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Abstract: We study the existence of undominated elements of acyclic and irreflexive relations. A sufficient condition for the existence is given in the general case without any topological assumptions. Sufficient conditions are also given when the relation in question is defined on a compact Hausdorff space. We study the existence of fixed points of acyclic correspondences, the existence of stable sets, and the possibility of representing the relation by a real valued function.

1. Introduction

Existence of undominated (or *maximal*) members of acyclic relations has interested economists for long. These kind of relations appear e.g. in preference theory (for references, see Alcantud [1]; Bergstrom [2]; Campbell and Walker [3]; Walker [4]). The interpretation of being undominated in such applications means that there exists a choice for which there exist no strictly better choices. Since acyclicity seems quite natural in consumer choice theory for example, it is clear why such relations interest economists.

Acyclic relations have applications also in equilibrium theory and in the analysis of dynamic systems. Equilibrium existence results are always some kind of fixed point theorems. Existence of fixed points and existence of undominated members of a relation are closely related problems, so it is not surprising that results in these areas would be potentially useful to economists.

A condition for the existence of undominated members is given in the general case without any topological assumptions (Proposition 1). Such conditions are also given when the relation in question is defined on a compact Hausdorff space. We show e.g. (Proposition 3) that if a closed, acyclic and irreflexive relation on a compact Hausdorff space has in every uncountable closed subset Z members that are undominated in Z, then this holds actually for every nonempty subset A, not just for uncountable closed subsets. We have a result about the possibility of representing the relation by a real valued function, when each closed subset Z has a member that is undominated in Z (Proposition 4). Existence of fixed points of acyclic correspondences and existence of Nash equilibrium are also analyzed

(Propositions 7– 10). Finally, we have a result about the existence of stable sets (Proposition 11).

The assumptions about relations on topological spaces are formulated in such a way, that they seem fit to game theoretical applications. For example, given an element x we declare the set of elements that are dominated by x closed. In applications in preference theory, it may often be more natural to declare these sets open (see Alcantud [1]). Anyway, it is fruitful to analyze the same problem (existence of undominated members) from different perspectives and with different topological assumptions.

The paper is organized in the following way. In Section 2, we give examples of situations when undominated members do not exist. Examples are simple but reveal something essential about what goes wrong when undominated members do not exist. Notation is introduced in Section 3, and there we give an example of a theorem from the existing literature, to facilitate comparison with our results. The main results are presented in Section 4 and 5. Section 5 is more application oriented, including the fixed point theorems and the result about stable sets.

2. Examples

Let us give next two examples of acyclic relations in which undominated members do not exist.

Example 1. Let X be the boundary of the closed unit ball in the two-dimensional plane with center at the origin of the plane. Define a relation R on X such that xRy, if the distance along X from x to y is 1, when we move from x to y clockwise. Since for all x there is an y such that xRy, no undominated members exist. Further R is acyclic: there are no points $x_1,...,x_n$ such that x_iRx_{i+1}, $i = 1,..., n-1$, and x_1Rx_n. [You can prove this easily by using the fact that X has length 2π.] Clearly, the relation R is also irreflexive and closed. In fact, there is a homeomorphism $f: X \to X$ such that R is the graph of f: $y = f(x)$ iff xRy.

So acyclicity and closedness alone do not guarantee the existence of undominated members, even when the relation is "single valued". But maybe the reason is that X in Example 1 was not convex? Consider the following example.

Example 2. Let $X = [0, 1]$. Let $a \in (1/2, 1)$ be irrational. Let R be a relation on X such that xRy, if $y = 1 - a + x$, when $x \leq a$, and xRy, if $y = x - a$, when $a \leq x$. Graphically, R consists of two disjoint line segments, one above the diagonal and the other below. Then R is closed and irreflexive. Since for all x there is an y such xRy, no undominated members exist. R is also acyclic. To see this, suppose there is a cycle $\{x_0,...,x_n\}$: x_iRx_{i+1}, $i = 0,...,n-1$, and $x_0 = x_n$. We may assume w.l.o.g. that the cycle is minimal in the sense that $x_i \neq x_j$ when $i,j < n$. Clearly $x_0 + ... + x_{n-1} = x_1 + ... + x_n =_{df} S$. Let B be the subset of the indices $i < n$ such that $x_i < a$, M the subset of the indices $i < n$ such that $x_i = a$, and T the subset of the indices $i < n$ such that $a < x_i$. Both B and T must be nonempty, and denote their cardinalities by b and t. M is either empty or contains only one member. Suppose first M is empty. Then $S = \sum_{i\in B}x_i + \sum_{i\in T}x_i = \sum_{i\in B}(x_i + 1 - a) + \sum_{i\in T}(x_i - a) = \sum_{i\in B}x_i + b(1 - a) + \sum_{i\in T}x_i - ta$. Hence $b = a(b + t)$, which is impossible since $t, b > 0$ and a is irrational. If M

contains one element, say j, that is, $x_j = a$, then either $x_{j+1} = 0$ or $x_{j+1} = 1$. We get that $S = \sum_{i\in B}x_i + \sum_{i\in T}x_i + a = \sum_{i\in B}(x_i + 1 - a) + x_{j+1} + \sum_{i\in T}(x_i - a) = \sum_{i\in B}x_i + b(1 - a) + \sum_{i\in T}x_i - ta + x_{j+1}$. Hence $a(b + t) = b + x_{i+1}$, a contradiction since $x_{j+1} = 0$ or $x_{j+1} = 1$.

So convexity of X does not guarantee the existence of undominated members of R, although R is acyclic, irreflexive and closed. In fact, we show in Proposition 1, that if X is countable and Hausdorff compact, then there exist undominated members if R is closed, acyclic and irreflexive. As a countable set, X cannot be convex. It cannot even be connected since it is Hausdorff and countable.

Examples 1 and 2 have one common feature. Starting from any point x in X, we can construct infinite sequences $\{x_n\}$, $x_0 = x$, such that x_nRx_{n+1} but x_iRx_{n+1} does *not* hold for any $i < n$. We call these kind of sequences *irreducible*. The existence of such irreducible sequences from any initial value x in fact precludes the possibility that there are undominated members. We show a partial converse in Proposition 1. Suppose there is some initial value x such that none of the sequences described above is irreducible. Then there are undominated members, if R is acyclic and irreflexive, and if every member is in relation to at most finitely many other members. In this result, no topological assumptions are needed.

3. Preliminaries

Let X be a nonempty set, and R a binary relation on X, so R is a subset of $X \times X$. The set X is called the *field* of R. We may denote $(x, y) \in R$ by xRy as usual. A finite subset $\{x_0,...,x_n\} \subset X$ such that x_iRx_{i+1} for $i = 0,..., n\text{-}1$, is called a *path in R*. If it is clear what relation R is in question, we may simply say that $\{x_0,...,x_n\}$ is a path. R is *acyclic*, if $x_0 \neq x_n$ for every path $\{x_0,...,x_n\} \subset X$ containing at least two different members. R is *irreflexive*, if xRx does not hold for any x. R is transitive, if xRy and yRz imply xRz, for all $x, y, z \in X$. The *transitive closure of a relation* R, denoted by R^{Tr}, is defined by $xR^{Tr}y$ iff there is path $\{x_0,...,x_n\}$ such that $x = x_0$ and $y = x_n$.

Given a relation on X and a nonempty subset Y of X, a member $y \in Y$ is *undominated in* Y, if there is no $y' \in Y$ such that yRy' and $y \neq y'$. If y is undominated in X, we say simply that y *is undominated*. We may also say in this case that R *has* (resp. has not) undominated members.

An infinite subset $\{x_n\} = \{x_0,...,x_n,...\} \subset X$ such that x_iRx_{i+1}, $x_i \neq x_{i+1}$, for $i = 0, 1, ...$, is called a *dominance sequence*. Note that if R is irreflexive, the requirements $x_i \neq x_{i+1}$ are automatically satisfied. If R is irreflexive and acyclic, also $x_i \neq x_n$ holds for all i and $n, i \neq n$, for any dominance sequence $\{x_n\}$.

If R is a relation on a nonempty set X, and Y is a nonempty subset of X, define *the restriction of* R *to* Y by $R_{|Y} = R \cap Y \times Y$. The field of the relation $R_{|Y}$ is Y. Then $R_{|Y}$ has no undominated members, if and only if R has no undominated members in Y.

Given a nonempty $Y \subset X$, let $RY = \{x \in X \mid xRy \text{ for some } y \in Y\}$, and $YR = \{x \in X \mid yRx \text{ for some } y \in Y\}$. RY is called the *inverse image of* Y, and

YR is called *the image of* Y. RX is the *domain* of R, and XR is the *range* of R. If R is irreflexive, it has undominated members if and only if the domain is not the whole X, and in this case $X \setminus RX$ is the set of undominated members. So undominated members of an irreflexive R exist precisely when the domain of R is a proper subset of the field of R.

If X is a topological space, R is *closed* if R is a closed subset of the product space $X \times X$ which is equipped with the product topology. The following is part of Theorem 4 by Alcantud (see Alcantud [1] for the whole theorem, and related earlier results).

Theorem. *Let R be an irreflexive and acyclic relation on X. There are undominated members, iff X has topology such that X is compact and $R\{x\}$ is open for each $x \in X$.*

It is well-known that if X is a compact Hausdorff space and R is closed, then YR and RY are closed for any nonempty closed subset Y of X. In particular, this holds for singletons $\{x\}$. Further, the *correspondence* $x \to \{x\}R$ defined on the domain of R is *upper semicontinuous* in this case. That is, the subset $\{x \mid \{x\}R \subset O\}$ is open for any open $O \subset X$. Since the function $f: X \times X \to X \times X$, $f((x, y)) = (y, x)$ is a homeomorphism, the *inverse of* R, $R^{-1} = \{(y, x) \mid (x, y) \in R\}$ is a closed if R is closed Then the correspondence $y \to \{y\}R^{-1} = R\{y\}$ is also upper semicontinuous.

4. Existence of Undominated Members

In this section a relation R on X will always be acyclic and irreflexive, unless otherwise explicitly stated.

We will first give conditions for the existence of undominated members without topological assumptions.

Given a relation R on a set X, a dominance sequence $\{x_n\}$ is called *reducible*, if there is $m > 0$ such that x_iRx_{m+1} holds for some $i < m$. If a dominance sequence is not reducible, then it is *irreducible.* Since R is acyclic, irreducibility means that x_kRx_n if and only if $n = k + 1$.

Proposition 1. *Let X be a nonempty set and R a nonempty relation on X such that $\{x\}R$ is finite for every $x \in X$. There exists undominated members in X, iff there exists $x_0 \in X$ such that every dominance sequence starting from x_0 is reducible.*

Proof. Necessity. Suppose x_0 is undominated. Then the set of dominance sequences starting from x_0 is empty, and therefore every dominance sequence starting from x_0 is reducible.

Sufficiency. Suppose no undominated members exists, and let x_0 be such that every dominance sequence starting from x_0 is reducible. Let T be the tree with root x_0 whose nodes are all paths $\{x_0,\dots,x_n\}$ such that x_iRx_k if and only if $k = i + 1$, where $i = 0,\dots, n - 1$. If $\{x_n\}$ is any dominance sequence starting from x_0, then by reducibility there exists a least index $k > 0$ such that x_iRx_{k+1} for some $i < k$. Hence all initial segments $\{x_0,\dots,x_m\}$ of $\{x_n\}$ with at most $k + 1$ members are nodes of T. So if there exists a node with m members, then there exist nodes with $k = 1,..,m - 1$ members.

We claim first that there exists a natural number M such that all nodes of T have at most M members. Suppose to the contrary that that for every M, there exists a node with more than M members. Then for every $M > 1$, there exists a node with exactly M members. Now T is a finite splitting tree: every node has at most a finite number of successor nodes. This follows because $\{x\}R$ is finite for every $x \in X$. Then it follows by the König's Lemma, that T has an infinite branch. That means that there is a dominance sequence $\{x_n\}$ starting from x_0 such that every initial segment $\{x_0,\ldots,x_m\}$ is a node of T. But this means $\{x_n\}$ is irreducible, a contradiction.

So every node of T has at most M members. The root x_0 has finitely many successors because $\{x_0\}R$ is finite. The subtree $T(x)$ of T whose root is $x \in \{x_0\}R$, is such that all its nodes have at most $M - 1$ members. Then it follows by induction that T has only a finitely many nodes. Since all nodes have only finitely many members, it follows that the subset $Y = \{x \in X \mid x$ *is a member of some node of* $T\}$ is finite. We show next that if $\{x_0,\ldots,x_m\}$ is *any* path starting from x_0, then every member of this path is a member of some node of T as well.

So let $\{x_0,\ldots,x_m\}$ be a path starting from x_0. Let $\{x_0,\ldots,x_k\}$ be the greatest initial segment of this path that is a node of T. That is, x_iRx_n if and only if $n = i + 1$, where $i = 0,..,k - 1$. Such a greatest initial segment clearly exists. Suppose $k < m$, so in particular, the path $\{x_0,\ldots,x_k, x_{k+1}\}$ is not a node of T. Let i be the least index such that x_iRx_{k+1}. Then the member x_{k+1} belongs to the node $\{x_0,\ldots,x_i, x_{k+1}\}$. If the path $\{x_0,\ldots,x_i, x_{k+1}, x_{k+2}\}$ is not a node of T, then this happens only because there exists $j \le i$ such that x_jRx_{k+2}. So let j be the least such index, and note that then x_{k+2} belongs to the node $\{x_0,...,x_j, x_{k+2}\}$. Continuing in this fashion we get that every member x of the path $\{x_0,...,x_m\}$ belongs to some node of T. Since $\{x_0,...,x_m\}$ was chosen arbitrarily, it follows that the subset $Y = \{x \in X \mid x$ *is a member of some node of* $T\}$ is the set of *all* members of X that belong to some path $\{x_0,...,x_m\}$ starting from x_0.

If $y \in Y$ is dominated, then there exists $y' \in Y$ such that yRy'. To see this, note that if yRx, then there exists a path starting from x_0 that contains y and ends to x. Since Y is finite and R is acyclic, not all members of Y can be dominated, a contradiction with the initial assumption that there are no undominated members. Q.E.D.

Now we turn to the case where X is a topological space.

Lemma 1. *If R is a nonempty and closed relation on a compact Hausdorff space X with no undominated members, then there is a minimal nonempty closed $Y \subset X$ such that every $y \in Y$ is dominated in Y.*

Proof. Partially order by set inclusion the set $\boldsymbol{C}$ of all nonempty closed subsets Z of X, such that all members of Z are dominated in Z. $\boldsymbol{C}$ is nonempty, since by assumption $X \in \boldsymbol{C}$. Let $\boldsymbol{T}$ be a maximal totally ordered subset of $\boldsymbol{C}$. Then $\boldsymbol{T}$ exists by the Hausdorff Maximality Principle. Let Y be the intersection of the members of $\boldsymbol{T}$. Then Y is nonempty and closed, since every $Z \in \boldsymbol{T}$ is nonempty and closed and X is a compact Hausdorff space. Since every $z \in Z$ is dominated in Z, we have $Z \subset RZ$ for all $Z \in \boldsymbol{T}$. Choose $y \in Y$. Since y is dominated in Z, $\{y\}R \cap Z$ is nonempty and closed for any

$Z \in \boldsymbol{T}$. Therefore $\{y\}R \cap Y$ is nonempty and closed, and y is dominated in Y. Since y was chosen arbitrarily, we are done. Q.E.D.

Recall that a nonempty closed subset Y of a metric space is called *perfect*, if Y contains no points that are isolated in Y, *i.e.* there is no $y \in Y$ such that for some open neighbourhood $V(y)$ of y, $Y \cap V(y) = \{y\}$. Perfect subsets are uncountable.

Lemma 2. *Let R be a nonempty and closed relation on a compact Hausdorff space X with no undominated members, and let $Y \subset X$ be as in Lemma 1. Then Y is homeomorphic to a compact perfect metric space, and $Y = R_{|Y}Y = YR_{|Y}$.*

Proof. By Lemma 1, $R_{|Y}$ has no undominated members, and therefore $Y = R_{|Y}Y$. Since $R_{|Y}$ is viewed as a relation on Y, we have $YR_{|Y} \subset Y$. Let $Z = YR_{|Y}$, and note that Z consists of all those members of Y that dominate some member of Y. Since every $z \in Z$ is dominated by some $y \in Y$, we must have $y \in Z$. Since Y is minimal and Z is closed, we have $Z = YR_{|Y} = Y$ by Lemma 1.

Let $\{x_n\} \subset Y$ be any dominance sequence, *i.e.* an infinite sequence such that x_iRx_{i+1}, $i = 0, 1, \ldots$. Such a sequence exists, since Y contains no undominated members. Since Y is closed, $cl\{x_n\} \subset Y$, where clZ means the closure of a set Z. Now $\{x_n\} = cl\{x_n\}$ is impossible. To see this, suppose $\{x_n\} = cl\{x_n\}$, that is, $\{x_n\}$ is closed. Then $\{x_n\}_{n\geq 1} = \{x_n\} \cap \{x_n\}R$ would be closed as well, since AR is closed for any nonempty closed $A \subset Y$. By induction, for all natural numbers k, the dominance sequence $\{x_n\}_{n\geq k}$ starting from x_k would be closed. Since $\{x_n\}_{n\geq m} \subset \{x_n\}_{n\geq k}$ when $m > k$, the intersection $\cap_k\{x_n\}_{n\geq k}$ must be nonempty, because Y is a compact Hausdorff space. Then for some m, $x_m \in \{x_n\}_{n\geq k}$ for all k, a contradiction with acyclicity. Therefore $\{x_n\}$ is a proper subset of $cl\{x_n\}$.

The set $cl\{x_n\}$ is a compact Hausdorff space having a countable dense subset $\{x_n\}$. Urysohn's metrizability theorem says that the topology of any normal topological space with a countable dense subset is metrizable. Compact Hausdorff spaces are normal (every pair of disjoint closed subsets have disjoint open neighbourhoods), and hence we may view $cl\{x_n\}$ as a compact metric space.

Take any $y \in cl\{x_n\} \setminus \{x_n\}$. There must be a subsequence $\{x_{n(k)}\}$ converging to y, since we may view $cl\{x_n\}$ as a compact metric space. For the same reason the sequence $\{x_{n(k)+1}\}$ has a subsequence converging to $z \in cl\{x_n\}$. Assume w.l.o.g. that $\{x_{n(k)+1}\}$ converges. Since R is a closed relation, and $x_{n(k)}Rx_{n(k)+1}$, it follows that yRz. Hence every member of $cl\{x_n\}$ is dominated by some member of $cl\{x_n\}$. Since $cl\{x_n\}$ is a subset of Y, we must have $Y = cl\{x_n\}$ by minimality of Y. Recall that $\{x_n\} \subset Y$ was an arbitrarily chosen dominance sequence.

If $y \in Y$ were isolated, we could take a dominance sequence $\{x_n\} \subset Y$ such that yRx_0. Then by acyclicity, y couldn't be a member of $\{x_n\}$. Hence y couldn't be a member of $cl\{x_n\}$, since y is isolated. But then $y \notin Y$ since $Y = cl\{x_n\}$, a contradiction. Hence Y is perfect. Q.E.D.

Proposition 2. *Let X be a countable, compact Hausdorff space and R a nonempty and closed relation on X. Then every nonempty closed $Z \subset X$ contains members that are undominated in Z.*

Proof. If there were no undominated members in X, then by Lemma 2 there would exist a nonempty subset $Y \subset X$ such that Y is homeomorphic to a a compact perfect metric space. But the cardinality of every nonempty perfect set is that of the continuum, a contradiction.

If $Z \subset X$ is nonempty and closed, then $R_{|Z}$ is a nonempty and closed relation on a countable, compact Hausdorff space Z, and the result follows from the first part of this proof. Q.E.D.

Next we drop the assumption that X is countable, and give some characterizations for the case that each nonempty closed subset Z of X has members that are undominated in Z.

Proposition 3. *Suppose R is a nonempty and closed relation on a compact Hausdorff space X. Each uncountable closed $Z \subset X$ contains members that are undominated in Z, iff each nonempty $A \subset X$ contains members that are undominated in A.*

Proof. Sufficiency. If each nonempty $A \subset X$ contains members that are undominated in A, then this holds for closed nonempty subsets as well.

Necessity. The mapping $Z \to RZ$ (take Z and see what members of X the members of Z dominate) is monotone on nonempty subsets of X. Namely, if $\varnothing \neq Z \subset Z'$, and xRz for $z \in Z$, then $x \in RZ$. But since $z \in Z'$, also $x \in RZ'$, and therefore $RZ \subset RZ'$.

Let $X(0) = X$, and define $X(n+1) = RX(n)$, $n \geq 0$. Then $X(n+1) \subset X(n)$ by monotonicity. Since each $X(n)$ is a closed subset of a compact space X, the intersection $Z = \cap_n X(n)$ is closed as well, and it is nonempty iff each $X(n)$ is nonempty.

Suppose indeed that Z is nonempty. If Z is uncountable, there is by assumption $y \in Z$ such that yRy' does not hold for any $y' \in Z$. If Z is countable, then also Z has a member y that is undominated in Z: $R_{|Z}$ is a closed (irreflexive and acyclic) relation on Z, and therefore has undominated members by Proposition 2.

Since $y \in X(1)$, y cannot be undominated in X. Hence $\{y\}R$, the set of members that dominate y, is nonempty and closed by closedness of R. Since $y \in X(n + 1)$ for every n, it follows that $\{y\}R \cap X(n)$ is nonempty and closed for every n. But then $\{y\}R \cap Z = \cap_n (\{y\}R \cap X(n))$ is a nonempty and closed subset of Z, a contradiction.

It follows that there exists a natural number n^* such that $X(n^* + 1)$ is empty but $X(n)$ is nonempty for all $n < n^* + 1$. That is, $RX(n^*)$ is empty although $X(n^*)$ is nonempty and closed, so members of $X(n^*)$ do not dominate anything. In particular, no member of $X(n^*)$ dominates another member of $X(n^*)$, and hence each x in $X(n^*)$ is undominated in $X(n^*)$.

Let $S(n) = X(n) \setminus X(n + 1)$, $n < n^*$, and $S(n^*) = X(n^*)$. Then $S(n)$ contains all members of $X(n)$ that are undominated in $X(n)$. Subsets $S(n)$ and $S(m)$ are disjoint when $n \neq m$. The union of all $S(n)$'s is X, so $\{S(n)\}$ is a partition of X.

Let $A \subset X$ be nonempty. Let n be the least index such that $A \cap S(n)$ is nonempty. Members of this nonempty intersection are undominated in A. Q.E.D.

We can give an analogous characterization in terms of functions $u : X \to \mathbf{R}$ ($\mathbf{R}$ is the set of real numbers) that preserve R "in one direction": xRy implies $u(x) < u(y)$.

Proposition 4. *Suppose R is a nonempty and closed relation on a compact Hausdorff space X. Each uncountable closed $Z \subset X$ contains members that are undominated in Z, iff there is a lower semicontinuous function $u : X \to \mathbf{R}$ with finite range such that xRz implies $u(x) < u(z)$ for all $x, z \in X$.*

Proof. Necessity. Suppose each uncountable closed $Z \subset X$ contains members that are undominated in Z. Then this holds for every nonempty closed Z by Proposition 2. Let $X(0) = X$, and define $X(n + 1) = RX(n), n \geq 0$. Then $X(n + 1) \subset X(n)$, each $X(n)$ is closed, and $X(n)$ is nonempty iff $n \leq n^*$ for some n^* (see the proof of Prop. 3).

Let $S(n) = X(n) \setminus X(n + 1)$, $n < n^*$, and $S(n^*) = X(n^*)$. Then $S(n)$ contains all members of $X(n)$ that are undominated in $X(n)$. Note that since each $X(n + 1)$ is closed, $n < n^*$, the union $S(0) \cup ... \cup S(n) = X \setminus X(n + 1)$ is open, and that subsets $S(n)$ and $S(m)$ are disjoint when $n \neq m$. The union of all $S(n)$'s is X, so $\{S(n)\}$ is a partition of X.

Define a function u on X by $u(x) = n^* - n$, where n is the unique number such that $x \in S(n)$. Now u represents R in the sense that xRz implies $u(x) < u(z)$. Namely, $u(x) = k$ iff $x \in S(n^* - k)$. Since x is undominated in $X(n^* - k)$, $z \in S(n)$ implies $n < n^* - k$. Hence $u(z) = n^* - n > k = u(x)$. Fix $a \in \mathbf{R}$, and note that $\{x \in X \mid a < u(x)\}$ is open, and so u is lower semicontinuous.

Sufficiency. Suppose there is a lower semicontinuous function $u : X \to \mathbf{R}$ with finite range such that xRz implies $u(x) < u(z)$ for all $x, z \in X$. We may assume w.l.o.g. that $u[X] = \{0,...,n^*\}$, the set of first $n^* + 1$ natural numbers. Let $T(n) = \{x \in X \mid n - 1 < u(x) \leq n\}$, and note that unions $T(k) \cup T(k + 1) \cup ... \cup T(n^*)$ are open, $k \leq n^*$, and that $\{T(n) \mid 0 \leq n \leq n^*\}$ is a partition of X.

Let Z be a nonempty closed subset of X, and let k be the largest number such that $Z \cap T(k)$ is nonempty. For each $x \in Z \cap T(k)$, $u(x) = k$, and if xRy then $u(x) < u(y)$. But then $y \notin Z$. So members of $Z \cap T(k)$ are undominated in Z. Q.E.D.

We will now formulate some sufficient conditions for the existence of undominated elements. In the first, we use the reducibility concept.

Let R a relation on X. We say that dominance sequences starting from $x \in X$ are *uniformly reducible*, if there is $M > 0$ such that to each dominance sequence $\{x_n\}$, $x_0 = x$, there is $n \leq M$ such that $x_i R x_{n+1}$ for some $i < n$.

Proposition 5. *Suppose R is a nonempty and closed relation on a compact Hausdorff space X. If every uncountable closed subset $Z \subset X$ contains a member z such that dominance sequences starting from z are uniformly reducible, then R has undominated members.*

Proof. Suppose R has no undominated members. Then there is a minimal closed subset Y by Lemma 1 such that every $y \in Y$ is dominated in Y. By Lemma 2, Y is homeomorphic to a compact perfect metric space, and so Y is uncountable.

Define $\{z\}R^n$ by $(\{z\}R^{n-1})R$, when $n > 0$ and $\{z\}R^0 = \{z\}$, $z \in X$. Then each $\{z\}R^n$ is closed, and so are the finite unions of these sets. By assumption, there is $y_0 \in Y$ such that dominance sequences starting from y_0 are uniformly reducible.

Claim. $\cup_n \{y_0\}R^n$ *is closed.*

Proof. Since the dominance sequences starting from y_0 are uniformly reducible, there is $M > 0$ such that for every dominance sequence $\{y_n\}$ that starts from y_0, there is $n \leq M$ such that $y_i R x_{n+1}$ and $i < n$.

Let $\{x_t\} \subset \cup_n \{y_0\}R^n$ be a sequence converging to x, not necessarily a dominance sequence. Since $x_t \in \cup_n \{y_0\}R^n$, there exists a shortest path $\{y_0,\ldots, y_k, x_t\}$. Since X contains no undominated members, this path extends to a dominance sequence $\{y_n\}$ starting from y_0 such that $y_{k+1} = x_t$. Since $\{y_0,\ldots, y_k, x_t\}$ is a shortest path from y_0 to $x_t = y_{k+1}$, we have that $y_i R y_j$ if and only if $j = i + 1$, when $j \leq k + 1$. By uniform reducibility, $k + 1 \leq M$.

Therefore $\{x_t\} \subset \{y_0\}R \cup \ldots \cup \{y_0\}R^M$, which is a closed set, and therefore the limit x of $\{x_t\}$ is in this set also, and so $x \in \cup_n \{y_0\}R^n$. *End.*

Now y_0 is such that y_0 does *not* dominate any $y \in \cup_n \{y_0\}R^n$. By Lemma 2 there is $y' \in Y$ such that y_0 dominates y', so $\cup_n \{y_0\}R^n \cap Y$ is a proper closed subset of Y. By Lemma 1, there is $z \in \cup_n \{y_0\}R^n \cap Y$ such that z is undominated in $\cup_n \{y_0\}R^n \cap Y$. Since $z \in Y$, zRy holds for some $y \in Y$, and then necessarily $y \notin \cup_n \{y_0\}R^n$. But $y \in \{z\}R \subset \cup_n \{y_0\}R^n$ by construction, a contradiction. Q.E.D.

If X is a topological space, we say that $A \subset X$ is *sequentially closed*, if for any sequence $\{a_n\} \subset A$ converging to a, also $a \in A$. Every closed subset is sequentially closed. In metric spaces, sequentially closed sets are closed.

Recall the definition of the transitive closure R^{Tr} of a relation R on X (see Section 3). Since R is acyclic and irreflexive, $\{x\}R^{Tr} = \cup_n \{x\}R^n \setminus \{x\}$ for all $x \in X$, where $\{x\}R^0 = \{x\}$ and $\{x\}R^{n+1} = (\{x\}R^n)R$ for $n \geq 0$.

Proposition 6. *Suppose R is a nonempty and closed relation on a compact Hausdorff space X. If every uncountable closed subset Z of X contains some z such that $\{z\}R^{Tr}$ is sequentially closed, then R has undominated members.*

Proof. Suppose there are no undominated members. Then there is a minimal closed set Y by Lemma 1 such that every $y \in Y$ is dominated in Y. By Lemma 2, Y is homeomorphic to a compact perfect metric space, and so Y is uncountable.

By assumption, $\{y\}R^{Tr}$ is sequentially closed for some $y \in Y$. Since Y can be viewed as a metric space, $Y \cap \{y\}R^{Tr}$ is closed. Since every member of Y is dominated in Y, $Y \cap \{y\}R^{Tr}$ is nonempty. Since y is not a member of $\{y\}R^{Tr}$, $Y \cap \{y\}R^{Tr}$ is a proper closed subset of Y. By Lemma 1, there is $z \in Y \cap \{y\}R^{Tr}$ such that z is undominated in $Y \cap \{y\}R^{Tr}$. Since $z \in Y$, zRx holds for some $x \in Y$, and then necessarily $x \notin Y \cap \{y\}R^{Tr}$. But $x \in \{z\}R \subset \{y\}R^{Tr}$ by construction, a contradiction. Q.E.D.

Corollary 1. *Suppose R is a nonempty and closed relation on a compact Hausdorff space X. R satisfies the conditions of Prop. 5 or Prop. 6, iff for each uncountable closed $Z \subset X$, there is $z \in Z$ which is undominated in Z.*

Proof. Necessity. If Z is countable, then by Prop. 1 Z contains a member z such that z is undominated in Z. If Z is uncountable, then apply Prop. 5 and 6 to the relation $R_{|Z}$ on Z.

Sufficiency. By Prop. 4, there is a function $u : X \rightarrow \mathbf{R}$ such that xRz implies $u(x) < u(z)$ for all x, z. If $\{x_n\} \subset X$ is a dominance sequence, then the image $u[\{x_n\}]$ is infinite. But $u[X]$ is finite by Prop. 4, so there are no dominance sequences. So the sufficient conditions of Prop. 5 and 6 are automatically satisfied. Q.E.D.

5. Fixed Points Theorems and Applications

Let $F: X \Rightarrow X$ be a correspondence: for each $x \in X$, $F(x) \subset X$. A correspondence F is nonempty valued (finite valued), if $F(x)$ is nonempty (finite) for all x. A correspondence F is acyclic (irreflexive), if its graph $grF = \{(x, y) \mid y \in F(x), x \in X\}$ is acyclic (irreflexive). We call a correspondence *reducible*, if for some x_0, all dominance sequences in grF starting from x_0 are reducible. In other words, if there exists an infinite sequence $\{x_n\}$ with first member x_0 such that $x_{i+1} \in F(x_i)$, $i = 0, 1, \ldots$, then for some $n > 0$, there exists $i < n$ such that $x_{n+1} \in F(x_i)$. If X is a topological space, then a correspondence F is closed, if its graph grF is closed in the product topology of $X \times X$.

We say that $x \in X$ is a fixed point of $F: X \Rightarrow X$, if $x \in F(x)$.

Proposition 7. *If X is a nonempty set, and $F : X \Rightarrow X$ is a nonempty and finite valued, acyclic and reducible correspondence, then F has a fixed point.*

Proof. If F has no fixed points, then F is irreflexive. By Proposition 1, the relation grF has then undominated members. That is, for some $x \in X$, there is no $y \in X$ such that $(x, y) \in grF$. Since F is nonempty valued, this is impossible. Q.E.D.

Proposition 8. *Suppose X is countable, compact Hausdorff space. If the correspondence $F : X \Rightarrow X$ is closed, nonempty valued and acyclic correspondence, then F has a fixed point.*

Proof. Apply Proposition 2 and the proof of Proposition 7. Q.E.D.

Given a correspondence $F: X \Rightarrow X$, let $F^0(x) = \{x\}$, and $F^n(x) = F[F^{n-1}(x)]$ for $n > 0$, for every $x \in X$.

Proposition 9. *Suppose X is a compact Hausdorff space, and the correspondence $F : X \Rightarrow X$ is closed, nonempty valued and acyclic correspondence. If every uncountable closed subset Z of X contains a member z such that $\cup_{n>0} F^n(z)$ is sequentially closed, then F has a fixed point.*

Proof. Note that $z \notin \cup_{n>0} F^n(z)$ since F is acyclic. Apply Proposition 6 and the proof of Proposition 7. Q.E.D.

An n *–person normal form game is* $G = \{S_1,...,S_n; u_1,...,u_n\}$, where S_i is a nonempty set of strategies of player $i = 1,...,n$, and u_i is a real valued utility function of player $i = 1,...,n$ defined on $S = S_1 \times ... \times S_n$, the set of strategy profiles. Given a strategy profile s and player i, denote $s = (s_i, s_{-i})$, where s_{-i} denotes the strategies chosen by i's opponents, and s_i denotes the strategy chosen by i. A strategy profile s^* is a *Nash equilibrium*, if the inequality

$$u_i(s^*) \geq u_i((s_i, s_{-i}^*))$$

holds for all i and all $s_i \in S_i$. Let $B_i(s) = \{x_i \mid u_i((x_i, s_{-i})) \geq u_i((y_i, s_{-i}))$, *for all* $y_i \in S_i\}$ denote the set of *best replies* of player i against a strategy profile $s \in S$. For all strategy profiles s, let $B(s) = B_1(s) \times ... \times B_n(s)$, and let $B : S \Rightarrow S$ denote the correspondence that to each s assigns the set $B(s)$. Then s^* is a Nash equilibrium, if and only if $s^* \in B(s^*)$. We say that B is the best reply correspondence of the game G.

Proposition 10. *Suppose* $G = \{S_1,...,S_n; u_1,...,u_n\}$ *is an* n*–person normal form game having a best reply correspondence* $B : S \Rightarrow S$. *If* B *is a nonempty and finite valued, acyclic and reducible correspondence, then* G *has a Nash equilibrium.*

Proof. By Proposition 7. Q.E.D.

A Nash equilibrium existence result based on Propositions 8 and 9 can be formulated analogously.

In game theoretic applications, dominance sequences are usually called best reply sequences. Acyclicity of a best reply correspondence B means that there cannot exist cycles in any best reply sequence (or path). Reducibility of B means that there exists a strategy profile s such that if $\{s^k\}$ is a best reply sequence starting from s, then for some k, s^{k+1} is a best reply not just against

s^k but also against some other s^i, $i < k$. Finite valuedness of B means simply that each player has only a finitely many best replies against every strategy profile.

As a last application, we show that stable sets exist, if a relation has undominated members in every nonempty closed subset.

Given a relation R on X, a nonempty $S \subset X$ is *stable*, if (i) $x, y \in S$ implies that neither xRy nor yRx holds; (ii) $x \in X \setminus S$ implies that xRy for some $y \in S$.

Proposition 11. *Suppose R is a nonempty, closed, irreflexive and acyclic relation on a compact Hausdorff space X, and assume that each nonempty closed Y ⊂ X contains a member y such that y is undominated in Y. Then a stable set exists.*

Proof. Let the subsets $S(n), X(n), n = 0,...., n^*$, be defined as in the proof of Prop. 3. So $X(0) = X$, and $X(n + 1) = RX(n) \subset X(n)$, and $S(n) = X(n) \setminus X(n + 1)$. It was shown in the proof of Prop. 3, that every member of $X(n^*)$ is undominated in $X(n^*)$ and $X(n^* + 1)$ is empty.

If $n^* = 0$, then $S(0) = X$ is a stable set. If $n^* > 0$, then $S(1)$ is nonempty. Since members of $S(1)$ are undominated in $X(1) = X \setminus S(0)$, we have that $S(1) \subset RS(0)$. Let k be the least index such that $S(k)$ is not a subset of $S(0) \cup RS(0)$. Let $T(1) = S(0) \cup (S(k) \setminus RS(0))$. Then $T(1)$ and $RT(1)$ are disjoint. If their union is X, then $T(1)$ is a stable set.

Suppose that $T(k)$ is defined such that $T(k)$ and $RT(k)$ are disjoint. If their union is X then $T(k)$ a stable set. If their union is not X, then define $T(k + 1)$ by $T(k + 1) = T(k) \cup (S(m) \setminus RT(k))$, where m is the least index such that $S(m)$ is not a subset of $T(k) \cup RT(k)$. Since there are only $n^* + 1$ subsets $S(n)$, there must exist k such that $T(k)$ is a stable set. Q.E.D.

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