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Correlated Equilibria of Classical Strategic Games with Quantum Signals

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Abstract

Correlated equilibria are sometimes more efficient than the Nash equilibria of a game without signals. We investigate whether the availability of quantum signals in the context of a classical strategic game may allow the players to achieve even better efficiency than in any correlated equilibrium with classical signals, and find the answer to be positive.

1   Introduction

Until recently, the most notable connection between game theory and quantum mechanics was the fact that von Neumann pioneered the mathematical foundations of both fields [10,11]. In recent years a few proposals have appeared in the literature, mostly coming from the theoretical physics community, aimed at finding a common ground for the two theories. Several recent papers have investigated the connections between games with classical mixed strategies and games with quantum strategies [3,8]. Generally, these results have had limited resonance in the game theory community. Unlike the vast majority of real-life strategic situations, in which the available strategies are predetermined and classical, a game with quantum strategies needs to be especially designed and implemented in order to be played. Moreover, it was soon pointed out that any such quantum game can be simulated by a suitably designed classical game (in which the players, say, communicate via telephone their choice of quantum strategy to a center), and hence interest in the field of quantum games has been so far relatively modest.

The aim of the present paper is also to bring together ideas from game theory and quantum mechanics, but following a significantly different approach, which is closest in spirit to [6] and [7]. We take the view that in many games the strategies available to the players are predetermined and classical, but following Aumann [1,2] we postulate the availability of a correlating device able to send

payoff-irrelevant private signals to the players before they play the game. Unlike Aumann's device, which can only send classical information, we also allow for devices able to send sequences of individual quantum signals (qubits) to the players. Today similar devices are amply available, as many practical implementations have been designed and demonstrated. A qubit may be variously encoded in the properties of a physical carrier, including the spin of a neutron, or the polarization of a photon.

Correlated equilibria are often more efficient than the Nash equilibria of a game without signals. This is because the correlation of the signals received by the players may allow them to coordinate towards Pareto-superior expected payoff combinations. We investigate whether allowing the use of quantum signals may allow the players to achieve even better coordination than is possible in any classical correlated equilibrium, and find the answer to be positive. This has some implications on the significance of several well-known impossibility results [9,5,12], which will need to be re-evaluated if quantum correlation is allowed.

The main contribution of this paper is to introduce a novel notion of correlated equilibrium in which the signals sent to the players are represented by quantum bits, and compare its properties with those of classical correlated equilibrium. Classical correlated equilibrium deals with information states which take the form of events in a suitably defined Boolean algebra. By contrast, the information states induced by quantum signals generally do not correspond to a Boolean algebra, but conform to a more general logic calculus in which the distributive laws of propositional logics appear in a weakened form.

Because of its unique features, the field of quantum information theory is currently an object of intense study, together with the rising fields of quantum computation, encryption and communication. The most distinguishing feature of quantum theory, which sets it clearly apart from classical mechanics, is the phenomenon of entanglement [4], by which the observed states of two particles may exibit correlations which cannot be rationalized in terms of classical probabilistic behavior. If two entangled particles are kept isolated from their local environment they will remain in a correlated state until some operation is performed on one of them (say, the spin is observed), in which case the states of both particles will change in a predictably correlated way, even if they have been meanwhile separated in space and time. This phenomenon is non-local: the states of the two particles are jointly affected, even though only one of them is observed.

2 An example

Consider the following game with imperfect information. Two players are randomly drawn from an i.i.d., uniform distribution over a set of three types, $\{A, B, C\}$. They privately observe their own type, then simultaneously decide whether to say "yes" or "no". If two players of the same type agree, i.e. they both say yes or they both say no, each of them receives a payoff of -900. If two players of different types agree they both receive a payoff of 9. If the two players

disagree they both get zero. Due to the catastrophic consequences of agreeing when of the same type, there are no efficient Nash equilibria which give positive probability to those events. All the possible pure strategy combinations which do not lead to catastrophes are (denoting "no" by 0 and "yes" by 1)

player 1	A	B	C	player2	A	B	C
a	0	0	0		1	1	1
b	0	0	1		1	1	0
c	0	1	0		1	0	1
d	0	1	1		1	0	0
e	1	0	0		0	1	1
f	1	0	1		0	1	0
g	1	1	0		0	0	1
h	1	1	1		0	0	0

Strategy combinations a and h lead to an expected payoff of zero. Any of the other strategy combinations leads to an expected payoff of 4. This game has 175 Nash equilibria, 8 of which are in pure strategies and correspond to strategy combinations $a, b, ..., h$ above. The only efficient equilibria are in pure strategies, and correspond to strategy combinations $b, c, ..., g$. Furthermore, $(4, 4)$ are not only the efficient Nash equilibrium payoffs, but also the efficient correlated equilibrium payoffs. In fact, it is well known that the set of correlated equilibrium payoffs which can be obtained with public signals is the convex hull of the set of Nash equilibrium payoffs. But this is a game of pure coordination, and hence any correlated equilibrium with private signals is still an equilibrium if the signals are made public (as the players can always ignore the extra information). It follows that $(4, 4)$ are also the efficient correlated equilibrium payoffs.

Now suppose that each player receives one half of a fully entangled qubit pair, and that three different measurements, x, y and z, are possible (say, spin measurements along three commonly known orthogonal axes). Then the following equilibrium improves on $(4, 4)$. If of type A, choose measurement x. Say "yes" if and only if the result is "up", and say "no" otherwise. Behave analogously, if of type B, after applying measurement y, and if of type C after applying measurement z. It follows from the rules of quantum mechanics that the conditional probabilities of agreement, depending on the combination of types, are those represented in the following table.

	A	B	C
A	0	3/4	3/4
B	3/4	0	3/4
C	3/4	3/4	0

It is easily checked that the strategy profile described above is an equilibrium, yielding an expected payoff of $6/9 * 3/4 * 9 = 9/2 > 4$ to both players.

Some remarks are in order at this point.

- The game is constructed in order to take advantage of the violation of one of Bell's inequalities by the entangled qubit pair. No distribution of classical signals can induce the conditional probabilities of agreement in the table above, and this the reason why no classical correlated equilibrium can achieve the same expected payoffs.

- In general, better coordination than in classical correlated equilibrium may ensue due to the rich patterns of positive and negative correlation which are allowed by the rules of quantum mechanics.

- Note that no actual information is exchanged between the two players. From the point of view of each player the probability that the opponent is of any given type is still 1/3, and the probability that he will say "yes" is still 1/2, even after performing the observation. Yet, the strategies of the two players have become effectively entangled, by making them contingent on the choice of measurement and on the observed state of the qubit pair.

- The correlated equilibrium described above makes use of a single entangled qubit pair, but possibly even better coordination may be achieved by transmitting finite sequences of entangled qubit pairs (q-bytes).

- Finally, note that in contrast with the classical case here observing is explicitly modelled as an act. This is necessary in the quantum-theoretic framework, where any observation intrinsically affects the state of the observed system.

3   Game-theoretic setup and a formal definition

The example described in the previous section can be conveniently modelled as a Bayesian game. In this section we introduce some standard game-theoretic notation and definitions, and then give a formal definition of quantum correlated equilibrium. We start with the definition of a strategic game.

Definition 1 *A strategic game* consists of a finite set N (the set of *players),
and for each player* $i \in N$

- *a set A_i (the set of* actions *available to i)*

- *a function $u_i : A \to \mathbb{R}$, where $A = \times_{j \in N} A_j$ (the* payoff *of player i).*

Denote by Σ_i the set of probability measures over A_i. These are player i's *mixed strategies*. We use the suffix $-i$ to denote all players except i. For instance, a profile $a_{-i} \in A_{-i}$ specifies an action for all players except i. A Nash equilibrium is a combination of probabilistic decisions (mixed strategies), one for each player, such that no single player can get a better expected payoff by choosing a different strategy.

Definition 2 *A (mixed-strategy) Nash equilibrium of a strategic game $(N, (A_i), (u_i))$ is a vector $(\pi_1, ..., \pi_n)$, with $\pi_i \in \Sigma_i$ for all $i \in N$, such that*

$$\sum_{a \in A} \pi_i(a_i)\pi_{-i}(a_{-i})u_i(a_{-i}, a_i) \geq \sum_{a \in A} \rho_i(a_i)\pi_{-i}(a_{-i})u_i(a_{-i}, a_i)$$

for all $\rho_i \in \Sigma_i$ and for all $i \in N$.

Pure-strategy Nash equilibria are those which only involve degenerate mixed strategies. In a Bayesian game, each player can make its choice of available action contingent on the observation of a private signal.

Definition 3 *A* Bayesian game *(in strategic form) consists of:*

- *a finite set N (the set of players)*

- *a finite set Ω (the set of* states*)*

and for each player $i \in N$

- *a set A_i (the set of actions available to i)*

- *a finite set T_i (the set of* signals *that may be observed by player i) and a function $\tau_i : \Omega \to T_i$ (the* signal function *of player i)*

- *a probability measure p_i on Ω (the* prior belief *of player i) for which $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$*

- *a function $u_i : A \times \Omega \to \mathbb{R}$, where $A = \times_{j \in N} A_j$ (the payoff of player i).*

Any Bayesian game can be reduced to a suitably defined strategic game. The Nash equilibria of a Bayesian game are defined as the Nash equilibria of the corresponding strategic game.

Definition 4 *A* Nash equilibrium *of a Bayesian game $(N, \Omega, (A_i), (T_i), (\tau_i), (p_i), (u_i))$ is a Nash equilibrium of the strategic game defined as follows.*

- *The set of players is the set of pairs (i, t_i) for $i \in N$ and $t_i \in T_i$.*

- *The set of actions of each player (i, t_i) is A_i.*

- *The payoff function $u_{(i,t_i)}$ of each player (i, t_i) is given by $u_{(i,t_i)}(a) = E[u_i(a, \omega)|t_i]$ (where $E[\ ]$ denotes the expectation operator)*

While a Nash equilibrium captures the notion of simultaneous, independent decisions, in a correlated equilibrium the players can make their choice of strategy conditional on some privately observed classical signals. Even though the signals are payoff-irrelevant, they influence the expected payoff by changing the players' information states.

Definition 5 *A* correlated equilibrium *of a strategic game* $(N, (A_i), (u_i))$ *consists of*

- *a finite probability space* (Ω, π) *(*Ω *is the set of states and* π *is a probability measure on* Ω*)*

- *for each player* $i \in N$ *a partition* $\mathcal{P}_i$ *of* Ω *(player* i*'s* information partition*)*

- *for each player* $i \in N$ *a function* $\sigma_i : \Omega \to A_i$ *with* $\sigma_i(\omega) = \sigma_i(\omega')$ *whenever* $\omega \in P_i$ *and* $\omega' \in P_i$ *for some* $P_i \in \mathcal{P}_i$ *such that*

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \rho_i(\omega))$$

  for any other function $\rho_i : \Omega \to M_i \times A_i$ *such that* $\rho_i(\omega) = \rho_i(\omega')$ *whenever* $\omega \in P_i$ *and* $\omega' \in P_i$ *for some* $P_i \in \mathcal{P}_i$.

As for the case of Nash equilibria, the correlated equilibria of a Bayesian game are simply the correlated equilibria of the associated strategic game.

We are now in the position to give a formal definition of quantum correlated equilibrium in a strategic game. The definition is similar to that of correlated equilibrium, but now the choice of measurement is explicitly modelled as an act.

Definition 6 *A* quantum correlated equilibrium *of a strategic game* $(N, (A_i), (u_i))$ *consists of*

- *a finite probability space* (Ω, π)

- *for each player* $i \in N$, *a set* M_i *(a subset of the set of Hermitian operators on a complex, separable Hilbert space) representing the available measurements*

- *for each player* $i \in N$, *and for each* $m_i \in M_i$, *a partition* $\mathcal{P}_i^{m_i}$ *of* Ω *representing player* i*'s information*

- *for each player* $i \in N$, *a function* $\sigma_i : \Omega \to M_i \times A_i$, *with* $\sigma_i^a(\omega) = \sigma_i^a(\omega')$ *whenever* $\sigma_i^m(\omega) = \sigma_i^m(\omega')$ *and* $\omega \in P_i$ *and* $\omega' \in P_i$ *for some* $P_i \in \mathcal{P}_i^{\sigma_i^m(\omega)}$ *(a* strategy *of player* i*), such that for any other strategy* ρ_i

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}^a(\omega), \sigma_i^a(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}^a(\omega), \rho_i^a(\omega)),$$

  where $\sigma_i^a(\omega)$ *and* $\sigma_i^m(\omega)$ *denote, respectively, the measurement and the action selected by strategy* σ_i *at state* ω.

Observe that the choice of measurement does not appear in the payoff functions u_i, but influences the expected payoff via the players' information partitions. The definition of quantum correlated equilibrium insures that there is no profitable unilateral strategy deviation after a choice of measurement has been performed and its result observed by a player, and also insures that there is no other choice of measurement which may lead to higher expected payoffs. Any correlated equilibrium is a quantum correlated equilibrium (with just a single measurement available to each player). The converse is not true in general, unless all the available measurements commute.

Note that a quantum correlated equilibrium always exists. In fact, if all the players choose to ignore the results of measurement and play a Nash equilibrium of the classical game this is always a quantum correlated equilibrium according to the above definition. More remarkably, the set of payoff combinations sustained by quantum correlated equilibria can be strictly larger than the set of classical correlated equilibrium payoffs, as the example in the previous section demonstrates.

It is not difficult to check that the equilibrium with two entangled qubits described in the example satisfies the definition of quantum correlated equilibrium. In the example, the set of states can be identified with the set of all combinations of type, measurement and action by each player. The information partition of player i captures which measurement (x, y or z) was chosen by i, and its outcome ("up" or "down"). A strategy specifies which of the three measurements is chosen, and restricts the subsequent choice of classical action (announcing "yes" or "not") to depend only on the type of the player, on the choice of measurement and on its outcome. In equilibrium, with probability $(6/9) * (3/4)$ the joint strategy yields a payoff of 9, and with the complementary probability yields a payoff of zero. Note that the equilibrium is strict: any other choice of action (for any given type, measurement and observation), as well as any other choice of measurement, would lead to a stricly lower expected payoff.

4   Discussion

We introduced a novel notion of correlated equilibrium which takes into explicit account the fact that any information received by the players prior to playing the game must be embodied in a physical carrier, and showed that the availability of a correlating device able to send quantum signals to the players may allow for equilibria which are more efficient than any classical equilibrium. One of the two approaches in the literature which comes closest to ours is the one in [7], which defines a class of games ('Quantum Correlation Games', or QCG) in which quantum correlation is exploited to yield more efficient equilibria if entangled quantum signals are available to the players. Yet, in QCGs the only available actions are choices of measurement, which makes QCGs a rather limited class of games. Unlike [7], we proposed an equilibrium notion which applies to any

classical strategic-form game, provided that quantum signals can be sent to the players and observed prior to their choice of classical action. Also, unlike [7], in our approach departures from classical correlated equilibrium are directly related to violations of Bell's inequalities.

The other approach which comes closest to ours is the one in [6], which also postulates the availability of quantum signals and demonstrates how entanglement can induce correlation in the players' strategies but fails to relate these ideas with the notion of correlated equilibrium, and to realize that quantum correlated equilibria can be strictly more efficient than classical correlated equilibria. In particular, in the game used in [6] to demonstrate quantum correlation, correlated equilibria based on classical private signals would be equally efficient. Unlike [6], we esplicitly related the notion of entanglement with that of correlated equilibrium in strategic games, and showed that in some games quantum correlated equilibria can be strictly more efficient than classical ones.

5   References

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