Are "Anti-Folk Theorems" in Repeated Games Nongeneric?

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Abstract

Folk Theorems in repeated games hold fixed the game payoffs, while the discount factor is varied freely. We show that these results may be sensitive to the order of limits in situations where players move asynchronously. Specifically, we show that when moves are asynchronous, then for a fixed discount factor close to one there is an open neighborhood of games which contains a pure coordination game such that every Perfect equilibrium of every game in the neighborhood approximates to an arbitrary degree the unique Pareto dominant payoff of the pure coordination game.

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‡The main result of this paper comprises Theorem 3 in earlier working paper: Lagunoff and Matsui (1995). Some other results in that paper were published in Lagunoff and Matsui (1997). Numerous helpful suggestions for that paper (and, by extension, the present one as well) came from an AT&T miniconference on Economic Theory. We would like to thank Peter Linhart, Roy Radner, and other participants of that conference. The first author, in particular, is indebted to Roy Radner for the many helpful suggestions and encouragement on this and other projects throughout the author's career.

1 Introduction

Results for most models of repeated strategic play have this key feature: payoffs of the game are fixed while the discount rate is freely varied by the modeller to achieve a desired result, namely the Folk Theorem.[1] The present paper shows that the Folk theorems may be sensitive to this particular order of limits.

We prove a result that pertains to a class of *asynchronously repeated games*. These are dynamic games with a certain payoff stationarity and where at most one player moves at each decision date. Lagunoff and Matsui (LM) (1995, 1997) and, more recently, Yoon (1999) examine Perfect equilibria of asynchronously repeated games.[2] LM show that if the stage game is one of pure coordination and if players are sufficiently patient, then every Perfect equilibrium payoff is arbitrarily close to the Pareto dominant payoff of the stage game. By contrast, Dutta (1995) and Yoon (1999) prove Folk Theorems for large classes of repeated games, including those with synchronous moves, asynchronous moves, and most everything in between, provided that stage games satisfy certain dimensionality restrictions.[3]

Since pure coordination games fail the dimensionality restrictions, Dutta's and Yoon's results suggest that the "Anti-Folk Theorem" of LM is nongeneric. More generally, the results suggest that multiplicity of equilibria in repeated situations is a generic property. This paper proves a result that suggests otherwise.

Specifically, we prove the following. Fix any stage game of pure coordination. Let u^* denote the unique, Pareto dominant Nash equilibrium vector of payoffs. We show that given some $\epsilon > 0$, if players are sufficiently patient, then there is an open neighborhood of the payoff vector of the pure coordination game such that every Pefect equilibrium payoff of the asynchronous repetition of every stage game in this neighborhood is within ϵ of u^*.

Hence, by fixing the level of discounting in advance, one can construct a neighborhood of games whose Perfect equilibria *all* approximate the unique Pareto dominant payoff the coordination game. The constructed neighborhood contains a positive measure of games that are full dimensional. In this sense, multiplicity of repeated games is not a generic phenomenon. The result also demonstrates that games that approximate team problems

[1] See Aumann (1981) and Fudenberg and Maskin (1986).

[2] Related models are found in Rubinstein and Wolinsky (1995) and Wen (1998) who examine repeated extensive form games. See also Benabou (1989), Maskin and Tirole (1987, 1988a,b), Haller and Lagunoff (1997), and Bhaskar and Vega-Redondo (1998) all of whom restrict attention to Markov Perfect equilibria of certain asynchronously repeated games.

[3] Strictly speaking, Dutta's result applies to all finite state stochastic games. While many asynchronously repeated games are stochastic games, Yoon's result applies to asynchronously repeated games with a, possibly, nonstationary process determining the set of movers each period.

described by Marshak and Radner (1972) have desirable outcomes.

The paper proceeds as follows. Section 2 describes the model and defines the class of asynchronously repeated games. In fact, we define more general class of games with repeated interaction, only some of which break the perfect synchronization of the standard model. We call games in this class *renewal games*. Originally described in Lagunoff and Matsui (1995), a renewal game is defined as a setting in which a stage game is repeated in continuous time, and at certain stochastic points in time, determined by an arbitrary renewal process, some set of players may be called upon to make a move. Both standard repeated games and asynchronously repeated games are special cases. Section 3 states the main result. Section 4 examines the order of limits sensitivity in the context of a 2×2 example. Section 5 gives the proof of the main result.

2 A Model of Asynchronously Repeated Interaction

2.1 Stage Game

Let $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ denote a normal form stage game where I is the finite set of players, S_i $(i \in I)$ is the finite set of actions for player i, and $u_i : S \equiv \times_{i \in I} S_i \to \Re$ is the payoff function for player $i \in I$. Without loss of generality, assume that $S_i \cap S_j = \emptyset$ for all $i \neq j$. We will call an element of $s = (s_1, \ldots, s_{|I|}) \in S$ a *behavior profile* (or simply "profile"). Given some $\hat{s}_i \in S_i$, let $s \backslash \hat{s}_i = (s_1, \ldots, s_{i-1}, \hat{s}_i, s_{i+1}, \ldots, s_{|I|})$. The tuple of payoff functions is denoted by $u = (u_i)_{i \in I}$. A mixed strategy for i will be denoted by σ_i and has the standard properties: $\sigma_i : S_i \to [0, 1]$ and $\sum_{s_i} \sigma_i(s_i) = 1$. A *mixed profile* is given by $\sigma = (\sigma_i)_{i \in I}$. Finally, a game G is a *coordination game* if its Nash equilibria are Pareto ranked and there is some Nash equilibrium that strictly Pareto dominates every other profile of the game.

2.2 Renewal Games and Asynchronously Repeated Games

In this Section, we introduce a framework that encompasses a wide variety of repeated strategic environments. Consider a continuous repetition of a stage game G. After the first decision node, which occurs for all players at time zero, all players' decision points are determined by a semi-Markov process with finitely many states.[4] In the following, revision

[4] We can formulate the problem in such a way that the first action profile is chosen by nature as in models of evolution. It will be clear that the following description and results will not be altered by specification of choice at time zero.

nodes refer to the decision nodes other than the first one at time zero.

A semi-Markov process is a stochastic process which makes transitions from state to state in accordance with a Markov chain, but in which the amount of time spent in each state before a transition occurs is random and follows a renewal process. For the sake of convenience, we separate the process into two parts, a renewal process and a Markov chain. Formally, let X_1, $X_2, \cdots$ be an infinite sequence of i.i.d. nonnegative random variables which follow a (marginal) probability measure ν with $E_\nu(X_1) < \infty$ and $\nu(X_1 > \epsilon) > \delta$ for some $\epsilon, \delta > 0$. It is also assumed that $\nu(X_1 = 0) = 0$ so that the orderliness condition for the renewals is guaranteed. Then let $T_0 = 0$ and $T_k = T_{k-1} + X_k = X_1 + \cdots + X_k$ $(k = 1, 2, \cdots)$. T_k is the time elapsed before the kth revision point. At each decision point a state ω is determined from a finite set Ω according to a Markov process $\{Y_k\}_{k=1}^{\infty} \in \Omega^\infty$ where $Y_k = \omega$ $(\omega \in \Omega)$ implies that state ω is reached at time T_k. We denote $p_{\omega\omega'} = Pr(Y_{k+1} = \omega' \mid Y_k = \omega)$ for $\omega, \omega' \in \Omega$. Let $\Omega_i \subseteq \Omega$ denote the nonempty set of states in which player i has a decision node. Let $\Omega_0 \subseteq \Omega$ be the set of "inertial" states in which no player has a decision node. By definition, $\Omega_0 = \Omega - (\cup_{i \in I} \Omega_i)$. We write $\vec{\Omega} = (\Omega_0, (\Omega_i)_{i\in I})$. We assume that the initial state, denoted by $\omega(0) \in \Omega$, is never reached again. By definition, $\omega(0) \in \Omega_i$ for all $i \in I$. Note that $\Omega_i \cap \Omega_j \setminus \{\omega(0)\}$ $(i \neq j)$ may or may not be empty. To summarize, the renewal process, ν, determines when the decision nodes (the "jumps") occur, while the Markov transition, p, determines who moves at each node.

Using this semi-Markov process, a typical play of the game is described as follows. In the beginning, a strategy profile s^0 is chosen. Deterministically or stochastically, the first revision time is reached at time T_1. Suppose that ω is chosen by the Markov process, and let $I(\omega) = \{i \in I : \omega \in \Omega_i\}$ denoting the players who can move if nature chooses ω. If player i $(i \in I(\omega))$ takes s_i^1, then the strategy profile changes to $s^1 = s^0 \backslash (s_i^1)_{i \in I(\omega)}$. That is, each time there is a renewal and revision, only the corresponding coordinate(s) of the previous strategy profile is replaced by the revised one, while other coordinates remain unchanged. If we define $(s^k)_{k \geq 0}$ this way, i.e., s^k $(k = 1, 2, \cdots)$ is the strategy profile between T_k and T_{k+1}, then the flow payoff is realized and the discounted payoff for player $i \in I$ will be given by

$$r \sum_{k=0}^{\infty} \int_{T_k}^{T_{k+1}} e^{-r\tau} u_i(s^k) d\tau. \tag{1}$$

Figure 1 illustrates the process for a two-person game.

Definition 1 A *renewal game* is a tuple

$$\Gamma = \langle G,\ \nu, \vec{\Omega},\ (p_{\omega\omega'})_{\omega,\omega' \in \Omega},\ r \rangle,$$

where $r > 0$ is a common discount rate, and for all $\omega \in \Omega$ and all $i \in I$, there exists a chain of states, $\omega^0, \omega^1, \omega^2, \ldots, \omega^M$ $(M < |\Omega|)$, with $\omega = \omega^0$ and $\omega^M \in \Omega_i$ such that $p_{\omega^{m-1}\omega^m} > 0$ $(m = 1, \cdots, M)$ (from any state, every player obtains a revision node).

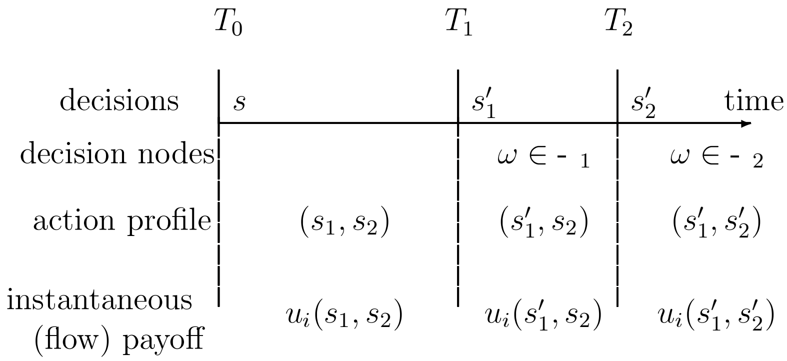

Figure 1: The play of the game

Since the number of states is finite, renewal games have the property that from any state, every player obtains revision nodes infinitely many times, and the expected time interval between two revision nodes is finite. Standard discounted repeated games are renewal games which may be described in several ways. One straightforward way is: $\Omega = \Omega_i$, $\forall i \in I$, and $\nu(X_1 = 1) = 1$. However, we wish to specialize further to only those renewal games in which choice is asynchronous.

Definition 2 An *asynchronously repeated game* is a renewal game in which $\Omega_i \cap \Omega_j \backslash \{\omega(0)\} = \emptyset$, $\forall i \neq j$

In asynchronously repeated games, no two individuals have simultaneous revision opportunities. When $\omega \in \Omega_i$, we will write $i(\omega) = i$. Some examples are:

Example 1 (alternating move game). Let $I = \{1, 2\}$, $\Omega = I$, $X_1 \equiv 1$, and $p_{ij} = 1$ if $i \neq j$. Then the decision points are deterministic, and players' revision nodes alternate.

An example of an alternating move game of pure coordination is a situation in which two firms in the same product group desire a uniform accounting standard to simplify their consolidation work. However, they have different closing dates due to the nature of their business, which makes the decision points alternate. Another example is one in which two offices of a company are located in New York and Tokyo, respectively, so that their business hours do not overlap.

Example 2 (Poisson revision process). Let $\Omega = I$. And let X_1 follow an exponential distribution with parameter $\lambda > 0$, i.e., $\nu(X_1 < x) = 1 - e^{-\lambda x}$, and let $p_{ij} = p_j$ for all $i, j \in I$. Then players' revision points are independent of each other, and player i's decision points ($i \in I$) follow a Poisson process almost surely with parameter λp_i.

An example of this type is a replacement process, common in evolutive models, in which a player is defined as a lineage rather than a single individual entity. A son inherits his father's position only after the father's death, at which time the son can take his own action and commit to it through the rest of his life.

Example 3 (ϵ-approximation of the standard repeated game). Let $I = \{1, 2\}$, and

$$- = \overbrace{\{\omega_0^1, \dots, \omega_0^M\}}^{0} \cup \overbrace{\{\omega_1^1, \omega_1^2\}}^{1} \cup \overbrace{\{\omega_2^1, \omega_2^2\}}^{2}$$

Then assume that Y_k's follow the process illlustrated in Figure 2 below. In the figure, the process proceeds through the inertial states until ω_0^M. At that time the process moves to either player 1's or player 2's decision node with probability 1/2 each. Let $X_1 \equiv 1/(M+2)$. Then if M is sufficiently large, the process approximates the standard repeated game in the sense that each player has a revision opportunity once in a unit of time, and that the two players' decision nodes are very close in timing.

$$\begin{array}{ccccc}
 & & \omega_0^1 & & \\
 & & \downarrow & & \\
 & & \omega_0^2 & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
 & & \omega_0^M & & \\
 & .5 \swarrow & & \searrow .5 & \\
\omega_1^1 & & & & \omega_2^1 \\
\downarrow & & & & \downarrow \\
\omega_2^2 & & & & \omega_1^2 \\
 & \searrow & & \swarrow & \\
 & & \omega_0^1 & & \\
 & & \downarrow & &
\end{array}$$

Figure 2: An ϵ-approximation of the standard repeated game.

2.3 Recursive Structure

One additional assumption that we will make will be to restrict the class of behavior strategies that individuals use. We assume that individuals only condition on the *sequences* of decision points and the actions taken at these points rather than on the time interval between them. This assumption does not restrict the strategy space at all if the renewal process is deterministic (the Markov process can be stochastic) as with Examples 1 and 3 in the previous section, and it significantly reduces the notational burden.

Restricting conditioning events to the "jumps" rather than on time intervals between jumps allows a straightforward recursive representation of individuals' continuation values in the asynchronous model. To formulate this, let $s(t) = (s_1(t), \dots s_{|I|}(t))$ denote the behavior profile at time t, and let $N_t \equiv \inf\{k : T_k > t\}$ denote the number of renewals up to time t. Due to the assumption on X_k's, $N_t < \infty$ holds almost surely. Then define the space of histories H such that a history $h^t \in H$ is given by $h^t = (y^t, s^t)$ wherever N_t is finite, and

$$y^t = \{Y_k\}_{k \leq N_t},$$

and

$$s^t = \{s(T_k)\}_{k \leq N_t}.$$

The null history is denoted by h^0. Since at most one player switches his action at a time after the initial profile we write $s^t = (s(h^0), s_{Y_1}, \cdots, s_{Y_{N_t}})$ whenever convenient. We let $s(h)$ and $\omega(h)$ denote the current behavior and state at history h, resp., and let $i(h)$ denote the last player whose decision node was reached. We also define $h^{t-} = (y^t, \{s(T_k)\}_{k<N_t})$ so that h^{t-} includes the same information as history h^t except for the behavior profile at time t which may include a new decision by a player. We write H^- for the set of all such *conditioning histories* and denote an element by h^-.

A *strategy* for player i is a history contingent action given by the function $f_i : H^- \to \Delta(S_i)$. Although this formulation appears to require that i is unable to condition on the current behavior profile, recall that decisions are only made at the "jumps" in the renewal process.

Given a strategy profile f, the play of the game proceeds as follows. At time zero, all the players in I simultaneously take actions, $f_i(h^{0-})$ for $i \in I$. Suppose $T_1 = t_1 = x_1$ and $Y_1 = y_1 \in$ - $_i$. If $i = 0$, nothing changes except the Markov "clock." If $i \in I$, then $h^{t_1-} = (f(h^{0-}); y_1)$, player i's revision node is reached at time t_1, and he takes action $f_i(h^{t_1-})$. History at time t_1 becomes $h^{t_1} = (y_1; f(h^{0-}), f(h^{0-}) \backslash f_i(h^{t_1-}))$. Given a strategy profile f and $h \in H^t$ (resp. $h^- \in H^{t-}$), $\{\tilde{s}(f|h)(\tau)\}_{\tau > t}$ (resp. $\{\tilde{s}(f|h^-)(\tau)\}_{\tau \geq t}$) denotes an induced (stochastic) path of action profile after h (resp. h^-).

Given a history $h^t \in H$ and a strategy profile $f = (f_i)_{i \in I}$, we define the conditional discounted expected payoff to player i at time t by

$$V_i(f|h^t) = r \int_t^\infty e^{-r(\tau-t)} E\left[u_i\left(\tilde{s}(f|h^t)(\tau)\right)\right] d\tau,$$

where $E[\cdot]$ is the expectation operator. A strategy profile f^* is called a *perfect equilibrium (PE)* if for each $i \in I$, f_i^* is a best response to $(f_j^*)_{j \neq i}$ after every history h^t, i.e.,

$$V_i(f^*|h^t) \geq V_i(f^* \backslash f_i|h^t)$$

for any of player i's strategies f_i.

One immediate result in the asynchronously repeated game is that for almost all histories, mixed strategies will not be used at any revision opportunity. Hence, the mixed strategy minimax payoff which is always an equilibrium of the stage coordination game is not the benchmark here necessarily. Hereafter, we will often denote a pure strategy by $f_i(h^-)$ as well as a mixed strategy.

Given a history $h = h^t \in H$, let $h^t \circ (\omega; s_j)$ denote the concatenated history in which, after h^t, the next state $\omega \in -_j$ is reached, at which player j takes s_j. Given $h^t \in H$, $h^t \circ \omega \in H^-$ is a path such that after h^t, state ω is reached (without specifying the revised action). Using this expression, the value after $h \circ \omega \in H^-$ is given by

$$V_i(f|h \circ \omega) \equiv V_i(f|h \circ (\omega; f_{i(\omega)}(h \circ \omega))).$$

The analysis will make extensive use of the following recursive formulation. The continuation value to i induced by f after history $h^t \in H$ with $\omega(h^t) = \omega$ may be expressed as

$$V_i(f|h^t) \;=\; (1 - \sum_{\omega' \in} \theta_{\omega\omega'}) u_i(s(h^t)) \;+\; \sum_{\omega' \in} \theta_{\omega\omega'} V_i(f|\; h^t \circ \omega'). \tag{2}$$

where

$$\begin{aligned} \theta_{\omega\omega'} \;&\equiv\; p_{\omega\omega'} \int_0^\infty e^{-rt} d\nu(t) \\ &=\; \text{expected discounted probability that } \omega' \\ &\quad\; \text{is the first state reached from } \omega. \end{aligned} \tag{3}$$

2.4 Existence

The following is a standard proof for the existence of perfect equilibria. What is proven here is actually the existence of Markov perfect equilibrium.

Theorem 0 *For any asynchronously repeated game* $\Gamma = \langle G, \nu, \vec{-}, p, r \rangle$ *there exists at least one perfect equilibrium.*

Proof Partition H^- into $\wp = \{H^-_{\omega s}\}_{s \in S, \omega \in}$ such that

$$H^-_{\omega s} = \{h^- \in H^- | \omega(h^-) = \omega, s(h^-) = s\}, \forall \omega \in - , \forall s \in S.$$

Observe that $\wp$ constitutes the "payoff relevent" set of states. Suppose that each player takes a $\sigma(\wp)$-measurable behavior strategy where $\sigma(\wp)$ is the σ-algebra generated by $\wp$. Then the play of the game follows a Markov process, and we can represent a strategy of player i by the "Markovian" function $\psi_i \in [\Delta(S_i)]^{\wp}$. The strategy represented by ψ_i is denoted by f_{ψ_i}.

For each $i \in I$ let $BR^i = \{BR^i_{\omega s}\}_{\omega \in \ , s \in S}$ satisfy

$$BR^i_{\omega(h)s}(\psi) = \arg\max_{f_{\psi_i}} V(f_\psi \backslash f_{\psi_i} | \, h) \tag{4}$$

Letting $BR = (BR_i)_{i \in I}$, equation (4) defines an upper hemicontinuous correspondence $BR : \times_i [\Delta(S_i)]^{\wp} \rightarrow \times_i [\Delta(S_i)]^{\wp}$ where $\times_i [\Delta(S_i)]^{\wp}$ is compact and convex. Therefore, by Kakutani's fixed point theorem, there exists ψ such that $\psi \in BR(\psi)$ holds. Standard arguments show that the corresponding strategy f_{ψ_i} is a best response to $(f_{\psi_j})_{j \neq i}$ within the class of *all* strategies after any history h (or, more precisely, after any h^-) since all j, $(j \neq i)$ only vary their behavior over "states" $\omega s \in H^-_{\omega s}$. Hence, f_ψ is a perfect equilibrium. □

As a special case, we will be interested in pure coordination games. A game G is a *pure coordination game* if $u_i = u_j = u$ for all i and j.[5] Let s^* denote the profile that gives each player his highest payoff u^*.

3 Main Result

3.1 A Theorem

The main result below demonstrates that order of limits matters. The typical Folk Theorem fixes the stage game and then varies r. By contrast, the hypothesis of the Theorem fixes r and then varies G. The former will surely be more familiar to those familiar with the Folk Theorem. However, the latter is more useful if it is the discount rate r, rather than the stage payoffs, which is pinned down by exogenous data.

[5] This can be weakened so that we require only that $u_i = \alpha u_j + \beta$ for $\alpha > 0$ and $\beta \in \Re$.

Theorem *Given $\epsilon > 0$ and any asynchronously repeated game $\langle G, \nu, (p_{\omega\omega'}), r\rangle$ in which $G = (I, S, u)$ is a pure coordination game with u^* as the unique Pareto efficient outcome, there exists $\bar{r} > 0$ such that for any $r \in (0, \bar{r})$, there exists an open subset $\mathcal{U} \subset \Re^{\times_{i \in I} S_i}$ with $u \in \mathcal{U}$ such that in an asynchronously repeated game $\langle (I, S, u'), \nu, (p_{\omega\omega'}), r\rangle$ with $u' \in \mathcal{U}$, every continuation value in any perfect equilibrium is at least $u^* - \epsilon$.*

3.2 The Order of Limits: An Example

We apply the Theorem to the class of two-player alternating move games where the stage game is of the form G_3 below. In the stage game G_3, $u^* > 2$ and $\alpha, \beta < 1$. Clearly, G_3 is an "impure" coordination game since the costs of miscoordination are not identical. Notice that G_3 satisfies full dimensionality. Using the full-dimensionality of this game, we first fix α and β then vary r to construct a perfect equilibrium of which payoff is bounded away from the Pareto efficient payoff pair, (u^*, u^*). Next, we apply our Theorem to show why the construction fails when, first fixing r, β is varied to be sufficiently close to α.

		2	
		s_2^*	$\bar{s}_2$
1	s_1^*	$u^*,\ u^*$	$\beta,\ \alpha$
	$\bar{s}_1$	$\alpha,\ \beta$	$1,\ 1$

G_2**:** $u^* \geq 2, \alpha, \beta < 1$

Figure 3

We construct such an equilibrium in the following way. First, we consider two phases, Phase I and Phase II. Phase I is divided into four subphases given by the following.

$$\overbrace{s^* \to \cdots \to s^*}^{k \text{ times}} \to \quad \overbrace{(\bar{s}_1, s_2^*) \to \cdots \to (\bar{s}_1, s_2^*)}^{\ell \text{ times}} \to \quad \bar{s} \to \quad \overbrace{(s_1^*, \bar{s}_2) \to \cdots \to (s_1^*, \bar{s}_2)}^{m \text{ times}} \to$$

Phase I.A *Phase I.B* *Phase I.C* *Phase I.D*

This is the prescribed path in Phase I. After the last stage of Phase I.D, the system returns to the first stage of Phase I.A. In this construction, k, ℓ, and m satisfy the following:

$$\frac{u^* k + \alpha\ell + 1 + \beta m}{k + \ell + 1 + m} > \frac{u^* k + \beta\ell + 1 + \alpha m}{k + \ell + 1 + m} > 1, \tag{5}$$

and

$$\frac{u^* + \beta\ell + 1}{\ell + 2} < 1. \tag{6}$$

Such k, ℓ, and m exist. Indeed, we choose ℓ large enough to satisfy (6). Then choose m so that $\alpha\ell + \beta m$ is greater than $\beta\ell + \alpha m$, i.e., $\ell > m$ if $\alpha > \beta$, and vice versa. This will guarantee the first inequality of (5). Note that we cannot find such a m if the game is pure coordination, i.e., $\alpha = \beta$. Finally, take a sufficiently large k to satisfy the second inequality of (5).

Phase II is the same as Phase I except that $(\bar{s}_1, s_2^*)$ (resp. $(s_1^*, \bar{s}_2)$) is replaced by $(s_1^*, \bar{s}_2)$ (resp. $(\bar{s}_1, s_2^*)$). That is, Phase II is a mirror image of Phase I with respect to the players. The play of the game begins with the first stage of Phase I.A and stays in Phase I, following the above arrows, unless there is a deviation. If player 1 deviates, then the system moves to an appropriate subphase of Phase II. For example, if player 1 deviates in Phase I.B to take s_1^*, then the system goes to some state corresponding to player 2's move in Phase II.A.

If player 2 deviates, then we have the following transitions:

1. If player 2 deviates in Phase I.A, then player 2's prescribed action in the next move is to return to s_2^*, and player 1 will keep s_1^* until player 2 takes s_2^*. After player 2 returns, the system goes to the last stage of Phase I.A.

2. If player 2 deviates to take $\bar{s}_2$ earlier than prescribed in Phase I.B, then player 1 will keep $\bar{s}_1$ until player 2 takes s_2^*, and then the system moves to the first stage of Phase I.B. If player 2 deviates to keep taking s_2^* in the last stage of Phase I.B, then the system moves to the second last stage of Phase I.B.

3. If player 2 deviates to take s_2^* in Phase I.D, then the system moves to the last stage of Phase I.A. If player 2 deviates to keep taking $\bar{s}_2$ in the last stage of Phase I.D, then the system moves to the second last stage of Phase I.D.

Note that player 2 will have no revision point in Phase I.C. Prescribed actions and the transition in Phase II are the same as those in Phase I except that the roles of the players are reversed.

Now, we are in a position to check that incentive constraints are satisfied for a sufficiently small discount rate $r > 0$. Note that this means we now vary r having fixed payoff parameters α and β. If player 1 deviates in Phase I, his expected payoff converges to

$$\frac{u^*k + \beta\ell + 1 + \alpha m}{k + \ell + 1 + m} \tag{7}$$

as r goes to zero. On the other hand, his expected payoff in Phase I converges to

$$\frac{u^*k + \alpha\ell + 1 + \beta m}{k + \ell + 1 + m},$$

which exceeds (7). Thus, for a sufficiently small $r > 0$, player 1 has no incentive to deviate. To check player's incentive to deviate, we examine three cases indicated in the above construction.

1. Phase I.A: If player 2 deviates, she will get α for a while instead of u^*. Therefore, she has no incentive to deviate there. Even if she keeps $\bar{s}_2$, she will get only $\alpha < 1$.
2. Phase I.B: If player 2 deviates to take $\bar{s}_2$ earlier than prescribed, she will get 1 and then some extra β before the system reaches the stage where she deviated. Since the expected payoff along the equilibrium path exceeds one, and $\beta < 1$, player 2 has no incentive to deviate. In the last stage of Phase I.B, if she deviates, she will get $\beta < 1$ for two more periods, which does not increase her payoff.
3. Phase I.D: If player 2 deviates to take s_2^* earlier than prescribed, then she gets u^* for one period, β for ℓ periods, 1 for one period, and some α's before the system reaches the original stage where player 2 deviates. From (6), the expected average payoff before the system reaches the same stage is less than one. Thus, player 2 has no incentive to deviate. Finally, in the the last stage of Phase I.D, if she deviates, she will get $\alpha < 1$ for two more periods, which does not increase her payoff.

Hence, the strategy profile constructed above is a perfect equilibrium. It should be noted that in a standard repeated game, we do not need this type of complicated construction since the strategy profile that prescribes $\bar{s}_i$ for player $i = 1, 2$ after any history is a subgame perfect equilibrium. On the other hand, it is shown that in an asynchronously repeated game, such a simple strategy does not constitute a subgame perfect equilibrium unless α is sufficiently larger than β. This type of construction is used in other Folk Theorems possibly without synchronous moves and without public randomizing devices. See, for example, Dutta (1995), Wen (1998), and Yoon (1999).

Now suppose that r is fixed in advance of fixing α and β. The problem with this and other constructions is the following. Returning to Inequalities (5) and (6), observe that they are satisfied with a judicious choice of phase lengths k, ℓ and m which depends, in turn, on values α and β. In particular, for a r bounded away from 0, payoffs in (5) only approximate actual dynamic payoffs in Phase 1. In fact, the LHS of (5) must exceed the RHS by more than $1 - e^{-r}$ times the minimum absolute stage payoff differential. For simplicity, let $z > 0$ denote this differential. Then, the incentive constraint for Phase 1 is given by

$$\frac{u^*k + \alpha\ell + 1 + \beta m}{k + \ell + 1 + m} > \frac{u^*k + \beta\ell + 1 + \alpha m}{k + \ell + 1 + m} + (1 - e^{-r})z$$

which we rewrite as

$$\alpha\ell + \beta m > \beta\ell + \alpha m + (k + \ell + 1 + m)(1 - e^{-r})z. \tag{8}$$

Instead of having to only satisfy $\alpha\ell + \beta m > \beta\ell + \alpha m$ as before, we now require that (8) be satisfied. Letting $\alpha = \beta + \varepsilon$, (8) becomes

$$\varepsilon(\ell - m) > (k + \ell + 1 + m)(1 - e^{-r})z$$

which is clearly violated for ε sufficiently small.

4 Proof of the Main Result

Take as given an asynchronously repeated game $\langle G, \nu, (p_{\omega\omega'}), r\rangle$ in which $G = (I, S, u)$ is a pure coordination game, and $s^* \in S$ is the unique Pareto efficient outcome. Also take $\epsilon > 0$ as given. Any affine transformation of u will give the same result in the following analysis. We set $\bar{r}$ so as to satisfy

$$\min_{\substack{(\omega_0,\omega_1,\cdots,\omega_N)\in \quad ^{N+1} \\ 2\le N\le |\quad| -1}} \left\{ \frac{\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}}{\theta(\omega_0,\cdots,\omega_N)} : \ \theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N} > 0 \right\} \ > \ 1 - \frac{\epsilon}{3|I|}\frac{1}{u^* - \min_s u(s)},$$

where

$$\begin{aligned} \theta(\omega_0,\cdots,\omega_N) &= 1 - \sum_{\ell=1}^{N} \Pi_{n=1}^{\ell-1}\theta_{\omega_{n-1}\omega_n} \sum_{\omega'\neq\omega_\ell} \theta_{\omega_{\ell-1}\omega'} \\ &= 1 - \sum_{\omega'\neq\omega_1}\theta_{\omega_0\omega'} - \theta_{\omega_0\omega_1}\sum_{\omega'\neq\omega_2}\theta_{\omega_1\omega'} - \cdots - \theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-2}\omega_{N-1}}\sum_{\omega'\neq\omega_N}\theta_{\omega_{N-1}\omega'}. \end{aligned}$$

Such an $\bar{r}$ can be found since both $\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}$ and $\theta(\cdot)$ converge to $p_{\omega_0\omega_1}\cdots p_{\omega_{N-1}\omega_N} > 0$ as r goes to zero. Also, let $\underline{\theta} = \min\{\theta(\omega_0,\cdots,\omega_N) : \theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N} > 0\}$. Fix $r \in (0, \bar{r})$.

Consider a neighborhood of u given by $\mathcal{U} = \{u' \in \Re^S |\ \forall s \in S,\ |u'_i(s) - u_i(s)| < \eta/4\}$ where η satisfies

$$0 < \eta < \frac{1}{2}\min\left\{ u^* - u^{**},\ \frac{\epsilon}{2|I||\text{-}|},\ \frac{\epsilon}{3(|I|+1)}[1 - \int_0^\infty e^{-r\tau}d\nu(\tau)] \right\},$$

where $u^{**} = \max_{s\neq s^*} u(s)$, i.e., the second highest payoff. In the following, we consider an asynchronously repeated game $\Gamma' = \langle (I, S, u'), \nu, \vec{-}, (p_{\omega\omega'}), r\rangle$ with $u' \in \mathcal{U}$.

Now, fix a perfect equilibrium f of Γ'. For each $i \in I$, each $s \in S$, and each $\omega \in -$, we let $\underline{V}_{i\omega}^{s} = \inf_{s(h)=s,\omega(h)=\omega} V_i(f|h)$, and $\underline{V}_i^s = \inf_{s(h)=s} V_i(f|h)$. Note that $\underline{V}_i^s = \min_{\omega\in} \ \underline{V}_{i\omega}^s$.

For any $\delta > 0$ there exists $h \in H$ such that $s(h) = s^*$ and

$$\underline{V}_i^{s^*} > V_i(f|h) - \delta, \ \forall i \in I. \tag{9}$$

The continuation value for player $i \in I$ after h is given by

$$V_i(f|h) = (1 - \sum_{\omega\in} \theta_{\omega(h)\omega})u_i'(s^*) \ + \ \sum_{\omega\in} \theta_{\omega(h)\omega} V_i(f| \ h \circ \omega). \tag{10}$$

If $\omega(h) \in -_0$, then $s(h \circ \omega) = s^*$ and therefore, $V_i(f|h \circ \omega) \geq \underline{V}_i^{s^*}$. Observe also that since f is a perfect equilibrium strategy profile, it must be the case that for all $\omega \in -_j$ $(j \in I)$,

$$V_j(f| \ h \circ \omega) \geq V_j(f| \ h \circ (s_j^*; \ \omega)) \geq \underline{V}_j^{s^*} \tag{11}$$

where the second inequality holds due to the definition of $\underline{V}_j^{s^*}$. Since $|u(s) - u_i(s)| < \eta/4$, $|V_j(f|h) - V_i(f|h)| < \eta/2$ holds for all $i, j \in I$, all f and all $h \in H$, and therefore, (11) implies

$$V_i(f| \ h \circ \ \omega) \geq \underline{V}_i^{s^*} - \eta. \tag{12}$$

Substituting (12) into (10) and using (9), we obtain

$$\underline{V}_i^{s^*} > (1 - \sum_{\omega'\in} \theta_{\omega(h)\omega'})u_i'(s^*) + \sum_{\omega'\in} \theta_{\omega(h)\omega'}\underline{V}_i^{s^*} - \delta - \eta \tag{13}$$

Inequality (13) implies

$$\underline{V}_i^{s^*} > u_i'(s^*) - \frac{\delta + \eta}{1 - \int_0^\infty e^{-r\tau} d\nu(\tau)}.$$

Since δ is arbitrary and independent of other variables, the definition of η implies $\underline{V}_i^{s^*} > u^* - \epsilon/(|I| + 1)$.

We will now show that for all $k = 1, 2, \cdots, |I|$, after h with $s(h) \in S^k$, its continuation value satisfies

$$V_i(f|h) > \underline{V}_i^{s'} - \frac{\epsilon}{|I| + 1}, \tag{14}$$

for some $s' \in S^{k-1}$ if $r < \bar{r}$. Backward induction implies that (14) holds for all $s' \in S^{k-1}$ if $r < \bar{r}$. Once we show (14) for all s' and k's, we verify that $0 < r < \bar{r}$ implies that for all history $h \in H$, $V_i(f|h) > u_i'(s^*) - \epsilon$. Moreover, recall that the choice of $\bar{r}$ is independent of f. Thus, the proof will be completed.

Fix $k = 1, 2, \cdots, |I|$ and $\hat{s} \in S^k$. By the definition of $\underline{V}_{i\omega}^{\hat{s}}$ ($\omega \in -$), there exists $h_\omega \in H$ such that $s(h_\omega) = \hat{s}$, $\omega(h_\omega) = \omega$, and

$$\underline{V}_{i\omega}^{\hat{s}} > V_i(f|h_\omega) - \frac{\epsilon\underline{\theta}}{3(|I| + 1)|-|}. \tag{15}$$

We have

$$V_i(f|h_\omega) = (1 - \sum_{\omega' \in} \theta_{\omega\omega'})u_i'(\hat{s}) \ + \ \sum_{\omega' \in} \theta_{\omega\omega'} V_i(f| \ h_\omega \circ \omega'). \tag{16}$$

Since f is a perfect equilibrium, for each $\omega' \in -_j$ with $j \in I$,

$$V_j(f| \ h_\omega \circ \omega') \geq V_j(f| \ h_\omega \circ (\hat{s}_j; \omega) \geq \underline{V}_{j\omega'}^{\hat{s}}, \tag{17}$$

where the second inequality holds by the definition of $\underline{V}_{j\omega}^{\hat{s}}$. Similarly, if $\hat{s}_j \neq s_j^*$, then

$$V_j(f| \ h_\omega \circ \omega') \geq V_j(f| \ h_\omega \circ (s_j^*; \omega)) \geq \underline{V}_{j\omega'}^{\hat{s}``s_j^*}, \tag{18}$$

Also, by the definition of $\underline{V}_{i\omega'}^{\hat{s}}$ for each $\omega \in -_0$, we have

$$V_i(f| \ h_\omega \circ \omega') \geq \underline{V}_{i\omega'}^{\hat{s}}, \ \forall \omega' \in -_0. \tag{19}$$

Since $|u(s) - u_i(s)| < \eta/4$, then $|V_j(f|h) - V_i(f|h)| < \eta/2$ holds for all $i, j \in I$, all f, all s, and all $h \in H$, and therefore, (17) and (18) imply

$$V_i(f| \ h_\omega \circ \omega') \geq \underline{V}_{i\omega'}^{\hat{s}} - \eta, \tag{20}$$

and

$$V_i(f| \ h_\omega \circ \omega') \geq \underline{V}_{i\omega'}^{\hat{s}\backslash s_j^*} - \eta, \tag{21}$$

respectively. Substituting these inequalities into (16), we obtain

$$V_i(f|h_\omega) \geq (1 - \sum_{\omega' \in} \theta_{\omega\omega'})u_i'(\hat{s}) \ + \ \sum_{\omega' \in} \theta_{\omega\omega'} \underline{V}_{i\omega'}^{\hat{s}} - \eta. \tag{22}$$

Inequalities (15) and (22) imply

$$\underline{V}_{i\omega}^{\hat{s}} \geq (1 - \sum_{\omega' \in} \theta_{\omega\omega'})u_i'(\hat{s}) \ + \ \sum_{\omega' \in} \theta_{\omega\omega'} \underline{V}_{i\omega'}^{\hat{s}} - \frac{\epsilon\underline{\theta}}{3(|I|+1)|- \ |} - \eta. \tag{23}$$

Since i was arbitrarily chosen, (23) holds for all $i \in I$. By definition, $\underline{V}_{i\omega_0}^{\hat{s}} = \underline{V}_i^{\hat{s}}$ for some ω_0. Take such ω_0. There exists $i \in I$ such that $\hat{s}_i \neq s_i^*$. Then there exists a chain $\omega_0, \omega_1, \ldots, \omega_N$ with $N < |- \ |$ and $\omega_N \in -_i$ such that $p_{\omega_{n-1}\omega_n} > 0$ for all $n = 1, \cdots, N$. Sequentially substituting ω_n ($n = 0, 1, \cdots, N$ in place of ω in (23) and applying (21), we obtain

$$\begin{aligned}
\underline{V}_i^{\hat{s}} \ \geq \ & (1 - \textstyle\sum_{\omega' \in} \theta_{\omega_0\omega'})u_i'(\hat{s}) + \displaystyle\sum_{\omega' \neq \omega_1} \theta_{\omega_0\omega'} \underline{V}_{i\omega'}^{\hat{s}} \\
& + \theta_{\omega_0\omega_1} \left[(1 - \sum_{\omega' \in} \theta_{\omega_1\omega'})u_i'(\hat{s}) + \sum_{\omega' \neq \omega_2} \theta_{\omega_1\omega'} \underline{V}_{i\omega'}^{\hat{s}} \right] \\
& + \cdots \\
& + \theta_{\omega_0\omega_1} \cdots \theta_{\omega_{N-2}\omega_{N-1}} \left[(1 - \textstyle\sum_{\omega' \in} \theta_{\omega_N\omega'})u_i'(\hat{s}) + \displaystyle\sum_{\omega' \neq \omega_N} \theta_{\omega_{N-1}\omega'} \underline{V}_{i\omega'}^{\hat{s}} \right] \\
& + \theta_{\omega_0\omega_1} \theta_{\omega_{N-1}\omega_N} \underline{V}_{i\omega_N}^{\hat{s}\backslash s_i^*} - \frac{\epsilon\underline{\theta}}{3(|I|+1)} - \eta|- \ |.
\end{aligned} \tag{24}$$

Using $\underline{V}_{i\omega}^{\hat{s}} \geq \underline{V}_i^{\hat{s}}$ $(\forall \omega \in -\ )$, we have

$$\underline{V}_i^{\hat{s}} \geq \left[1 - \frac{\theta_{\omega_0\omega_1} \cdots \theta_{\omega_{N-1}\omega_N}}{\theta(\omega_0, \cdots, \omega_N)}\right] u_i'(\hat{s}) \; + \; \frac{\theta_{\omega_0\omega_1} \cdots \theta_{\omega_{N-1}\omega_N}}{\theta(\omega_0, \cdots, \omega_N)} \underline{V}_i^{\hat{s}\backslash s_i^*} - \frac{2\epsilon}{3(|I|+1)}$$

Thus, for all $r < \bar{r}$,

$$\underline{V}_i^{\hat{s}} > \underline{V}_i^{\hat{s}\backslash s_i^*} - \frac{\epsilon}{|I|+1},$$

where, by construction, $\hat{s}\backslash s_i^* \in S^{k-1}$. $\square$

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