

On the Existence of Pure Strategy Nash Equilibria in Large Games*

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Abstract

We consider an asymptotic version of Mas-Colell's theorem on the existence of pure strategy Nash equilibria in large games. Our result states that, if players' payoff functions are selected from an equicontinuous family, then all sufficiently large games have an ε -pure, ε -equilibrium for all $\varepsilon > 0$. We also show that our result is equivalent to Mas-Colell's existence theorem, implying that it can properly be considered as its asymptotic version.

1 Introduction

Nash (1950)'s celebrated existence theorem asserts that every finite normal form game has a mixed strategy equilibrium. However, in many contexts mixed strategies are unappealing and hard to interpret, leading naturally to the question of the existence of pure strategy Nash equilibria.

Regarding games with a finite action space, considered by Nash (1950), Schmeidler (1973) was successful in obtaining an answer to the above question. He showed that in a special class of games — in which each player's payoff depends only his choice and on the average choice of the others — a pure strategy Nash equilibrium exists in every such game with a continuum of players.

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Later, Mas-Colell (1984) extended Schmeidler's Theorem to a broader class of games by allowing players' payoff to depend on the distribution of choices made by the others.¹ At almost the same time, Rashid (1983) obtained an asymptotic version of Schmeidler's theorem, according to which all sufficiently large games of the same class considered by Schmeidler (1973) have an approximately pure, approximate equilibrium provided that players' payoff functions are selected from an equicontinuous family.²

Asymptotic results are important because, after all, real-world games have finitely many players. Therefore, we ask whether or not there is an asymptotic version for Mas-Colell's Theorem that parallels Rashid's version of Schmeidler's Theorem.

Our Theorem 1 answers the above question affirmatively. It states that if players' payoff functions are selected from an equicontinuous family, then all sufficiently large games of the same class considered by Mas-Colell (1984) have an ε - pure, ε - equilibrium for all $\varepsilon > 0$.

This result thus provides an approximate version of Mas-Colell's theorem. Despite the similarities, there are nevertheless some differences: our assumption on the number of players is less extreme, but our conclusion is weaker and requires an equicontinuity assumption. So, is our result really a version of Mas-Colell's?

Again, the answer is yes: our Theorem 3 shows that the two results are equivalent, i.e., that each one follows from the other. This means that they are simply two different ways of expressing the same phenomenon: the existence of pure strategy Nash equilibria can be addressed either in its exact version in games with a continuum of players or in an approximate version in large, equicontinuous games.

This remark reinforces the main conclusion we obtained in Carmona (2004a). There we showed that in the same class of games we consider here, Nash equilibria of games with a continuum of players are the limit points of approximate equilibria of similar large finite games. Given this, the

¹More precisely, Mas-Colell's Theorem asserts that an equilibrium distribution exists whenever the action space is a compact metric space. In the special case considered by Schmeidler (1973), an equilibrium strategy can be obtained from the equilibrium distribution.

²More precisely, Rashid's Theorem shows that all Nash equilibria of sufficiently large games of the same class can be approximately purified. Note that, as we have shown in Carmona (2004b), Rashid's Theorem requires that players' payoff functions be selected from an equicontinuous family.

equivalence asserted by our Theorem 3 becomes quite natural.

It is useful to compare our asymptotic result to others obtained in the literature. The equivalence between it and Mas-Colell's implies that it is more general than Schmeidler's Theorem. It is also more general than Rashid's result on the existence of approximately pure, approximate equilibria in large, equicontinuous games. In both cases, this follows simply because we consider a more general framework.

The existence of approximately pure, approximate equilibria in large, equicontinuous games follows also from the work of Kalai (2001) and Cartwright and Wooders (2003). Although the class of games considered by them differs from ours, we can nevertheless approximate equicontinuous games in their class by equicontinuous games in ours (this approximation requires that we focus on a sub-class in which all players have the same finite action space and there is no move by nature, assumptions that are not made by those authors). This implies that the existence of approximately pure, approximate equilibria in large, equicontinuous games in their framework satisfying the above assumptions follows also from our Theorem 1.

We consider several possible extensions for our work. Motivated by the above relationship with the papers of Rashid (1983), Kalai (2001) and Cartwright and Wooders (2003) we ask whether or not we can approximately purify all Nash equilibria of large, equicontinuous games, as they can. Our Theorem 4 shows that, in general, we cannot. This implies that our Theorem 1 is not more general than their main results, but only to the corollaries on the existence of approximately pure, approximate equilibria in large, equicontinuous games that can be obtained from them.³

Finally, we provide examples that show that neither can we obtain an exact version for our Theorem 1 (i.e., with $\varepsilon = 0$), nor can we dispense with the equicontinuity assumption.

2 Notation and Definitions

In the class of normal-form games we consider, all players have the same strategy space: the standard unit simplex Δ_m on \mathbb{R}^m , $m > 1$. As usual,

³It is worth emphasizing that both Kalai (2001) and Cartwright and Wooders (2003) allow moves by nature and different players to have different action spaces. Furthermore, Cartwright and Wooders (2003) allow players to have denumerable action spaces. These are aspects in which their analysis is more general than ours.

the interpretation is that there are m pure strategies which are identified with the vertices $E = \{e_1, \dots, e_m\}$ of Δ_m , and players can randomize over it. Since the focus is on a property that depends on the number of players, we will index any game by the number of its players. Thus, G_n is a normal-form game in which the set of players is $T_n = \{1, \dots, n\}$, and each has Δ_m as its choice set. A strategy is then a function $x : \{1, \dots, n\} \rightarrow \Delta_m$. Obviously, a strategy x can also be thought of as the vector (x_1, \dots, x_n) in Δ_m^n .

A game G_n is then specified by the vector of payoff functions, one for each player. We denote player t 's payoff function by U_t , for any $t \in T_n$; therefore, G_n is described by the vector $(U_t)_{t \in T_n}$. If we let \mathcal{G}_n denote the set of all functions $g : \Delta_m^n \rightarrow \mathbb{R}$ and $\mathcal{G} = \cup_{n=1}^{\infty} \mathcal{G}_n$, then a game G_n can be thought of as a subset of \mathcal{G}_n , and so of \mathcal{G} .

In this paper we will focus on a special class of games in which each player's payoff depends on his strategy and on the distribution of strategies chosen by the other players. Let \mathcal{M} be the set of Borel probability measures on Δ_m endowed with the Prohorov metric ρ .⁴ Given a strategy x in a game with n players and $t \in T_n$, we define $\nu_{n,t} \circ x^{-1} \in \mathcal{M}$ as follows:

$$\nu_{n,t} \circ x^{-1}(B) = \frac{|\{j \in T_n \setminus \{t\} : x_j \in B\}|}{n-1} \quad (1)$$

for any Borel set $B \subseteq \Delta_m$. To each player t , we associate m continuous functions $V_{tk} : \mathcal{M} \rightarrow \mathbb{R}$, $1 \leq k \leq m$; then player t 's payoff function is

$$U_t(x) = x_{t1}V_{t1}(\nu_{n,t} \circ x^{-1}) + \dots + x_{tm}V_{tm}(\nu_{n,t} \circ x^{-1}), \quad (2)$$

for any strategy x . We denote this class of games by \mathcal{H} .

Note that a game $G_n \in \mathcal{H}$ is described by the vector $(V_{tk})_{t,k}$. We let $V(G_n) = \{V_{tk} : 1 \leq t \leq n, 1 \leq k \leq m\}$ and so $V(G_n)$ is a subset of \mathcal{U} , the space of all continuous, real-valued functions on \mathcal{M} .

The class of games considered by Rashid (1983) is easily seen to be a special case of our framework. Let $\bar{f} : \mathcal{M} \rightarrow \Delta_m$ be defined by $\bar{f}(\mu) = \int_{\Delta_m} id\mu$, where $i : \Delta_m \rightarrow \Delta_m$ denoted the identity function. Assume that G_n is such that for every $k \in \{1, \dots, m\}$ and $t \in T_n$, there exists a function $R_{tk} : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$V_{tk}(\mu) = R_{tk} \left(\frac{n-1}{n} \bar{f}(\mu) \right), \quad (3)$$

⁴See Appendix A.1 for the definition of this and other mathematical concepts.

for all $\mu \in \mathcal{M}$. In such a game, then

$$\bar{f}(\nu_{n,t} \circ x^{-1}) = \sum_{j \neq t} \frac{x_j}{n-1}, \quad (4)$$

and so

$$U_t(x) = x_{t1} R_{t1} \left(\sum_{j \neq t} \frac{x_j}{n} \right) + \dots + x_{tm} R_{tm} \left(\sum_{j \neq t} \frac{x_j}{n} \right), \quad (5)$$

for any strategy x . We will denote this class of games by \mathcal{R} .

For any $\varepsilon \geq 0$, we say that $x^* = (x_1^*, \dots, x_n^*)$ is an ε - *equilibrium of a game* G_n if, for all $t \in \{1, \dots, n\}$,

$$U_t(x^*) \geq U_t(x_t, x_{-t}^*) - \varepsilon \text{ for all } x_t \in \Delta_m. \quad (6)$$

Thus, in an ε - equilibrium all players are close to their optimum by choosing according to x^* . A strategy x^* is a *Nash equilibrium of* G_n if x^* is an ε - equilibrium of G_n for $\varepsilon = 0$.

For any strategy x in a game G_n , let $P_n(x) = \{t \in T_n : x_t \in E\}$. We say that x is ε - *pure* if

$$\frac{|P_n^c(x)|}{n} < \varepsilon. \quad (7)$$

That is, in an ε - pure strategy all but a small fraction of players play a pure strategy.

Let x^* be a Nash equilibrium of a game G_n . Then, we say that x is an ε - *purification of* x^* if x is an ε - pure, ε - equilibrium and

$$\|U(x) - U(x^*)\| < \varepsilon. \quad (8)$$

Let K be a subset of \mathcal{U} . We say that K is *equicontinuous* (or, that the family K of functions is equicontinuous) if for all $\eta > 0$ there exists a $\delta > 0$ such that

$$\|V(x) - V(y)\| < \eta$$

whenever $\rho(x, y) < \delta$, $x, y \in \mathcal{M}$ and $V \in K$ (see Rudin (1976, p. 156)). In our framework, equicontinuity can be interpreted as placing “a bound on the diversity of payoffs” (see Khan, Rath, and Sun (1997)).

3 Existence of Approximately Pure, Approximate Equilibria in Large, Equicontinuous Games

In this section we state our first main result. It says that all sufficiently large games have an ε -pure, ε -equilibrium, provided that players' payoff functions are selected from an equicontinuous family.

Theorem 1 *Let K be an equicontinuous subset of \mathcal{U} . Then, for all $\varepsilon > 0$ and all $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $n \geq N$ and $V(G_n) \subseteq K$ implies that G_n has an ε -pure, ε -equilibrium.*

Theorem 1 is an approximation result: it guarantees the existence of an approximate equilibrium that is approximately pure if the game is sufficiently large. Essentially, we are approximating not only pure strategy Nash equilibria, but also games with a continuum of players, in which the conclusion of Theorem 1 holds exactly. This is how we proceed: we associate, to any sufficiently large finite game, a game with a continuum of players having the same distribution of payoff functions. In such a game, we restrict players to play pure strategies and, by Mas-Colell (1984)'s existence theorem, we obtain an equilibrium distribution. We then use this equilibrium distribution to construct an approximately pure, approximate equilibrium for the original finite game.⁵

Our construction shows that we can think of Theorem 1 as being an asymptotic version of Mas-Colell (1984)'s existence theorem. As we have emphasized in Carmona (2004a), such asymptotic results are natural since we can characterize the Nash equilibria of games with a continuum of players in terms of approximate equilibria of games with a finite number of players.

Another aspect that stresses the asymptotic nature of Theorem 1 is that it requires a class of payoff functions for which an equilibrium is sure to exist whenever the game has a continuum of players. In particular, it is false for games in which players have general payoff functions. As a simple example, consider $\varepsilon = 1/4$, and the following games G_n with $m = 2$ and n even: players' $2k$ and $2k - 1$, $1 \leq k \leq n/2$, play the matching pennies, i.e., their payoff depend only on their choices and is given by:

⁵See Appendix A.3 for the proof of Theorem 1.

$2k - 1 \backslash 2k$	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Table 1: Payoff Function for the Matching Pennies

Then, if x is a $1/4$ – pure strategy, it follows that there is $1 \leq k \leq n/2$ such that both players $2k - 1$ and $2k$ play a pure strategy. This implies that x cannot be an $1/4$ – equilibrium. Therefore, G_n has no $1/4$ – pure, $1/4$ – equilibrium for all n even.

In general, Theorem 1 is also false if we insist on $\varepsilon = 0$, or if we drop the equicontinuity assumption. This will be established by the example in Section 6.

It is also useful to compare Theorem 1 to other results in the literature. It follows from Rashid (1983)’s Theorem that the conclusion of Theorem 1 holds for games in \mathcal{R} . Since we do not require that payoffs depend only on the average choice of the other players, Theorem 1 is more general in this respect.⁶

Theorem 1 also provides similar generalizations for other classes of games that have been considered. Before we can state them, we need a slight generalization of the theorem.

Let $C \subseteq \mathcal{G}$ and $K \subseteq \mathcal{U}$. Then C can be approximated by (\mathcal{H}, K) if the following property holds: For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ and $F_n = (U_{tk})_{t,k} \in C$ there exists $G_n \in \mathcal{H}$ satisfying $V(G_n) \subseteq K$ and

$$\|V(x) - U(x)\| < \varepsilon \tag{9}$$

for all strategies $x : \{1, \dots, n\} \rightarrow \Delta_m$. The following corollary is an obvious, but useful, consequence of the above definition.

Corollary 1 *Let K be an equicontinuous subset of \mathcal{U} and assume that $C \subseteq \mathcal{G}$ can be approximated by (\mathcal{H}, K) . Then, Theorem 1 holds for C .*

That is, for all $\varepsilon > 0$ and all $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $n \geq N$ and $G_n \in C$ implies that G_n has an ε – pure, ε – equilibrium.

⁶Note that although not explicitly stated, Rashid’s Theorem requires that players’ payoff functions are selected from an equicontinuous family. For details, see Carmona (2004b).

Corollary 1 is useful for extending Theorem 1 to other classes of games. A class in which we are interested consists of a particular case of the one considered in Kalai (2001) and Cartwright and Wooders (2003) (see footnote 3). We will let \mathcal{K} be the class of the following games: A game $G_n \in \mathcal{K}$ is defined by the vector $(f_{tk})_{t,k}$ where $f_{tk} : \Delta_m \rightarrow \mathbb{R}$ is continuous for all $1 \leq t \leq n$ and $1 \leq k \leq m$. For $c = (c_1, \dots, c_n) \in E^n$, $e \in E$ and $t \in T_n$, define

$$\text{emp}_{c_{-t}}(e) = \frac{|\{i \neq t : c_i = e\}|}{n-1}. \quad (10)$$

Then, for any strategy x , define

$$U_t(x) = x_{t1}U_{t1}(x_{-t}) + \dots + x_{tm}U_{tm}(x_{-t}), \quad (11)$$

and

$$U_{tk}(x_{-t}) = \sum_{c_{-t} \in \prod_{i \neq t} E_i} \left(\prod_{i \neq t} x_i(c_i) \right) f_{tk}(\text{emp}_{c_{-t}}). \quad (12)$$

Let \mathcal{V} be the space of all continuous functions $f : \Delta_m \rightarrow \mathbb{R}$ and $F(G_n) = \{f_{tk} : 1 \leq t \leq n, 1 \leq k \leq m\} \subseteq \mathcal{V}$ for any $G_n \in \mathcal{K}$.

Let D be an equicontinuous subset of \mathcal{V} . Define $K \subseteq \mathcal{U}$ as follows $K = \{f \circ \bar{f} : f \in D\}$. Then, K is an equicontinuous subset of \mathcal{U} . Let C be the subset of \mathcal{K} induced by D , i.e., all the games $G_n = (f_{tk})_{t,k}$ satisfying $f_{tk} \in D$ for all $1 \leq t \leq n$ and $1 \leq k \leq m$. It then follows from Kalai (2001, Lemma 4) that C can be approximated by (\mathcal{H}, K) . Hence, the conclusion of Theorem 1 holds for the class \mathcal{K} , provided that the payoff functions are derived from an equicontinuous family of functions.

We conclude this section with another class of games. This class of games is very similar to the one we consider in Theorem 1, since games in the two classes differ only on the definition of the distribution of actions that affects players' payoffs.

Let G_n be a game and x a strategy. We let ν_n denote the uniform measure on T_n , and so we define $\nu_n \circ x^{-1} \in \mathcal{M}$ as follows:

$$\nu_n \circ x^{-1}(B) = \frac{|\{t \in T_n : x_t \in B\}|}{n} \quad (13)$$

for any Borel set $B \subseteq \Delta_m$. To G_n we associate the game \bar{G}_n defined as follows: it has the same set of players T_n and each has the same action space Δ_m . For $t \in T_n$, his payoff function is now

$$\bar{U}_t(x) = x_{t1}V_{t1}(\nu_n \circ x^{-1}) + \dots + x_{tm}V_{tm}(\nu_n \circ x^{-1}), \quad (14)$$

for any strategy x . Thus, in \overline{G}_n player t 's payoff depends on the distribution of actions played by all players, including t himself, whereas in G_n it depends only on the distribution induced by the choices of the other players. Note, however, that $V(\overline{G}_n) = V(G_n)$. We let $\overline{\mathcal{H}}$ denote this class of games.

The following remark states that the conclusion of Theorem 1 also holds for $\overline{\mathcal{H}}$. In fact, Lemma 3 in Appendix A.2 shows that it is equivalent to Theorem 1.

Remark 1 *Let K be an equicontinuous subset of \mathcal{U} . Then, for all $\varepsilon > 0$ and all $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $n \geq N$ and $V(\overline{G}_n) \subseteq K$ implies that \overline{G}_n has an ε -pure, ε -equilibrium.*

Note that in a game $G_n \in \mathcal{H}$, each player's payoff function is linear in his own choice. Since this is not necessarily the case in \overline{G}_n , Remark 1 shows that the existence of approximately pure, approximate equilibria in large games does not depend on that assumption.

4 On the Equivalence to Mas-Colell's Theorem

We have emphasized that Theorem 1 can be thought of as an asymptotic version of Mas-Colell (1984)'s existence theorem. Moreover, we have used it in our proof, and so it follows that Theorem 1 is implied by it. In this section we show that the two results are equivalent.

We next describe Mas-Colell (1984)'s Theorem in our context, i.e., in games with a finite number of pure strategies. We note that his theorem holds more generally, and therefore, our Theorem 1 is really equivalent to a particular version of it.

Let \mathcal{E} be the space of the continuous functions $u : E \times \mathcal{M}(E) \rightarrow \mathbb{R}$. A game with a continuum of players is described by a Borel probability measure on \mathcal{E} .

Given a Borel probability measure τ on $\mathcal{E} \times E$, we denote by $\tau_{\mathcal{E}}$ and τ_E the marginal distributions of τ on \mathcal{E} and E respectively. The expression $u(e, \tau) \geq u(E, \tau)$ means $u(e, \tau) \geq u(e', \tau)$ for all $e' \in E$.

Given a game μ , a Borel probability measure τ on $\mathcal{E} \times E$ is an *equilibrium distribution for μ* if

1. $\tau_{\mathcal{E}} = \mu$, and

$$2. \tau(\{(u, e) \in \mathcal{E} \times E : u(e, \tau_E) \geq u(E, \tau_E)\}) = 1.$$

Mas-Colell (1984)'s Theorem 1 asserts the existence of an equilibrium distribution in games with a continuum of players.

Theorem 2 (Mas-Colell) *An equilibrium distribution exists for all games $\mu \in \mathcal{M}(\mathcal{E})$.*

Theorem 3 below shows that this version of Mas-Colell (1984)'s Theorem is equivalent to Theorem 1.⁷

Theorem 3 *Theorem 1 holds if and only if Theorem 2 holds.*

5 On the Purification of Nash Equilibria

We have shown that we can generalize the existence of approximately pure, approximate equilibria in large games from the classes \mathcal{R} and \mathcal{K} . Existence of such equilibria in those classes is easy to establish, since there all Nash equilibria can be approximately purified in large games. In this section, we ask whether or not it is possible to purify all Nash equilibria of sufficiently large games in \mathcal{H} . Theorem 4 shows that the answer is no.

Theorem 4 *There exists an equicontinuous subset K of \mathcal{U} , $\varepsilon > 0$ and $m \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exist a game G_n with $n \geq N$, $V(G_n) \subseteq K$ and a Nash equilibrium x of G_n that has no ε -purification.*

Proof. Let $\varepsilon = 1/4$ and $m = 2$. Define $g : \mathcal{M} \rightarrow \mathbb{R}$ as follows:

$$g(\mu) = \frac{\rho(\mu, B_{1/4}^c(1_{1/2}))}{\rho(\mu, B_{1/4}^c(1_{1/2})) + \rho(\mu, 1_{1/2})}. \quad (15)$$

Note that g is continuous, $g(1_{1/2}) = 1$ and $g(\mu) = 0$ for all $\mu \notin B_{1/4}(1_{1/2})$. Let $K = \{g\}$.

Let $N \in \mathbb{N}$ and let $n = N + 2$. Define G_n by letting $V_{tk}(\mu) = g(\mu)$ for all $t \in \{1, \dots, n\}$ and $k = 1, 2$. It follows that any strategy in G_n is a Nash equilibrium.

⁷See Appendix A.4 for its proof.

Consider x^* defined by $x_t^* = 1/2$ for all $t \in T_n$. Then $U_t(x^*) = 1$ for all $t \in T_n$. We claim that any $1/4$ -pure strategy x satisfies $\rho(\nu_{n,t} \circ x^{-1}, 1_{1/2}) \geq 1/4$ for all $t \in T_n$, from which the theorem follows because then $U_t(x) = 0$ for all $t \in T_n$ and so

$$\|U(x^*) - U(x)\| = 1 > \frac{1}{4}. \quad (16)$$

So, it remains to prove that if x is a $1/4$ -pure strategy, then $\rho(\nu_{n,t} \circ x^{-1}, 1_{1/2}) \geq 1/4$ for all $t \in T$. For convenience, denote $\nu_{n,t} \circ x^{-1}$ by τ . Let $D = \{1/2\}$. Then obviously $1_{1/2}(D) = 1$ and $\bar{B}_{1/4}(D) = [1/4, 3/4]$. It follows that

$$\tau(\bar{B}_{1/4}(D)) \leq \nu_{n,t}(P_n^c(x)) < \frac{n}{4(n-1)}. \quad (17)$$

Since $n \geq 2$, we obtain that

$$1_{1/2}(D) = 1 > \frac{n}{4(n-1)} + \frac{1}{4} > \tau(\bar{B}_{1/4}(D)) + \frac{1}{4}. \quad (18)$$

Thus, $1/4 \notin \Sigma(\tau, 1_{1/2})$ and so $\rho(\tau, 1_{1/2}) \geq 1/4$. ■

6 On the Non-Existence of Equilibria

In this section, we show that in general, we cannot obtain a pure Nash equilibrium even in equicontinuous large games. The exact version of Theorem 1 (i.e., with $\varepsilon = 0$) therefore holds only in the limit case of games with a continuum of players. As an easy consequence, we also show that it is not possible to dispense with the equicontinuity assumption used in Theorem 1.

Theorem 5 *There exists an equicontinuous subset K of \mathcal{U} and $m \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists a game G_n with $n \geq N$ and $V(G_n) \subseteq K$ which has no pure strategy Nash equilibrium.*

Proof. We let $m = 2$, and for any $N \in \mathbb{N}$, we let $n \geq N$ be a multiple of 4. Define G_n as follows: let $T_1 = \{1, \dots, n/2\}$ and $T_2 = \{n/2 + 1, \dots, n\}$. For $t \in T_1$, let

$$\begin{aligned} V_{t_1} \left(\sum_{j \neq t} \frac{x_{j1}}{n} \right) &= \frac{1}{2} - \frac{n}{n-1} \sum_{j \neq t} \frac{x_{j1}}{n} \text{ and} \\ V_{t_2} \left(\sum_{j \neq t} \frac{x_{j1}}{n} \right) &= -V_{t_1}, \end{aligned} \quad (19)$$

while for $t \in T_2$, let $V_{ti} = -V_{1i}$, $i = 1, 2$. Thus, for $t \in T_1$, we have that player t 's optimal choice is:

$$x_t = \begin{cases} 1 & \text{if } \sum_{j \neq t} \frac{x_{j1}}{n-1} < \frac{1}{2}, \\ x \in [0, 1] & \text{if } \sum_{j \neq t} \frac{x_{j1}}{n-1} = \frac{1}{2}, \\ 0 & \text{if } \sum_{j \neq t} \frac{x_{j1}}{n-1} > \frac{1}{2}. \end{cases} \quad (20)$$

For any player $t \in T_2$, the optimal choice is just the opposite, that is, player t 's optimal choice is:

$$x_t = \begin{cases} 0 & \text{if } \sum_{j \neq t} \frac{x_{j1}}{n-1} < \frac{1}{2}, \\ x \in [0, 1] & \text{if } \sum_{j \neq t} \frac{x_{j1}}{n-1} = \frac{1}{2}, \\ 1 & \text{if } \sum_{j \neq t} \frac{x_{j1}}{n-1} > \frac{1}{2}. \end{cases} \quad (21)$$

Let $K = \{V_{11}, V_{12}, -V_{11}, -V_{12}\}$.

We claim that G_n has no pure strategy Nash equilibrium. Let

$$\bar{x} = \sum_{t \in T} \frac{x_t}{n},$$

and

$$\bar{x}_t = \sum_{j \neq t} \frac{x_j}{n-1} = \frac{n\bar{x} - x_t}{n-1},$$

for any $t \in T$. We consider the cases $\bar{x} < 1/2$, $\bar{x} = 1/2$ and $\bar{x} > 1/2$.

Assume that x is pure and that $\bar{x} = 1/2$. If $t \in T_2$ is such that $x_t = 0$, then $\bar{x}_t = n/(2n-2) > 1/2$. Therefore, the gain from deviating to $\tilde{x}_t = 1$ is

$$2\bar{x}_t - 1 = \frac{1}{n-1} > 0. \quad (22)$$

Similarly, if $t \in T_2$ is such that $x_t = 1$, then $\bar{x}_t = (n-2)/(2n-2) < 1/2$, and player t is not optimizing.

If $\bar{x} < 1/2$, then there exists $t \in T_1$ such that $x_t = 0$, which is clearly not optimizing. Finally, if $\bar{x} > 1/2$, then there exists $t \in T_1$ such that $x_t = 1$, which again is not optimizing. Thus, G_n has no pure strategy Nash equilibrium. ■

The following result shows that we cannot dispense with the equicontinuity assumption in Theorem 1. In its proof, we consider the same family of games as in Theorem 5, and we show that the corresponding n -players game has no $1/4$ -pure, ε_n -equilibrium, for some $0 < \varepsilon_n \leq 1/(n-1)$.

This implies that the game obtained by dividing players' payoff functions by ε_n has no $1/4$ – pure, 1 – equilibrium, and so no $1/4$ – pure, $1/4$ – equilibrium. Since $\varepsilon_n \rightarrow 0$, the family of games obtained in this way fails to be equicontinuous.

Theorem 6 *There exist $\varepsilon > 0$ and $m \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists a game G_n with $n \geq N$ which has no ε – pure, ε – equilibrium.*

Proof. Let $\varepsilon = 1/4$ and $m = 2$. Let $N \in \mathbb{N}$ and let $n \geq N$ be a multiple of 4. Define \tilde{G}_n as follows: let $T_1 = \{1, \dots, n/2\}$ and $T_2 = \{n/2 + 1, \dots, n\}$. For $t \in T_1$, let

$$\begin{aligned} \tilde{V}_{t1} \left(\sum_{j \neq t} \frac{x_{j1}}{n} \right) &= \frac{1}{2} - \frac{n}{n-1} \sum_{j \neq t} \frac{x_{j1}}{n} \text{ and} \\ \tilde{V}_{t2} \left(\sum_{j \neq t} \frac{x_{j1}}{n} \right) &= -\tilde{V}_{t1}, \end{aligned} \tag{23}$$

while for $t \in T_2$, let $\tilde{V}_{ti} = -\tilde{V}_{ti}$, $i = 1, 2$. Thus, \tilde{G}_n equals the game G_n used in the proof of Theorem 5.

We will show that there exists an $\varepsilon_n > 0$ with the property that no $1/4$ – pure strategy is an ε_n – equilibrium. We then define G_n by letting

$$V_{ti} = \frac{\tilde{V}_{ti}}{\varepsilon_n} \tag{24}$$

for all $t \in T$ and $i = 1, 2$. Then no $1/4$ – pure strategy is an 1 – equilibrium of G_n and so, G_n has no $1/4$ – pure, $1/4$ – equilibrium.

We will show that there is no $1/4$ – pure Nash equilibrium of \tilde{G}_n . This implies the existence of $\varepsilon_n > 0$ with the property that no $1/4$ – pure strategy is an ε_n – equilibrium of \tilde{G}_n as follows: Note that a strategy can be thought of as a vector $x \in [0, 1]^n$ and consider the set X of strategies that are $1/4$ – pure. We can show that X is compact: let $X_0 = [0, 1]^n$ and for $0 < l \leq n/4 - 1$, let X_l be defined as follows: first, define \mathcal{T}_l as the collection of all sets with l elements from T ; second, for $\tau \in \mathcal{T}_l$, define $X_{l,\tau}$ as the set of strategies in which players outside of τ must play a pure strategy, i.e., $X_{l,\tau} = X_{l,\tau,1} \times \dots \times X_{l,\tau,n}$ with

$$X_{l,\tau,t} = \begin{cases} [0, 1] & \text{if } t \in \tau, \\ \{0, 1\} & \text{otherwise;} \end{cases} \tag{25}$$

third, and finally, define $X_l = \cup_{\tau \in \mathcal{T}_l} X_{l,\tau}$ — the set X_l is the set of strategies in which exactly l players are allowed to play a mixed strategy. Let $0 \leq l \leq n/4 - 1$. Clearly, $X_{l,\tau}$ is compact for all $\tau \in \mathcal{T}_l$. It then follows that X_l is compact since it equals a finite union of compact sets. Then $X = \cup_{l=0}^{n/4-1} X_l$ is compact. Hence, if our conclusion is false, we could find an $1/4$ – pure, $1/k$ – equilibrium x_k of \tilde{G}_n for every $k \in \mathbb{N}$. Taking a subsequence if necessary, we can assume that x_k converges, and so $x = \lim_k x_k$ is an $1/4$ – pure Nash equilibrium of \tilde{G}_n .

We have shown in Theorem 5 that \tilde{G}_n has no pure strategy Nash equilibrium. It now remains to show that \tilde{G}_n has no $1/4$ – pure Nash equilibrium. Let x be $1/4$ – pure. Consider the case in which $\bar{x} = 1/2$. Then, there is $t \in T_2$ playing a pure strategy and we can show exactly as in the proof of Theorem 5 that he cannot be optimizing.

Consider now the case $\bar{x} < 1/2$. By the above, we may assume that there is at least one player playing a mixed strategy. For any such player t to be optimizing, it must be that $\bar{x}_t = 1/2$, which is equivalent to

$$x_t = n\bar{x} - \frac{n-1}{2} < \frac{1}{2}. \quad (26)$$

If $x_t \leq 0$, we get a contradiction. So assume that $x_t > 0$. Then, letting $P_1(x) = \{t \in T : x_t = 1\}$, it follows that

$$\bar{x} = \frac{|P^c(x)|}{n} \left(n\bar{x} - \frac{n-1}{2} \right) + \frac{|P_1(x)|}{n} \quad (27)$$

which implies that

$$|P^c(x)| = \frac{n\bar{x} - |P_1(x)|}{n\bar{x} - \frac{n-1}{2}}. \quad (28)$$

Since obviously $|P^c(x)| < 1$, it follows that $|P_1(x)| > (n-1)/2$ and so $|P_1(x)| \geq n/2$. But this implies that $\bar{x} \geq 1/2$, a contradiction.

Finally, consider the case $\bar{x} > 1/2$. Assume, in order to get a contradiction, that x is a $1/4$ – pure equilibrium. Then define y by $y_t = 1 - x_t$ for all $t \in T$. Then one easily computes that $\bar{y} = 1 - \bar{x}$ and that $\bar{y}_t = 1 - \bar{x}_t$. This implies that y is a $1/4$ – pure equilibrium with $\bar{y} < 1/2$, a contradiction. ■

We remark that the above example does not contradict Theorem 1. In the game G_n , we can write, for any $t \in T$ and $k = 1, 2$,

$$V_{tk}(\mu) = \frac{1}{\varepsilon_n} \left(\frac{1}{2} - \bar{f}(\mu) \right), \quad (29)$$

Let $\omega_{\bar{f}}(\delta)$ denote the modulus of continuity of \bar{f} . Then, the modulus of continuity $\omega(\delta)$ of V_{tk} is, for all $1 \leq t \leq T_n$ and $1 \leq k \leq m$, $\frac{\omega_{\bar{f}}(\delta)}{\varepsilon_n} \geq (n-1)\omega_{\bar{f}}(\delta)$ since it follows by 22 that $\varepsilon_n \leq \frac{1}{n-1}$. This implies that $\omega(\delta)$ converges to infinity with n , which shows that the family of payoff functions we used in the example is not equicontinuous. This accounts for the failure of the conclusion of Theorem 1.

Finally, we note that the game G_n used in both theorems belongs to the class \mathcal{R} . This shows that even if we restrict attention to games in \mathcal{R} , we can obtain neither a pure strategy Nash equilibrium nor an ε -pure, ε -equilibrium for all $\varepsilon > 0$ without equicontinuity for all sufficiently large games. This last conclusion generalizes our result in Carmona (2004b): since for the non-equicontinuous game in our example there is no ε -pure, ε -equilibrium, for any $\varepsilon > 0$ small enough, then it follows that no mixed strategy equilibrium can be ε -purified.

7 Concluding Remarks

The main objective of this paper is to obtain an asymptotic version of Mas-Colell's Theorem on the existence of a pure strategy Nash equilibrium. Our main results show that:

1. for any equicontinuous family of payoff functions and any $\varepsilon > 0$, all sufficiently large games have an ε -pure, ε -equilibrium, and,
2. the above asymptotic result is equivalent to Mas-Colell's.

These results imply that the existence of pure strategy Nash equilibria in games with a continuum of players can be studied equivalently to the existence of approximately pure, approximate equilibria in large, equicontinuous games.

Furthermore, they provide an additional example of the type of results that show that approximate equilibria of finite games have the same, or approximately the same, properties as equilibria of continuum games. In this way, it stresses the close relationship between equilibria of games with a continuum of players and approximate equilibria of games with a finite number of players.

A Appendix

A.1 Mathematical Appendix

In this section we provide the definition of some of the mathematical concepts we have used.

Let $z \in \mathbb{R}^n$. If $n = 1$, then $\|z\|$ denotes the usual absolute value of z , while if $n > 1$, then $z = (z_1, \dots, z_n)$ and $\|z\| = \max_{1 \leq i \leq n} \|z_i\|$.

If Y is a subset of some space X , then Y^c denotes the complement of Y in X , i.e., $Y^c = \{x \in X : x \notin Y\}$. When Y is finite, $|Y|$ denotes the number of points belonging to Y . If X is a metric space with metric d , $x \in X$ and $\varepsilon > 0$, then $B_\varepsilon(x) = \{z \in X : d(x, z) < \varepsilon\}$ is the open ball of radius ε around x . If D is a subset of X , then $\text{dist}(x, D) = \inf_{z \in D} d(x, z)$ and $\overline{B}_\varepsilon(D) = \{x \in X : \text{dist}(x, D) \leq \varepsilon\}$.

Let X be a metric space. Then $\mathcal{M}(X)$ denotes the space of all Borel probability measures on X . For any $x \in X$, 1_x denotes the measure concentrated in x , i.e., $1_x(B) = 1$ for all Borel subsets B of X that contain x , while $1_x(D) = 0$ for all Borel subsets D of X that do not contain x . If X is compact, then $\mathcal{M}(X)$, when endowed with the weak convergence topology, is also a compact metric space (Parthasarathy (1967, Theorem II.6.4)). The Prohorov metric ρ on $\mathcal{M}(X)$, which is known to metricize the weak convergence topology, is defined as follows: let $\mu, \tau \in \mathcal{M}(X)$ and define

$$\begin{aligned} \Sigma(\mu, \tau) = \{ \varepsilon > 0 : \tau(D) \leq \mu(\overline{B}_\varepsilon(D)) + \varepsilon \text{ and} \\ \mu(D) \leq \tau(\overline{B}_\varepsilon(D)) + \varepsilon \text{ for all Borel } D \subseteq X \}. \end{aligned} \quad (30)$$

Then, $\rho(\mu, \tau) = \inf \Sigma(\mu, \tau)$. If $\tau \in \mathcal{M}(X)$ and $\{\tau_n\}_n \subseteq \mathcal{M}(X)$, we write $\tau_n \Rightarrow \tau$ whenever $\rho(\tau_n, \tau) \rightarrow 0$.

Throughout the paper we have had the opportunity to explore the relationship between games with a continuum of players and games with finitely many players. Let A be a non-empty, compact metric space (representing the space of actions) and let $\tilde{\mathcal{U}}$ denote the space of continuous utility functions $u : A \times \mathcal{M}(A) \rightarrow \mathbb{R}$ endowed with the supremum norm (and which represents the space of players' payoff functions). A *game with a continuum of players* is characterized by a measurable function $U : [0, 1] \rightarrow \tilde{\mathcal{U}}$, where the unit interval $[0, 1]$ is endowed with the Lebesgue measure λ on the Lebesgue measurable sets and represents the set of players. A strategy is a measurable function $x : [0, 1] \rightarrow A$ and the payoff for player $t \in [0, 1]$ is

$$U(t)(x(t), \lambda \circ x^{-1}). \quad (31)$$

We denote such game by $G = ([0, 1], \lambda, U, A)$.

A *game with a finite number of players* is characterized by a function $U : T \rightarrow \tilde{\mathcal{U}}$, where T is a finite set. The set T represents the set of players and it is endowed with the uniform measure ν . A strategy is a function $x : T \rightarrow A$ and the payoff for player $t \in T$ is

$$U(t)(x(t), \nu \circ x^{-1}). \quad (32)$$

Given a strategy x , $a \in A$, and $t \in T$, let $x \setminus_t a$ denote the strategy y defined by $y(t) = a$, and $y(\tilde{t}) = x(\tilde{t})$, for all $\tilde{t} \neq t$ — this is the strategy that results if player t deviates from x by playing a . We denote such a game by $G = ((T, \nu), U, A)$.

Note that for any $m, n \in \mathbb{N}$, a vector $(V_{tk})_{1 \leq t \leq n, 1 \leq k \leq m}$ of functions $V_{tk} : \mathcal{M}(\Delta_m) \rightarrow \mathbb{R}$ defines a game with a finite number of players in the above sense by equations 13 and 14. The above notion of a game with a finite number of players is useful because we have shown in Carmona (2004a) (see Theorem 1 and its proof) that: if G is a game with a continuum of players G_n is a sequence of games with a finite number of players, $|T_n| \rightarrow \infty$, τ_n is an equilibrium distribution of G_n , $\tau_n \Rightarrow \tau$ and $\tau_{\tilde{\mathcal{U}}} = \lambda \circ U^{-1}$, then τ is an equilibrium distribution of G .

A.2 Lemmata

In this appendix, we prove three results needed for Theorem 1. Lemma 1 deals with measures with a finite support. For such measures, we will sometimes write μ_l instead of $\mu(\{l\})$, whenever l is a point in the space in which μ is strictly positive. This notation also suggests that a measure with a finite support can be thought of as a vector in some Euclidean space. Roughly, Lemma 1 says that the Prohorov distance between two measures whose support is contained in some finite set is proportional to their Euclidean distance.

Lemma 1 *Let $\tau, \mu \in \mathcal{M}(X)$ be such that $\text{supp}(\tau) \cup \text{supp}(\mu) \subseteq \Psi$, where Ψ is a finite set. If there exists $\varepsilon > 0$ such that $|\tau_l - \mu_l| \leq \varepsilon$ for all $1 \leq l \leq |\Psi|$, then $\rho(\tau, \mu) \leq |\Psi|\varepsilon$.*

Proof. Let $\varepsilon > 0$ and $B \subseteq X$ be Borel measurable. Then,

$$\begin{aligned} \tau(B) &= \sum_{l \in \Psi \cap B} \tau(\{l\}) \leq \sum_{l \in \Psi \cap B} (\mu(\{l\}) + \varepsilon) \leq \\ &\sum_{l \in \Psi \cap B} \mu(\{l\}) + |\Psi| \varepsilon \leq \\ &\mu(\overline{B}_{|\Psi| \varepsilon}(B)) + |\Psi| \varepsilon. \end{aligned} \tag{33}$$

Similarly, we can show that $\mu(B) \leq \tau(\overline{B}_{|\Psi| \varepsilon}(B)) + |\Psi| \varepsilon$. This implies that $\rho(\tau, \mu) \leq |\Psi| \varepsilon$. ■

Lemma 2 shows that in large games a player deviation has a small impact on the distribution of actions.

Lemma 2 *Let G_n be a game and let x be a strategy. If y is another strategy that differs from x in exactly one point, then*

$$\rho(\nu_n \circ x^{-1}, \nu_n \circ y^{-1}) \leq \frac{1}{n}, \tag{34}$$

Proof. Let $\tau = \nu_n \circ x^{-1}$ and $\tau' = \nu_n \circ y^{-1}$. We have that

$$\tau(\{a\}) = \frac{|\{t \in T_n : x(t) = a\}|}{n}$$

for all $a \in \Delta_m$, and similarly for τ' . Since there is just one $t \in T_n$ such that y and x differ, there are only two points a_1 and a_2 in Δ_m such that $\tau(\{a_i\})$ differs from $\tau'(\{a_i\})$. Note that we have $\|\tau(\{a_i\}) - \tau'(\{a_i\})\| = 1/n$ for $i = 1, 2$. Then, for any Borel subset B of Δ_m , it follows that $\|\tau(B) - \tau'(B)\| = 0$ if either a_1 and a_2 belong to B or a_1 and a_2 belong to B^c , while $\|\tau(B) - \tau'(B)\| = 1/n$ otherwise. This implies $\rho(\tau, \tau') \leq 1/n$. ■

The following lemma is the analog to Rashid (1983)'s observation 1. Roughly, it says that for approximate equilibria of large games, it does not matter whether or not we include a player's own action when defining the distribution of actions that determines her payoff.

Lemma 3 *Let K be an equicontinuous subset of \mathcal{U} . Then, for all $\eta > 0$ and all $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, $V(G_n) \subseteq K$ and \overline{G}_n has an η - equilibrium, then G_n has an 4η - equilibrium. Furthermore, if $n \geq N$, $V(G_n) \subseteq K$ and G_n has an η - equilibrium, then \overline{G}_n has an 4η - equilibrium.*

Proof. Let $\eta > 0$. Since K is equicontinuous, let $\delta > 0$ be such that

$$\|V(\tau) - V(\mu)\| < \eta \quad (35)$$

whenever $\rho(\tau, \mu) < \delta$, $\tau, \mu \in \mathcal{M}$ and $V \in K$. Finally, let $N \in \mathbb{N}$ be such that $1/(n-1) < \delta$ whenever $n \geq N$.

Let $n \geq N$, $V(G_n) \subseteq K$ and x be a η -equilibrium of \overline{G}_n . Let $t \in T_n$ and let $\mu = \nu_n \circ x^{-1}$ and $\tau = \nu_{n,t} \circ x^{-1}$. We claim that

$$\rho(\mu, \tau) \leq \frac{1}{n-1}. \quad (36)$$

Let D be a Borel measurable subset of Δ_m and $d = |\{i \in T_n : x_i \in D\}|$. Then

$$\mu(D) = \frac{d}{n}, \quad (37)$$

and

$$\tau(D) = \begin{cases} \frac{d}{n-1} & \text{if } x_t \notin D, \\ \frac{d-1}{n-1} & \text{otherwise.} \end{cases} \quad (38)$$

Then, in order to have $\mu(D) \leq \tau(\overline{B}_\varepsilon(D)) + \varepsilon$ for all D , it is enough that

$$\frac{d}{n} \leq \frac{d-1}{n-1} + \varepsilon \quad (39)$$

for all $0 \leq d \leq n$, $d \in \mathbb{N}$. One easily sees that 39 is satisfied if $\varepsilon \geq 1/(n-1)$. Also, in order to have $\tau(D) \leq \mu(\overline{B}_\varepsilon(D)) + \varepsilon$ for all D , it is enough that

$$\frac{d}{n-1} \leq \frac{d}{n} + \varepsilon \quad (40)$$

for all $0 \leq d \leq n$, $d \in \mathbb{N}$. Again, one easily sees that 40 is satisfied if $\varepsilon \geq 1/(n-1)$. Therefore, $\rho(\mu, \tau) \leq \frac{1}{n-1}$ as we claimed.

Hence,

$$\begin{aligned} U_t(x_t, \tau) &\geq U_t(x_t, \mu) - \eta \\ &\geq U_t(\bar{a}, \nu_{n,t} \circ (x \setminus_t \bar{a})) - 2\eta \\ &\geq U_t(\bar{a}, \mu) - 3\eta \\ &\geq U_t(\bar{a}, \tau) - 4\eta, \end{aligned} \quad (41)$$

for all $\bar{a} \in \Delta_m$. Hence, x is a 4η -equilibrium of G_n .

Finally, note that the above computations also imply that if $n \geq N$, $V(G_n) \subseteq K$ and x be a η -equilibrium of G_n , then x is a 4η -equilibrium of \overline{G}_n . ■

A.3 Proof of Theorem 1

Let $\varepsilon > 0$ and let $\eta > 0$ be such that $4\eta < \varepsilon$. Since K is equicontinuous, let $\delta > 0$ be such that

$$\|V(\tau) - V(\mu)\| < \eta/3 \quad (42)$$

whenever $\rho(\tau, \mu) < \delta$, $\tau, \mu \in \mathcal{M}$ and $V \in K$. Finally, let $N \in \mathbb{N}$ be such that $2m^2/n < \delta$ and $m/n < \eta$ whenever $n \geq N$.

Let $n \geq N$ and G_n be a game. By Lemma 3, it is enough to show that \bar{G}_n has an η -pure, η -equilibrium. Consider the game $G_\lambda = (([0, 1], \lambda), E, V)$ where $V(t) = U_i$ if $t \in T_i := [\frac{i-1}{n}, \frac{i}{n})$ for $1 \leq i \leq n-1$ and $V(t) = U_n$ if $t \in T_n = [\frac{n-1}{n}, 1]$. Then G_λ has Nash equilibrium distribution τ on $\mathcal{U} \times E$ by Mas-Colell (1984, Theorem 1).

Let $\tau_{i,j} = \tau(\{(u_i, e_j)\})$ and $\tau_j = \sum_{i=1}^n \tau_{i,j}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. The equilibrium distribution can be represented in the following way:

$$\begin{array}{cccc} \tau_{1,1} & \tau_{1,2} & \cdots & \tau_{1,m} \\ \tau_{2,1} & \tau_{2,2} & & \tau_{2,m} \\ \vdots & & & \\ \tau_{n,1} & \tau_{n,2} & & \tau_{n,m} \\ \hline \tau_1 & \tau_2 & & \tau_m \end{array}$$

Note that $\sum_{j=1}^m \tau_{i,j} = 1/n$ for all $1 \leq i \leq n$.

Let

$$S_i = \left\{ \frac{1}{n} e_j : \tau_{i,j} > 0 \right\}.$$

Then, we have that $(\tau_{i,1}, \dots, \tau_{i,m}) \in \text{co}(S_i)$ for every $1 \leq i \leq n$, since

$$(\tau_{i,1}, \dots, \tau_{i,m}) = \sum_{j:\tau_{i,j}>0} \tau_{i,j} e_j = \sum_{j:\tau_{i,j}>0} n\tau_{i,j} \frac{e_j}{n}, \quad (43)$$

$n\tau_{i,j} \geq 0$ for all j and $\sum_{j:\tau_{i,j}>0} n\tau_{i,j} = 1$. This implies that $(\tau_1, \dots, \tau_m) \in \text{co}(\sum_{i=1}^n S_i) = \sum_{i=1}^n \text{co}(S_i)$, and so by the Shapley-Folkman Theorem, it follows that there are n points $(\alpha_{i,1}, \dots, \alpha_{i,m}) \in \text{co}(S_i)$, $i = 1, \dots, n$, such that $(\tau_1, \dots, \tau_m) = \sum_{i=1}^n (\alpha_{i,1}, \dots, \alpha_{i,m})$ and $|\{i : (\alpha_{i,1}, \dots, \alpha_{i,m}) \notin S_i\}| \leq m$.

In order to simplify notation, we write Δ for Δ_m . Since τ is an equilibrium distribution, it follows that if $\tau_{i,j} > 0$ then $U_i(e_j, \tau_\Delta) \geq U_i(e_l, \tau_\Delta)$ for all $l = 1, \dots, m$. Let $\bar{a} \in \Delta$. By the expected utility hypothesis ($a = \sum_{l=1}^m a_l e_l$

implies that $U_t(a, \mu) = \sum_{l=1}^m a_l U_t(e_l, \mu)$, it follows that $\tau_{i,j} > 0$ implies that $U_i(e_j, \tau_\Delta) \geq U_i(\bar{a}, \tau_\Delta)$. Furthermore, we conclude that

$$U_i(a, \tau_\Delta) \geq U_i(\bar{a}, \tau_\Delta) \quad (44)$$

if $a \in n\text{co}(S_i) = \text{co}(nS_i) = \text{co}(\{e_j : \tau_{i,j} > 0\})$.

Define the following strategy x in \bar{G}_n : $x_i = (n\alpha_{i,1}, \dots, n\alpha_{i,m})$, $i = 1, \dots, n$. Let $\mu = \nu_n \circ (U, x)^{-1}$. By construction, at least $n - m$ players are playing a pure strategy, and so $\nu_n(P_n^c(x)) \leq m/n < \eta$. Thus x is η -pure.

Let Ψ be a finite set with at most $2m$ elements containing $\{1_e : e \in E\}$ and $\text{supp}(\mu_\Delta)$: since at most $m = n - (n - m)$ players play a mixed strategy, then $|\text{supp}(\mu_\Delta) \setminus \{1_e : e \in E\}| \leq m$. Write $\Psi = \{\psi_1, \dots, \psi_{|\Psi|}\}$ with $\psi_1 = 1_{e_1}, \dots, \psi_m = 1_{e_m}$. We claim that $\|\tau_l - \mu_l\| \leq m/n$ for all $1 \leq l \leq |\Psi|$: If $1 \leq l \leq m$, we have that $\mu_l = \sum_{i \in P_n(x)} \alpha_{i,l}$ and $\tau_l = \sum_{i=1}^n \tau_{i,l} = \sum_{i=1}^n \alpha_{i,l}$; therefore, $\|\tau_l - \mu_l\| \leq \sum_{i \notin P_n(x)} \alpha_{i,l} \leq m/n$. If $m + 1 \leq l \leq |\Psi|$, then, $\|\tau_l - \mu_l\| = \mu_l \leq m/n$.

By Lemma 1, it follows that $\rho(\tau_\Delta, \mu_\Delta) \leq 2m^2/n < \delta$. Hence, for all $t \in \{1, \dots, n\}$ and $\bar{a} \in \Delta$, we obtain

$$\begin{aligned} U_t(x_t, \mu_\Delta) &\geq U_t(x_t, \tau_\Delta) - \frac{\eta}{3} \\ &\geq U_t(\bar{a}, \tau_\Delta) - \frac{\eta}{3} \\ &\geq U_t(\bar{a}, \mu_\Delta) - \frac{2\eta}{3} \\ &\geq U_t(\bar{a}, \nu_n \circ (x \setminus_t \bar{a})) - \eta, \end{aligned} \quad (45)$$

where the first and third inequality follow from 42, the second from 44 and the last from Lemma 2. Therefore, x is an η -equilibrium of \bar{G}_n .

A.4 Proof of Theorem 3

Since we have established Theorem 1 using Theorem 2, it is enough to show that we can prove Theorem 2 with Theorem 1.

Let μ be a game with a continuum of players. Then, there exists a sequence $\mu_n \in \mathcal{M}(\mathcal{E})$ satisfying $\mu_n \Rightarrow \mu$, $\text{supp}(\mu_n)$ is finite and $\mu_n(\{u\}) \in \mathbb{Q}$ for all $u \in \text{supp}(\mu_n)$. Let $\text{supp}(\mu_n) = \{u_n^1, \dots, u_n^{K_n}\}$, where $u_n^k \in \mathcal{E}$ and

$$\frac{\beta_n^k}{t_n} = \mu_n(\{u_n^k\}). \quad (46)$$

Let $f : \mathcal{M}(E) \rightarrow \mathcal{M}(\Delta)$ be defined by $f(\phi)(B) = \sum_{e \in B \cap E} \phi(\{e\})$ for all Borel subset B of Δ_m . Then, f is a homeomorphism between $\mathcal{M}(E)$ and $E' = f(\mathcal{M}(E))$. Furthermore, note that E' is a closed subset of $\mathcal{M}(\Delta)$, a compact metric space. Therefore, for each $n \in \mathbb{N}$ and $1 \leq k \leq K_n$ $u_n^k : E \times \mathcal{M}(E) \rightarrow \mathbb{R}$ can be extended to a continuous function $V_n^k : E \times \mathcal{M}(\Delta) \rightarrow \mathbb{R}$, by Tietze's extension theorem (see Kahn (1975, Theorem 6.1, p. 99)). The space of all the continuous functions $u : E \times \mathcal{M}(\Delta) \rightarrow \mathbb{R}$ is denoted by \mathcal{U}' and $\tilde{\mathcal{U}}$ denotes the space of all continuous functions $u : \Delta \times \mathcal{M}(\Delta) \rightarrow \mathbb{R}$.

Let $n \in \mathbb{N}$ be fixed. Then $\{V_n^k(e_i, \cdot)\}_{1 \leq k \leq K_n, 1 \leq i \leq m}$ is an equicontinuous subset of \mathcal{U} . Define the following set of games $\bar{G}_{\gamma t_n} = ((T_{\gamma t_n}, \nu_{\gamma t_n}), U_{\gamma t_n}, \Delta)$ for all $\gamma \in \mathbb{N}$. That is, $\bar{G}_{\gamma t_n}$ has γt_n players, each has Δ as his choice set and their payoff functions is defined in the following way: let $V_{\gamma t_n} : T_{\gamma t_n} \rightarrow \mathcal{U}'$ be a function with the property that it associates V_n^k to $\gamma \beta_n^k$ players, for all $1 \leq k \leq K_n$; then let $U_{\gamma t_n}(t)$ be the mixed extension of $V_{\gamma t_n}(t)$, i.e.,

$$U_{\gamma t_n}(t)(\sigma, \tau) = \sum_{k=1}^m \sigma(e_k) V_{\gamma t_n}(t)(e_k, \tau) \quad (47)$$

for all $t \in T_{\gamma t_n}$, $\sigma \in \Delta$ and τ in $\mathcal{M}(\Delta)$. Then, $U_{\gamma t_n}$ is a function from $T_{\gamma t_n}$ to $\tilde{\mathcal{U}}$ and let $\{U_n^1, \dots, U_n^{K_n}\} = U_{\gamma t_n}(T_{\gamma t_n}) \subseteq \tilde{\mathcal{U}}$. That is, $\bar{G}_{\gamma t_n}$ can also be thought of as a normal-form game with payoffs defined by equation 14.

By Remark 1 (which is equivalent to Theorem 1), $\bar{G}_{\gamma t_n}$ has a $1/n$ - pure, $1/n$ - equilibrium $f_{\gamma t_n} : T_{\gamma t_n} \rightarrow \Delta$ if γ_n is sufficiently large. Let $\tau_n = \nu_{\gamma t_n} \circ (U_{\gamma t_n}, f_{\gamma t_n})^{-1} \in \mathcal{M}(\tilde{\mathcal{U}} \times \Delta)$. We may assume that $\gamma t_n > n$, by choosing γ_n large enough.

Since $\mathcal{M}(\tilde{\mathcal{U}} \times \Delta)$ is compact, we may assume that $\{\tau_n\}$ converges. Let $\tau = \lim_n \tau_n$. Then, by Theorem 1 in Carmona (2004a) it follows that τ is an equilibrium distribution of $\tau_{\tilde{\mathcal{U}}}$. Furthermore, since E is a closed subset Δ and $\tau_n(\tilde{\mathcal{U}} \times E) \geq 1 - 1/n$, then $\tau(\tilde{\mathcal{U}} \times E) = 1$.

Let $g : \tilde{\mathcal{U}} \times \Delta \rightarrow \mathcal{E} \times \Delta$ be defined by $g = (R, i)$, where $i : \Delta \rightarrow \Delta$ is the identity and $R : \tilde{\mathcal{U}} \rightarrow \mathcal{E}$ is defined by

$$R(u)(e, \phi) = u(e, f(\phi)). \quad (48)$$

One easily checks that g is continuous. Therefore, we can define a measure ξ on $\mathcal{M}(\mathcal{E} \times E)$ as follows:

$$\xi(B) = \tau(g^{-1}(B)) = \tau \circ g^{-1}(B), \quad (49)$$

for every Borel subset B of $\mathcal{E} \times E$.

We claim that ξ is an equilibrium distribution of μ . We will start by showing that $\xi_{\mathcal{E}} = \mu$. We have that $\xi_{\mathcal{E}} = \tau_{\tilde{\mathcal{U}}} \circ R^{-1}$, so it is enough to show that $\mu = \tau_{\tilde{\mathcal{U}}} \circ R^{-1}$. Note that $\mu_n = \tau_{\tilde{\mathcal{U}},n} \circ R^{-1}$ since

$$\begin{aligned}\tau_{\tilde{\mathcal{U}},n} \circ R^{-1}(A) &= \frac{\sum_{k:U_n^k \in A} \beta_n^k}{t_n} \\ \mu_n(B) &= \frac{\sum_{k:u_n^k \in B} \beta_n^k}{t_n}\end{aligned}\tag{50}$$

and

$$u_n^k \in B \Leftrightarrow U_n^k \in A,\tag{51}$$

for every Borel subset A of $\tilde{\mathcal{U}}$ and B of \mathcal{E} . Then, since R is continuous, $\tau_{\tilde{\mathcal{U}},n} \Rightarrow \tau_{\tilde{\mathcal{U}}}$ and $\mu_n \Rightarrow \mu$, it follows that $\mu = \tau_{\tilde{\mathcal{U}}} \circ R^{-1}$.

Note also that $f(\xi_E) = \tau_{\Delta}$: Since $\tau_{\Delta}(E) = 1$, then there exists a_i for all $i = 1, \dots, m$ such that $\tau_{\Delta}(\{e_i\}) = a_i$, $a_i \geq 0$ and $\sum_{i=1}^m a_i = 1$. Thus, it is enough to show that $\xi_E(\{e_i\}) = a_i$ for all $i = 1, \dots, m$. Since $R^{-1}(\mathcal{E}) = \tilde{\mathcal{U}}$, it follows that

$$\xi_E(\{e_i\}) = \xi(\mathcal{E} \times \{e_i\}) = \tau(R^{-1}(\mathcal{E}) \times \{e_i\}) = \tau_{\Delta}(\{e_i\}) = a_i.\tag{52}$$

Finally, we show that $\xi(B_{\xi}) = 1$, where $B_{\xi} = \{(u, e) \in \mathcal{E} \times E : u(e, \xi_E) \geq u(E, \xi_E)\}$. Let $B_{\tau} = \{(u, \sigma) \in \tilde{\mathcal{U}} \times \Delta : u(\sigma, \tau_{\Delta}) \geq u(\Delta, \tau_{\Delta})\}$ and $C_{\tau} = B_{\tau} \cap (\tilde{\mathcal{U}} \times E)$. Since $\tau(B_{\tau}) = 1$ and $\tau(\tilde{\mathcal{U}} \times E) = 1$, it follows that $\tau(C_{\tau}) = 1$. Hence, it is enough to show that $C_{\tau} \subseteq g^{-1}(B_{\xi})$.

Let $(u, e) \in C_{\tau}$. Then

$$R(u)(e, \xi_E) = u(e, f(\xi_E)) = u(e, \tau_{\Delta}) \geq u(E, \tau_{\Delta}) = R(u)(E, \xi_E).\tag{53}$$

That is, $(R(u), e) \in B_{\xi}$ and so $(u, e) \in g^{-1}(B_{\xi})$.

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