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On the existence of Nash equilibrium in electoral competition games: The hybrid case*

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Abstract

This paper generalizes previous existence results on unidimensional electoral competition, by extending the traditional two-party electoral game to the case where parties have mixed motivations, in the sense that they are interested in winning the election, but also in the policy implemented after the contest. Although this game has discontinuous payoffs, it satisfies payoff security and reciprocally upper semi-continuity. However, conditional payoffs might violate quasi-concavity. Hence, our first result shows that the existence of a pure-strategy Nash equilibrium can be guaranteed only if parties' interests are symmetric. Instead, we prove that the mixed extension satisfies better reply security and, therefore, that a mixed-strategy equilibrium always exists. We also characterize the set of equilibria for a tractable version of the model. This shows that the interaction between the electoral uncertainty, the aggregate level of opportunism and its distribution among parties shape the equilibrium strategies. In particular, when the opportunism is large and asymmetrically distributed, the support of each mixed-strategy equilibrium is a closed interval located on one side of the median. Further, as the uncertainty increases, the probability distributions concentrate on the extremes of the support. And the mixed-strategy equilibrium vanishes above a critical level, over which each party plays a pure strategy in its own ideological side.

Keywords: electoral competition, mixed motivations, discontinuous games, Nash equilibrium.

JEL Code: D70, D72, C72.

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1 Introduction

The electoral competition between two or more political parties is probably one of the most common and important features of every modern democracy. In economics, there is a large literature about this process.[1] This paper contributes to this literature by extending the traditional unidimensional, two-party model of Downs [9] and Wittman [27] and [28] to the case where parties have *mixed motivations*, in the sense that they are interested in winning the election, but also in the policy implemented after the contest.

Economic models of electoral competition originated in the famous location model of Hotelling [15] and Downs [9]. In the most simple version of this model, two parties (or candidates) announce simultaneously a platform before the election, voters cast their votes, and the winner implements its announced policy. The basic idea captured by this model is that, in a unidimensional setting, each player can increase the number of votes by moving closer to the other player.

Thus, if parties care only about the outcome of the election, which is the *Downsian* assumption, there is a unique Nash equilibrium in which both announce the same policy. In addition, if voters' preferences admit the existence of a Condorcet winner, as it is the case, for example, when they satisfy single-peakedness or single-crossing, then this policy coincides with the median of the distribution of the voters' most preferred policies. This result also holds if there exists uncertainty regarding individual preferences, as long as parties share a common prior about the location of the median ideal policy.

Contrary, consider the consequence of assuming that each party has an *exogenous* ideology, and cares only about how close the policy of the winner is relative to its own ideological position. This is the famous departure proposed by Wittman [27] and [28]. As Roemer [23] has shown, if there exists uncertainty about voters' preferences, this game has also a Nash equilibrium in pure strategies, but party platforms do not necessarily converge. This is because, as one party moves closer to the other, it becomes worse off in the event that it wins, but at the same time it increases its probability of winning the contest. Hence, it potentially faces a trade-off between its ideology and the electoral success, which results in an equilibrium where parties' positions are different.

Interestingly enough, the *hybrid* case, where parties have preferences over policies, but also on the office itself, has not received enough attention in the literature. To our best knowledge, only Ball [3] has analyzed the implications of this assumption, referred to as the *mixed motivations assumption* (MMA), for the existence of Nash equilibria in electoral competition games. He

[1] See Calvert [6], Coughlin [7], Shepsle [26], Osborne [21], Roemer [24] and Duggan [10] for excellent surveys on this subject.

shows an example where pure-strategy Nash equilibrium fails to exist, and he provides sufficient conditions for the existence of mixed-strategy Nash equilibria. However, his model has the disadvantage that the probability of winning the election of each party is given by an exogenous function. That is, it is taken as a primitive, instead of being derived endogenously from the parties' uncertainty about the median ideal point.[2]

Apart from this work, the MMA appears in few other papers. For instance, Roemer [23] and Duggan [10] mention this possibility, but they do not go beyond that. On the other hand, Aragones and Palfrey [1] apply this assumption in a model of electoral competition with a valence advantage. They focus on a unidimensional and *discrete* model where parties' ideal points and their trade-off between power and ideology are private information. They show that pure-strategy equilibria always exist, and that it approaches the mixed equilibrium of Aragones and Palfrey [2] when both candidates' weights on policy preferences go to zero. This paper extends Groseclose [14], which also uses the MMA, but in a complete information setting.

Although the MMA is not frequently observed, it is clearly more appealing than the traditional hypotheses about party motivation, i.e. the office motivation and the policy motivation assumptions. This is because the latter restrict party preferences in two different ways. On one hand, they choose a particular target for the party, namely power or ideology. On the other, they assume that parties' interests are perfectly symmetric, in the sense that either both are opportunistic or both are ideological in the same *magnitude*.

Regarding these restrictions, the MMA might arise naturally if, for example, professional politicians are the leaders of the parties.[3] Since politicians may be namely interested in their career and, therefore, in winning the election, while regular party members may care more about policy outcomes, it is natural to assume that both objectives will enter into the party payoff function with some weights. Of course, these weights need not be the same across parties. Therefore, asymmetric payoff functions can arise quite naturally as well.

But the fact that the MMA is attractive from an empirical viewpoint is not the only reason nor the more interesting one for studying its implications. There is also a technical reason to focus on this assumption. It is well known in the literature of political competition under uncertainty that the probability of winning function is discontinuous on the diagonal. This may preclude the existence of best replies, and therefore the best reply correspondence need not be nonempty valued, let alone upper hemi-continuous.

Of course, this is not a problem for the Downsian game, which always

[2]We will return to Ball's analysis later in the paper, to compare his results with ours.

[3]See Ortuño Ortín [20] for a model in this line.

has an equilibrium in pure strategies. Furthermore, it does not affect the Wittman model neither. Roemer [23] has shown that pure ideological parties have continuous payoff functions, in spite of the discontinuities of the probability of winning function, so that best reply correspondences are always well defined. However, it may be a problem for the hybrid case.

In this paper, we deal with this problem of equilibrium existence. We show first that, under the MMA, parties' payoff functions are neither continuous nor semicontinuous on the diagonal. This implies that, unless both parties have symmetric motivations, the less ideological party may have incentives to undercut the other party's position, by moving its own position to an alternative infinitely close to the platform of the other; i.e., by moving itself to the diagonal. Under certain conditions, that the paper tries to specify, this implies that best reply correspondences are empty and, therefore, that pure-strategy Nash equilibrium does not exist. In particular, this is the case if one of the parties is office motivated and the other is entirely ideological.

Then, we prove that the hybrid game satisfies payoff security and reciprocally upper semi-continuity. So, following Reny [22], we conclude that the blame for the non-existence of an equilibrium in pure strategies for all parametric conditions can be fully assigned to the violation of quasi-concavity, rather than to the discontinuity of the probability of wining function itself. Furthermore, we show that problems of quasi-concavity are relevant only if parties' interests are asymmetric. Contrary, for the symmetric case, regardless of whether the motivations are mixed or not, we are able to prove that a pure-strategy Nash equilibrium always exists, being the Downsian and Wittman equilibria two particular corollaries of this result.

Finally, to overcome the non-existence in the case of asymmetric interests, we move on to the mixed-strategy analysis.[4] We prove that the mixed extension satisfies better reply security and, therefore, that the hybrid electoral competition game has always a Nash equilibrium, though probably in mixed strategies. This, together with the result above, generalizes previous existence results for unidimensional electoral games.

We also characterize the set of Nash equilibria for a tractable version of the model. The characterization shows that the interaction between the electoral uncertainty, the aggregate level of opportunism and its distribution among parties shape the equilibrium strategies. In particular, when the opportunism is large compared to the electoral uncertainty, and it is asymmetrically distributed, the support of each mixed-strategy equilibrium is a closed interval located on one side of the median agent. Moreover, as the uncertainty increases, the probability distributions concentrate on

[4] See, among others, Kramer [16], Laffont et. al [18], Dutta and Laslier [12], Laslier [19], Aragones and Palfrey [2], Duggan and Jackson [11] and Bernhardt et. al [4] and [5] for further applications and interpretations of mixed strategies in electoral competition.

the extremes of the support. And the mixed-strategy equilibrium vanishes above a critical level, over which each party plays a pure strategy in its own ideological side.

The rest of the paper is organized as follows. Section 2 present the model, the notation and the main definitions. Section 3 focuses on the pure-strategy analysis. In Section 4 we study equilibrium existence for the mixed extension. Section 5 presents the characterization for the uniform distribution case. Final remarks are made in Section 6.

2 The electoral game

Consider the following electoral competition game. Assume there is a continuum of voters, indexed by a type $\theta \in \Theta = [0, 1]$, where θ is distributed according to a continuous distribution function F on Θ. Let $X = [0, 1]$ be the policy space and $u(x, \theta)$ a utility function representing the preferences of a type θ over X. We assume voters' preferences are continuous, single-peaked and symmetric on X. More formally,

Assumption 1 *(a) $u(x, \theta)$ is continuous in x and θ; (b) $u(\cdot, \theta)$ is strictly quasi-concave in x; and (c) $u(\cdot, \theta)$ is symmetric about $x(\theta) = \arg\max_{x \in X} u(x, \theta)$.*

Like in the ordinary electoral competition game, there are two political parties (or candidates), noted by 1 and 2. These parties simultaneously announce a policy platform on X. We denote $A_i = X$ party i's strategy set, with generic action $x_i \in X$, and $A = A_1 \times A_2$ the set of all strategy profiles. We refer to $D(A) = \{(x_1, x_2) \in A : x_1 = x_2\}$ as the diagonal of the product of the strategy sets; or, for conciseness, just as "the diagonal".

Given the proposal profile $(x_1, x_2) \in A$, each voter votes for the platform (party) he likes the most, being $S(x_1, x_2) = \{\theta \in \Theta : u(x_1, \theta) > u(x_2, \theta)\}$ the set of types that support x_1. The party that obtains more than half of the votes wins the election, and its proposed policy is implemented. Ties are broken by a random draw, so that each party wins with probability one half in the case of a tie in votes.

Apart from the uncertainty due to the possibility of a tie, parties are also uncertain about the position of the median voter. Following Roemer [24], we assume that the uncertainty over voters' preferences follows the so called *error distribution model.* That is, let ξ be an error term, distributed according to a continuous distribution function H on $[-\beta, \beta]$, $\beta > 0$, with density h and $H(0) = 1/2$:

Assumption 2 *For all $(x_1, x_2) \in A$, parties believe the fraction of types supporting x_1 is $F(S(x_1, x_2)) + \xi$.*[5]

[5]By A1, for each pair $(x_1, x_2) \in A$, if $u(x_1, \theta) = u(x_2, \theta)$ has a solution in θ, it is unique. Therefore, $F(I(x_1, x_2)) = 0$, where $I(x_1, x_2) = \{\theta \in \Theta : u(x_1, \theta) = u(x_2, \theta)\}$.

Thus, given a pair $(x_1, x_2) \in A$, the probability that party 1 attaches to winning the election is

$$p(x_1, x_2) = \begin{cases} 1 - H(1/2 - F(S(x_1, x_2))) & \text{if } (x_1, x_2) \notin D(A), \\ 1/2 & \text{if } (x_1, x_2) \in D(A). \end{cases} \tag{1}$$

where it is understood that $H(y) = 0$ if $y \leq -\beta$ and $H(y) = 1$ if $y \geq \beta$.

From (1), it is immediate that $p(x_1, x_2)$ is discontinuous at every point on $D(A)$, except at (x_m, x_m), where $x_m = \text{median}\{x(\theta) \in X : \theta \in \Theta\}$ is the median ideal point. This is because, first, assumptions A1.b (strictly quasi-concavity) and A1.c (symmetry) imply that voters' preferences are single-crossing on X (Lemma 3, Saporiti and Tohmé [25]). Single-crossing in turn implies that the function $p(x_1, x_2)$ satisfies Ball's monotonicity.[6] That is, $p(x_1, x_2)$ is non-decreasing in x_i, if $x_1 < x_2$, and non-increasing if $x_1 > x_2$ (Lemma 3.3, Roemer [24]). Finally, Proposition 1 in Ball [3] shows that monotonicity and continuity are incompatible.

Let $\chi_i \geq 0$ be the intrinsic value that party i places on being in office, and θ^i the preferences on X that it represents. Define the function $\psi(x, y, \theta) = u(x, \theta) - u(y, \theta)$. Following Ball [3] and Duggan [10], we make the following assumption regarding to parties' preferences:

Assumption 3 *Parties' payoff functions are*

$$\begin{aligned} \pi_1(x_1, x_2) &= p(x_1, x_2)\left(\psi(x_1, x_2, \theta^1) + \chi_1\right), \\ \pi_2(x_1, x_2) &= (1 - p(x_1, x_2))\left(\psi(x_2, x_1, \theta^2) + \chi_2\right). \end{aligned}$$

That is, parties have preferences over policy, but also on the office itself. Moreover, the payoff functions are separable in these arguments. Of course, this need not be the case, and one can imagine a model in which parties care more about policy if they win the election than if they lose. In addition, there might be other motivations to consider. For example, a party may have preferences over its margin of victory, apart from policies and the office. However, A3 seems the simplest case to begin with.

It is easy to see that the payoff functions of the traditional models of electoral competition are particular instances of A3. In effect, if $\chi_i = 0$, then $\pi_i(x)$ corresponds to the objective function of the Wittman game, where parties maximize the expected utility of winning the contest. On the other hand, if $\chi_i \to +\infty$, then each party maximizes the probability of winning, without caring at all about policies, which is the classical assumption of the Downsian model.[7] The *hybrid case*, where $\chi_i \in \Re_{++}$ for all i, and χ_1 is

[6] In words, this property, which reflects the intermediate structure of voters' preferences, says that if one party moves its proposal toward that of its opponent, then its probability of winning the election cannot decrease. Similarly, if it moves the platform away, then its probability of winning cannot increase.

[7] Notice that $\tilde{\pi}_i(x) = 1/\chi_i\, \pi_i(x)$ represents the same preferences that $\pi_i(x)$, and that $\lim_{\chi_1 \to \infty} \tilde{\pi}_1(x) = p(x)$ and $\lim_{\chi_2 \to \infty} \tilde{\pi}_2(x) = 1 - p(x)$.

not necessarily equal to χ_2, is somewhere between these two extreme and symmetric cases.

With respect to the specification of the MMA adopted in A3, notice that we follow Ball [3] and Duggan [10], who take the idea of Ferejohn [13] that policy-makers enjoy some rents χ from being in power. But there is an alternative way to capture the mixed motivations. Aragones and Palfrey [1] and Groseclose [14] both assume that the objective function of each party is a linear combination of the probability of winning the election and a second component corresponding to its policy preferences. That is,

$$\begin{aligned}\hat{\pi}_1(x_1,x_2) &= \lambda_1 p(x_1,x_2)+(1-\lambda_1)p(x_1,x_2)\psi(x_1,x_2,\theta^1),\\ \hat{\pi}_2(x_1,x_2) &= \lambda_2(1-p(x_1,x_2))+(1-\lambda_2)(1-p(x_1,x_2))\psi(x_2,x_1,\theta^2).\end{aligned}$$

where λ_j is the weight j assigns on holding office. Simple algebraic manipulation shows that for the main purpose of our analysis these two specifications are absolutely equivalent. In effect, denoting $\chi_j=\frac{\lambda_j}{1-\lambda_j}$, it follows that

$$\begin{aligned}\hat{\pi}_1(x_1,x_2) &= (1-\lambda_1)p(x_1,x_2)\left(\psi(x_1,x_2,\theta^1)+\chi_1\right),\\ &= (1-\lambda_1)\,\pi_1(x_1,x_2),\\ \hat{\pi}_2(x_1,x_2) &= (1-\lambda_2)(1-p(x_1,x_2))\left(\psi(x_2,x_1,\theta^2)+\chi_2\right),\\ &= (1-\lambda_2)\,\pi_2(x_1,x_2).\end{aligned}$$

Thus, since $\lambda_j\in(0,1)$, the equations above show that $\hat{\pi}_j$ is a continuous and strictly increasing transformation of π_j for all $j=1,2$. Therefore, all the results for pure-strategy and mixed-strategy analysis in the next sections extend directly to this alternative specification of the MMA.[8]

Finally, regarding to the technical consequences of A3, notice that together with (1) it implies that parties' payoff functions are discontinuous at every point on $D(A)$, except at (x_m,x_m). Furthermore, π_i is neither upper semi-continuous nor lower semi-continuous on A.[9] And, apart from the case where $\chi_1=\chi_2$, the sum of the payoffs $\Pi=\pi_1+\pi_2$ is also discontinuous on $D(A)$. This is illustrated in Example 1 and Lemma 1 below, respectively.

Example 1 Suppose ξ and θ are uniformly distributed, and assume that $\theta^1<x_m<\theta^2$. Take a point $x_\delta=(x_m+\delta,x_m+\delta)\in D(A)$, $\delta>0$, and a sequence $(x_1^n,x_2^n)=(x_m+\delta-\frac{1}{n},x_m+\delta)$, that converges to x_δ from the left. Since $x_1^n<x_2^n$, $\pi_1(x_1^n,x_2^n)=\left(\frac{\delta}{2\beta}-\frac{1}{4\beta n}+\frac{1}{2}\right)(\chi_1+\frac{1}{n})$. Then, $\lim_{n\to\infty}\pi_1(x_1^n,x_2^n)=\frac{\chi_1}{2}+\frac{\delta\chi_1}{2\beta}>\frac{\chi_1}{2}=\pi_1(x_\delta)$. Similarly, consider now

[8] As we explain later, in the case of mixed strategies, our existence result holds under the alternative specification because we use better reply security, instead of restrictions on the sum of payoff functions.

[9] A function $f:Y\to\Re$ on a nonempty and compact subset $Y\subset\Re^m$ is upper semi-continuous (u.s.c.) if for any sequence $\{y^n\}\subseteq Y$ such that $y^n\to\bar{y}$, $\limsup_{n\to\infty}f(y^n)\leq f(\bar{y})$. On the other hand, f is lower semi-continuous (l.s.c) if $-f$ is u.s.c.

a sequence $(\hat{x}_1^n, \hat{x}_2^n) = (x_m + \delta + \frac{1}{n}, x_m + \delta)$ that converges to x_δ from the right. Since $\hat{x}_1^n > \hat{x}_2^n$, $\pi_1(\hat{x}_1^n, \hat{x}_2^n) = \left(\frac{1}{2} - \frac{\delta}{2\beta} - \frac{1}{4\beta n}\right)(\chi_1 - \frac{1}{n})$. Then, $\lim_{n\to\infty} \pi_1(\hat{x}_1^n, \hat{x}_2^n) = \frac{\chi_1}{2} - \frac{\delta\chi_1}{2\beta} < \frac{\chi_1}{2} = \pi_1(x_\delta)$. Hence, π_1 is neither l.s.c. nor u.s.c. at x_δ. □

Lemma 1 $\Pi = \pi_1 + \pi_2$ *is continuous on* A *only if* $\chi_1 = \chi_2$.

Proof. By A3, $\Pi(x_1, x_2) = \phi(x_1, x_2, \theta^1, \theta^2) + \psi(x_2, x_1, \theta^2) + \chi_2$, where $\phi(x_1, x_2, \theta^1, \theta^2) = p(x_1, x_2)[\psi(x_1, x_2, \theta^1) - \psi(x_2, x_1, \theta^2) + (\chi_1 - \chi_2)]$. Since $u(\cdot, \theta^2)$ is continuous on x and χ_2 is a constant, $\psi(x_2, x_1, \theta^2) + \chi_2$ is continuous on A. To see that $\phi(x_1, x_2, \theta^1, \theta^2)$ is also continuous on A, suppose by contradiction there is a point $(\bar{x}_1, \bar{x}_2) \in A$ of discontinuity.

If $\bar{x}_1 \neq \bar{x}_2$, then $p(x_1, x_2) = 1 - H(1/2 - F(S(x_1, x_2)))$ is continuous at $(\bar{x}_1, \bar{x}_2)$, since H and F are continuous functions. Moreover, A1.a implies $\psi(\cdot, \theta^1) - \psi(\cdot, \theta^2)$ is continuous everywhere on A. Therefore, $\phi(\cdot, \theta^1, \theta^2)$ must be continuous as well. Contrary, if $\bar{x}_1 = \bar{x}_2$, then $p(\bar{x}_1, \bar{x}_2) = 1/2$. Consider a sequence $(x_1^n, x_2^n) \subseteq A$, such that $(x_1^n, x_2^n) \to (\bar{x}_1, \bar{x}_2)$. Notice that $\phi(\bar{x}_1, \bar{x}_2, \theta^1, \theta^2) = (\chi_1 - \chi_2)/2$. Thus, if $\chi_1 = \chi_2$, then $\phi(\bar{x}_1, \bar{x}_2, \theta^1, \theta^2) = 0$ and $\phi(x_1^n, x_2^n, \theta^1, \theta^2) = p(x_1^n, x_2^n)[\psi(x_1^n, x_2^n, \theta^1) - \psi(x_2^n, x_1^n, \theta^2)]$. Taking the limit of $\phi(x_1^n, x_2^n, \theta^1, \theta^2)$ with $(x_1^n, x_2^n) \to (\bar{x}_1, \bar{x}_2)$, we have that

$$\lim_{x^n \to \bar{x}} \phi(x_1^n, x_2^n, \theta^1, \theta^2) = 0,$$

since $p(x_1^n, x_2^n)$ is bounded and $\psi(x_1^n, x_2^n, \theta^1) - \psi(x_2^n, x_1^n, \theta^2)$ converges to zero when $x^n \to \bar{x}$. Therefore, Π is continuous on A.

On the other hand, if $\chi_1 \neq \chi_2$, then the term $p(x_1^n, x_2^n)(\chi_1 - \chi_2)$ does not converge to $(\chi_1 - \chi_2)/2$, except at (x_m, x_m). Hence, in this case the discontinuity of the probability of winning the election produces discontinuities not only in each payoff function, but also in the sum.[10] □

Thus, this stands in sharp contrast with Roemer [23], where it is shown that in spite of the discontinuities of the probability of winning, *pure ideological* parties have continuous payoff functions. More importantly, it implies that even if only one party assigns an arbitrary small weight on winning the election, this completely invalidates Roemer's [23] and [24] equilibrium analysis, leaving the question of equilibrium existence without any answer. This observation constitutes the main motivation for the current research.

Let $\mathcal{G} = [(A_i, \pi_i); i = 1, 2]$ be the two-party *hybrid* electoral competition game, where each π_i satisfies A1-A3.

Definition 1 *A pure-strategy Nash equilibrium (PNE) for* $\mathcal{G}$ *is a strategy profile* $(x_1^*, x_2^*) \in A$ *such that* $\pi_1(x_1^*, x_2^*) \geq \pi_1(x_1, x_2^*)$ *and* $\pi_2(x_1^*, x_2^*) \geq \pi_2(x_1^*, x_2)$, *for all* $(x_1, x_2) \in A$.

[10] This is because the discontinuities in the probability of winning function entail a shift of the electorate from one party's platform to the other's. So, if platforms are not equally profitable for parties, total payoff changes discontinuously.

We denote $NE(\mathcal{G})$ the set of all PNE of $\mathcal{G}$. For the particular case where $\chi_1 = \chi_2 = \infty$, that is, for the Downsian game, and for $\chi_1 = \chi_2 = 0$, which represents the Wittman model, it is well known that this set is nonempty. We will not go further on this, but the interested reader can find a comprehensive analysis in Roemer [24].

On the other hand, for the hybrid case where $\chi_i \in \Re_{++}$ for all $i = 1, 2$, the discontinuity of the payoff functions does not allow to apply a traditional analysis of equilibrium existence, based on the direct application of a fixed point theorem. However, it is possible to circumvent this difficulty by invoking recent sufficient conditions given by Reny [22]. To do that, we introduce the following notation and definitions. Let $d : \Re^2 \to \Re_+$ be the usual distance function on the real line and $B_\delta(y) = \{x \in X : d(x, y) < \delta\}$ the open ball about y with radius $\delta > 0$.

Definition 2 *Party 1 can secure a payoff* $\alpha \in \Re$ *at* $(x_1, x_2) \in A$ *if there exists* $\tilde{x}_1 \in A_1$ *and* $\delta > 0$ *such that* $\pi_1(\tilde{x}_1, x_2^{'}) \geq \alpha$ *for all* $x_2^{'} \in B_\delta(x_2)$.

This definition can be extended in the obvious way to party 2. That is, party 2 can *secure* a payoff $\alpha \in \Re$ at $(x_1, x_2) \in A$ if there exists $\tilde{x}_2 \in A_2$ and $\delta > 0$ such that $\pi_2(x_1^{'}, \tilde{x}_2) \geq \alpha$ for all $x_1^{'} \in B_\delta(x_1)$. In words, party i can secure a payoff α at x if it has a strategy that guarantees at least that payoff even if the other deviates slightly from x.

Definition 3 $\mathcal{G}$ *is payoff secure if for all* $x \in A$ *and all* $\epsilon > 0$, *each party* i *can secure a payoff of* $\pi_i(x) - \epsilon$ *at* x.

Payoff security requires that for every strategy profile $x \in A$, each party has a strategy that virtually guarantees the payoff he receives at x, even if the other party deviates slightly from x.

Let $\pi : A \to \Re^2$ be the vector payoff function of $\mathcal{G}$, defined by $\pi(x) = (\pi_1(x), \pi_2(x))$ for all $x \in A$. The graph of π is a subset of $A \times \Re^2$. That is, $gr(\pi) = \{(x, \pi) \in A \times \Re^2 : \pi_i(x) = \pi_i \, \forall i\}$. Let $cl(gr(\pi))$ be the closure of the graph of the vector payoff function of $\mathcal{G}$.[11]

Definition 4 $\mathcal{G}$ *is reciprocally upper semi-continuous (r.u.s.c.) if whenever* $(x, \pi) \in cl(gr(\pi))$ *and* $\pi_i(x) \leq \pi_i \; \forall i$, *then* $\pi_i(x) = \pi_i \; \forall i$.

Reciprocally upper semi-continuity generalizes the condition introduced by Dasgupta and Maskin [8] that the sum of the players' payoffs be upper semi-continuous. In effect, suppose Π is u.s.c. on A, and assume, by contradiction, there exists $(\bar{x}, \bar{\pi}) \in cl(gr(\pi))$ such that $\pi_i(\bar{x}) \leq \bar{\pi}_i \, \forall i$, and $\pi(\bar{x}) \neq \bar{\pi}$. Then, $\pi_1(\bar{x}) + \pi_2(\bar{x}) < \bar{\pi}_1 + \bar{\pi}_2$. Consider a sequence $(x_1^n, x_2^n) \subseteq A$, such that $(x_1^n, x_2^n) \to (\bar{x}_1, \bar{x}_2)$. Since $(\bar{x}, \bar{\pi}) \in cl(gr(\pi))$,

[11] Recall that $(x^*, \pi^*) \in cl(gr(\pi))$ if and only if $B_\epsilon(x^*, \pi^*) \cap gr(\pi) \neq \emptyset \; \forall \epsilon > 0$.

$\limsup_{x^n \to \bar{x}} \pi_i(x_1^n, x_2^n) = \bar{\pi}_i$.[12] Then, $\limsup_{x^n \to \bar{x}} \Pi(x^n) = \bar{\pi}_1 + \bar{\pi}_2$. On the other hand, $\limsup_{x^n \to \bar{x}} \Pi(x^n) \leq \pi_1(\bar{x}) + \pi_2(\bar{x})$, because Π is u.s.c. on A. Hence, $\bar{\pi}_1 + \bar{\pi}_2 \leq \pi_1(\bar{x}) + \pi_2(\bar{x})$. Contradiction. Thus, if Π is u.s.c., then the game must be r.u.s.c.

Now, we move on to the equilibrium analysis.

3 Pure-strategy analysis

In this section, we show that, although $\mathcal{G}$ is payoff secure and reciprocally upper semi-continuous for all $\chi_i \in \Re_{++}$, the existence of a PNE can be guaranteed only if parties have symmetric interests; i.e., if $\chi_1 = \chi_2$. The reason is the hybrid game is intrinsically badly behaved, in the sense that in general conditional payoff functions are not quasi-concave.

Proposition 1 *$\mathcal{G}$ is payoff secure.*

Proof. Suppose, by contradiction, there exists $(\bar{x}_1, \bar{x}_2) \in A$ and $\bar{\epsilon} > 0$ such that for some i the payoff $\pi_i(\bar{x}) - \bar{\epsilon}$ cannot be secured at $\bar{x}$. Without loss of generality, assume that $i = 1$. The fact that party 1 cannot secure $\pi_1(\bar{x}) - \bar{\epsilon}$ at $\bar{x}$ implies that

$$\forall\, \tilde{x}_1 \in A_1 \text{ and all } \delta > 0,\ \exists\, x_2^{'} \in B_\delta(\bar{x}_2) \,:\, \pi_1(\tilde{x}_1, x_2^{'}) < \pi_1(\bar{x}_1, \bar{x}_2) - \bar{\epsilon}. \quad (*)$$

If $(\bar{x}_1, \bar{x}_2) \notin D(A)$, then π_1 is continuous at $\bar{x}$. Therefore, $\exists \bar{\delta} > 0$ such that $\pi_1(\bar{x}_1, x_2^{'}) > \pi_1(\bar{x}_1, \bar{x}_2) - \bar{\epsilon}$, $\forall x_2^{'} \in B_{\bar{\delta}}(\bar{x}_2)$. Thus, if $\bar{x}_1 \neq \bar{x}_2$, we have an strategy for party 1, $\bar{x}_1$, and an interval for x_2, determined by $\bar{\delta} > 0$, such that $(*)$ does not hold. Contradiction.

On the contrary, assume $(\bar{x}_1, \bar{x}_2) \in D(A)$. If $\chi_1 = 0$, then π_1 is continuous on A, (see Roemer [23], Lemma 1). Hence, using the same argument than before, we have a contradiction with $(*)$. Suppose then $\chi_1 > 0$. Since $\bar{x}_1 = \bar{x}_2$, $\pi_1(\bar{x}) = \chi_1/2$. Three cases are possible, according to the position of $\bar{x}_1$, θ^1 and x_m on X:[13]

Case 1: $\theta^1 < \bar{x}_1 < x_m$. Consider an alternative $\tilde{x}_1 > \bar{x}_1$, $\tilde{x}_1 \in A_1$, close enough to $\bar{x}_1$. Define $\bar{\delta} = d(\tilde{x}_1, \bar{x}_1)$ and $B_{\bar{\delta}}(\bar{x}_2)$. We choose $\tilde{x}_1$ such that $\theta^1 < \bar{x}_1 - \bar{\delta}$ and $\bar{x}_1 + \bar{\delta} < x_m$. Since $\theta^1 < \bar{x}_1 < x_m$, this number always exists. By A1.b and A1.c, for all $x_2^{'} \in B_{\bar{\delta}}(\bar{x}_2)$, $\pi_1(\tilde{x}_1, x_2^{'}) = p(\tilde{x}_1, x_2^{'})\ (\chi_1 - (1-\alpha)\bar{\delta})$, where $\alpha \in (-1, 1)$. Notice that p is discontinuous at $\bar{x}$, so $p(\tilde{x}_1, x_2^{'})$ is well above 1/2. On the other hand, p is bounded. Therefore, there exists $\bar{\delta} > 0$ small enough such that $\pi(\tilde{x}_1, x_2^{'}) > \chi_1/2 - \epsilon$, for all $x_2^{'} \in B_{\bar{\delta}}(\bar{x}_2)$ and all $\epsilon > 0$, which contradicts $(*)$.[14]

[12] Recall that, for a sequence $\{x^n\}$, $\limsup_{n\to\infty} x^n = \inf_{n\geq 1} \sup_{k \geq n} x^k$.

[13] The reader can verify that all remaining situations are variants of these three cases.

[14] Concretely, this is true for any $\bar{\delta} < \frac{\left(p(\tilde{x}_1, x_2^{'}) - 1/2\right)\chi_1}{p(\tilde{x}_1, x_2^{'})(1-\alpha)}$.

Case 2: $\bar{x}_1 < \theta^1 < x_m$. Like in the previous case, consider $\tilde{x}_1$ and $\bar{\delta}$, such that $\bar{x}_1 + \bar{\delta} < \theta^1$. By A1.b and A1.c, $\pi_1(\tilde{x}_1, x_2^{'}) = p(\tilde{x}_1, x_2^{'})\,((1-\alpha)\bar{\delta} + \chi_1)$, $\alpha \in (-1, 1)$. Again, $p(\tilde{x}_1, x_2^{'}) > 1/2$. Moreover, $(1-\alpha)\bar{\delta} > 0$. Hence, $\pi(\tilde{x}_1, x_2^{'}) > \chi_1/2 - \epsilon$, for all $x_2^{'} \in B_{\bar{\delta}}(\bar{x}_2)$ and all $\epsilon > 0$. Contradiction.

Case 3: $\theta^1 < \bar{x}_1 = x_m$. Since the probability of winning function $p(x_1, x_2)$ is continuous at (x_m, x_m), $\pi_1(x_m, x_2)$ is continuous in x_2. Therefore, by the same argument employed above, $(*)$ cannot be true.

Thus, $\mathcal{G}$ is payoff secure. $\square$

Proposition 2 *For all* χ_1, $\chi_2 \in \Re_{++}$, $\mathcal{G}$ *is r.u.s.c.*

Proof. Suppose, by contradiction, there exists $(\bar{x}, \bar{\pi}) \in cl(gr(\pi))$ such that $\bar{\pi}_i \geq \pi_i(\bar{x})\,\forall i$, and $(\pi_1(\bar{x}), \pi_2(\bar{x})) \neq (\bar{\pi}_1, \bar{\pi}_2)$. Then, $(\bar{x}_1, \bar{x}_2) \in D(A)$. By hypothesis, there exists i such that $\bar{\pi}_i > \pi_i(\bar{x})$. Without loss of generality, suppose $i = 1$. Consider a sequence $(x_1^n, x_2^n) \subseteq A$, such that $(x_1^n, x_2^n) \to (\bar{x}_1, \bar{x}_2)$. Notice that $\bar{\pi}_1 = \inf_{n \geq 1} \sup_{k \geq n} \pi_1(x_1^k, x_2^k) = p(x_1^n, x_2^n)\,\chi_1$. On the other hand, $\pi_1(\bar{x}_1, \bar{x}_2) = 1/2\,\chi_1$. Therefore, $[1 - p(x_1^n, x_2^n)]\,\chi_2 < 1/2\,\chi_2$. But this implies that $\bar{\pi}_2 < \bar{\pi}_2(\bar{x})$. Contradiction. Hence, $\mathcal{G}$ is r.u.s.c. $\square$

By Lemma 1, $\mathcal{G}$ is also r.u.s.c. if $\chi_1 = \chi_2 = 0$. However, r.u.s.c. is violated if $\chi_i = 0$ and $\chi_j = +\infty$. To see this, suppose that $\chi_1 = 0$ and $\chi_2 \to +\infty$, with $x(\theta^1) > x_m$. Take a sequence $x^n = (x_1^n, x_2^n) \subseteq A$, such that $(x_1^n, x_2^n) \to (\bar{x}, \bar{x}) \in D(A)$. Suppose $x_m < \bar{x} < x(\theta^1)$, and assume that x^n converges to $\bar{x}$ from below. For instance, take $(x_1^n, x_2^n) = (\bar{x}, \bar{x} - \frac{1}{n})$, where $n > 0$. Then, as n raises, party 2's payoff, $1 - p(x_1^n, x_2^n)$, increases above 1/2, and it drops down to 1/2 at $(\bar{x}_1, \bar{x}_2)$. On the other hand, party 1's payoff, $p(x_1^n, x_2^n)\,\psi(x_1^n, x_2^n, \theta^1)$, converges to 0 as n goes to infinity. Therefore, $\pi_i(\bar{x}) \leq \bar{\pi}_i = \limsup_{x^n \to \bar{x}} \pi_i(x^n)$ for all i. However, $\pi_1(\bar{x}) = \bar{\pi}_1$, while $\bar{\pi}_2 > 1/2$. Hence, since $(\bar{x}, \bar{\pi}) \in cl(gr(\pi))$, we have that $\mathcal{G}$ violates r.u.s.c.[15]

The next proposition confirms that, under the assumptions above, $\mathcal{G}$ has no equilibrium in pure strategies. Let $\theta_m = x^{-1}(x_m)$. By A1, $x(\theta)$ is strictly increasing on Θ, so that θ_m is well defined.

Assumption 4 *Party 1 is policy motivated (i.e.,* $\chi_1 = 0$*), and it represents a type* $\theta^1 \neq \theta_m$*. Party 2 is office motivated (i.e.,* $\chi_2 \to +\infty$*).*

Proposition 3 *Suppose* $\mathcal{G}$ *satisfies A1-A4. Then,* $NE(\mathcal{G}) = \emptyset$*.*

Proof. We make the proof in three steps:

Step 1. $(x_m, x_m) \notin NE(\mathcal{G})$. Consider a deviation for party 1, from x_m to $x_1 \neq x_m$. By A1.b and A1.c, $F(S(x_1, x_m)) < 1/2$. Then, $p(x_1, x_m) = 1 - H[1/2 - F(S(x_1, x_m))] < 1/2 = p(x_m, x_m)$. However, by continuity of F and H, $\exists\ \delta > 0$ such that $p(x_1, x_m) > 0$ for all $x_1 \in B_\delta(x_m)$. Moreover,

[15] It violates r.u.s.c because when the point in the diagonal is reached, parties' payoffs all jump in the same direction.

let $\epsilon = d(x_m, x(\theta^1))$. $\epsilon > 0$ because $\theta^1 \neq \theta_m$. By A1.b, $\psi(x_1, x_m, \theta^1) > 0$ for all $x_1 \in B_\epsilon(x(\theta^1))$. Therefore, for all $x_1 \in B_\epsilon(x(\theta^1)) \cap B_\delta(x_m)$, $\pi_1(x_1, x_m) > \pi_1(x_m, x_m)$, meaning that the deviation is profitable.

Step 2. For any $x \neq x_m$, $(x, x) \notin NE(\mathcal{G})$. Trivial. Let $\delta = d(x, x_m)$. Consider any deviation of player 2 to an alternative $\tilde{x} \in B_\delta(x_m)$. By the assumptions on $u(\cdot, \theta)$, $F(S(x, \tilde{x})) < 1/2$. Then, $\pi_2(x, \tilde{x}) = 1 - p(x, \tilde{x}) > 1/2 = \pi_2(x, x)$.

Step 3. Suppose, by contradiction, there exists $(x_1, x_2) \notin D(A)$ such that $(x_1, x_2) \in NE(\mathcal{G})$. Without loss of generality, let $x(\theta^1) > x_m$. Then, $x_1 \geq x_m$. Suppose not. Then, party 1 can deviate to $\tilde{x}_1 \in B_\delta(x_m)$, $\delta = d(x_1, x_m)$, and increase both ψ and p. Moreover, if $x_1 = x_m$, then $(x_m, x_2) \notin NE(\mathcal{G})$, since 2 can deviate to x_m and increase its probability of being in power. Hence, $x_1 > x_m$. But then, for any $x_2 \in X$, $x_2 \neq x_1$, there exists $\delta > 0$ such that $p(x_1, \tilde{x}_2) < p(x_1, x_2)$ for all $\tilde{x}_2 \in B_\delta(x_1)$.[16] Therefore, $NE(\mathcal{G}) = \emptyset$. $\square$

Although the message of Proposition 3 is quite negative, Example 2 below shows that the fact that parties have asymmetric interests (i.e., $\chi_1 \neq \chi_2$) does not necessarily implies that PNE always fail to exist. Contrary, the example highlights that the existence of an equilibrium in pure strategies depends on the relationship between the electoral uncertainty, the aggregate level of opportunism and its distribution among political parties.

Example 2 Consider the electoral game introduced above. Suppose ξ is uniformly distributed on $[-\beta, \beta]$. Let $\theta^1 < x_m < \theta^2$ and $\chi_i \in \Re_{++}$ for all i. Consider an equilibrium candidate (x_1^*, x_2^*) for this game, with the property that $\theta^1 < x_1^* < x_2^* < \theta^2$. Near the equilibrium, parties' payoff functions can be written as

$$\begin{aligned} \pi_1(x_1, x_2) &= p(x_1, x_2)\,(x_2 - x_1 + \chi_1)\,, \\ \pi_2(x_1, x_2) &= (1 - p(x_1, x_2))\,(x_2 - x_1 + \chi_2)\,, \end{aligned}$$

where $p(x_1, x_2) = \frac{1}{2\beta}[F(S(x_1, x_2)) - \frac{1}{2} + \beta]$ if $x_1 \neq x_2$, and $p(x_1, x_2) = 1/2$ otherwise. Taken the first order conditions, we have

$$f(\hat{x})\,(x_2 - x_1 + \chi_1) - 2\beta = 2\,(F(\hat{x}) - 1/2)$$

$$-f(\hat{x})\,(x_2 - x_1 + \chi_2) + 2\beta = 2\,(F(\hat{x}) - 1/2)\,, \qquad (2)$$

where $\hat{x} = \frac{x_1 + x_2}{2}$ and f is the density function of θ. Assuming that θ is also uniformly distributed on Θ and solving (2), it follows that $x_1^* = 1/2 - \beta + \chi_1/2$ and $x_2^* = 1/2 + \beta - \chi_2/2$. It is easy to verify that, for $2\beta - 1 < \chi_i < 2\beta$, the pair

$$(x_1^*, x_2^*) = \left(\frac{1}{2} - \beta + \frac{\chi_1}{2}, \ \frac{1}{2} + \beta - \frac{\chi_2}{2}\right)$$

[16] That is, if $x_1 > x_m$, party 2 would like to choose the largest platform that is less than x_1. However, since the policy space is continuous, this value is not well defined.

constitutes indeed a Nash equilibrium of the game. The first condition, $2\beta - 1 < \chi_i$, ensures that $x_1^* > 0$ and $x_2^* < 1$. On the other hand, in order to guarantee that $x_1^* < x_2^*$, we need

$$\chi_1 + \chi_2 < 4\beta, \tag{3}$$

which is fulfilled if $\chi_i < 2\beta$ for all i. In fact, this implies $x_1^* < x_m < x_2^*$.[17]

Contrary, if (3) is not satisfied, that is, if the aggregate level of opportunism, measured by $\chi_1 + \chi_2$, is high regarding to the electoral uncertainty, represented by β, then (x_1^*, x_2^*) is not an equilibrium. Is there any other candidate? Since $\theta_1 < x_m < \theta_2$, it is clear that $x_1 > x_2$ cannot be a Nash equilibrium. Hence, the only remaining possibility is $x_1 = x_2$.

Assume that $\chi_1 + \chi_2 \geq 4\beta$. Consider first the case where $\chi_1 = \chi_2 = \chi$. Take the pair $(x_1, x_2) = (x_m, x_m)$ as the equilibrium candidate. (Recall that $x_m = 1/2$). Then, $\pi_i(x_m, x_m) = \chi/2$ for all i. Consider a deviation $x_1^{'} = x_m - \delta$, $\delta > 0$, for party 1. (Deviations to the right of x_m or below θ^1 are not profitable for 1.) Then,

$$\pi_1(x_1^{'}, x_m) = \frac{-\delta^2}{4\beta} + \left(\frac{1}{2} - \frac{\chi_1}{4\beta}\right)\delta + \frac{\chi_1}{2}.$$

The deviation is profitable, that is, $\pi_1(x_1^{'}, x_m) > \pi_1(x_m, x_m)$, if and only if $\delta < 2\beta - \chi$. But, since $\chi \geq 2\beta$, this requires $\delta < 0$. Contradiction.

In the same way, consider a deviation $x_2^{'} = x_m + \delta$, $\delta > 0$, for party 2. Following the same reasoning than before, $\pi_2(x_m, x_2^{'}) > \pi_2(x_m, x_m)$ if and only if $\delta < 2\beta - \chi$, which contradicts again the initial hypothesis. Therefore, since 2 cannot improve its payoff by deviating to the left of x_m or above θ^2, it follows that (x_m, x_m) is a pure-strategy Nash equilibrium.

What about if $\chi_1 \neq \chi_2$? Is (x_m, x_m) a PNE? As before, a deviation is profitable for party i if $\delta < 2\beta - \chi_i$. Suppose $\chi_1 = 3\beta/2$ and $\chi_2 = 5\beta/2$. Notice that we still have $\chi_1 + \chi_2 \geq 4\beta$. Therefore, there is no Nash equilibrium with $x_1 < x_2$. Moreover, for $\delta < 1/4$, any deviation $x_1^{'} = x_m - \delta$ is such that $\pi_1(x_1^{'}, x_m) > \pi_1(x_m, x_m)$. So, (x_m, x_m) is not a Nash equilibrium either. Finally, since $x_1 > x_2$ and $x_1 = x_2 \neq x_m$ are not equilibrium strategies, it follows that, for these values of χ_1 and χ_2, the game has no PNE. □

The previous example provides several interesting insights about electoral competition. First of all, it shows that, although $\chi_i > 0$ creates discontinuities in the payoff function, this does not necessary preclude the existence of a PNE. At least for the uniform distribution case, no matter how different χ_1 and χ_2 are, if the aggregate level of opportunism is *low* regarding to the electoral uncertainty, then a PNE always exists. In the example, sufficient conditions are $2\beta - 1 < \chi_i < 2\beta$ for all i. Moreover,

[17]Second order conditions also hold, because $\partial^2\pi_i/\partial x_i^2 = -2 < 0$ for all i.

this equilibrium is such that $x_1^* < x_2^*$. That is, equilibrium platforms do not converge, though $x_i^* \to x_m$ as $\chi_i \to 2\beta$.

On the other hand, when the aggregate level of opportunism is *high* (i.e., when (3) does not hold), and it is relatively more concentrated in one party, then a PNE may not exist. In the example, the game has no equilibrium in pure strategies if (i) $\chi_1 + \chi_2 \geq 4\beta$, and (ii) $\chi_1 < 2\beta$. This result is obtained by the combination of two forces. On one hand, the relatively more opportunistic party destroys equilibria with policy differentiation. On the other, the relatively more ideological party, due to its *excessive* policy orientation, impedes an equilibrium at the median position.

Remarkably, the non-existence of a PNE is not related to the dimensionality of the policy space, which is usually view in the literature of electoral competition as the main source of existence problems. Regarding to this, notice that our model satisfies two extremely nice properties, frequently invoked to guarantee the existence of equilibria: (i) the policy space is unidimensional; and (ii) voters' preferences are single-peaked and symmetric, so that they admit the existence of a strict Condorcet winner.

Furthermore, contrary to Ball's [3] explanation, we argue that it not *directly* related neither with the discontinuity of the probability of winning function. As we proved before, $\mathcal{G}$ satisfies payoff security and r.u.s.c. for all $\chi_1, \chi_2 \in \Re_{++}$. Thus, according to Reny's [22] Corollary 3.3, our game should always possess a pure-strategy Nash equilibrium, provided that each party's payoff function is quasi-concave in its own strategy.

In other words, Proposition 1 and 2 show that the blame for the failure of the hybrid electoral competition game to possess an equilibrium in pure strategies for all parametric conditions can be completely assigned to the violation of quasi-concavity, rather than to the discontinuity of the probability of wining function itself. Example 3 below illustrates this point.

Example 3 Consider the uniform case studied in Example 2. Let $\beta = 0.5$ and $\chi_1 = 0.75$. Suppose $x_2 = 0.49$. Then,

$$\pi_1(x_1, 0.49) = \begin{cases} -0.5x_1^2 + 0.375x_1 + 0.3038 & \text{if } \theta^1 < x_1 < 0.49, \\ 0.375 & \text{if } x_1 = 0.49, \\ 0.5x_1^2 - 1.375x_1 + 0.9362 & \text{if } x_1 > 0.49. \end{cases}$$

Notice that $\pi_1(0.375, 0.49) = 0.3741125$, $\lim_{x_1 \to^+ 0.49} \pi_1(x_1, 0.49) = 0.3675$ and $\lim_{x_1 \to^- 0.49} \pi_1(x_1, 0.49) = 0.3825$. Therefore, the conditional payoff $\pi_1(\cdot, 0.49)$ is not quasi-concave in x_1, as we can take the convex combination $x_1^\lambda = (1 - \lambda)0.375 + \lambda 0.491$ and get $\pi_1(x_1^\lambda, 0.49) < \min\{\pi_1(0.375, 0.49), \pi_1(0.491, 0.49)\}$ for all $\lambda \in (0, 1)$. □

Summarizing, the previous analysis shows that, in general, it is impossible to guarantee that the hybrid electoral competition game has an equilibrium in pure strategies. This negative result was previously noted by Ball

[3]. However, we argue here that it is not because of the mixed motivations, nor the discontinuities themselves. The main reason behind it is that the hybrid model allows parties' interests to be asymmetric. This is the *real* driving force of the aforementioned non-existence result.

As the next proposition shows, when political parties have mixed, but symmetric motivations, (i.e., when $\chi_1 = \chi_2$), the two-party electoral competition game always has a PNE. For the uniform distribution, this was already illustrated in Example 2, where $\chi_1 = \chi_2$ implies that the game has a PNE at either (x_m, x_m) or at $(x_1^*, x_2^*) = \left(\frac{1}{2} - \beta + \frac{\chi_1}{2},\ \frac{1}{2} + \beta - \frac{\chi_2}{2}\right)$. Now, we generalize this for any distribution.

Assumption 5 $\theta^1 < \theta_m < \theta^2$.[18]

Assumption 6 $\log(p(x_1, x_2))$ and $\log(1 - p(x_1, x_2))$ are concave in x_1 and x_2, respectively.[19]

Theorem 1 *Let $\chi_1 = \chi_2$ and assume $\mathcal{G} = [(A_i, \pi_i);\ i = 1, 2]$ satisfies A1-A3, A5 and A6. Then, $NE(\mathcal{G}) \neq \emptyset$.*

Proof. To prove Theorem 1 we proceed as follows. Consider the following restricted game $\hat{\mathcal{G}} = [(\hat{A}_i, \log(\pi_i|_{\hat{A}}));\ i = 1, 2]$, where $\hat{A}_1 = [0, x_m]$, $\hat{A}_2 = [x_m, 1]$, $\hat{A} = \hat{A}_1 \times \hat{A}_2$ and $\pi_i|_{\hat{A}}$ is the corresponding restriction of π_i to $\hat{A}$. $\hat{A}_i$ is a nonempty, compact and convex subset of $\Re$. The payoff function $\log(\pi_i|_{\hat{A}})$ is continuous on $\hat{A}$, because $p(x_1, x_2)$ is continuous on $\hat{A}$. Furthermore,

$$\begin{aligned}
\log(\pi_1|_{\hat{A}}(x_1, x_2)) &= \log(p(x_1, x_2)) + \log(\psi(x_1, x_2, \theta^1) + \chi_1) \\
\log(\pi_2|_{\hat{A}}(x_1, x_2)) &= \log(1 - p(x_1, x_2)) + \log(\psi(x_2, x_1, \theta^2) + \chi_2).
\end{aligned}$$

Therefore, $\log(\pi_i|_{\hat{A}})$ is concave on x_i. Thus, standard application of Kakutani's fixed point theorem shows that $\hat{\mathcal{G}}$ has a pure-strategy Nash equilibrium. That is, there exists a pair (x_1^*, x_2^*) such that $\pi_1|_{\hat{A}}(x_1^*, x_2^*) \geq \pi_1|_{\hat{A}}(x_1, x_2^*)$ and $\pi_2|_{\hat{A}}(x_1^*, x_2^*) \geq \pi_2|_{\hat{A}}(x_1^*, x_2)$ for all $(x_1, x_2) \in \hat{A}$.[20]

Now, we show that (x_1^*, x_2^*) is indeed a PNE in the original game $\mathcal{G} = [(A_i, \pi_i);\ i = 1, 2]$. To do that, we expand the set of strategies of both parties, and we study the incentives for unilateral deviations from the equilibrium candidate. For brevity, with carry out the analysis for party 1, but a similar reasoning also applies for the other party.

There are two possibilities to consider, depending on the features of the equilibrium in $\hat{\mathcal{G}}$.

[18]In words, A5 simply means that party 1 is left-oriented and party 2 right-oriented.

[19]Although this assumption is not nice, it is standard in the literature. The point is that, as it happens in the Wittman model, no simple conditions can be stated to guarantee the concavity of these functions. For more on this, see Roemer [24].

[20]Recall that $\log(\cdot)$ is a continuous and strictly increasing function.

Case 1: If $x_1^* = x_2^* = x_m$, then it is immediate to see that there is no deviation $\tilde{x}_1 > x_m$ such that $\pi_1(\tilde{x}_1, x_m) > \pi_1(x_m, x_m) = \chi_1/2$. In effect, this requires $(p(\tilde{x}_1, x_m) - 1/2)\chi_1 + p(\tilde{x}_1, x_m)(x_m - \tilde{x}_1) > 0$. However, $p(\tilde{x}_1, x_m) < 1/2$, $x_m < \tilde{x}_1$, and $\chi_1 \geq 0$. Therefore, such alternative $\tilde{x}_1$ does not exist. Repeating the argument for the other party, it follows that, if $(x_m, x_m) \in NE(\hat{\mathcal{G}})$, then it also belongs to $NE(\mathcal{G})$.

Case 2: If $x_1^* < x_2^*$, then three cases are possible. Suppose $x_1^* < x_m = x_2^*$. Then, the conditional payoff $\log(\pi_1(x_1, x_m))$ is continuous and concave on A_1, and

$$\arg\max_{A_1} \ \pi_1(x_1, x_m) = x_1^*. \tag{$*$}$$

That is, party 1's best response to x_m does not change in going from $\hat{A}_1 = [0, x_m]$ to $A_1 = [0, 1]$. On the other hand, if $x_1^* < x_m < x_2^*$, then $(*)$ holds because $\log(\pi_1(x_1, x_m))$ is continuous everywhere except at x_2^*, and it drops down to the right of x_2^*. The conditional payoff is not longer concave, but it is strictly quasi-concave, so that its unique maximum on A_1 coincides again with its restricted maximum on $\hat{A}_1$.

Finally, suppose $x_1^* = x_m < x_2^*$. In this case, we cannot immediately conclude that $(*)$ holds, because x_1^* is a corner solution in the restricted game and, therefore, $\pi_1(x_1, x_2^*)$ could increase to the right of x_m. However, we prove below that such a pair of strategies cannot be an equilibrium in $\hat{\mathcal{G}}$. The argument is as follows. Assume, by contradiction, $(x_1^*, x_2^*) \in NE(\hat{\mathcal{G}})$. Then, taking $\partial\pi_i/\partial x_i$ and evaluating it at (x_1^*, x_2^*), we have

$$\frac{1}{2}\left(x_2^* - x_1^* + \chi\right) \ h\left(\frac{1}{2} - F\left(\frac{x_1^* + x_2^*}{2}\right)\right) \ f\left(\frac{x_1^* + x_2^*}{2}\right) \geq p(x_1^*, x_2^*),$$

and

$$\frac{1}{2}(x_2^* - x_1^* + \chi) \ h\left(\frac{1}{2} - F\left(\frac{x_1^* + x_2^*}{2}\right)\right) \ f\left(\frac{x_1^* + x_2^*}{2}\right) = 1 - p(x_1^*, x_2^*).$$

But $p(x_1^*, x_2^*) > 1/2$. Hence, these two conditions cannot be simultaneously satisfied. Contradiction. This complete the proof. □

The result above generalizes the well known existence results for the Downsian and Wittman electoral competition games. It clearly shows that the blame for the non-existence of pure-strategy Nash equilibria for *all* parametric conditions does not lie on the discontinuities of the payoff functions nor the mixed motivations assumption, but on the asymmetric interests of political parties.

To overcome this difficulty of the hybrid model, in the next section we examine whether the mixed extension of $\mathcal{G}$ possesses a Nash equilibrium. After that, in Section 5, we provide a complete characterization of the set of equilibria for the uniform distribution case.

4 Mixed-strategy analysis

Consider the two-party hybrid electoral competition game $\mathcal{G}$ introduced in Section 2. Let $\Delta(A_i)$ be the space of all (Borel) probability measures on $A_i = [0,1]$, $i = 1,2$. A mixed strategy for party i is an element $\mu_i \in \Delta(A_i)$. In what follows, we endow $\Delta(A_i)$ with the topology of weak convergence. That is, we say that a sequence of measures $\{\mu_i^n\} \subseteq \Delta(A_i)$ converges to $\mu_i^* \in \Delta(A_i)$ if $\int_{A_i} g(x_i)d\mu_i^n \to \int_{A_i} g(x_i)d\mu_i^*$ for all real-valued and continuous function g defined on A_i. We denote $\Delta(A) = \Delta(A_1) \times \Delta(A_2)$.

Since the discontinuity of the parties' payoff functions is the main problem to prove the existence of a PNE in $\mathcal{G}$, before continuing it is necessary to show that each π_i is in fact a Borel measurable function on A, so that it will always make sense to talk about mixed strategies and the expected value of π_i under $\mu = (\mu_1, \mu_2) \in \Delta(A)$. Observe that this cannot be guaranteed by invoking neither continuity nor semi-continuity of the payoff functions. As we showed in Example 1, both conditions are violated in our game.

Let $(A, \mathcal{B}(A), \mu)$ be a Borel probability space, where $\mathcal{B}(A)$ is a Borel σ-algebra on A and $\mu : \mathcal{B}(A) \to \Re_+$ a probability measure on $\mathcal{B}(A)$. A real-valued function $z : A \to \Re$ is measurable with respect to $\mathcal{B}(A)$ if, for all $\alpha \in \Re$, $\{x \in A : z(x) < \alpha\} \in \mathcal{B}(A)$.

Lemma 2 *Each party payoff function $\pi_i : A \to \Re$ is a Borel measurable function on A.*

Proof. We prove the lemma for π_1. The argument for π_2 is identical. Define the function $\pi_1^*(x_1,x_2) = [1-H(1/2-F(S(x_1,x_2))]\,(\psi(x_1,x_2,\theta^1)+\chi_1)$ for all $(x_1,x_2) \in A$. Let $\bar{D}(A) = \{(x_1,x_2) \in A : x_1 = x_2 \text{ and } x_i \neq x_m\}$. It is clear that $\{(x_1,x_2) \in A : \pi_1(x_1,x_2) \neq \pi_1^*(x_1,x_2)\} = \bar{D}(A)$. Moreover, $\mu(\bar{D}(A)) = 0$, since $\bar{D}(A)$ can be covered by countably many open rectangles, each of which of measure zero. Hence, π_1 and π_1^* are equivalent with respect to the measure μ, (Kolmogorov and Fomin [17], Definition 2, pp. 288), because they coincide at all points except on a set of measure zero. Furthermore, π_1^* is continuous on A. Therefore, it is measurable. But, since a function equivalent to a measurable function is itself measurable (Kolmogorov and Fomin [17], Theorem 12, pp. 289),[21] it follows that π_1 is Borel measurable on A as well. □

Now we can complete the description of the mixed extension of the hybrid electoral competition game. To do that, we extend parties' payoff functions to the domain of mixed-strategy profiles in the usual way:[22]

$$U_i(\mu_1,\mu_2) = \int_A \pi_i(x)\, d(\mu_1,\mu_2), \quad \mu_i \in \Delta(A_i)\,\forall i$$

[21] This is because, for any $\alpha \in \Re$, the sets $\{x \in A : \pi_1(x) < \alpha\}$ and $\{x \in A : \pi_1^*(x) < \alpha\}$ can differ only by a set of measure zero. Hence, if the latter is measurable, so is the former.

[22] Notice that π_i is bounded, because $p(x)$ and $\psi(x,\theta^i)$ are bounded on A. Therefore, by Lemma 2, π_i is μ-integrable and its expected value is well-defined.

Observe that each $U_i : \Delta(A) \to \Re$ is well-defined, since any Borel measurable function on A is measurable in the associated product measure space.

Definition 5 *A mixed-strategy Nash equilibrium of* $\mathcal{G} = [(A_i, \pi_i); i = 1, 2]$ *is a pair of probability measures* $(\mu_1^*, \mu_2^*) \in \Delta(A)$ *such that* $U_1(\mu_1^*, \mu_2^*) \geq U_1(\mu_1, \mu_2^*)$ *and* $U_2(\mu_1^*, \mu_2^*) \geq U_2(\mu_1^*, \mu_2)$ *for all* $(\mu_1, \mu_2) \in \Delta(A)$.

That is, a mixed-strategy Nash equilibrium of $\mathcal{G}$ is a Nash equilibrium of the mixed extension $\overline{\mathcal{G}} = [(\Delta(A_i), U_i); i = 1, 2]$. We denote $ME(\mathcal{G})$ the set of all such equilibria.

To prove that $ME(\mathcal{G}) \neq \emptyset$ is not a trivial matter, since one is required to study infinite-action games with discontinuous payoffs. In order to solve this problem, we will use Reny's [22] sufficient condition, according to which a (compact and Hausdorff) game possesses a mixed-strategy Nash equilibrium if its mixed extension satisfies a property called *better reply security.*

Let $U : \Delta(A) \to \Re^2$ be the mixed extension's vector payoff function, defined as $U(\mu) = (U_1(\mu), U_2(\mu))$ for all $\mu \in \Delta(A)$.[23]

Definition 6 $\overline{\mathcal{G}} = [(\Delta(A_i), U_i); i = 1, 2]$ *is better reply secure if whenever* $(\mu^*, U^*) \in cl(gr(U))$ *and* μ^* *is not an equilibrium, some player* i *can secure a payoff strictly above* U_i^* *at* μ^*.[24]

Proposition 4 $\overline{\mathcal{G}}$ *is better reply secure.*

Proof. Suppose not. That is, assume by contradiction that there exists $(\mu^*, U^*) \in cl(gr(U))$, $\mu^* \notin ME(\mathcal{G})$, such that no party $i \in \{1, 2\}$ can secure a payoff strictly above U_i^* at μ^*.

This is equivalent to say that, for all i, all $\bar{\mu}_i \in \Delta(A_i)$, and all $\delta > 0$, there exists $\mu'_{-i}(\delta)$ such that $\|\mu'_{-i}(\delta) - \mu^*_{-i}\| < \delta$ and

$$U_i(\bar{\mu}_i, \mu'_{-i}(\delta)) \leq U_i^*, \tag{4}$$

where $\| \cdot \|$ is the norm in $\Delta(A)$. We disprove (4) in the following way.

Step 1. Since μ^* is not an equilibrium profile, that is, since $\mu^* \notin ME(\mathcal{G})$, there must exist $j \in \{1, 2\}$, and $\hat{\mu}_j \in \Delta(A_j)$ such that

$$U_j(\hat{\mu}_j, \mu^*_{-j}) > U_j(\mu_j^*, \mu^*_{-j}). \tag{5}$$

Step 2. If μ^*_{-j} is atomless on $[0, 1]$, then $\mu^*(D(A)) = 0$. Since π_j is continuous on $A - D(A)$, $U_j^* = U_j(\mu_j^*, \mu^*_{-j})$. Thus, (5) implies $U_j(\hat{\mu}_j, \mu^*_{-j}) > U_j^*$. Moreover, by continuity of U_j in μ^*_{-j}, for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $U_j(\hat{\mu}_j, \tilde{\mu}_{-j}) > U_j(\hat{\mu}_j, \mu^*_{-j}) - \epsilon$, for all $\tilde{\mu}_{-j}$ such that $\|\tilde{\mu}_{-j} - \mu^*_{-j}\| <$

[23] The graph of U and its closure are defined as before.

[24] As before, party i can secure a payoff $\alpha \in \Re$ at $\mu \in \Delta(A)$ if there exists $\bar{\mu}_i \in \Delta(A_i)$ such that $U_i(\bar{\mu}_i, \mu'_{-i}) \geq \alpha$ for all μ'_{-i} in some open neighborhood of μ_{-i}.

δ_ϵ. Hence, for ϵ small enough, j can secure a payoff strictly above U_j^*. Contradiction.

Step 3. On the contrary, suppose μ_{-j}^* is not atomless. Since μ_{-j}^* is a probability measure, it has at most countably many atoms. Denote $\mathcal{A}(\mu_{-j}^*) = \{x \in [0,1] : \mu_{-j}^*(\{x\}) > 0\}$. Consider $\bar{x}_j \in supp(\hat{\mu}_j)$, such that

$$\int_{A_{-j}} \pi_j(\bar{x}_j, x_{-j})\, d\mu_{-j}^* \geq U_j(\hat{\mu}_j, \mu_{-j}^*). \tag{6}$$

Step 4. If $\bar{x}_j \notin \mathcal{A}(\mu_{-j}^*)$, then π_j is continuous in x_{-j}. Thus, for all $\epsilon > 0$ there exists δ_ϵ such that

$$\int_{A_{-j}} \pi_j(\bar{x}_j, x_{-j})\, d\tilde{\mu}_{-j} > \int_{A_{-j}} \pi_j(\bar{x}_j, x_{-j})\, d\mu_{-j}^* - \epsilon \tag{7}$$

for all $\tilde{\mu}_{-j}$ such that $\|\tilde{\mu}_{-j} - \mu_{-j}^*\| < \delta_\epsilon$. Therefore, for ϵ small enough, (5), (6) and (7) imply that

$$\int_{A_{-j}} \pi_j(\bar{x}_j, x_{-j})\, d\tilde{\mu}_{-j} > U_j(\mu_j^*, \mu_{-j}^*), \tag{8}$$

for all $\tilde{\mu}_{-j}$ such that $\|\tilde{\mu}_{-j} - \mu_{-j}^*\| < \delta_\epsilon$.

Then, if $U_j(\mu_j^*, \mu_{-j}^*) \geq U_j^*$, we are done. By (8), it follows that j can secure a payoff strictly above U_j^*. Hence, $U_j^* > U_j(\mu_j^*, \mu_{-j}^*)$, which means that μ_j^* is not atomless.

Step 5. Assume that $U_i^* < U_i(\mu_i^*, \mu_{-i}^*)$, $i \neq j$, and that

$$\exists\, \hat{\mu}_i \in \Delta(A_i) \text{ such that } U_i(\hat{\mu}_i, \mu_{-i}^*) \geq U_i(\mu_i^*, \mu_{-i}^*). \tag{9}$$

Then, there must exist $\bar{x}_i \in supp(\hat{\mu}_i)$ such that $\int_{A_{-i}} \pi_i(\bar{x}_i, x_{-i})\, d\mu_{-i}^* > U_i^*$.

Step 6. If $\bar{x}_i \notin \mathcal{A}(\mu_{-i}^*)$, then π_i is continuous in x_{-i}. Repeating the argument in Step 4, it follows that for ϵ small enough,

$$\int_{A_{-i}} \pi_i(\bar{x}_i, x_{-i})\, d\tilde{\mu}_{-i} > U_i^*,$$

for all $\tilde{\mu}_{-i}$ such that $\|\tilde{\mu}_{-i} - \mu_{-i}^*\| < \delta_\epsilon$. Contradiction.

Step 7. On the other hand, if $\bar{x}_i \in \mathcal{A}(\mu_{-i}^*)$, consider $\tilde{x}_i$ close to $\bar{x}_i$, such that $\tilde{x}_i \notin \mathcal{A}(\mu_{-i}^*)$, and

$$\int_{A_{-i}} \pi_i(\tilde{x}_i, x_{-i})\, d\mu_{-i}^* \geq \int_{A_{-i}} \pi_i(\bar{x}_i, x_{-i})\, d\mu_{-i}^*. \tag{10}$$

Since μ_{-i}^* has at most countably many atoms, this alternative $\tilde{x}_i$ always exists. In effect, following the argument in Proposition 1's proof, we know that there exists $\bar{\delta}$ and $\tilde{x}_i = \bar{x}_i + \bar{\delta}$ such that $\pi_i(\tilde{x}_i, x_{-i}') \geq \pi_i(\bar{x}_i, x_{-i})$ for all $x_{-i}' \in B_{\bar{\delta}}(x_{-i})$. Hence,

$$\int_{B_{\bar{\delta}}(x_{-i})} \pi_i(\tilde{x}_i, x_{-i}) d\mu_{-i}^* \geq \int_{B_{\bar{\delta}}(x_{-i})} \pi_i(\bar{x}_i, x_{-i}) d\mu_{-i}^*. \tag{11}$$

Moreover, by continuity of $\int_{x_{-i}\notin B_{\bar{\delta}}(x_{-i})} \pi_i(\bar{x}_i, x_{-i}) d\mu^*_{-i}$ in x_i, for all $\epsilon > 0$

$$\int_{x_{-i}\notin B_{\bar{\delta}}(x_{-i})} \pi_i(\tilde{x}_i, x_{-i}) d\mu^*_{-i} > \int_{x_{-i}\notin B_{\bar{\delta}}(x_{-i})} \pi_i(\bar{x}_i, x_{-i}) d\mu^*_{-i} - \epsilon, \tag{12}$$

for all $\tilde{x}_i$ in some open neighborhood of $\bar{x}_i$. Therefore, for ϵ small enough, (11) and (12) imply (10). Finally, carrying out the reasoning in Step 6 for $\tilde{x}_i$, instead of for $\bar{x}_i$, we conclude that

$$\int_{A_{-i}} \pi_i(\tilde{x}_i, x_{-i})\, d\tilde{\mu}_{-i} > U_i^*,$$

for all $\tilde{\mu}_{-i}$ such that $\|\tilde{\mu}_{-i} - \mu^*_{-i}\| < \delta_\epsilon$. Contradiction.

Step 8. Therefore, (9) cannot be true. That is, if $U_i^* < U_i(\mu_i^*, \mu_{-i}^*)$, then

$$U_i(\hat{\mu}_i, \mu^*_{-i}) < U_i(\mu_i^*, \mu^*_{-i}) \text{ for all } \hat{\mu}_i \in \Delta(A_i). \tag{13}$$

But, remember that $\mu_i^* \,(= \mu^*_{-j})$ is not atomless. Moreover, μ^* must assign positive probability mass to some pairs in the diagonal, because $U_i(\mu_i^*, \mu_{-i}^*) > U_i^*$ and $U_j(\mu_j^*, \mu_{-j}^*) < U_j^*$. Therefore, Step 7 indicates that i can improve its payoff by reassigning an arbitrary small amount of probability mass to an alternative out of the diagonal. This increases i's expected payoff discretely, contradicting (13).

Step 9. Notice that the complement of the case analyzed in Step 4, (that is, the case where $\bar{x}_j \in \mathcal{A}(\mu^*_{-j})$), is ruled out by applying the same argument than in Step 7.

Step 10. Thus, in order to complete the proof, the only remaining possibility to consider is that $U_i^* \geq U_i(\mu_i^*, \mu_{-i}^*)$. Since we have assumed that $(\mu^*, U^*) \in cl(gr(U))$, and $U_j^* > U_j(\mu_j^*, \mu_{-j}^*)$, this means that $\overline{\mathcal{G}}$ is not reciprocally upper semi-continuous.[25]

Then, by hypothesis, for any sequence $\mu^n \to \mu^*$, we have that [26]

$$\lim_{\mu^n \to \mu^*} \left(\int_A \Pi(x)\, d\mu^n \right) > \int_A \Pi(x)\, d\mu^*.$$

The above inequality can be rewritten as

$$\begin{aligned} &\lim_{\mu^n \to \mu^*} \int_{A-D(A)} \Pi(x)\, d\mu^n - \int_{A-D(A)} \Pi(x)\, d\mu^* > \\ &> \frac{\chi_1 + \chi_2}{2} \left(\int_{D(A)} d\mu^* - \lim_{\mu^n \to \mu^*} \int_{D(A)} d\mu^n \right). \end{aligned} \tag{14}$$

[25]If the sum of the party payoff functions $\Pi = \pi_1 + \pi_2$ is u.s.c. on A, Proposition 5.1 in Reny [22] implies that the mixed extension $\overline{\mathcal{G}}$ is r.u.s.c., because $U_1 + U_2$ is u.s.c. in μ on $\Delta(A)$. Hence, $\chi_1 \neq \chi_2$. (Recall that, by Lemma 1, if $\chi_1 = \chi_2$, then Π is continuous on A.)

[26]Observe that, by the boundedness of π_i and the compactness of A_i, when $\mu^n \to \mu^*$, the sequence $\{\int_A \Pi(x)\, d\mu^n\}$ always converges. Thus, without loss of generality, we assume that the limit, $\lim_{\mu^n \to \mu^*} \left(\int_A \Pi(x)\, d\mu^n\right)$, always exists.

But, since Π is continuous on $A - D(A)$, and $\chi_i \in \Re_+$ for all i, by the weak convergence, both the LHS and the RHS of (14) tends to zero. Contradiction.

Therefore, $\overline{\mathcal{G}}$ is better reply secure. □

Now, we present the main result of the paper:

Theorem 2 *The two-party hybrid electoral competition game has a mixed-strategy Nash equilibrium; i.e., $ME(\mathcal{G}) \neq \emptyset$.*

Proof. Immediate from Proposition 4 and Reny's [22] Corollary 5.2 to Theorem 3.1. □

Compared to Ball [3], our result in Theorem 2 have two main differences. First, in our model the probability of winning the election is endogenously derived, instead of being given by an exogenous function. This is important because the properties required to ensure the existence of a MNE are not imposed on this function, but on more fundamental primitives of the model. Furthermore, by modeling explicitly the electoral uncertainty, we also get a better understanding of the game, which allowed to prove in Section 3 the existence of a PNE when parties have symmetric motivations, and to carry out comparative statics in the next section.

On the other hand, our existence analysis is also different because it is based on Reny's conditions (namely, on better reply security). Instead, Ball's [3] analysis relies on Dasgupta and Maskin's [8] approach (namely, on Theorem 5b). Related to this, it is important to emphasize that better reply security is *virtually* an ordinal property, (Reny [22], pp. 1034), in the sense that, if $f_i : \Re \to \Re$ is continuous and strictly increasing for every $i = 1, 2$, then $[(\Delta(A_i), U_i); i = 1, 2]$ is better reply secure if and only if $[(\Delta(A_i), f_i \circ U_i); i = 1, 2]$ is. So, our result in Proposition 4 and Theorem 2 hold for every continuous and strictly monotone transformation of the payoff functions. In particular, it extends to the alternative specification of the MMA discussed in Section 2. We now shift to the characterization analysis.

5 Equilibrium characterization

The result in Theorem 2 is of no help in finding or describing equilibria of $\mathcal{G}$. However, it is a useful fact to know, as it indicates that this task is not meaningless. Next, we focus on the equilibrium characterization of the most simple case, the uniform case. We hope this will highlight features of electoral competition that can be extended to more general settings.

Suppose the preference parameter θ and the error term ξ are both uniformly distributed, and assume that $\theta^1 < 1/2 < \theta^2$, so that party 1 is left-oriented and 2 right-oriented. Denote $\mathcal{G}^u$ the resulting electoral competition game. To fix the notation, let $\Delta(A_i)$ be the set of probability distributions on $[0,1]$, and $\mathcal{S}_i$ the support of the equilibrium mixed strategy for party i, with $\underline{x}_i = \inf \mathcal{S}_i$ and $\overline{x}_i = \sup \mathcal{S}_i$.

The following lemma summarizes the results found in Example 2:

Lemma 3 *If either $\chi_1 + \chi_2 < 4\beta$, or $\chi_i \geq 2\beta$ for all $i = 1, 2$, then $\mathcal{G}^u$ has a unique Nash equilibrium, given by $(x_1, x_2) = (\frac{1}{2} - \beta + \frac{\chi_1}{2},\ \frac{1}{2} + \beta - \frac{\chi_2}{2})$ and $(x_1, x_2) = (\frac{1}{2},\ \frac{1}{2})$, respectively.*

Proof. Immediate from the analysis in Example 2. □

Lemma 4 *If $\chi_i < 2\beta$ for some i, there is no symmetric equilibrium in $\mathcal{G}^u$.*

Proof. Without loss of generality, assume that $\chi_1 < 2\beta$. If $\chi_1 + \chi_2 < 4\beta$, then the claim follows immediately from Lemma 3. Thus, suppose $\chi_1 + \chi_2 \geq 4\beta$. Without the benefit of each step being explained, we claim that, if $(G_1, G_2) = (G, G)$ is a symmetric MNE, the support of G is a closed interval.[27] Moreover, the upper bound $\overline{x} = 1/2$.[28] Then, if G has an atom on $1/2$, that is, if both parties assign positive probability mass to the pair $(1/2, 1/2)$ on the diagonal, then party 1 can increase its expected payoff by reassigning all this probability mass to a point arbitrary close to $1/2$. Instead, if G assigns zero probability mass on $1/2$, then party 2 can transfer some probability from the left to $1/2$. In either case, (G, G) is not an equilibrium profile. □

Proposition 5 *If $\chi_1 + \chi_2 \geq 4\beta$ and $\chi_i < 2\beta$ for some i, then $\mathcal{G}^u$ has an asymmetric mixed-strategy Nash equilibrium $(G_1, G_2) \in \Delta(A_1) \times \Delta(A_2)$, and every such equilibrium satisfies the following properties:*

1. *If $\chi_1 < 2\beta$, then $\mathcal{S}_1 = \mathcal{S}_2 = [\underline{x}, \overline{x}] \subseteq [\theta^1, 1/2]$,*
2. *If $\chi_2 < 2\beta$, then $\mathcal{S}_1 = \mathcal{S}_2 = [\underline{x}, \overline{x}] \subseteq [1/2, \theta^2]$,*
3. *Each G_j is atomless and differentiable on $(\underline{x}, \overline{x})$, with density g_j,*
4. *G_1 has an atom on $\underline{x}$, and G_2 on $\overline{x}$,*
5. *g_1 is strictly decreasing and g_2 is strictly increasing on $(\underline{x}, \overline{x})$, and*
6. *Each G_j converges weakly to the point mass on $1/2$, as $\chi_i \to 2\beta$.*

Proof. Under the hypothesis of Proposition 5, the existence of an asymmetric MNE follows from Theorem 2 and Lemma 4. Regarding to its properties, we prove them in the following way. Suppose $\chi_1 < 2\beta$. (The other case is similar.)

[27] See Claim 1 below for a similar argument.

[28] Recall that (G_1, G_2) is a MNE if and only if, for each player i, (1) $U_i(x, G_j) = U_i(z, G_j)$ for all $x, z \in \mathcal{S}_i$, and (2) $U_i(x, G_j) \geq U_i(z, G_j)$ for all $x \in \mathcal{S}_i$ and all $z \notin \mathcal{S}_i$.

Claim 1: $\mathcal{S}_1 = \mathcal{S}_2 = [\underline{x}, \overline{x}] \subseteq [\theta^1, 1/2]$. The fact that $\mathcal{S}_1 \subseteq [\theta^1, 1/2]$ is immediate, since every alternative greater than $1/2$, or smaller than θ^1, is strictly dominated for 1 and, therefore, it is never played with positive probability in a MNE. On the other hand, $\overline{x}_1 = \overline{x}_2$. Contrary, if $\overline{x}_1 > \overline{x}_2$, it is easy to see that $\theta^1 < \overline{x}_2 < 1/2$. But then, for $\epsilon > 0$ small enough, $U_1(\overline{x}_1, G_2) < U_1(\overline{x}_1 - \epsilon, G_2)$, contradicting that G_1 is an equilibrium strategy. Similarly, if $\overline{x}_1 < \overline{x}_2$, it turns out that, for $\epsilon > 0$ sufficiently small, $U_2(G_1, \overline{x}_2) < U_2(G_1, \overline{x}_2 - \epsilon)$. Thus, $\overline{x}_1 = \overline{x}_2 = 1/2$, where the last equality follows from the fact that, if $\overline{x}_1 = \overline{x}_2 < 1/2$, then 2 can assign a positive probability mass on $\overline{x}_2 + \epsilon$, $\epsilon > 0$, and increase its expected payoff given G_1.

By a similar reasoning and after many calculations, it also follows that $\underline{x}_1 = \underline{x}_2$. The main argument is that, if $\underline{x}_1 < \underline{x}_2$, there will exist policies $x, y \in (\underline{x}_1, \underline{x}_2)$ among which party 1 cannot be indifferent. That means that there will be alternatives in $(\underline{x}_1, \underline{x}_2)$ that do not belong to $\mathcal{S}_1$. But then party 2 can increase its probability of winning by moving a positive mass to these points. Thus, $\underline{x}_1 = \underline{x}_2$ and, therefore, $\mathcal{S}_2 \subseteq [\theta^1, 1/2]$.

Finally, we prove $\mathcal{S}_j$ is a closed interval. Consider first $\mathcal{S}_2$. Let $x \in (\underline{x}, \overline{x})$ and assume by contradiction that $x \notin \mathcal{S}_2$. If $x \notin \mathcal{S}_1$, then party 2 can increase its expected payoff by reassigning an arbitrary small amount of probability mass to x (which increases its probability of winning). Thus, $x \in \mathcal{S}_1$. Moreover, since x was arbitrary chosen, it turns out that $\mathcal{S}_1 = [\underline{x}, \overline{x}]$. That means that, for $\epsilon > 0$ conveniently chosen, we have that $U_1(x, G_2) = U_1(x + \epsilon, G_2)$, and that the probability mass assigned by G_2 to the right of x and $x + \epsilon$ is exactly the same. But then party 1 should strictly prefer x to $x + \epsilon$, because the former is closer to its ideal point. Contradiction. Therefore, $\mathcal{S}_2 = [\underline{x}, \overline{x}]$. A similar argument shows $\mathcal{S}_1 = [\underline{x}, \overline{x}]$.

Claim 3: G_j is atomless and differentiable on $(\underline{x}, \overline{x})$. To show that G_j is differentiable on $(\underline{x}, \overline{x})$, it is sufficient to prove that it is continuous on $(\underline{x}, \overline{x})$.[29] Denote $G_j(x^-) = \lim_{y \to^- x} G_j(y)$ (respectively, $G_j(x^+) = \lim_{y \to^+ x} G_j(y)$) the left-limit (respectively, the right-limit) of G_j at x. Consider first the case where $j = 2$. Assume, by contradiction, that G_2 is discontinuous at some $x \in (\underline{x}, \overline{x})$.

If $G_2(x) - G_2(x^-) > 0$, G_2 has an atom at x. But then, for any $\epsilon > 0$ arbitrary close to zero, party 1 cannot be indifferent between x and $x - \epsilon$. This is because the utility derived from these policies is almost the same, but the probability of winning is much lower at $x - \epsilon$, since the discontinuity of G_2 at x implies that those policies greater than or equal to x are played with higher probability mass by 2. That is, for $\epsilon > 0$ sufficiently small, $U_1(x, G_2) > U_1(x - \epsilon, G_2)$. But this contradicts that $\mathcal{S}_1$ is a closed interval. Therefore, $G_2(x) = G_2(x^-)$.[30] Moreover, since x was arbitrary chosen, it

[29]If G_j is differentiable on $(\underline{x}, \overline{x})$, then we can write $G_j(x) = \int_{\underline{x}}^{x} g_j(y)\, dy$, where g_j is said to be the density of G_j.

[30]$G_2(x)$ cannot be smaller than $G_2(x^-)$ because G_2 is non-decreasing on $(\underline{x}, \overline{x})$.

follows that G_2 is left continuous and atomless on $(\underline{x}, \overline{x})$.

So, it must be that $G_2(x^+) > G_2(x)$. However, following the same reasoning than before, for $\epsilon > 0$ small enough, $U_1(x+\epsilon, G_2) > U_1(x, G_2)$. This implies that $x \notin \mathcal{S}_1$. Contradiction. Hence, $G_2(x) = G_2(x^+)$. This, together with left continuity, shows that G_2 is continuous and, therefore, differentiable on $(\underline{x}, \overline{x})$. A similar argument applies for G_1.

Claim 4: G_1 has an atom on $\underline{x}$, and G_2 on 1/2. Consider a pair $y < z$ in $(\underline{x}, \overline{x})$. By definition, $U_1(y, G_2) = U_1(z, G_2)$. However, given that $\chi_1 < 2\beta$, if the expected probability of winning of party 1 is higher or even the same at y than at z, then 1 will always prefer to be close to its ideal point; i.e., at y. Thus, if party 1 is indifferent between y and z, its chances of winning must be lower at y than at z. That is, player 2 must assign higher probability mass to the right of y. Since this must be true for every pair in the support, no matter how close they are to $1/2$, it follows that G_2 must have an atom on $1/2$.[31] A similar argument proves G_1 has an atom on $\underline{x}$.

Claim 5: g_1 is strictly decreasing and g_2 is strictly increasing on $(\underline{x}, \overline{x})$. We conjecture that this is because, when party 1 moves to the right along the support of G_1, its expected probability of winning the contest must raise by a strictly increasing amount, to compensate the utility lost generated by being further from its most preferred ideological position. Thus, G_2 must be strictly convex and, therefore, g_2 strictly increasing.

Similarly, when party 2 moves to the left in the support, it must also increase its expected probability of winning in a strictly increasing way, since it has to be compensated for being farther of its ideology and the median. (Recall that the expected payoff given G_1, $U_2(G_1, x)$, must be constant on $\mathcal{S}$.) Hence, G_1 must be concave, implying that g_1 is strictly decreasing.

Claim 6: G_j converges weakly to the point mass on $1/2$, as $\chi_i \to 2\beta$. This is because $\underline{x} = x_1^* = \frac{1}{2} - \beta + \frac{\chi_1}{2}$ whenever $x_1^* \geq \theta^1$. Thus, $\underline{x} \to 1/2$ as $\chi_1 \to 2\beta$. □

Regarding to the results of this section, notice first that, although in our model political parties have different ideologies, which in principle could be quite distinct, the fact that they also care about the election itself provides new and interesting predictions. For example, the previous analysis shows that, when the opportunism is high and relatively more concentrated in one party, say the rightist one, not only a PNE might not exist, but also the support of each mixed-strategy equilibrium is such that no party proposes policies to the right of the median voter. That is, regardless of their ideologies, both parties announce policies on the left of the political spectrum. As we said before, this is simply because, no matter how far its ideology is from

[31]In fact, it must assign at least half of the probability mass on 1/2. As we said, this is because the expected probability of winning of party 1 should be always smaller at policies close to its ideal point.

the median position, the relatively more opportunistic party always moves on to the political side of the other party to undercut its platform.

Finally, another interesting result arising from Proposition 5 and Lemma 3 is that the equilibrium correspondence $\Gamma : \{\mathcal{G}^u(\beta)\} \rightrightarrows ME(\mathcal{G}^u(\beta))$ is in fact discontinuous in β. To see this, suppose $\chi_1 + \chi_2 \geq 4\beta$, and let $\chi_1 < 2\beta$, so that 2 is the relatively more opportunistic party. Then, as the uncertainty increases, party 1's probability distribution concentrates on the lower bound of $\mathcal{S}$, and G_2 on the upper bound. But, by Lemma 3, there exists β^* such that every mixed-strategy equilibrium disappears above this critical value. That is, for $\beta > \beta^*$, parties' incentives to randomize vanish. And, in the unique equilibrium of the game, each party proposes a policy on its own ideological side. Similar results arise if the aggregate level of opportunism decreases.

6 Final remarks

This paper deals with a unidimensional, two-party electoral competition game. Instead of assuming that parties have single and symmetric motivations, we suppose here that they are interested in winning the election, but also in the policy implemented after the contest.

The main implications of this assumption are the following. First, payoff functions are neither continuous nor semi-continuous on the strategy space. The game satisfies payoff security and reciprocally upper semi-continuity, but conditional payoffs might violate quasi-concavity. As a consequence, the existence of a pure-strategy Nash equilibrium can be guaranteed only if political parties have symmetric motivations, being the Downsian and Wittman equilibria two particular corollaries of this result.

On the other hand, for the case of asymmetric motivations, a Nash equilibrium always exists, but it is probably one in mixed strategies. The existence of such equilibrium is guaranteed because the mixed extension of the hybrid electoral competition game satisfies Reny's [22] better reply security for all parametric conditions. This result extends Ball's [3] analysis to the case where the probability of winning the election is endogenously derived; and, together with the result above, it generalizes previous existence results on unidimensional electoral competition.

Finally, the characterization of the set of equilibria for the uniform distribution case shows that, when parties have mixed and probably asymmetric motivations, four variables shape the electoral outcomes. These variables are: parties' ideology, the opportunism of each party, the aggregate level opportunism, and the electoral uncertainty. Depending on the values of these parameters, we might have PNE or MNE. More importantly, we might end up with both parties proposing platforms at the median position, to the right of the median, to the left, or with one party in each side of the political spectrum.

Interestingly, these results stand in sharp contrast with the standard prediction of the Downsian game, where regardless of the level of electoral uncertainty, both parties locate at the median position. Moreover, they substantially differ as well from the Wittman equilibria, where parties' ideologies constitute the main driven force of electoral outcomes. It is left for a future work to explore their validity for more general models of electoral uncertainty.

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