Realizing efficient outcomes in cost spanning problems*†

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February 26, 2004

Abstract

We propose a simple non-cooperative mechanism of network formation in cost spanning tree problems. The only subgame equilibrium payoff is efficient. Moreover, we extend the result to the case of budget restrictions. The equilibrium payoff can them be easily adapted to the framework of Steiner trees.

Keywords: efficiency, cost spanning tree problem, cost allocation, network formation, subgame perfect equilibrium, budget restrictions, Steiner trees

1 Introduction

Many problems involving network formation have been studied in the operational research and the economic literature. The most explored issues in operational research are the design of efficient algorithms and their computational complexity. The economic literature focuses on aspects such like the cost sharing of the network and the design of mechanisms trying to explain the way in which the network forms.

In this paper we focus in the cost sharing aspect. In particular, we study cost spanning tree problems (*cstp*). Consider that a group of agents, located at different geographical places, want some particular service which can only be provided by a common supplier, called the source. Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly to the source.

There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence

*Latest version at `http://webs.uvigo.es/vidalpuga/`

†Finantial support from the Spanish Ministerio de Educación y Ciencia and FEDER through grant BEC2002-04102-c02-01 and Xunta de Galicia through grant PGIDIT03PXIC30002PN is gratefully acknowledged.

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they have to share cost of the distribution network. This example appears in Dutta and Kar (2002). Bergantiños and Lorenzo (2004) study a real situation where villagers should pay the cost of constructing pipes to a the water supplier.

The literature about *cstp* starts by defining algorithms for constructing minimal cost spanning trees (*mcst*). We can mention, for instance, the algorithms of Kruskal (1956) and Prim (1957). Once the *mcst* is constructed, the next issue is how to allocate the cost associated to the *mcst* between the agents.

Bird (1976) studies this problem using game theory. Bird associates to each *cstp* a cooperative game and proposes a cost allocation rule based on Prim (1957) algorithm. This paper has generated more literature. For instance, Grannot and Huberman (1981, 1984) study the core and the nucleolus of the game and Kar (2002) the Shapley value. Following Bird's approach, Feltkamp, Tijs, and Muto (1994) propose a rule based on Kruskal algorithm, and Dutta and Kar (2004) propose another rule based on Prim algorithm. Bergantiños and Vidal-Puga (2004) show that the rule defined by Feltkamp et al. can also be defined as the Shapley value of a different cooperative game. We call this rule $\check{\varphi}$.

Bergantiños and Vidal-Puga (2004) show that $\check{\varphi}$ satisfies a number of appealing properties. For example:

Core selection No subset of agents can reduce their cost by themselves.

Strong cost monotonicity If the cost of an arc increases, no agent is better-off.

Population monotonicity No agent is worse when a new node joins the network.

Equal share of extra costs In problems in which the most expensive arcs are those adjacent to the source, an increase in the cost of these arcs is equally shared by all the agents.

Equal contributions The impact of the connection of agent j on agent's i cost coincides with the impact of the connection of agent i on agent's j cost.

For a detailed description of these properties, the reader is referred to Bergantiños and Vidal-Puga (2004).

Moreover, Bergantiños and Vidal-Puga (2004) show that $\check{\varphi}$ is characterized by strong cost monotonicity, population monotonicity, and equal share of extra costs. Moreover, it is the only rule that satisfies equal contributions.

Another problem is to find non-cooperative mechanisms in which players, by acting as utility-maximizers, agree on how to share the cost of an efficient graph. This problem is not trivial. Bergantiños and Lorenzo (2004) study a real-life problem in which the agents connect inefficiently to the source. Moreover, players may have budget restrictions. The problem with budget restrictions in addressed in Bergantiños and Lorenzo (2003).

In this paper, we design a non-cooperative mechanism in which players agree on an optimal tree. Moreover, the equilibrium cost is given by $\check{\varphi}$.

The idea of the mechanism is quite simple: in a first stage, agents offer prices to each other. These prices represent the amount that the agents are willing to pay to other agents if they connect. Then, the agent with maximum net offer is asked to connect to the source. In order to get efficiency, we still allow the agent to connect to the source through other nodes. Thus, the chosen agent is allowed to propose a graph that connects him to the source. If all the affected agents agree, these nodes connect to the source and the process is repeated with the rest of the players. Otherwise, the proposer should connect to the source on his own.

The choice of a particular agent by means of his net offer has been previously used in the literature of implementation. For example, Pérez-Castrillo and Wettstein (2001, 2002), Mustuswami, Pérez-Castrillo and Wettstein (2004), Vidal-Puga and Bergantiños (2003), Vidal-Puga (2002), and Porteiro (2003). In these paper, the offers are bids for the right to make a 'take-it-or-leave-it-offer'. Thus, the proposer should pay to the other players. As opposed, in this paper the offers establish a *status quo* that points the node that should connect. Moreover, the chosen agent receives in exchange the proposal offered by the other agents.

This model can be easily extended to the case in which players have budget restrictions. By budget restrictions, we mean a wide range of reasons under which players whould not pay any price to be connected. This may happen because some agents have not enough money to pay high cost. Another possible reason is that the utility of being connected might not be enough to compensate agents if they have to pay high connection costs.

In order to deal with cases in which agents have budget restrictions, we give the proposer the choice to declare himself as insolvent. When a player is declared insolvent, he remains as a passive player. A passive player cannot make offers, but he still can receive them. The negotiation goes on with a set of active players who make offers and proposals and vote them, and a set of passive players who vote them when affected.

The paper is organized as follows. In Section 2 we present the notation. In Section 3 we introduce the mechanism. In Sections 4 and 5 we study the case with budget restrictions and prove the main result. In Section 6 we show that Steiner trees can be considered as a special case of problems with budget restrictions.

2 Notation

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. We are interested in networks whose nodes are elements of a set $N_0 = N \cup 0$, where $N \subset \mathcal{N}$ is finite and 0 is a distinguished node called the *source*. Our interest lies on networks where each node in N is (directly or indirectly) connected to the source.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$, and $c_{ii} = 0$ for each $i \in N_0$. The pair (N_0, C) determines a *(cost spanning) problem* for N.

We denote by $\mathcal{C}^N$ the set of all cost matrices for N.

We denote by $\mathcal{G}^N$ the set of all networks whose nodes are N_0. The elements of $g \in \mathcal{G}^N$ are called *arcs*. Given a graph g and a pair of nodes i and j, a *path* from i to j is a sequence $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$ and $j = i_l$. We say that the node i is *connected* in the graph g if there exists a path from i to the source. If $(i, j) \in g$, we say that i and j are *directly linked* in g. Players $i, j \in N$ are *linked* if there exists a path from i to j which do not include the source.

We denote by $\mathcal{G}_0^N \subset \mathcal{G}^N$ the set of *spanning graphs* in N, i.e. the set of graphs such that every agent in N is connected to the source.

Let $g \in \mathcal{G}^N$. We denote by $S(g) \subset N$ the set of nodes which are connected in g. We denote by $D(g)$ the set of nodes which are not connected in g.

Given $g \in \mathcal{G}^N$, we define the *cost* of C associated to g as

$$c(C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there are no ambiguities, we write $c(g)$ instead of $c(C, g)$.

Given $C \in \mathcal{C}^N$, we define the cost associated to C as

$$c(C) = \min_{g \in \mathcal{G}_0^N} c(g).$$

We are interested in assigning the cost of forming a network among the players. This motivates the following definition: A *(cost allocation) rule* for N is a function $\psi : \mathcal{C}^N \to \mathbb{R}^N$ satisfying $\sum_{i \in N} \psi_i(C) \geq c(C)$. Each $\psi_i(C)$ represents the cost assigned to player i for building the network.

A *minimum cost spanning tree* (*mcst*) for C is a tree $t \in \mathcal{G}_0^N$ such that $c(t) = c(C)$. It is clear that the *mcst* exists for each C, even though it does not need to be unique.

Given $C \in \mathcal{C}^N$ and $S \subset N$, we denote by $C^S = \left(c_{ij}^S\right)_{i,j \in S}$ the cost matrix obtained from C assuming that agents of $T = N \backslash S$ have left. This means that C^S is the restriction of C to S_0; i.e. $c_{ij}^S = c_{ij}$ for all $i, j \in S_0$.

Two interesting properties of a rule are:

Efficiency A rule ψ is *efficient* if $\sum_{i \in N} \psi_i(C) = c(C)$.

Core selection A rule ψ satisfies *core selection* if $\sum_{i \in S} \psi_i(C) \leq c\left(C^S\right)$ for all $S \subset N$.

It is known that there always are allocations $x \in \mathbb{R}^N$ satisfying $\sum_{i \in S} x_i \leq c\left(C^S\right)$ for all $S \subset N$. If a rule always selects one of these elements, no subset of players would ever have incentives to refuse the proposed allocation and find another one decreasing their cost.

Or course, core selection implies efficiency.

Given $C \in \mathcal{C}^N$ and $S \subset N$, we denote by $C^{+S} = \left(c_{ij}^{+S}\right)_{i,j \in S}$ the cost matrix obtained from C assuming that agents of S have already connected (and thus,

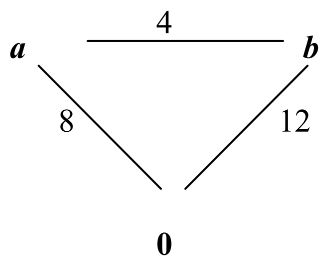

Figure 1: $\hat{v}^C(a) = 8, \hat{v}^C(b) = \hat{v}^C(ab) = 12; \check{v}^C(a) = \check{v}^C(b) = 4, \check{v}^C(ab) = 12$

they have become sources themselves). This means that $c_{ij}^{+S} = c_{ij}$ for each $i, j \in N \backslash S$ and $c_{i0}^{+S} = \min_{j \in S_0} c_{ij}$ for each $i \in N \backslash S$.

We associate to C the TU games $(N, \hat{v}^C)$ and $(N, \check{v}^C)$ where

$$\begin{aligned} \hat{v}^C(S) &= c\left(C^S\right) \\ \check{v}^C(S) &= c\left(C^{+(N \backslash S)}\right). \end{aligned}$$

Note that $\hat{v}^C(S)$ is the minimal cost of connecting all agents to the source without counting on players in $N \backslash S$ (see Figure 1). On the other hand, $\check{v}^C(S)$ is the cost of connecting all agents to the source assuming that agents of $N \backslash S$ are already connected. We denote by $\hat{\varphi}(C)$ and $\check{\varphi}(C)$ the Shapley value (Shapley, 1953) of the games $(N, \hat{v}^C)$ and $(N, \check{v}^C)$, respectively. Of course, both $\hat{\varphi}$ and $\check{\varphi}$ are efficient rules for N.

For simplicity, when there are no other cost matrices present, we denote $\hat{v}, \check{v}, \hat{\varphi}$ and $\check{\varphi}$ instead of $\hat{v}^C, \check{v}^C, \hat{\varphi}(C)$ and $\check{\varphi}(C)$, respectively.

3 The non-cooperative mechanism

In each round, there exists a set A of active players and a cost matrix $C \in \mathcal{C}^N$. In the first round, $A = N$. We denote by $\mathbb{M}_0(A, C)$ the mechanism played with these elements.

In each round, each active player $i \in A$ proposes a vector $\left(x_i^j\right)_{j \in A \backslash i}$ such that $x_i^j \in \mathbb{R}_+$ represents the payoff that player i is willing to pay to player j so that player j connects to the source. We define the *net proposal* of player i as the difference between what other players offer to player i and what player i offers to other players, namely

$$X(i) := \sum_{j \neq i} x_j^i - \sum_{j \neq i} x_i^j.$$

One of the players with the highest net proposal is then randomly chosen as proposer. Assume player α is chosen. Then, player α receives x_i^α from each

$i \in A \backslash \alpha$. Now, player α proposes a (maybe incomplete) graph $g \in \mathcal{G}^N$ such that $\alpha \in S(g)$, and a vector of $y \in \mathbb{R}^A$ such that $\sum_{i \in A} y_i = c(C, g)$. The vector y represents the contribution that each player in A should give for the construction of the graph g.

If all players in A accept this proposal (they are asked in some prespecified order), then the graph g is formed and each player $i \in A$ pays y_i. These players play the mechanism $\mathbb{M}_0\left(D(g), C^{+S(g)}\right)$.

Thus, the final payoff for player $i \in A$ is as follows:

- if $i = \alpha$, he pays $y_\alpha - \sum_{j \in A \backslash \alpha} x_j^\alpha$,
- if $i \in S(g)$, he pays $x_i^\alpha + y_i$,
- if $i \in D(g)$, he pays $x_i^\alpha + y_i$ plus the payoff associated to playing $\mathbb{M}_0\left(D(g), C^{+S(g)}\right)$.

If at least a player in A rejects the proposal, then player α should form a link with the source and leave the game. The rest of the players play the game $\mathbb{M}_0(A \backslash \alpha, C^{+\alpha})$. Thus, the final cost for player $i \in A$ is as follows:

- if $i = \alpha$, he pays $c_{\alpha 0} - \sum_{j \in A \backslash \alpha} x_j^\alpha$
- if $i \neq \alpha$, he pays x_i^α plus the cost associated to playing the game $\mathbb{M}_0(A \backslash \alpha, C^{+\alpha})$.

The mechanism goes on until all the players are connected, i.e. $D(g) = \emptyset$.

Theorem 1 *There exists a unique subgame perfect equilibrium payoff in the bargaining mechanism, and it is given by $\check{\varphi}$.*

Proof. It is an immediate consequence of Theorem 7 and Proposition 8 in Section 5. ■

4 The problem with budget restrictions

In the previous sections we implicitly assumed that players value the connection to the source enough so that individual rationality is guaranteed, i.e. if player i values in $v_i \in \mathbb{R}$ the connection to the source, then $v_i \geq \check{\varphi}_i$. There may be situations, however, in which the connection to the source is not profitable for some players. This may be due to lack of liability by the players, or that they simply do not value the connection to the source enough so that it is not efficient for them to connect. However, it may also be efficient that some players connect to the source, even if they do not value this connection. For example, see the example in Figure 2. The numbers in brackets indicate each player's value to connection to the source. The rule for this problem is $(6, 6)$ but player a is not willing to pay it. However, player b still has incentives to pay to player a and use his link to the source.

In any case, we should focus the situation as a problem of assignment of value, and not of cost. The value that a player $i \in N$ assigns to the graph

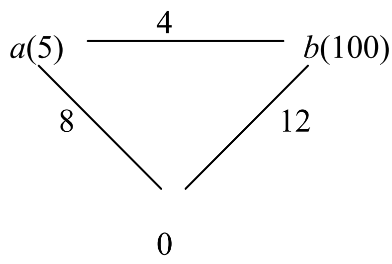

Figure 2: A problem with budget restrictions.

$g \in \mathcal{G}^N$ is given by a function $v_i : \mathcal{G}^N \to \mathbb{R}$. In the case of the cost spanning problem, this function is given by

$$v_i(g) = \begin{cases} v_i & \text{if } i \in S(g) \\ 0 & \text{if } i \in D(g). \end{cases}$$

for some $v \in \mathbb{R}^N$. Given $i \in N$, we will assume that $v_i \neq c_{ij}$ for all $j \in N_0$.

Instead of assigning costs, we assign utilities. For example, let $\breve{w}^C$ be TU game defined by

$$\breve{w}^C(S) = \max_{g \in \mathcal{G}^S} \left[\sum_{i \in S} v_i(g) - c\left(C^{+(N \setminus S)}, g \right) \right] \tag{1}$$

for all $S \subset N$. When agents have no budget restrictions, this game is equivalent (up to a linear transformation of utilities) to the dual of $\check{v}$, and thus their Shapley values are also equivalent. However, in the example of Figure 2 (with budget restrictions) the game $\breve{w}$ is given by $\breve{w}^C(a) = 1$, $\breve{w}^C(ab) = 96$, and $\breve{w}^C(ab) = 93$. Thus, the Shapley value assigns -1 to player a, i.e. player a should pay 6 for the connection, even though he only values it in 5.

In this section, we propose a modification of the value so that this problem is solved.

A *solution* is a function that assigns to each (generalized) cost spanning problem (N, C, v) a vector in $\mathbb{R}^N$. We are interested in solutions which satisfy at least the following properties:

Feasibility For each problem (N, C, v), there exists a graph $g \in G^N$ such that

$$\sum_{i \in N} f_i = \sum_{i \in N} v_i(g) - c(C, g)$$

Efficiency For each problem (N, C, v),

$$\sum_{i \in N} f_i = \max_{g \in \mathcal{G}^N} \left[\sum_{i \in N} v_i(g) - c(C, g) \right].$$

Individual rationality For each problem (N, C, v) and each $i \in N$,

$$f_i \geq 0.$$

For example, let $\breve{\varphi}$ be the Shapley value of the TU-game $\left(N, \breve{w}^C\right)$ defined by (1). Then, $\breve{\varphi}$ is a feasible and efficient solution. However, it does not satisfy individual rationality.

Consider again the example in Figure 2. In the optimistic version of the Shapley value (games $\breve{v}$ and $\breve{w}$), player b would connect to the source at a cost of 4, because it is assumed that player a had already connected to the source. In the pessimistic version (game $\hat{v}$) this cost would be 12. Since player a has now limited liability, we cannot assume that player a would connect to the source without player b. Thus, we take an intermediate approach. We will assume that the cost of a coalition is computed assuming that the other players had connected to the source (like in the optimistic version) *but only if they find it profitable.*

For example, in Figure 2, player b would connect to the source at a cost of 7. This value is computed as follows: player a values its connection in 5, and player b finds it profitable to pay the cost 12 of the two links joining him and the source through player a. Then, 7 is the difference between 12 and 5. On the other hand, the cost for player a would be 4, because player b finds it profitable to connect even when player a is not present. In terms of utilities, the game would be defined as follows:

$$\begin{aligned}\bar{w}(a) &= 1\\ \bar{w}(b) &= 93\\ \bar{w}(ab) &= 93\end{aligned}$$

and the Shapley value would be $\bar{\varphi} = (0.5, 92.5)$; i.e. the *mcst* is formed at a price of 12, player a pays 4.5 and player b pays 7.5.

Notice that, even though $\bar{w}$ is a more realistic game than $\breve{w}$, they share the same optimistic philosophy. The cost of a player is computed assuming that the other player, if not connected to the source, will allow not only to connect to the source through him, but he will also pay his value to be connected. We will call this kind of players *passive players.* The rest of the players are called *active players.*

We will now define $\bar{w}$ formally. In the general case, there may be many passive players. We define $\bar{w}$ depending on the partition $\{A, P\}$ of N, where A is the set of active players and P is the set of passive players.

Given $A, P \subset N$, we define

$$\mathcal{G}_A^N = \left\{g \in \mathcal{G}^N : A \cap S(g) \neq \emptyset\right\} \cup \{\emptyset\}$$

and

$$\bar{\omega}(A, P, C, v) = \max\left\{\sum_{i \in N} v_i(g) - c(C, g) : g \in \mathcal{G}_A^N\right\}.$$

Note that the maximizer graph may be the empty graph. In particular, if $A = \emptyset$, then $\mathcal{G}_A^N = \{\emptyset\}$ and thus $\bar{\omega}(A, P, C, v) = 0$.

We now define $\bar{\varphi}$ by induction on the number of active players. If $A = \emptyset$, then $\bar{\varphi}_i(A, P, C, v) = 0$ for all $i \in N$.

Assume we have defined $\bar{\varphi}_i(A', P', C', v)$ for all A' with less than $|A|$ active players.

Given $A, P \subset N$ and $C \in \mathcal{C}^N$ we will compute $\bar{\varphi}(A, P, C, v)$ in terms of equal contributions. The property of equal contributions implies that, given any pair of players $i, j \in A$, the gains of a player i when player j leaves the game equal the gains of player i when player i leaves the game[1]. A player leaves the game in two possible forms: He may be connected to the source (as in the optimistic value) or he becomes passive. He would only connect to the source if it is profitable to do so. Formally, let $d_j^i(A, P, C) \in \mathbb{R}^{S \setminus i}$ be the value of player $j \in N \setminus i$ when player $i \in A$ leaves in one of the above ways. Then,

$$d^i(A, P, C, v) = \begin{cases} \bar{\varphi}\left(A \setminus i, P, C^{+i}, v\right) & \text{if } c_{i0} \leq v_i \\ \bar{\varphi}(A \setminus i, P \cup i, C, v) & \text{if } c_{i0} > v_i \end{cases}$$

Note that both $\bar{\varphi}\left(A \setminus i, P, C^{+j}\right)$ and $\bar{\varphi}(A \setminus i, P \cup j, C)$ are well-defined by induction hypothesis.

For simplicity, we denote $d^i \in \mathbb{R}^{N \setminus i}$ instead of $d^i(A, P, C)$.

Now, we define $\bar{\varphi}(A, P, C, v)$ as

$$\bar{\varphi}_i(A, P, C, v) = \frac{1}{|A|}\left[\bar{w}(A, P, C, v) + \sum_{j \in A \setminus i} d_i^j - \sum_{j \in A \setminus i} d_j^i\right] \tag{2}$$

for all $i \in A$, and

$$\bar{\varphi}_i(A, P, C, v) = 0$$

for all $i \in P$.

Remark 2 *The formula given by (2) resembles those of the Shapley value that appears in Maschler and Owen (1989) and Hart and Mas-Colell (1989). However, $\bar{\varphi}$ is not the Shapley value of a game form. It depends on the set of active and passive players.*

For simplicity, we denote $\bar{\varphi}$ or $\bar{\varphi}(C)$ instead of $\bar{\varphi}(A, P, C, v)$.

We know state our property of equal contributions as follows:

Generalized Equal Contributions (GEC) For any A, P, and any $i, j \in A$,

$$d_i^j - \bar{\varphi}_i = d_j^i - \bar{\varphi}_j.$$

The next property is a generalization of efficiency:

[1] We consider that a player leaves the game in an optimistic way, that is why there are gains.

Generalized Efficiency (GEff) For any generalized problem (A, P, C, v),

$$\sum_{i \in N} f_i = \bar{w}(A, P, C, v).$$

A tuple (A, P, C, v) is called a *generalized problem*, and it generalizes the problems with budget restrictions (when $P = \emptyset$). A *generalized value* is a function which assign to each (A, P, C, v) a vector of utilities $f \in \mathbb{R}^N$. Notice that a generalized value f in the class of generalized problems induces a value in the class of problems with budget restrictions: Take $g(N, C, v) = f(N, \emptyset, C, v)$.

GEC is a very strong property and no value satisfies both GEff and GEC. We consider then the following weaker form of GEC:

Weak Generalized Equal Contributions (WGEC) For any A, P, and any $i \in A$,

$$\sum_{j \in A \backslash i} \left(d_i^j - \bar{\varphi}_i\right) = \sum_{j \in A \backslash i} \left(d_j^i - \bar{\varphi}_j\right).$$

Lemma 3 *Let $i \in A$ be such that there exists an optimal graph g with $i \notin S(g)$. Then,*

$$\bar{\varphi}(A, P, C, v) = \bar{\varphi}(A \backslash i, P \cup i, C, v).$$

Proof. Note first that $c_{i0} > v_i$. Otherwise, any optimal graph should include player i.

Given $i, j \in A$, we define

$$d^{ij} := \begin{cases} d^i\left(A \backslash j, P, C^{+ij}\right) & \text{if } c_{j0} \leq v_j \\ d^i\left(A \backslash j, P \cup j, C^{+i}\right) & \text{if } c_{j0} > v_j. \end{cases}$$

We proceed by induction on $|A|$. If $A = \{i\}$, then $\bar{w}(A, P, C) = 0$ and the result holds. Assume now the result is true for less than $|A|$ active players. We need to prove that, for all $k \in A \backslash i$,

$$\frac{1}{|A|}\left[\bar{w}(A, P, C, v) + \sum_{j \in A \backslash k} d_k^j - \sum_{j \in A \backslash k} d_j^k\right] = \bar{\varphi}_k(A \backslash i, P \cup i, C, v).$$

Let g be an optimal graph such that $i \notin S(g)$. Given any $j \in A \backslash i$, we see two cases:

- If $c_{j0} < v_j$, then $j \in S(g)$ and $d^j = \bar{\varphi}\left(A \backslash j, P, C^{+j}, v\right)$. Moreover, $c_{ij} > v_i$. Now, the graph g has a cycle in C^{+j}. By removing the most expensive arc of this cycle, we get an optimal graph g' such that $i \notin S(g)$. By induction hypothesis, $d^j = d^{ji}$.

- If $c_{j0} > v_j$, then $d^j = \bar{\varphi}(A \backslash j, P \cup j, C, v)$ and g is still a *mcst* such that $i \notin S(g)$. By induction hypothesis, $d^j = d^{ji}$.

Moreover, $d_i^k = 0$ and $d_k^i = \bar{\varphi}_k (A \backslash i, P \cup i, C)$. Thus,

$$
\begin{aligned}
& \frac{1}{|A|}\left[\bar{w}(A, P, C, v) + \sum_{j \in A \backslash k} d_k^j - \sum_{j \in A \backslash k} d_j^k\right] \\
= & \frac{1}{|A|}\left[\bar{w}(A, P, C, v) + \sum_{j \in A \backslash k, i} d_k^{ji} + d_k^i - \sum_{j \in A \backslash k, i} d_j^{ki} - d_i^k\right] \\
= & \frac{1}{|A|}\left[\bar{w}(A \backslash i, P \cup i, C, v) + \sum_{j \in A \backslash k, i} d_k^{ji} - \sum_{j \in A \backslash k} d_j^{ki} + \bar{\varphi}_k (A \backslash i, P \cup i, C, v)\right] \\
= & \frac{1}{|A|}\left[(|A| - 1)\, \bar{\varphi}_k (A \backslash i, P \cup i, C, v) + \bar{\varphi}_k (A \backslash i, P \cup i, C, v)\right] \\
= & \bar{\varphi}_k (A \backslash i, P \cup i, C, v).
\end{aligned}
$$

■

Proposition 4 *$\bar{\varphi}$ satisfies individual rationality, i.e. $\bar{\varphi}_i (A, P, C, v) \geq 0$ for all $i \in N$.*

Proof. We proceed by induction on the number of active players $|A|$. If $|A| = 0$, then the result is trivial. Assume the result is true for less than $|A|$ active players. Let $i \in N$. If $i \in P$, it is clear that $\bar{\varphi}_i = 0$. Assume then $i \in A$. We see two cases,

- if $c_{i0} < v_i$, then $d_j^i = \bar{\varphi}_j \left(A \backslash i, P, C^{+i}, v\right)$ for all $j \in N \backslash i$. Since $\bar{\varphi}$ satisfies GEff, we know that $\sum_{j \in A \backslash i} d_j^i = \bar{w}\left(A \backslash i, P, C^{+i}, v\right)$. But $\bar{w}(A, P, C, v) \geq \bar{w}\left(A \backslash i, P, C^{+i}, v\right)$ and thus

$$
\begin{aligned}
\bar{\varphi}_i &= \frac{1}{|A|}\left[\bar{w}(A, P, C, v) + \sum_{j \in A \backslash i} d_i^j - \sum_{j \in A \backslash i} d_j^i\right] \\
&\geq \frac{1}{|A|} \sum_{j \in A \backslash i} d_i^j \geq 0
\end{aligned}
$$

- if $c_{i0} > v_i$, then $d_j^i = \bar{\varphi}_j (A \backslash i, P \cup i, C, v)$ for all $j \in N \backslash i$ and moreover $\bar{w}(A, P, C, v) = \bar{w}(A \backslash i, P \cup i, C, v)$. Since $\bar{\varphi}_i (A \backslash i, P \cup i, C, v) = 0$,

$$
\begin{aligned}
\bar{\varphi}_i &= \frac{1}{|A|}\left[\bar{w}(A, P, C, v) + \sum_{j \in A \backslash i} d_i^j - \sum_{j \in A \backslash i} d_j^i\right] \\
&= \frac{1}{|A|}\left[\bar{w}(A \backslash i, P \cup i, C, v) + \sum_{j \in A \backslash i} d_i^j - \sum_{j \in N} \bar{\varphi}_j (A \backslash i, P \cup i, C, v)\right] \\
&= \frac{1}{|A|} \sum_{j \in A \backslash i} d_i^j \geq 0.
\end{aligned}
$$

■

Our value is characterized by GEff, WGEC and individual rationality:

Proposition 5 *There exists a unique generalized value which satisfies GEff, WGEC and individual rationality, and it is* $\bar{\varphi}$*.*

Proof. We know that $\bar{\varphi}$ satisfies individual rationality. We check that $\bar{\varphi}$ satisfies GEff:

$$\sum_{i\in N}\bar{\varphi}_i=\sum_{i\in A}\bar{\varphi}_i=\bar{w}\left(A,P,C,v\right)+\frac{1}{|A|}\left(\sum_{i,j\in A}d_i^j-\sum_{i,j\in A}d_j^i\right)=\bar{w}\left(A,P,C,v\right).$$

We check now that $\bar{\varphi}$ satisfies WGEC.

Given $i\in A$, by definition

$$\begin{aligned}
|A|\bar{\varphi}_i &= \bar{w}\left(A,P,C,v\right)+\sum_{j\in A\setminus i}d_i^j-\sum_{j\in A\setminus i}d_j^i\\
\left(|A|-1\right)\bar{\varphi}_i-\sum_{j\in A\setminus i}d_i^j &= \bar{w}\left(A,P,C,v\right)-\bar{\varphi}_i-\sum_{j\in A\setminus i}d_j^i\\
\sum_{j\in A\setminus i}\bar{\varphi}_i-\sum_{j\in A\setminus i}d_i^j &= \sum_{j\in A\setminus i}\bar{\varphi}_j-\sum_{j\in A\setminus i}d_j^i
\end{aligned}$$

We check now that $\bar{\varphi}$ is the unique generalized value satisfying GEff and WGEC. Let $i\in N$. Let σ a generalized value satisfying GEff, WGEC and individual rationality. Then,

$$\begin{aligned}
\sum_{j\in A\setminus i}\sigma_i-\sum_{j\in A\setminus i}d_i^j &= \sum_{j\in A\setminus i}\sigma_j-\sum_{j\in A\setminus i}d_j^i\\
\left(|A|-1\right)\sigma_i-\sum_{j\neq i}d_i^j &= \bar{w}\left(A,P,C,v\right)-\sigma_i-\sum_{j\neq i}d_j^i\\
|A|\sigma_i &= \bar{w}\left(A,P,C,v\right)+\sum_{j\neq i}d_i^j-\sum_{j\neq i}d_j^i
\end{aligned}$$

which is the formula given by (2). Moreover, $\sum_{i\in A}\sigma_i=\bar{w}\left(A,P,C,v\right)$ and thus GEff implies $\sum_{i\in P}\sigma_i=0$. By individual rationality, $\sigma_i\geq 0$ for all $i\in P$ and hence $\sigma_i\geq 0$ for all $i\in P$. ■

Remark 6 *The three properties are independent. Let f be the value $f_i=0$ for all $i\in N$. Let g be the value $g_i=\bar{w}\left(A,P,C,v\right)/n$ for all $i\in N$. Let h be the value $h_i=\bar{\varphi}_i$ for all $i\in A$, $h_i=1-|P|$ if $i=\min P$, and $h_i=1$ if $i\in P$ and $i>\min P$. Then, it is straightforward to check the following table:*

	GEff	*WGEC*	*ind.rat.*
f	*NO*	*YES*	*YES*
g	*YES*	*NO*	*YES*
h	*YES*	*YES*	*NO.*

5 The generalized mechanism

We restate the mechanism as follows: In each round, there exists a set A of active players, a set P of passive players, and a cost matrix $C \in \mathcal{C}^N$. In the first round, $A = N$ and $P = \emptyset$. We denote by $\mathbb{M}(A, P, C)$ or $\mathbb{M}(C)$ the mechanism played with these elements.

In each round, each active player $i \in A$ offers a vector $\left(x_i^j\right)_{j \in A \backslash i}$ such that $x_i^j \in \mathbb{R}_+$ represents the payoff that player i is willing to pay to player j so that player j connects to the source. We define the *net offer* of player i as the difference between what other players offer him and what he offers to other players, namely

$$X(i) = \sum_{j \in A \backslash i} x_j^i - \sum_{j \in A \backslash i} x_i^j.$$

One of the players with the highest net proposal is then randomly chosen as proposer. Assume player α is chosen. Then, player α receives x_i^α from each $i \in A \backslash \alpha$. Now, player α proposes a (maybe incomplete) graph $g \in \mathcal{G}^N$ such that $\alpha \in S(g)$, and a vector of $y \in \mathbb{R}^{A \cup S(g)}$ such that $\sum_{i \in A \cup S(g)} y_i = c(C, g)$. The vector y represents the contribution that each player in $A \cup S(g)$ should give for the construction of the graph g.

If all players in $A \cup S(g)$ accept this proposal (they are asked in some pre-specified order), then the graph g is formed and each player $i \in A \cup S(g)$ pays y_i. The players in $D(g)$ play the mechanism $\mathbb{M}\left(C^{+S(g)}\right)$.

Thus, the final payoff for player $i \in N$ is as follows:

- if $i = \alpha$, he gets $v_\alpha + \sum_{i \in A \backslash \alpha} x_i^\alpha - y_\alpha$;
- if $i \in A \backslash \alpha$ and $i \in S(g)$, he gets $v_i - x_i^\alpha - y_i$;
- if $i \in A \backslash \alpha$ and $i \in D(g)$, he gets $-x_i^\alpha - y_i$ plus the payoff associated with playing $\mathbb{M}\left(C^{+S(g)}\right)$;
- if $i \in P$ and $i \in S(g)$, he gets $v_i - y_i$;
- if $i \in P$ and $i \in D(g)$, he gets he payoff associated with playing $\mathbb{M}\left(C^{+S(g)}\right)$.

If at least a player in $A \cup S(g)$ rejects the proposal, then player α can declare himself as solvent or insolvent.

If player α declare himself as solvent, then he forms a link with the source and leaves the game. The rest of the players play the game $\mathbb{M}(A \backslash \alpha, P, C^{+\alpha})$. Thus, the final payoff for player $i \in N$ is as follows:

- if $i = \alpha$, he gets $v_\alpha + \sum_{i \in A \backslash \alpha} x_i^\alpha - c_{\alpha 0}$;
- if $i \in A \backslash \alpha$, he gets the payoff associated with playing the game $\mathbb{M}(A \backslash \alpha, P, C^{+\alpha})$ minus x_i^α;
- if $i \in P$, he gets the payoff associated with playing the game $\mathbb{M}(A \backslash \alpha, P, C^{+\alpha})$.

If player α declare himself as insolvent, then he becomes passive and the game $\mathbb{M}(A\backslash\alpha, P\cup\alpha, C)$ is played. The final payoff for player $i\in N$ is as follows:

- if $i=\alpha$, he gets $\sum_{j\neq\alpha} x_j^\alpha$ plus the payoff associated with playing the game $\mathbb{M}(A\backslash\alpha, P\cup\alpha, C)$;
- if $i\in A\backslash\alpha$, he gets the payoff associated with playing the game $\mathbb{M}(A\backslash\alpha, P\cup\alpha, C)$ minus x_i^α;
- if $i\in P$, he gets the payoff associated with playing the game $\mathbb{M}(A\backslash\alpha, P\cup\alpha, C)$.

The mechanism goes on until there are no more active players. In this case, no additional links are formed, and each passive player remains disconnected. The payoff is then zero for everyone.

Theorem 7 *There exists a unique subgame perfect equilibrium payoff in the bargaining mechanism, and it is given by* $\bar{\varphi}(N,\emptyset,C,v)$.

Proof. We will prove the following stronger result:

There exists a unique subgame perfect equilibrium payoff in the subgame $\mathbb{M}(A,P,C)$, and it is given by $\bar{\varphi}(A,P,C,v)$.

The proof proceeds by induction on the number of active players $|A|$. The result trivially holds for $|A|=0$.

We now assume that the theorem holds for less than $|A|$ active players and show that it also holds for $|A|$ active players. We first prove that $\bar{\varphi}$ is indeed an equilibrium outcome. We explicitly construct an equilibrium that yields $\bar{\varphi}$ as outcome.

Consider the following set of strategies for the players in $\mathbb{M}(A,P,C)$:

If $i\in A$ and there does not exist any efficient graph including player i, he proposes $x_i^j=0$ to each $j\in A\backslash i$ and he declares insolvent should he be chosen as proposer.

Assume $i\in S(g)$ for some optimal graph g. If $i\in A$, then he proposes $x_i^j = d_i^j-\bar{\varphi}_i$ to each $j\in A\backslash i$. If player i is chosen as proposer, he proposes (g,y) such that $y\in\mathbb{R}^{A\cup S(g)}$ is given by $y_j=v_j(g)-d_j^i$ if $j\neq i$ and $y_i=c(g,C)-\sum_{j\neq i} y_j$.

If player $i\in A$ is chosen as proposer and his proposal is rejected, he declares solvent iff $c_{i0}\leq v_i$.

Assume player $\alpha\neq i$ is chosen as proposer and makes a proposal (g,y). If $i\in A\cup S(g)$, then player i accepts the proposal iff $v_i(g)-y_i\geq d_i^a$.

Assume the proposer is α and proposes an optimal graph g. Given $i\in N\backslash\alpha$, we check that these strategies yield $\bar{\varphi}_i$ for player i.

- If $i\in(A\cap S(g))\backslash\alpha$, then he gets $v_i-y_i-x_i^\alpha=v_i-(v_i-d_i^\alpha)-(d_i^\alpha-\bar{\varphi}_i)=\bar{\varphi}_i$;
- if $i\in P\cap S(g)$, then he gets $v_i-y_i=v_i-(v_i-d_i^\alpha)=d_i^\alpha=0=\bar{\varphi}_i$;
- if $i\in A\cap D(g)$, then by Lemma 3 he gets $\bar{\varphi}_i\left(C^{+S(g)}\right)-x_i^\alpha=0=\bar{\varphi}_i$;

- if $i \in P \cap S(g)$, then by Lemma 3 he gets $\bar{\varphi}_i \left(C^{+S(g)}\right) = 0 = \bar{\varphi}_i$.

Moreover, given that following the strategies an optimal graph is formed, the proposer also obtains $\bar{\varphi}_\alpha$.

We now show that all net offers $X(i)$ are equal to zero. Following the above mentioned strategies, by WGEC,

$$X(i) = \sum_{j \in A \backslash j} x_i^j - \sum_{j \in A \backslash j} x_j^i = \sum_{j \in A \backslash j} \left(d_i^j - \bar{\varphi}_i\right) - \sum_{j \in A \backslash j} \left(d_j^i - \bar{\varphi}_j\right) = 0.$$

To check that the previous strategies constitute an equilibrium note, first, that the strategies after the proposer α is chosen are best responses. By induction hypothesis, the final payoff after rejection is d_i^α for all $i \in N \backslash \alpha$. If the proposal is accepted, the final payoff is $v_i(g) - y_i = d_i^\alpha$ for all $i \in N$. Thus, the responders act optimally.

We now check that the proposer has no profitable deviation.

The payoff for the proposer is

$$v_\alpha - y_\alpha = v_\alpha - c(g, C) + \sum_{i \in A \cup S(g) \backslash \alpha} \left[v_i(g) - d_i^\alpha\right].$$

We see two cases:

- If $c_{\alpha 0} < v_\alpha$, then $d^\alpha = \bar{\varphi}(A \backslash \alpha, P, C^{+\alpha})$ and thus

$$\begin{aligned} v_\alpha - y_\alpha &= v_\alpha - c(g, C) + \sum_{i \in A \backslash \alpha} \left[v_i(g) - \bar{\varphi}_i \left(A \backslash \alpha, P, C^{+\alpha}\right)\right] + \sum_{i \in P \cap S(g)} v_i \\ &= \sum_{i \in N} v_i(g) - c(g, C) - \sum_{i \in A \backslash \alpha} \bar{\varphi}_i \left(A \backslash \alpha, P, C^{+\alpha}\right) \\ &= \bar{w}(A, P, C) - \bar{w}\left(A \backslash \alpha, P, C^{+\alpha}\right). \end{aligned}$$

  Assume player α deviates and makes an unacceptable offer. Then, his payoff is $v_\alpha - c_{\alpha 0}$. By induction hypothesis, players in $A \backslash \alpha$ create an optimal tree g' in $C^{+\alpha}$. Let $g'' = g' \cup \{(0, \alpha)\}$. Then, g'' is a tree in C. Thus,

$$\begin{aligned} \bar{w}(A, P, C) &\geq \sum_{i \in N} v_i(g'') - c(g'', C) \\ &= \sum_{i \in N \backslash \alpha} v_i(g') - c\left(g', C^{+\alpha}\right) + v_\alpha - c_{\alpha 0} \\ &= \bar{w}\left(A \backslash \alpha, P, C^{+\alpha}\right) + v_\alpha - c_{\alpha 0} \end{aligned}$$

  from where we deduce that $v_\alpha - y_\alpha \geq v_\alpha - c_{\alpha 0}$ and player α does not improve.

- If $c_{\alpha 0} > v_\alpha$, then $d^\alpha = \bar{\varphi}(A\backslash\alpha, P\cup\alpha, C)$ and thus

$$\begin{aligned} v_\alpha - y_\alpha &= v_\alpha - c(g,C) + \sum_{i\in A\backslash\alpha}\left[v_i(g) - \bar{\varphi}_i(A\backslash\alpha, P\cup\alpha, C)\right] + \sum_{i\in P\cap S(g)} v_i \\ &= \sum_{i\in N} v_i(g) - c(g,C) - \sum_{i\in A\backslash\alpha}\bar{\varphi}_i(A\backslash\alpha, P\cup\alpha, C) \end{aligned}$$

since $c_{\alpha 0} > v_\alpha$ and $\alpha \in S(g)$, we conclude that $\{\alpha\} \subsetneq S(g)$, thus

$$= \sum_{i\in N} v_i(g) - c(g,C) - \bar{w}\left(A\backslash\alpha, P, C^{+\alpha}\right) = 0.$$

Assume player α deviates and makes an unacceptable offer. Then, his final payoff is $v_\alpha - c_{\alpha 0}$. Since $v_\alpha - y_\alpha = 0 > v_\alpha - c_{\alpha 0}$, player a does not improve.

Assume now that player $i \in A$ changes his offers x_i^j. If he decreases his aggregate offer $\sum_{j\in A\backslash i} x_i^j$, he will be chosen as the proposer with certainty, and his final outcome would still equal $\bar{\varphi}_i$. Moreover, any increase in one of his offers will mean that, if he is not the proposer, his final payoff decreases.

We now show that any equilibrium yields $\bar{\varphi}$ as final outcome. We first prove the following claim:

Claim: Assume the proposer is α and he proposes (g,y) such that

$$y_i < v_i - d_i^\alpha$$

for all $i \in S(g)\backslash\alpha$ and

$$y_i < \bar{\varphi}_i\left(C^{+S(g)}\right) - d_i^\alpha$$

for all $i \in A \cap D(g)$. Then, this offer is accepted.

By induction hypothesis, the payoff after rejection for player $i \in A \cup S(g)\backslash\alpha$ is d_i^α. Moreover, if the offer is accepted, the payoff is $v_i - y_i$ for all $i \in S(g)\backslash\alpha$, and $\bar{\varphi}_i\left(C^{+S(g)}\right) - y_i$ for all $i \in A\cap D(g)$. Thus, it is straightforward to check by backward induction that in equilibrium the offer is accepted.

We will prove that, given any $\varepsilon > 0$, any player i can assure himself a final payoff of at least $\bar{\varphi}_i - \varepsilon$.

By Lemma 3, this is true if there exists an optimal graph such that $i \notin S(g)$. Assume then $i \in S(g)$ for each optimal graph g. We consider now the following strategy:

If $i \in A$, then he proposes $x_i^j = d_i^j - \bar{\varphi}_i$ to each $j \in A\backslash i$.

If player $i \in A$ is chosen as proposer, he proposes (g,y) such that g is an optimal graph. Moreover, $y \in \mathbb{R}^{A\cup S(g)}$ is given by

$$y_j = v_j - d_j^i - \frac{\varepsilon}{|A \cup S(g) - 1|}$$

for all $j \in S(g) \backslash i$ and

$$\begin{aligned} y_j &= \bar{\varphi}_j\left(C^{+S(g)}\right) - d_j^i - \frac{\varepsilon}{|A \cup S(g) - 1|} \\ &= -d_j^i - \frac{\varepsilon}{|A \cup S(g) - 1|} \end{aligned}$$

for all $j \in A \cap D(g)$.

Assume player α is chosen as proposer and makes a proposal (g, y). Then, player i rejects the proposal.

Following this strategy, two things may happen:

1. Player i is not chosen as proposer. Then, he pays $x_i^\alpha = d_i^\alpha - \bar{\varphi}_i$ to the proposer α. Moreover, he rejects the offer made by the proposer and, by induction hypothesis, his payoff afterwards is d_i^α. His final payoff is then $d_i^\alpha - x_i^\alpha = \bar{\varphi}_i$.

2. Player i is chosen as proposer. Then, $\sum_{j \in A \backslash i} x_j^i \geq \sum_{j \in A \backslash i} x_i^j$. Moreover, by the Claim, his offer is accepted. His final payoff is then $v_i + \sum_{j \in A \backslash i} x_j^i - y_i \geq$

$$\begin{aligned} & v_i + \sum_{j \in A \backslash i} x_i^j - y_i \\ = \; & v_i + \sum_{j \in A \backslash i} \left(d_i^j - \bar{\varphi}_i\right) - c(g, C) + \sum_{j \in A \cup S(g) \backslash i} y_j \\ = \; & v_i + \sum_{j \in A \backslash i} \left(d_i^j - \bar{\varphi}_i\right) - c(g, C) + \sum_{j \in S(g) \backslash i} \left(v_j - d_j^i\right) + \sum_{j \in A \cap D(g)} \left[-d_j^i\right] - \varepsilon \\ = \; & \sum_{j \in N} v_i(g) + \sum_{j \in A \backslash i} \left[d_i^j - \bar{\varphi}_i\right] - c(g, C) - \sum_{j \in S(g) \backslash i} d_j^i - \sum_{j \in A \cap D(g)} d_j^i - \varepsilon \\ = \; & \sum_{j \in N} v_i(g) + \sum_{j \in A \backslash i} d_i^j - (|A| - 1)\bar{\varphi}_i - c(g, C) - \sum_{j \in A \cup S(g) \backslash i} d_j^i - \varepsilon \\ = \; & \bar{w}(A, P, C, v) + \sum_{j \in A \backslash i} d_i^j - |A|\bar{\varphi}_i + \bar{\varphi}_i - \sum_{j \in A \cup S(g) \backslash i} d_j^i - \varepsilon \end{aligned}$$

   by definition of $\bar{\varphi}_i$

$$\begin{aligned} = \; & \bar{w}(A, P, C, v) + \sum_{j \in A \backslash i} d_i^j - \bar{w}(A, P, C, v) - \sum_{j \in A \backslash i} d_i^j + \sum_{j \in A \backslash i} d_j^i + \bar{\varphi}_i - \sum_{j \in A \cup S(g) \backslash i} d_j^i - \varepsilon \\ = \; & \bar{\varphi}_i - \sum_{j \in P \cap S(g)} d_j^i - \varepsilon = \bar{\varphi}_i - \varepsilon. \end{aligned}$$

■

Proposition 8 *Assume players have no budget restrictions. Then, the equilibria of the mechanisms* $\mathbb{M}_0$ *and* $\mathbb{M}$ *coincide.*

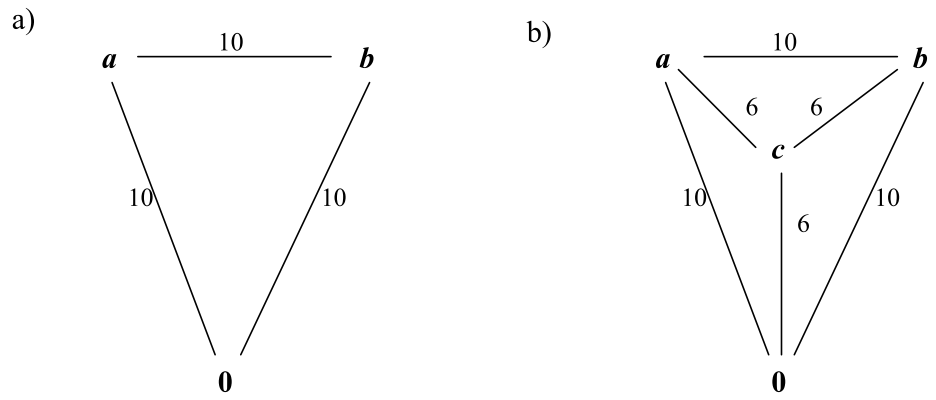

Figure 3: A Steiner tree problem.

Proof. In $\mathbb{M}$, any passive player always get a payoff of 0. This means a negative payoff when players have no budget restrictions. Thus, any strategy that involves to declare oneself as insolvent is strictly dominated by the same strategy but declaring oneself as solvent and connecting to the source. ■

6 Steiner tree problems

The Steiner tree problems arise as cost spanning tree problems in which the arcs can join in any place. For example, in Figure 3a), the cost of connection is 20. However, if the arcs do not need to join in one of the nodes (for example, they are roads or water pipes), we can restate the problem as in Figure 3b). Here, the additional node does not need to be connected, but it would be advisable, because it decreases the minimal cost of connection (from 20 to 18).

In this example, nodes a and b are called *terminal nodes*, and node c is a *Steiner node*.

Formally, a Steiner tree is defined as a minimum-weight tree connecting the terminal nodes, such that the tree may include Steiner nodes.

In fact, we can see a Steiner tree problem as a generalized cost spanning tree problem. We just need to assign a big value to the terminal nodes and 0 to the Steiner nodes (see Figure 4).

Moreover, a possible cost allocation in Steiner trees may be given by $\bar{\varphi}$, with A the set of terminal nodes, and P the set of Steiner nodes. Note that $\bar{\varphi}$ assigns 0 to passive agents.

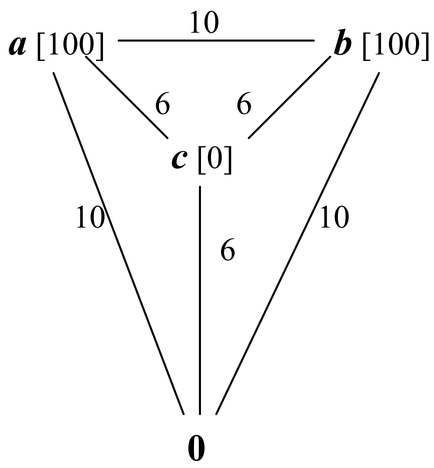

Figure 4: A Steiner tree problem as a generalized cost spanning tree problem.

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