncooperative games as a special case.

a *dynamic consistency requirement*. Dynamic consistency, long taken for granted in decision models, presumes that future decision makers' points of view coincide with that of the present decision maker if the latter were to called upon to make the decision in that future period. A large literature has emerged recently to evaluate dynamically *in*consistent decision-making from an individual's perspective. Nevertheless, most dynamic models of government policy maintain assumptions of dynamically consistency.[12]

For our purposes, the following definition will be used in the analysis. Consider any time separable payoff functions of the form, $v(p, \theta') = (1-\delta)v^1(p) + \delta \int_{\omega'} v^2(\omega', \theta')d\eta$. (Clearly, dynamic payoffs in the present model satisfy this requirement). A class of political rules, C, will be said to be *dynamically consistent* if it is partially rationalized by a continuous social welfare function F that satisfies: in every state $s = (\omega, \theta)$,

$$F\left((1-\delta)v^1(p) + \delta \int_{\omega'} v^2(\omega', \theta')d\eta,\ s\right) = (1-\delta)F\left(v^1(p),\ \theta\right) + \delta \int_{\omega'} F\left(v^2(\omega', \theta'),\ \theta\right) d\eta$$

This definition embodies two critical properties. First, F is time and state separable and linearly homogeneous in the discount weights $(1-\delta, \delta)$.[13] Second, F does not vary directly with the economic state, i.e, $F(\cdot,\ s) = F(\cdot,\ \theta)$. Consequently, each political state θ is uniquely associated with a political rule. Both properties are needed to guarantee that the rate of intertemporal substitution between the current period payoff and next period coincides with the rate of substitution between any other pair of successive payoff-dates. Example (i) in the previous Section is clearly dynamically consistent. However, Examples (ii)-(iv) are only dynamically consistent in particular circumstances. The results below provide greater detail about the nature of these circumstances.

The following result asserts that dynamic consistency of the political rule is sufficient for existence of political fixed points.

Theorem 1 *Suppose in a dynamic political game G, either Assumption (A1) or (A1') holds, and the class C is dynamically consistent. Then for any profile, σ, of private sector decision rules, the game admits political fixed points.*

The proof below shows first that political fixed points correspond to fixed points in value function space of the associated "Bellman's equation." A solution to the Bellman's equation exists iff there is a political fixed point. A Contraction Mapping argument then completes the proof.

[12]The two exceptions I am aware of are Krusell, Kuruscu, and Smith (2002) and Lagunoff (2004a).

[13]Broader definitions of dynamic consistency do not require linear homogeneity in the discount factor. The results here may also be amenable to a more general definition, but at a cost in transparency.

Proof Let $W : S \to [0, K]^n$ be a profile of bounded, measurable continuation payoff functions. We call a continuation function W *feasible* if $W = V(\cdot, \pi)$ for some strategy profile, π. Let $\mathcal{W}$ denote the set of feasible continuations. By a slight abuse of our previous notation, we define the recursive public payoff with an arbitrary continuation W in (7) to

$$U_i(s;\ \sigma, W)(p, \theta') \ = \ (1-\delta)u_i(\omega, \sigma(s), p) \ + \ \delta \int W_i(\omega', \theta') dq(\omega' |\ \omega, \sigma(s), p\ ) \tag{10}$$

We prove first that political fixed points are associated with the fixed points (in value function space) of the associated Bellman's mapping. Specifically, let G be a dynamic political game in which rule C is partially rationalized by a continuous function F. For each $\sigma \in \Sigma$, suppose that $\hat{W}$ is a fixed point of the "Bellman's" operator $B : \mathcal{W} \to \mathcal{W}$ defined by

$$(BW)(s) = U(s;\ \sigma, W)(\psi(s), \mu(s)) \tag{11}$$

where $(\psi(s), \mu(s))$ solves

$$\max_{p,\theta'} F\left(U(s;\ \sigma, W)(p, \theta'),\ s\right) \tag{12}$$

Fix σ and let $\hat{W}$ be a fixed point of the Bellman's operator defined in (11). Then we show that the pair $(\hat{\psi}, \hat{\mu})$ that solves (12) for $W = \hat{W}$, and the pair constitutes a political fixed point. Let $(\hat{\psi}, \hat{\mu})$ denote the solution to (12) under $\hat{W}$, i.e,

$$F\left(\hat{W}(s),\ s\right) = F\left(U(s;\ \sigma, \hat{W})(\hat{\psi}(s), \hat{\mu}(s)),\ s\right),\ \forall s$$

By our earlier (abuse of) notation, $U(s;\ \sigma, \hat{W}) = U(s;\ \sigma, \hat{\psi}, \hat{\mu})$. If C is either partially or fully rationalized by F, then $(\hat{\psi}, \hat{\mu}) \in C(U(s;\ \sigma, \hat{\psi}, \hat{\mu}),\ s)$, and so $(\hat{\psi}, \hat{\mu})$ is a political fixed point.

Next, we prove that if C is fully rationalized by F, then the converse holds: namely, for every political fixed point $(\hat{\psi}, \hat{\mu})$, the corresponding value $\hat{W}$ of the Bellman's operator is a fixed point of (11). Suppose then that C is fully rationalized by F and let $(\hat{\psi}, \hat{\mu})$ be a political fixed point. Then

$$(\hat{\psi}, \hat{\mu}) \in C(U(s; \sigma, \hat{\psi}, \hat{\mu}),\ s) = \arg\max_{p,\theta'} F\left(U(s;\ \sigma, V(\cdot; \sigma, \hat{\psi}, \hat{\mu}))(p, \theta'),\ s\right)$$

By definition, $V(\cdot; \sigma, \hat{\psi}, \hat{\mu})$ is a fixed point of the Bellman operator.

We prove the remainder of the result as follows. Fix an arbitrary σ and let W be any bounded continuation profile, and by our abuse of notation, write $U(s, \sigma, W)$. Since C is dynamically consistent, the Bellman's operator defined in (11) is, in this case, given by

$$\begin{aligned}(BW)(s) &= \max_{p,\theta'} F\left(U(s, \sigma, W)(p, \theta'),\ \theta\right) \\ &= \max_{p,\theta'} \left\{(1-\delta)F\left(u(\omega, \sigma(\omega, \theta), p),\ \theta\right) + \delta \int F\left(W(\omega', \theta'),\ \theta\right) dq(\omega' | \omega', \sigma(\omega, \theta), p)\right\}\end{aligned} \tag{13}$$

Notice that under either (A1) or (A1'), (BW) in nonempty valued. It is easy to very that B satisfies two sufficient conditions, discounting and monotonicity, in a well known result of Blackwell (1965), implying that B is a contraction. Applying the Contraction Mapping Theorem, B has a fixed point, $\hat{W}$. By our previous argument relating fixed points in value space with those in strategy space, the conclusion follows. ∎

Although dynamic consistent rules solve the political fixed point problem, most common rules are not dynamically consistent without further restrictions. Significantly, the Condorcet rule in Section 2, for example, is only dynamically consistent when its policies coincide with those of a median voter whose identity does not vary across states. What guarantees such a condition? We find that a simple restriction on stage game payoffs — a restriction similar to Grandmont's (1978) Intermediate Preference Assumption — suffices for any voting rule to be dynamic consistent.

Formally, C is a *class of voting rules* if there is a nonempty collection $\mathcal{D} \subseteq 2^I \times 2^I$ satisfying: for each v, θ, and (p, θ'), $\;(p, \theta') \in C(v, \theta)$ iff $\forall(\hat{p}, \hat{\theta}')$, $\exists(D_1, D_2) \in \mathcal{D}$ such that,

$$v_i(p, \theta') > v_i(\hat{p}, \hat{\theta}'),\ \forall i \in D_1$$

$$\text{and} \quad v_i(p, \theta') \geq v_i(\hat{p}, \hat{\theta}'),\ \forall i \in D_2$$

Informally, $\mathcal{D}$ denotes the ordered sets consisting of strictly and weakly decisive coalitions of individuals. These coalitions define the sets of individuals that have veto power over all public decisions. Politically feasible decisions require the consent of each member of some decisive coalition. This definition is fairly standard although there are a number of, mostly equivalent, definitions. (See Austen-Smith and Banks (1999)).

Theorem 2 *Suppose that G is a dynamic political game in which either Assumption (A1) or (A1') holds. Suppose that C is a class of voting rules and that the stage game payoffs admit the affine preference representation,*

$$u_i(\omega, e, p) = k(i)h(\omega, e, p) + g(\omega, e, p) \tag{14}$$

all i, where k is an increasing, real valued function. Then C is dynamically consistent.

It is a simple exercise to show that recursive preferences admit an affine representation if stage game payoffs do. Moreover, the ordering on I as given by $k(\cdot)$ is preserved across all states. Hence, these preferences satisfy the order restriction property of Rothstein (1990). Rothstein's Theorem can therefore be applied to show that C is partially rationalized by at least one individual's recursive payoff function. The details of the argument are in the

Appendix. Combining Theorems 1 and 2, if G is a DPG in which (14) is satisfied and C is a class of voting rules, then G admits a political fixed point.

Even with the restriction to affine preferences, a strong case can be made that many of the political rules observed in the world today are not voting rules and, in fact, are dynamically *inconsistent*. Two sources of dynamic inconsistency are of particular interest. First, dynamically inconsistent choices arise because the political rules vary with economic states such as the income distribution. Wealth-weighted "voting" rules satisfy this property. A pure example of this is the wealth-is-power rule examined by Jordan (2002). In its simplest form, policies are entirely determined by those with the most aggregate wealth.

(v) *Wealth-is-Power* Let Θ be a degenerate singleton $\{\theta\}$. For each state $s = (\omega, \theta)$, let the economic state determine distribution of wealth, i.e, $\omega = (\omega_1, \ldots, \omega_n)$. The Wealth-is-Power rule is given by: $(p, \theta) \in C(v, s)$ if for all $(\hat{p}, \theta)$,

$$\sum_{i \in M} \omega_i \; < \; \sum_{i \notin M} \omega_i$$

where $M = \{i \in I : \; v_i(\hat{p}, \theta) > v_i(p, \theta)\}$.

Jordan (2002) shows that outcomes of the wealth-is-power rule correspond to the core of a certain cooperative game. He characterizes the set of wealth distributions that generate nonempty core, or in our context, generate political fixed points. It is easy to see that under certain types of technological shocks, dynamic inconsistencies can occur. Reversals over time in the aggregate wealth of two mutually exclusive groups of individuals create conflicts between the present and future incarnations of the political aggregation process.

A second form of dynamic inconsistency arises because the political rules are not separable. An example is a weighted Rawlsian social choice rule under which society wishes to maximize the welfare of the person whose weighted payoff makes him least well off.

(vi) *The Rawlsian Rule* Let Θ be a finite subset of $\{\theta \in \mathbb{R}^n_+ : \sum_k \theta_k = 1\}$. For all states $s = (\omega, \theta)$,

$$C(v, s) = \arg\max_{(p, \theta')} \min_i \{\theta_1 v_1, \ldots, \theta_n v_n\}$$

Even if the economic state is degenerate, i.e., $\Omega = \{\omega\}$, it is not generally true under the weighted Rawlsian rule that the least well off individual also has the least well off continuation payoff next period. Hence, the rate of substitution between today and period $t + 1$ differs possibly from rate of substitution between $t + 1$ and $t + 2$.

Clearly, each of these rules may induce dynamically inconsistent public sector decisions. It less clear whether, and to what extent, they pose difficulties in the political fixed point problem. Of the two, wealth-weighted voting seems more problematic. Jordan (2002) shows, for instance, that fairly extreme wealth distributions are required to avoid cycles in the wealth-is-power rule even in static problems and with affine preferences. For now, we leave this for future research and proceed to the general existence problem.

4.2 Extension to Mixed Strategies: A Finite Existence Theorem

Unfortunately, even if the DPG admits political fixed points, a solution to the general existence problem is not guaranteed. The simplest way to resolve the existence issue is to assume that all sets, E, P, Ω and Θ are finite (i.e., Assumption (A1')). Then the analysis must be extended to allow for mixed strategies. The extension is fairly straightforward. Note that since the political rule C makes a joint determination of p and θ', it must be extended to the set $\Delta(P \times \Theta)$ of correlated distributions.

First, represent the public sector strategies for p and θ' as a pair ψ^* and μ^* such that $\mu^* : S \to \Delta(\theta)$ where $\mu^*(\theta'|s)$ is the probability of θ' given s, and $\psi^* : S \times \Theta \to \Delta(P)$ where $\psi^*(p|s, \theta')$ denotes the conditional probability of p given s and the realized θ'. Given s, the associate mixed action in $\Delta(P \times \Theta)$ is expressed as $(\psi^* \times \mu^*)(s)$. Next, let $\sigma_i^*(e_i|s)$ denote the conditional probability of private decision e_i given s. A profile of mixed Markov strategies is then defined by $\pi^* = (\sigma^*, \psi^* \times \mu^*)$. The payoff (6) is extended to mixed Markov strategies in the following way:

$$V_i(s;\ \pi^*) \ = \ E\left[(1-\delta)u_i(\omega, e, p\ ) \ + \ \delta \sum_{\omega'} V_i(\omega', \theta'; \pi) q(\omega'|\ \omega, e, p\ ) \ \Big|\ (\psi^* \times \mu^*)(s), \sigma^*(s)\right] \tag{15}$$

Theorem 3 *Suppose in any dynamic political game satisfying (A1') the class C of political rules is dynamically consistent. Then there exists an equilibrium $\pi^* = (\sigma^*, \psi^* \times \mu^*)$ in mixed Markov strategies.*

In the particular case of a voting rule, equilibria exist if stage game preferences admit affine representations. The argument is straightforward. Since F is dynamically consistent, then for any political state θ, we can treat the social welfare function $F(\cdot, \theta)$ in state θ as a "player" in a standard dynamic game. This player has a set of feasible pure actions equal to $P \times \Theta$ if θ is the current rule, and is equal to the empty set $\emptyset$ if θ is not. This player then has dynamic preferences (in pure strategies) given by,

$$(1-\delta)F\left(u(\omega, e, p),\ \theta\right) + \delta \sum_{\omega'} F\left(V(\omega', \theta';\ \pi),\ \theta\right) f(\omega'|\omega, e, p)$$

Viewed in this way, the DPG can be transformed into a standard, finite stochastic game with $n + |\Theta|$ players. At this point, the Theorem 3 is just an application of a standard result, namely, that stochastic games with finite actions sets and finite states admit in mixed strategies Markov Perfect equilibria. We therefore omit remainder of the proof.

4.3 A Smooth Existence Theorem

While the finite existence theorem is useful in many contexts, it is limited in a number of ways. First, it does not apply to many economically relevant environments. In many such environments, the natural economic states are capital stocks which may be unboundedly infinite.

Second, practical applications demand more structure. In most dynamic models of policy, equilibria are characterized by their Euler equations. These Euler equations are more elaborate than those in single agent problems. Roughly, Euler equations in DPGs have extra terms due the intertemporal effects that other agents' future policy functions have on a decision maker's policy decision in the current period. These extra terms involve higher order differential equations of future policy functions.[14] Euler equation characterizations in dynamic political models are found in Klein, Krusell, and Rios-Rull (2002) and Jack and Lagunoff (2003). The latter examines a particular class of DPG in which voting rights are endogenously determined. The former examine a politico-economic model of government fiscal policy. Klein, Krusell, and Rios-Rull also develop computationally tractable techniques for obtaining optimal policies from "Generalized" Euler equations in their model.[15]

For these reasons, it is highly desirable to find conditions on DPGs for which smooth Markov equilibria exist. Until recently, not much was known about smooth Markov equilibria even in standard stochastic games. An elegant result of Horst (2003) asserts existence of Lipschitz-continuous (hence, almost everywhere smooth) Markov Perfect equilibria in dynamic games. His result makes use of a "moderate social influence" (MSI) assumption whereby the interactions between players are sufficiently weak. The MSI assumption apparently originates from a restriction on payoffs in a local interaction model of Horst and Scheinkman (2002). The idea, roughly, is that one's own actions have a relatively greater effect on one's own marginal dynamic payoff than those of all other individuals combined. MSI overcomes a common problem in dynamic games. Generally, continuity of each player's "Bellman's" operator fails because it conflicts with conditions required for compactness of the function space to which the operator applies. One can recover continuity of the Bellman operator if there are uniform,

[14]See Basar and Olsder (1999) for a general formulation.

[15]Strictly speaking, politico-economic models are hybrids in the sense that the price mechanism in these models endogenously constrains one's *feasible* actions, and so out-of-equilibrium behavior is not defined. Nevertheless, their techniques are likely adaptable to dynamic game environments if smooth Markov equilibria exist.

Lipschitzian bounds on the players' marginal best replies in each state in the dynamic game. The MSI assumption establishes existence of such bounds.

We combine elements of Horst's result, including the MSI assumption, with earlier results on political fixed points to show that dynamic political games with dynamically consistent rules admit equilibria that are almost everywhere smooth in the economic state, ω.

To make sense of formal assumptions below, we adopt the following definitions and notational conventions. First, endow the class of (smooth) C^∞ functions, $H : \mathbb{R}^\ell \to \mathbb{R}^k$, with the topology of C^∞-uniform convergence on compacta. Formally, $H^m \to H$ in this topology if, for any compact set $Y \subset \mathbb{R}^\ell$, $\{H^m\}$ converges to H C^∞-uniformly on Y (i.e., for each r and each rth partial derivative, $||D^r H^m - D^r H||_r \to 0$ on Y).[16] The function H is C^∞-uniformly bounded if it is smooth and there is some some finite number $L > 0$ that uniformly bounds H and bounds all its higher order derivatives in sup norm.

Next, define a real valued function, $g : \mathbb{R}^\ell \to \mathbb{R}$ to be α*-concave* with $\alpha > 0$ if $g(x)+\frac{1}{2}\alpha||x||^2$ is concave.[17] Notice that $\alpha = 0$ corresponds to the standard definition of concavity. When $\alpha > 0$, then α-concavity is obviously a stronger curvature condition. α-concavity is used elsewhere in the literature to bound higher order derivatives via the Implicit Function Theorem. It is used here for a similar purpose.

Finally, given some $\epsilon > 0$, we let E^ϵ and P^ϵ denote interior neighborhoods of E and P, respectively, such that any point $e \in E^\epsilon$ or $p \in P^\epsilon$ is ϵ in distance away from the respective boundaries in E and P.

In addition to Assumption (A1), the following assumptions on the dynamic political game will be used.

(A2) (Concavity) There is an $\alpha_i > 0$ such that for each ω, u_i is α_i-concave in the private and policy decision pair, (e_i, p) pair.

(A3) (Uniform Bounds) There is an $L > 0$ such that for each i, the payoff function u_i is smooth and C^∞-uniformly bounded by $L > K$ (recall that K is the bound on u_i). There is also an $M > 0$ such that for each ω' and each ω, the conditional density $f(\omega'|\ \omega, \cdot)$, as a function of decisions (e, p), is assumed to be C^∞-uniformly bounded by M.

[16]The sup norm, $||\cdot||_r$ on the r^{th} derivative $D^r H : \mathbb{R}^\ell \to \mathbb{R}^{kr\ell}$ is defined by

$$||D^r H||_r = \sup_{x'} \sup_{j_1,\ldots,j_r} ||\frac{\partial^r H}{\partial x_{j_1} \cdots \partial x_{j_r}}(x')||.$$

In this notation, $||\cdot||_0$ is the standard sup norm on H.

[17]An equivalent definition is: g is α-concave if the matrix $D^2 g + \alpha I$, with I denoting the identity matrix, is negative semi-definite.

(A4) (Moderate Social Influence) There exists a $0 < \gamma < 1$ such that for all i, and all $s = (\omega, \theta)$,

$$\frac{(1-\delta)L + \delta KM}{\alpha_i} \leq \gamma(1-\delta).$$

(A5) (Interiority) There exists an $\epsilon > 0$ such that for each i, each e_{-i}, and each pair of economic states ω and ω', both $u_i(\omega, e_{-i}, \cdot)$, and $f(\omega'|\omega, \cdot)$ achieve their upper bounds on $E^\epsilon \times P^\epsilon$.

Assumptions (A2) and (A3) are used to bound the marginal best replies uniformly over all states and continuations. Assumption (A4) is the Moderate Social Influence (MSI) assumption adapted from Horst (2003) and Horst and Scheinkman (2002). Assumption (A5) ensures interior solutions in best replies.

Theorem 4 *Let G be any dynamic political game satisfying (A1)-(A5). Suppose the class of political rules C is dynamically consistent and rationalized by a smooth function F on $\mathbb{R}^n$. Then G admits an equilibrium, $\pi = (\sigma, \psi, \mu^*)$, in which σ and ψ are pure Markov strategies and μ^* is a mixed Markov strategy, all of which are almost everywhere smooth in the economic state ω.*

The Proof below has two main steps. First, we establish existence of a smooth Markov equilibrium for the game restricted to only private decisions. With this restriction, the game conforms to an ordinary stochastic game. To an extent (though not completely), this step adapts many of the arguments from Horst's proof. In particular, we adapt the methodology of showing that the dynamic game reduces to a series of one-shot auxiliary games all of which have smooth Nash equilibria in the economic state.[18] The smoothness requires both the MSI assumption and an older result of Montrucchio (1987) who shows that dynamic, single agent decision problems have solutions that are Lipschitzian functions of the state.

Horst first applies a fixed point argument for bounded state spaces. These fixed points are Markov equilibria of the game with bounded states. He then extends the result by taking a uniform limit of fixed points as the bound is relaxed. The present argument, however, takes a direct approach by applying the fixed point argument directly on the full state space.

However, the truly distinctive step of our proof is to show that any dynamic political game with a dynamically consistent class C maps into an ordinary stochastic game. By using the set of rules to augment the set of players, public sector decisions may be transformed into private decisions taken by these additional players. Assumptions (A1)-(A5) in the original game are

[18]This same methodology is common in other stochastic game equilibrium existence results. See, for example, Mertens and Parthasarathy (1987), Dutta and Sundaram (1994), Amir (1996), and Curtat (1996).

shown to apply to the new stochastic game. It is not obvious that this transformation can be done if political rules are dynamically *in*consistent.

Proof of Theorem 4 The steps outlined informally above are formalized here.

Step 1. The Restriction to Private Decisions

We first prove an existence Theorem without public decisions. That is, consider the standard dynamic game with only private decisions, e_i. In what follows, we exclude the public decision component from notation altogether. That is, we first assume that stage game payoffs are given by $u_i(\omega, e)$ while the density is given by $f(\omega'|\,\omega, e)$. Let $\bar{G} = \langle (u_i)_{i\in I}, \Omega, q, E, \omega_0; \rangle$ denote the game with only private decisions. Restating the result, we first wish to prove

THEOREM A *Let $\bar{G}$ denote a dynamic game with only private decisions. Suppose $\bar{G}$ satisfies (A1)-(A5). Then the game has a Markov Perfect equilibrium σ that is smooth on Ω.*

The Proof of Theorem A constitutes the remainder of Step 1. Let $\mathcal{X}$ denote the set of all uniformly bounded, Lipschitz continuous functions, $x : \Omega \to [0, K)^n$ with uniform Lipschitz bound given by L. Standard results show that $\mathcal{X}$ is compact in the topology of uniform convergence on compacta (see, for example, Mas Colell (1985, Theorem K.2.2). For each such function $x \in \mathcal{X}$, define a one shot game by the payoffs,

$$H_i(\omega, e,\ x) \ = \ (1-\delta)u_i(\omega, e) \ + \ \delta \int x_i(\omega')f(\omega'|\,\omega, e)d\eta \tag{16}$$

for each i. Then let $H = (H_i)_{i=1}^n$ be the vector valued function with components defined by (16).

Lemma 1 *For each state ω and each continuation value $x \in \mathcal{X}$, the one shot game defined by payoff profile, $H(\omega, \cdot,\ x)$, has a unique pure strategy Nash equilibrium profile,*

$$(\bar{\sigma}_1(\omega, x), \ldots, \bar{\sigma}_n(\omega, x)\ )$$

of private decisions. The profile $\bar{\sigma}$ is smooth with uniformly bounded first derivatives in ω, and is uniformly bounded and continuous in x.

The proof of this and all subsequent Lemmatta are contained in the Appendix. Using Lemma 1, let $\bar{\sigma}$ be the map that defines the unique Nash equilibrium $\bar{\sigma}(\omega, x)$ for the one shot game with payoffs, $H_i(\omega, e, x),\ i = 1, \ldots, n$.

Lemma 2 *The equilibrium payoff function $H_i(\cdot, \bar{\sigma}(\cdot), x)$ is smooth with a uniformly bounded first derivative in ω, and with the uniform bound applying across all x.*

Now define the operator, T defined on $\mathcal{X}^n$ by

$$(Tx)_i(\omega) = H_i(\omega, \bar{\sigma}(\omega, x), x)$$

for each $i = 1, \ldots, n$, or, in other words,

$$(Tx)(\omega) = (H_1(\omega, \bar{\sigma}(\omega, x), x), \ldots, H_n(\omega, \bar{\sigma}(\omega, x), x)\ ) \tag{17}$$

Clearly, from Lemma 2, the function $(Tx)(\cdot)$ is smooth in ω with uniformly bounded first derivative in ω over all ω and x. This implies, in particular, that Tx has uniform Lipschitz bound. Consequently, $Tx \in \mathcal{X}^n$ for all $V \in \mathcal{X}^n$.

Lemma 3 *T is a continuous operator.*

Using Lemma 3, T maps continuously from the compact set $\mathcal{X}^n$ into $\mathcal{X}^n$. By Schauder's Fixed Point Theorem (see, for example, Aliprantis and Border (1999)), T has a fixed point, x^*:

$$x^* = Tx^*.$$

Therefore, the profile, σ^* defined by $\sigma^* \equiv \bar{\sigma}(\cdot, x^*)$ is a smooth Markov Perfect equilibrium of the dynamic political game without public decisions.

Step 2. The Extension to Public Decisions

Fix a dynamic political game G satisfying (A1)-(A5) with C dynamically consistent. We now define a simple transformation of the full game, G, with public decisions to one with only private decisions. Call this transformed game $\bar{G}$ and let $\bar{G} = \langle (\bar{u}_j)_{j \in J}, f, \bar{E}, \Omega, \omega_0 \rangle$.

The transformed game is defined as follows. Suppose, without loss of generality that Θ and I are disjoint sets, and let $J = I \cup \Theta$. There are $|J| \times |\Theta|$ players, each doubly indexed by $j\theta$, with $j \in J$ and $\theta \in \Theta$. The interpretation of index $j\theta$ is that of Player j whose political "type" (state) is θ. Formally, if $\tau = |\Theta|$ then there are $(n + \tau)\tau$ players. The private decision space for Player $j\theta$ is

$$\bar{E}_{j\theta} = \begin{cases} E & if \quad j \in I \\ \\ P \times \Delta(\Theta) & if \quad j \in \Theta \end{cases}$$

Notationally, let $\bar{e}_I = (\bar{e}_{j\theta})_{j \in I}$ denoting the profile of private actions of the "original" players. Then for each $j \in \Theta$, $\bar{e}_{j\theta} = (\bar{p}_{j\theta}, \beta_{j\theta})$, where $\beta_{j\theta}$ is the (possibly) mixed strategy that assigns probability $\beta_{j\theta}(\theta')$ to θ' by Player $j\theta$.

By the dynamic consistency of C, each player $j\theta$'s stage payoff $\bar{u}_{j\theta}$ may be defined by:

$$\bar{u}_{j\theta}(\omega, \bar{e}) = \begin{cases} u_j(\omega, \bar{e}_I, \bar{p}_{\theta\theta}) & if \quad j \in I \\ \\ F(u(\omega, \bar{e}_I, \bar{p}_{\theta\theta}),\ j) & if \quad j \in \Theta \end{cases}$$

Without loss of generality, we can express a player's stage game payoff as $\bar{u}_{j\theta}(\omega, \bar{e}_I, \bar{p}_{\theta\theta})$. Fix a vector $\beta = (\beta_{\theta\theta})_{\theta\in\Theta}$. Treating β as a part of the state, i.e, (ω, β), it is straightforward to verify that the assumptions (A1)-(A5) are satisfied for the private decisions game $\bar{G}$ if they are satisfied in the underlying game G.

Now define $\bar{x} : \Omega \times \Delta(\Theta) \to [0, K]^J$ by

$$\bar{x}_{j\theta}(\omega, \beta) = \sum_{\theta' \in \Theta} x_{j\theta'}(\omega)\beta_{\theta\theta}(\theta')$$

Observe that $\bar{x}$ is uniformly bounded and Lipschitz continuous on $\Omega \times \Delta(\Theta)$ with uniform bound of, as before, L. It is also linear in β. Therefore, we can write the dynamic payoff function H from Theorem A as

$$H_{j\theta}(\omega, \beta, \bar{e}_I, \bar{p}_{\theta\theta}, \bar{x}) \ \equiv \ (1-\delta)\bar{u}_{j\theta}(\omega, \bar{e}_I, \bar{p}_{\theta\theta}) \ + \ \delta \int \bar{x}_{j\theta}(\omega', \beta) f(\omega' | \omega, \bar{e}_I, \bar{p}_{\theta\theta}) d\eta$$

By Lemma 1, the payoff $H_{j\theta}(\omega, \beta, \cdot, \bar{x})$ has a unique pure strategy Nash equilibrium, $(\bar{\sigma}(\omega, \beta, \bar{x}), \bar{\psi}(\omega, \beta, \bar{x})\ )$ in "state" $(\omega, \beta, \bar{x})$ where $\bar{\sigma}_{j\theta}(\omega, \beta, \bar{x}) = \bar{e}_{j\theta}$ is the private decision of $j\theta$ when $j \in I$, and $\bar{\psi}_{j\theta}(\omega, \beta, \bar{x}) = \bar{p}_{j\theta}$ is the "private" decision when $j \in \Theta$. Moreover, as in the Lemma 2 Proof (see the Appendix), the pair $(\bar{\sigma}, \bar{\psi})$ is smooth in ω and in β.

Next define

$$\bar{\mu}_{j\theta}(\omega, \bar{e}_I, \bar{p}_{\theta\theta}, \bar{x}) = \arg\max_{\beta} H_{j\theta}(\omega, \beta, \bar{e}_I, \bar{p}_{\theta\theta}, \bar{x})$$

Notice that H is linear in β, and is a smooth and C^∞-uniformly bounded function of $(\omega, \bar{e})$ (see Proof of Lemma 1 in the Appendix). Consequently, $\bar{\mu}$ is upperhemicontinuous and almost everywhere smooth in $\bar{e}$ and ω, and so by Kakutani's Theorem, the map

$$(\omega, \bar{e}_I, \bar{p}_{\theta\theta}, \beta, \bar{x}) \mapsto (\ (\bar{\sigma}_{j\theta}(\omega, \beta, \bar{x}))_{j\in I}, (\bar{\psi}_{j\theta}(\omega, \beta, \bar{x}))_{j\in\Theta}, (\bar{\mu}_{j\theta}(\omega, \bar{e}_I, \bar{p}_{\theta\theta}, \bar{x}))_{j\in\Theta}\ )_{\theta\in\Theta}$$

has a fixed point

$$(\bar{\sigma}^*_{j\theta}(\omega, \bar{x}))_{j\in I}, (\bar{\psi}^*_{j\theta}(\omega, \bar{x}))_{j\in\Theta}, (\bar{\mu}^*_{j\theta}(\omega, \bar{x}))_{j\in\Theta}\ )_{\theta\in\Theta}$$

which is almost everywhere smooth in Ω. Let $\bar{\sigma}^*(\omega, \bar{x}) \equiv (\bar{\sigma}^*_{j\theta}(\omega, \bar{x}))_{j\in I, \theta\in\Theta}$, $\bar{\psi}^*(\omega, \bar{x}) \equiv (\bar{\psi}^*_{j\theta}(\omega, \bar{x}))_{j\in\Theta, \theta\in\Theta}$, and $\bar{\mu}^*(\omega, \bar{x}) \equiv (\bar{\mu}^*_{j\theta}(\omega, \bar{x}))_{j\in\Theta, \theta\in\Theta}$.

The remainder of the argument now mimics that of Theorem A. Specifically, there exists a fixed point $\bar{x}^*$ of an operator T defined by

$$(T\bar{x})_{j\theta}(\omega) = H_{j\theta}(\omega, \bar{\sigma}^*(\omega, \bar{x}), \bar{\psi}^*(\omega, \bar{x}), \bar{\mu}^*(\omega, \bar{x}), \bar{x})$$

Applying the same argument as in Theorem A, a Markovian equilibrium exists which is given by

$$(\sigma^*, \psi^*, \mu^*) = (\bar{\sigma}^*(\cdot, \bar{x}^*), \bar{\psi}^*(\cdot, \bar{x}^*), \bar{\mu}^*(\cdot, \bar{x}^*)\ )$$

This equilibrium is smooth in ω. We conclude the proof. ▮▮

5 Conclusion

This paper introduces a class of stochastic games, called dynamic political games, in which political institutions are instrumental objects of choice each period. In dynamic political games, public sector decisions are determined by rules that aggregate profiles of dynamic recursive preferences of individuals. The public sector decisions include parameters of future political rules. Hence, rules used in date $t + 1$ are a part of the decision determined by rules in date t.

We specify an equilibrium concept requiring that political aggregation be recursively consistent. Although this concept differs somewhat from Nash equilibrium, we argue that it is a more canonical starting point for an analysis of dynamically endogenous institutions.

In this framework, Markov equilibria exist if all sets are finite, or if certain "smooth uniformity" conditions are satisfied. Under the last types of conditions, we obtain a smooth existence result: Markov equilibria exist that are smooth functions of the economic state.

The last result is of particular interest to applied political economists who characterize dynamic equilibria by their Euler equations. Since Euler equations in dynamic games contain higher order differentials of policy functions, existence of smooth equilibria, at least locally, is important.

To obtain the existence results we show that DPGs must confront a difficult "political fixed point problem" that arises in virtually any model of dynamic political aggregation. Because a political rule must operate on dynamic preferences which, themselves, depend endogenously on future strategies and rules, aggregation is well defined now only if it is well defined in all future states and for all future rules. Along with standard technical conditions, dynamically consistent rules evidently solve this problem.

Unfortunately, many historically relevant political institutions — wealth-weighted voting for instance — are not dynamically consistent. Moreover, even simple rules such as voting rules are consistent in only under restrictive conditions such as preferences that admit affine representations. It is an open question whether, and to what extent dynamically inconsistent classes of rules admit political fixed points. We leave this issue for future research.

6 Appendix

Proof of Theorem 2 Fix an arbitrary behavior profile π. Then, at each date t, for each state s_t,

$$\begin{aligned} V_i(s_t;\pi) &= E\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}(1-\delta)[u_i(\omega_\tau,\sigma(s_\tau),\psi(s_\tau))\ ]\Big|s_t\right] \\ &= E\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}(1-\delta)[k(i)h(\omega_\tau,\sigma(s_\tau),\psi(s_\tau))+g(\omega_\tau,\sigma(s_\tau),\psi(s_\tau))\ ]\Big|s_t\right] \\ &= (1-\delta)\left\{k(i)E\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}[h(\omega_\tau,\sigma(s_\tau),\psi(s_\tau))\ ]\Big|s_t\right]+E\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}[g(\omega_\tau,\sigma(s_\tau),\psi(s_\tau))\ ]\Big|s_t\right]\right\} \\ &\equiv k(i)H(s_t)+G(s_t) \end{aligned}$$

Clearly, the continuation profile $V(\cdot,\pi)$ has an affine representation of the same form as stage payoffs in (14) for each π. Therefore,

$$\begin{aligned} U_i(s,\pi)(p,\theta') &= (1-\delta)u_i(\omega,\sigma(s),p)+\delta\int V_i(\omega',\theta';\pi)dq(\omega'|\omega,\sigma(s),p) \\ &= (1-\delta)(h(\omega,\sigma(s),p)k(i)+g(\omega,\sigma(s),p)\ ) \qquad (18) \\ &\quad +\delta\int[k(i)H(\omega',\theta')+G(\omega',\theta')]dq(\omega'|\omega,\sigma(s),p) \end{aligned}$$

The profile $U(s,\pi)$ has also has an affine representation. Because k is strictly increasing, the collection of profiles, $U(s,\pi)(\cdot)$, in which $U_i(s,\pi)(\cdot)$ satisfies (18) for each i is order restricted in the sense of Rothstein (1990), with respect to linear order on i induced by k (i.e., $i \succ j$ iff $k(i) > k(j)$). Rothstein's Theorem therefore implies that there exists at least one individual m such that C is partially rationalized by the recursive preference function U_m. That is, Rothstein's Theorem implies the existence of some $m \in I$ such that $F(U(s,\pi),s) = U_m(s,\pi)$. Notice that F satisfies the requirements of dynamic consistency. Moreover, by either Assumption (A1) or Assumption (A1'),

$$\arg\max_{p,\theta'} U_m(s,\pi)(p,\theta')$$

has a solution, and so C is nonempty valued. We conclude the proof. ∎

Proof of Lemma 1

Observe, first, that by Assumptions (A1)-(A3), for each i, H_i is a smooth and C^∞-uniformly bounded function of (ω, e) (in the relative topology), with uniform bound given by

$$(1-\delta)L + \delta KM \tag{19}$$

Clearly, this bound is independent of x since x is itself uniformly bounded by c. Consequently, H_i is uniform bounded on its entire domain.

Next, we show that for each state ω, H_i is $\bar{\alpha}_i$-concave in e_i where $\bar{\alpha}_i = (1-\delta)\alpha_i - \delta KM > 0$. To show this, we must show that for each ω, $D^2_{e_i} H_i(\omega, \bar{e}\ x) + \bar{\alpha}_i I$ is negative semi definite. To this end, fix ω. Observe that by α-concavity on stage utility functions u_i (Assumption (A2)), and uniform boundedness of the conditional densities, f (Assumption (A3)), we have for every pair of profiles, $\bar{e}$ and e,

$$\begin{aligned}
& e_i^T \cdot D^2_{e_i} H_i(\omega, \bar{e}\ x) \cdot e_i, \\
\leq\ & e_i^T \cdot \left[D^2_{e_i} (1-\delta) u_i(\omega, \bar{e}) + \delta K \int D^2_{e_i} f(\omega' \middle|\, \omega, \bar{e})\ d\eta \right] \cdot e_i \\
\leq\ & -\alpha_i (1-\delta) ||e_i||^2 \ + \ \delta KM ||e_i||^2 \\
=\ & -\bar{\alpha}_i ||e_i||^2
\end{aligned}$$

Since, by the Moderate Social Influence condition (Assumption (A4)), $\bar{\alpha}_i > 0$, it follows that H_i is $\bar{\alpha}$-concave. Consequently, by compactness of E (Assumption (A1)), and by the smoothness and strict concavity of H_i in e_i, the best response

$$g_i(\omega, e_{-i}, x) \ \equiv \ \arg\max_{e_i \in E} H_i(\omega, e, x)$$

for each i is nonempty and single valued.

Consider the best response function, g_i. By the Assumption (A5), $g_i(\omega, e_{-i}, x)$ defines a critical point, i.e.,

$$D_{e_i} H_i(\omega, g_i(\omega, e_{-i}, x), e_{-i}, x) = 0.$$

Then by strict concavity of H_i, the Implicit Function Theorem implies that g_i is a locally smooth function in a neighborhood of (ω, e_{-i}) (in the relative topology). In this neighborhood, the Implicit Function Theorem implies

$$Dg_i \ = \ -[D^2_{e_i} H_i]^{-1} \cdot [D_{\omega, e_{-i}} D_{e_i} H_i]$$

Given the C^∞-uniform bound on H_i given by (19), the $\bar{\alpha}_i$-concavity of H_i implies that there is a uniform bound on Dg_i given by $\frac{1}{\bar{\alpha}_i}[(1-\delta)L + \delta KM]$. Finally, since the choice of (ω, e_{-i}) was arbitrary, every such point is a regular point and so g_i is smooth with uniformly bounded first derivative.

We now show that there is a unique Nash equilibrium, $\bar{\sigma}(\omega, x)$ of the game with payoffs, $H(\omega, \cdot, x)$. Fixing ω and x, consider the best response map

$$e \mapsto (g_1(\omega, e_{-1}, x), \ldots g_n(\omega, e_{-n}, x)\ ).$$

By the arguments above, the conditions for Brouwer's Theorem are met and so this map has a fixed point. Since all best responses are interior — as shown above — the fixed point must be an interior point in E^n. To verify that this fixed point is unique, it suffices to show that the best response difference map

$$e \mapsto e - (g_1(\omega, e_{-1}, x), \ldots g_n(\omega, e_{-n}, x)\ ) \tag{20}$$

has no critical points. It suffices then to show that the Jacobian of this map at differentiable points is nonsingular. In turn, the Jacobian is nonsingular if it has a dominant diagonal. The Jacobian has a dominant diagonal if

$$||D_{e_{-i}} g_i||_1 < 1,\ \forall i, \tag{21}$$

at points (ω, e_{-i}, x) of the best response map, g_i. To verify (21), consider the best response map, g_i. Then we have:

$$\begin{aligned} ||D_{e_{-i}} g_i||_1 &= ||-[D^2_{e_i} H_i]^{-1} \cdot [D_{e_{-i}} D_{e_i} H_i]||_2 \\ &\leq \frac{1}{\bar{\alpha}_i} ||(1-\delta) D_{e_{-i}} D_{e_i} u_i + \delta D_{e_{-i}} D_{e_i} \int V_i(\omega') f(\omega' \Big| \omega, e) d\eta||_2 \\ &\leq \frac{1}{\bar{\alpha}_i}[(1-\delta)L + \delta KM] \\ &< 1 \end{aligned} \tag{22}$$

The first equality is the Implicit Function Theorem,[19] the first *inequality* follows from the definition of $\bar{\alpha}$-concavity, the second follows from the Uniform Bounds Assumption (Assumption (A3)), and the last follows from the MSI condition (Assumption(A4)).

[19] if e_i is one dimensional, then the sup norm picks out one such term,

$$\frac{\partial g_i}{\partial e_j} = (\frac{\partial^2 H_i}{\partial e_i^2})^{-1} \frac{\partial^2 H_i}{\partial e_j \partial e_i},\ \forall j \neq i$$

Next, we show that the profile $\bar{\sigma}$ is smooth with uniformly bounded first derivatives in ω with the bound uniform across all x as well. Observe that the nonsingularity of (20) implies that the unique Nash equilibrium $\bar{\sigma}(\omega, x)$ is implicitly defined by the Implicit Function Theorem (IFT). In turn, the IFT also implies that $D_\omega \bar{\sigma}$ is smooth and defined by

$$D_\omega \bar{\sigma} \ = \ [D_e g]^{-1} [D_\omega g]$$

Note that the inverse $[D_e g]^{-1}$ exists and is uniformly bounded over all ω and all x by the dominance diagonal condition, (22). Consequently, $D_\omega \bar{\sigma}$ exists everywhere and is uniformly bounded over all ω and x.

Finally, we now prove that σ is Lipschitz continuous in x with uniform Lipschitz constant. This follows from a result of Montrucchio (1987, Theorem 3.1). In particular, their result implies that for each i, each ω, and any pair x, x',

$$||g_i(\omega, \cdot, x) - g_i(\omega, \cdot, x')||_0 < \gamma ||x - x'||_0 \tag{23}$$

where γ is the MSI bound in Assumption (A4). Using the difference map in (20) to define the fixed points, the Implicit Function Theorem again implies that (23) applies to the fixed point, $\bar{\sigma}$, as well to the best response map g. ∎

Proof of Lemma 2 By definition,

$$H_i(\omega, \bar{\sigma}(\omega, x), x) \ = \ (1-\delta) u_i(\omega, \bar{\sigma}(\omega, x) \ ) \ + \ \delta \int V_i(\omega') f(\omega' | \, \omega, \bar{\sigma}(\omega, x) \ ) d\eta$$

The smoothness therefore follows from the smooth of H in ω directly and from the smoothness of $\bar{\sigma}(\omega, x)$ in ω established in Lemma 1. The uniform boundedness of first derivatives in ω follows from the C^∞-uniform boundedness of H and the uniform boundedness of first derivatives of $\bar{\sigma}$ established in Lemma 1. ∎

Proof Lemma 3 Let $\{x^\ell\}$ be a sequence such that $x^\ell \in \mathcal{X}$ for all ℓ and $x^\ell \to x \in \mathcal{X}$ with the convergence uniform on each compact set $Y \subset \Omega$ as $\ell \to \infty$. By Lemma 2, we also know that by Lipschitz continuity of $\bar{\sigma}$ in x, $||\bar{\sigma}(\cdot, x^\ell) - \bar{\sigma}(\cdot, x)|| \to 0$ uniformly in ω. Consequently, by the smoothness properties of u_i for each i and of f we can fix $\epsilon > 0$ and let $\bar{\ell}$ satisfy for all $\ell \geq \bar{\ell}$, all ω' and all i, $||u_i(\cdot, \bar{\sigma}(\cdot, x^\ell) \ ) - u_i(\cdot, \bar{\sigma}(\cdot, x) \ )||_0 < \epsilon$, and $|f(\omega' \Big| \omega, \bar{\sigma}(\omega, x) \ ) - f(\omega' \Big| \omega, \bar{\sigma}(\omega, x^\ell) \ )| < \epsilon$.

Now by Lemma 5.2 in Horst (2003), given $\epsilon > 0$, for all i, and for all $\ell \geq \bar{\ell}$,

$$|| \int x_i(\omega') f(\omega' \Big| \cdot, \bar{\sigma}(\cdot, x) \ ) d\eta \ - \ \int x_i^\ell(\omega') f(\omega' \Big| \cdot, \bar{\sigma}(\cdot, x^\ell) \ ) d\eta ||_0 \ < \ \epsilon$$

With these results we see that:

$$
\begin{aligned}
& ||(Tx^\ell)_i(\cdot) - (Tx)_i(\cdot)||_0 \\
= \;& ||H_i(\cdot, \bar\sigma(\cdot, x^\ell), x^\ell) - H_i(\cdot, \bar\sigma(\cdot, x), x)||_0 \\
\le \;& (1-\delta)||u_i(\cdot, \bar\sigma(\cdot, x^\ell)\ ) - u_i(\cdot, \bar\sigma(\cdot, x)\ )||_0 \\
& \quad + \; \delta|| \int x_i(\omega') f(\omega' \Big| \cdot, \bar\sigma(\cdot, x)\ ) d\eta \; - \; \int x_i^\ell(\omega') f(\omega' \Big| \cdot, \bar\sigma(\cdot, x^\ell)\ ) d\eta||_0 \\
< \;& (1-\delta)\epsilon \; + \; \delta\epsilon \; = \; \epsilon
\end{aligned}
$$

Hence T is continuous. ∎

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