ON CONCAVIFICATION AND CONVEX GAMES

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August 12, 2004

Abstract. We propose a new geometric approach for the analysis of cooperative games. A cooperative game is viewed as a real valued function u defined on a finite set of points in the unit simplex. We define the *concavification* of u on the simplex as the minimal concave function on the simplex which is greater than or equal to u.

The concavification of u induces a game which is the *totally balanced cover* of the game. The concavification of u is used to characterize well-known classes of games, such as balanced, totally balanced, exact and convex games. As a consequence of the analysis it turns out that a game is convex if and only if each one of its sub-games is exact.

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1. Introduction

Geometric methods have been found fruitful in the analysis of cooperative games. Two major directions have been taken so far. The first is identifying a game with a vector in $\mathbb{R}^{2^n-1}$. This is done by ordering all non-empty coalitions $(S_1, S_2, \ldots, S_{2^n-1})$ and identifying the game v with the $2^n - 1$ dimensional vector, whose the i'th coordinate is $v(S_i)$. In this case, the set of all n-players cooperative games is identified with $\mathbb{R}^{2^n-1}$. This approach allows to use geometric and algebraic techniques for the analyzes. It is possible to find a set of relatively simple games that forms a basis (in the algebraic meaning) for the entire space. Such an analysis may facilitate the analysis significantly. For instance, the Shapley value, a solution concept which respects additivity, is determined by its behavior on the basis games.

The second geometric approach is to identify every coalition $S \subseteq N$ with the indicator of S, that is with the n-vector whose i'th coordinate equals 1 if $i \in S$ and 0 otherwise. In this case, every coalition corresponds to an extreme point of the unit cube in $\mathbb{R}^n$. Therefore, a game is a real valued function defined on the set of extreme points of the unit cube. Since the function is defined only on the set of extreme points, it is natural to consider an extension of the domain to the entire cube. A natural way of doing so is the multi-linear extension (Owen, 1972).

In this paper we propose a new geometric interpretation of a cooperative game. Every coalition S is identified with a point in the n-dimensional unit simplex. The coalition S is identified with the vector $C_S = \frac{I_S}{|S|}$ where I_S is the indicator of S. Thus, the coalition is identified with the uniform distribution over the members of S. A game v is converted to a function u defined over the points $C_S, S \subseteq N$. The value that the function u is assumes at the point C_S is the average of the worth of S, that is $u(C_S) = \frac{v(S)}{|S|}$.

Given such a function u, we consider the concavification of u, denoted **cav**u, which is a function defined on the entire simplex. The concavification of a function u is defined as the minimum of all concave functions that are greater than or equal to u. Since the minimum of a family of concave functions is concave, $\mathbf{cav}u$ is the minimal concave function which is greater than or equal to u.

The following argument might give the intuition for the reason why considering $\mathbf{cav}u$ is beneficial. One of the most intuitive solution concepts of a cooperative game is the core, defined by $\text{core}(v) = \{x \in \mathbb{R}^n;\ \sum_{i=1}^n x_i = v(N), \sum_{i\in S} x_i \geq v(S), S \subseteq N\}$. Let x be a particular vector in $\text{core}(v)$ and define the function[1] $f_x(q) = xq$ for $q \in \Delta$. f_x is a linear function on Δ and therefore concave. Since $x \in core(v)$ it follows that $f_x(C_S) \geq u(C_S)$ for every coalition S, with equality for $S = N$. Thus, since $\mathbf{cav}u$ is the minimal concave function which is greater than or equal to u, $f_x \geq \mathbf{cav}u$ on Δ and $f_x(C_N) = \mathbf{cav}u(C_N)$. In other words, f_x is a linear support for $\mathbf{cav}u$ at the point C_N. This argument suggests that there is a correspondence between core vectors of the game v and linear supports of $\mathbf{cav}u$ at C_N.

The paper contains results of three kinds. All demonstrate relations between certain properties of the game v and the structure of the functions u and $\mathbf{cav}u$. The first kind of results deals with the core of v and its sub-games. It turns out that v has a non-empty core if and only if $\mathbf{cav}u$ and u coincide on the center of the simplex, C_N. Moreover, the core of every sub-game of v is non-empty if and only if $\mathbf{cav}u$ and u coincide on all the points $C_S, S \subseteq N$.

The second kind of results refers to exact games. Exact games (Schmeidler, 1972) are characterized in terms of the concavification of u. Furthermore, a condition similar to that of Shapley-Bondareva (see Shapley, 1967 and Bondareva, 1962) theorem that characterizes exact games is provided.

The third kind of results refers to convex games (Shapley, 1971). It turns out that a game is convex if and only if each one of its sub-games

[1] xq denotes the inner product of x and q

is exact. In addition a convex game is characterized by a property of its concavification.

The paper is organized as follows. In section 2 we formally introduce the function u and its concavification. Section 3 is devoted to the core and Section 4 to exact games. The paper ends with Section 5 where convex games are discussed.

2. Concavification of a cooperative game

Let v be a cooperative game with N being the set of players, $|N| = n$. We denote by Δ the unit simplex of $\mathbb{R}^n$, that is $\Delta = \{(q_1, q_2, \ldots, q_n);\ \sum_{i=1}^{n} q_i = 1, q_i \geq 0, i = 1, 2, \ldots, n\}$. For any non-empty coalition $R \subseteq N$ and a player $i \in N$ define C_R^i to be $\frac{1}{|R|}$ if $i \in R$ and 0 otherwise. Denote $C_R = (C_R^1, ..., C_R^n)$. Notice that C_R is in Δ for every R.
For any non-empty coalition R define $u(C_R) = \frac{v(R)}{|R|}$. u is a function over a set of $2^n - 1$ points in the n dimensional simplex. $u(C_R)$ is the average of the worth of coalition R.

Definition 1. *The* concavification *of u, denoted $\textbf{cav}u$, is defined as the minimum of all concave functions $f : \Delta \to \mathbb{R}$ such that $f(C_R) \geq u(C_R)$ for every non-empty coalition R.*

Remark 1. *Since the minimum of a family of concave functions over Δ is concave, $\textbf{cav}u$ is concave. Thus, $\textbf{cav}u$ is the minimal concave function that is greater than or equal to u on every point of the type C_R.*

Lemma 1. *For every $q \in \Delta$,*

$$\textbf{cav}u(q) = \max\{\sum_{R\subseteq N} \alpha_R u(C_R);\ \sum_{R\subseteq N} \alpha_R C_R = q,\ \alpha_R \geq 0 \text{ and} \sum_{R\subseteq N} \alpha_R = 1\}.$$

Proof. Denote, $w(q) = \max\big\{\sum_{R\subseteq N} \alpha_R u(C_R);\ \sum_{R\subseteq N} \alpha_R C_R = q,\ \alpha_R \geq 0$ and $\sum_{R\subseteq N} \alpha_R = 1\big\}$. Since w is concave and $w \geq u$, $\textbf{cav}u \leq w$. On the other hand, if $\sum_{R\subseteq N} \alpha_R C_R = q$ where $\alpha_R \geq 0$ and $\sum_{R\subseteq N} \alpha_R = 1$, then by concavity of $\textbf{cav}u$, $\textbf{cav}u(q) \geq \sum_{R\subseteq N} \alpha_R \textbf{cav}u(C_R) \geq \sum_{R\subseteq N} \alpha_R u(C_R)$. Thus, $\textbf{cav}u \geq w$. ■

3. Convcavification and the core

We will use the following standard definition:

Definition 2. *For a function $f : \Delta \to \mathbb{R}$ and a point $p \in \Delta$, a vector $x \in \mathbb{R}^n$ is a* linear support *for f at p, if $xp = f(p)$ and $xq \geq f(q)$ for any $q \in \Delta$.*

The following proposition provides a simple characterization of games with non-empty core.

Proposition 1. *v has a non-empty core iff $\mathbf{cav}u(C_n) = u(C_n)$.*

Proof. Assume first that v has a non-empty core and let $x \in core(v)$. Consider the linear (and in particular concave) function on Δ defined by $f(q) = xq$. Since x is in the core, for every non-empty coalition $R \subseteq N$, $f(C_R) = xC_R = \frac{x(R)}{|R|} \geq \frac{v(R)}{|R|} = u(C_R)$. It follows that $f(q) \geq \mathbf{cav}u(q)$ for every $q \in \Delta$. By a similar argument, $f(C_N) = xC_n = u(C_N)$. Therefore, $u(C_N) \leq \mathbf{cav}u(C_N) \leq xC_N = u(C_N)$, so $u(C_N) = \mathbf{cav}u(C_N)$.

In the other direction, assume that $\mathbf{cav}u(C_n) = u(C_n)$. Since $\mathbf{cav}u$ is concave it has a linear support at the point C_N, call it x. By assumption, $xC_N = \mathbf{cav}u(C_N) = u(C_N)$. Also, for every $R \subseteq N$, $\frac{x(R)}{|R|} = xC_R \geq u(C_R) = \frac{v(R)}{|R|}$, so $x(R) \geq v(R)$. Therefore, $x \in core(v)$.■

Remark 2. *Shapley-Bondareva Theorem asserts that v has a non-empty core iff the equation*

$$\sum_R \alpha_R C_R = C_N, \tag{1}$$

where $\alpha_R \geq 0$ and $\sum_R \alpha_R = 1$ implies

$$\sum_R \alpha_R u(C_R) \leq u(C_N). \tag{2}$$

The non-trivial part of this statement is the "if" direction, which turns out to be a simple consequence of Proposition 1 and Lemma 1. Indeed, due to Lemma 1 and the fact that equation (1) implies equation

(2), $\boldsymbol{cav}u(C_N) \leq u(C_N)$*. Thus,* $\boldsymbol{cav}u(C_N) = u(C_N)$ *and by Proposition 1, the core of* v *is not empty.*

Corollary 1. *Assume that* v *has a non-empty core. Then,*
(a) x *is in the core of* v *iff* x *is a linear support of* $\boldsymbol{cav}u$ *at* C_N*.*
(b) *The dimension of the core is* $n - d$*, where*

$$d = \max\Big\{k; \quad (i) \quad \sum_{\ell=1}^{k} \alpha_\ell C_{R_\ell} = C_N, \quad (ii) \quad \sum_{\ell=1}^{k} \alpha_\ell u(C_{R_\ell}) = u(C_N), \quad (iii) \quad \alpha_\ell > 0 \ \text{ for every } \ell = 1, ..., k; \text{ and} \quad (iv) \quad C_{R_2} - C_{R_1}, ..., C_{R_k} - C_{R_1} \text{ are linearly independent}\Big\}.$$

Proof. (a) Follows from the proof of Proposition 1.
(b) The dimension of the set of supports at a certain point is the dimension of the domain (here, Δ, whose dimension is $n-1$) minus the dimension of the facet of the graph of **cav**u that contains this point in its relative interior. The dimension of this facet at C_N is $d-1$ and therefore the dimension of the set of supports at this point, which is the core, is $n - 1 - (d-1) = n - d$. ■

Proposition 1 reveals the relation between non-emptiness of the core and the concavification of u. It seems natural at this point to ask whether the same relation holds for the sub-games of v. We denote by v_R the sub-game of v where the set of players is restricted to R. The following Lemma asserts that the previous result holds for sub-games as well.

Lemma 2. *For any coalition* R*, the core of* v_R *is non-empty iff* $\boldsymbol{cav}u(C_R) = u(C_R)$*.*

Proof. Fix some coalition R. Denote by Δ_R the vectors in Δ that vanish outside of R (i.e., the vectors whose support is R). **cav**u restricted to Δ_R is a concave function. Assume that **cav**u coincides with u on

C_R. Since C_R is in the relative interior of Δ_R it has a linear support $x_R \in I\!R^n$, whose coordinates out of R vanish. By the argument of Proposition 1 the vector x_R, restricted to R, is in the core of v_R.

Conversely, suppose that $x_R = (x^i_R)_{i \in R}$ is in the core of v_R. Define $y_R = (y^i_R)_{i \in N} \in I\!R^n$ as follows. If $i \in R$, then $y^i_R = x^i_R$. Otherwise, $y^i_R = M$, where M is a large number to be determined later. Note, that if $T \subseteq R$, then $y_R C_T = x_R C_T$. Since x_R is in the core of v_R, then for every $T \subseteq R$, $y_R C_T \geq \frac{v_R(T)}{|T|} = \frac{v(T)}{|T|} = u(C_T)$, with equality when $T = R$. If M is large enough, then $y_R C_S \geq \frac{v(S)}{|S|} = u(C_S)$ for every S. Therefore, y_R defines a linear function (in particular, concave) over Δ which attains the value $u(C_R)$ on C_R and values which are greater than or equal to $u(C_S)$ on C_S, for other coalitions S. It follows that $\mathbf{cav}u(C_R) \leq y_R C_R$. Since, $\mathbf{cav}u(C_R) \geq u(C_R) = y_R C_R$ we have $\mathbf{cav}u(C_R) = u(C_R)$ as needed. ■

Corollary 2. *v is a market game iff $\mathbf{cav}u = u$.*

Proof. It is well known (see Shapley and Shubik ,1969) that v is a market game iff the core of every sub-game of v is not empty. By Lemma 2 it is equivalent to $\mathbf{cav}u = u$. ■

Remark 3. *For every game v, the corresponding $\mathbf{cav}u$ induces a game $\bar{v}$ defined as follows. For every coalition R, $\bar{v}(R) = |R|\mathbf{cav}u(C_R)$. By Corollary 2, $\bar{v}$ is a market game. This is the smallest market game which is greater than v itself. $\bar{v}$ is the* totally-balanced-cover *of the game v.*

Remark 4. *Kalai and Zemel (1982) assert that a market game is a minimum of finitely many linear functions. Indeed, if v is a market game, then $\mathbf{cav}u = u$, as Corollary 2 states. Thus, $\mathbf{cav}u$ is the minimum of its supports at the points of the sort C_R. Since there are finitely many of those, the assertion follows.*

4. Convcavification and exact games

Definition 3. *(Schmeidler, 1972) The game v is* exact *if for every coalition R there is x in the core of v such that $v(R) = x(R)$.*

Proposition 2. *The following are equivalent*
(a) *v is exact.*
(b) *u is the minimum of a family of linear functions $\{f_\ell\}_\ell$ over Δ such that $f_\ell(C_N) = u(C_N)$ for every ℓ.*
(c) *The equation*

$$\sum_R \alpha_R C_R = \beta C_T + (1-\beta)C_N, \tag{3}$$

where $\alpha_R \geq 0$, $\sum_R \alpha_R = 1$, T is a coalition and $\beta \in [0,1]$, implies

$$\sum_R \alpha_R u(C_R) \leq \beta u(C_T) + (1-\beta)u(C_N). \tag{4}$$

Proof. To show that (a) implies (b) assume that v is exact. For every coalition R let $x_R \in \mathbb{R}^n$ be such that $x_R C_S = \frac{x_R(S)}{|S|} \geq \frac{v(S)}{|S|} = u(C_S)$ for every S, with equality for $S = R, N$. Define $f_R(q) = x_R q$, and let $w(q) = \min_R\{f_R(q)\}$. It is not hard to see that $w(C_R) = u(C_R)$ for every coalition R, so (b) follows.

Next, assume that (b) holds. Then for every R there is a linear function f_R such that $f_R \geq u$ and $f_R(q) = u(q)$, $q = C_R, C_N$. Denote the segment connecting the points $(C_R, u(C_R))$ and $(C_N, u(C_N))$ by L. L is on the graph of f_R. Since $\mathbf{cav}u$ is concave, L is below the graph of $\mathbf{cav}u$. As $\mathbf{cav}u \leq f_R$, L is above the graph of $\mathbf{cav}u$. Thus, L is on the graph of $\mathbf{cav}u$. In particular, $\beta\mathbf{cav}u(C_R) + (1-\beta)\mathbf{cav}u(C_N) = \beta u(C_R) + (1-\beta)u(C_N)$ for every $\beta \in [0,1]$.

Now, assume that equation (3) holds. Due to concavity of $\mathbf{cav}u$, $\sum_R \alpha_R \mathbf{cav}u(C_R) \leq \beta\mathbf{cav}u(C_T) + (1-\beta)\mathbf{cav}u(C_N)$. Thus, $\sum_R \alpha_R u(C_R) \leq \sum_R \alpha_R \mathbf{cav}u(C_R) \leq \beta u(C_T) + (1-\beta)u(C_N)$, which proves (c).

Finally, assume (c). Lemma 1 and (c) imply that $\mathbf{cav}u(\beta C_R + (1-\beta)C_N) \leq \beta u(C_R) + (1-\beta)u(C_N)$. Concavity and the fact that $\mathbf{cav}u \geq u$

imply that

$$\mathbf{cav}u(\beta C_R + (1-\beta)C_N) = \beta u(C_R) + (1-\beta)u(C_N).$$

Thus, L is on the graph of $\mathbf{cav}u$. Therefore, there is x which is a linear support of $\mathbf{cav}u$ at both, C_N and C_R. By corollary 1 $x \in core(v)$ and since it is a linear support at C_R, $x(R) = v(R)$. This proves that v is exact. ■

Remark 5. *Note the similarity between the condition of Shapley-Bondareva Theorem (i.e., (1) implies (2) – see Remark 2) and Proposition 2 (c) (i.e., (3) implies (4)).*

5. Convcavification and convex games

Definition 4. *(Shapley 1971) The game v is convex if for any two coalitions S and T, $v(S) + v(T) \le v(S \cap T) + v(S \cup T)$.*

Notation 1. *(a) For a permutation π over N, denote by R^i_π the coalition $\{\pi(1), \pi(2), \ldots, \pi(i)\}$, $i = 1, \ldots, n$.*
(b) Let v be a game and π an order over the set of players N. Then the vector of marginal contributions, with respect π is $x_\pi = (x^1_\pi, \ldots, x^n_\pi)$, where $x^i_\pi = v(R^i_\pi) - v(R^{i-1}_\pi), i \in N$.
(c) Let $q = (q_1, \ldots, q_n) \in \Delta$. π_q denotes a permutation of the players such that $q_{\pi_q(1)} \ge q_{\pi_q(2)} \ge \ldots \ge q_{\pi_q(n)}$. When there is more than one such permutations, i.e., $q_i = q_j$ for some $i \neq j$, π_q is any one of them.
(d) For $q = (q_1, \ldots, q_n) \in \Delta$, let $R^i_q = R^i_{\pi_q}$ and $x_q = x_{\pi_q}$.

Remark 6. *The assertion that v is convex implies that v is exact is well known. It follows from the fact that in a convex game the vector of marginal contributions, with respect to an order π, x_π, is in the core of v. Thus, if according to π the players of R are ordered first and then all the rest, $x_\pi(R) = v(R)$. Thus, v is exact.*

The next proposition relates convex games to the structure of the function u.

Proposition 3. *The following are equivalent*

(a) *v is convex.*

(b) *v_R is exact for every coalition R.*

(c) *The equation*

$$\sum_R \alpha_R C_R = \beta C_T + (1-\beta) C_S,$$

where $\alpha_R \geq 0$, $\sum_R \alpha_R = 1$, $T \subseteq S$ are two coalitions and $\beta \in [0,1]$ implies

$$\sum_R \alpha_R u(C_R) \leq \beta u(C_T) + (1-\beta) u(C_S).$$

Proof. The fact that (a) and (c) of Proposition 2 are equivalent implies that (b) and (c) are equivalent.

Assume (c) and we will show (a). Let S and T be two coalitions. Then, $\frac{|S|}{|S|+|T|}C_S + \frac{|T|}{|S|+|T|}C_T = \frac{|S\cap T|}{|S|+|T|}C_{S\cap T} + \frac{|S\cup T|}{|S|+|T|}C_{S\cup T}$. (c) implies that $\frac{|S|}{|S|+|T|}u(C_S) + \frac{|T|}{|S|+|T|}u(C_T) \leq \frac{|S\cap T|}{|S|+|T|}u(C_{S\cap T}) + \frac{|S\cup T|}{|S|+|T|}u(C_{S\cup T})$. This is equivalent to $v(S) + v(T) \leq v(S\cap T) + v(S \cup T)$. Thus, v is convex.

If v is convex, then every v_R is convex, and therefore, by Remark 6, every v_R is exact. Thus, (a) implies (b). ■

Remark 7. *Proposition 2 (c) and Lemma 1 imply that when v is exact and q is on the segment connecting C_N and C_R (for some coalition R), **cav**$u(q)$ is equal to corresponding weighted average of $u(C_N)$ and $u(C_R)$. Proposition 3 (c) asserts that, when v is convex, this property holds also for other coalitions than the grand one: if q is on the segment connecting C_S and C_T and $T \subseteq S$, then **cav**$u(q)$ is equal to corresponding weighted average of $u(C_S)$ and $u(C_T)$.*

We conclude with two propositions which provide different characterizations of convex games. The first one uses **cav**u to describe the set of convex games, while the second asserts that a game is convex if and only if u has a certain consistency property. However, both these propositions heavily relies on the fact that in a convex game the vector of marginal contributions of the players (for any order) is in the core.

For the next proposition recall Notation 1 (d).

Proposition 4. *v is convex if and only if for every $q \in \Delta$,*

$$\boldsymbol{cav}u(q) = qx_q.$$

Proof. Assume first that v is convex, and fix some $q \in \Delta$. Without loss of generality we may assume that $q_1 \geq q_2 \geq \ldots \geq q_n$, so $\pi = \pi_q$ is the identity, and $R^i = R^i_q = \{1, 2, \ldots, i\}$. Since v is convex, $x_q \in core(v)$. Thus, by Corollary 1 (a), x_q is a linear support of $\mathbf{cav}u$ at the point C_N. In particular, $qx_q \geq \mathbf{cav}u(q)$.
For $k = 1, \ldots, n$ define $\alpha_k = k(q_k - q_{k+1})$ (with the convention that $q_{n+1} = 0$). Notice that $\alpha \in \Delta$. Indeed, $\alpha_k \geq 0$ for every k since $q_k \geq q_{k+1}$, and $\sum_{k=1}^n \alpha_k = \sum_{k=1}^n k(q_k - q_{k+1}) = \sum_{k=1}^n q_k = 1$. Consider the convex combination $\sum_{k=1}^n \alpha_k C_{R^k}$. For any coordinate $1 \leq j \leq n$ we have,

$$\sum_{k=1}^n \alpha_k C^j_{R^k} = \sum_{k=j}^n k(q_k - q_{k+1})\frac{1}{k} = \sum_{k=j}^n (q_k - q_{k+1}) = q_j$$

It follows that $\sum_{k=1}^n \alpha_k C_{R^k} = q$. By Lemma 1,

$$\begin{aligned}\mathbf{cav}u(q) &\geq \sum_{k=1}^n \alpha_k u(C_{R^k}) = \sum_{k=1}^n k(q_k - q_{k+1})\frac{v(R^k)}{k} \\ &= \sum_{k=1}^n (q_k - q_{k+1})v(R^k) = \sum_{k=1}^n q_k \left(v(R^k) - v(R^{k-1})\right) = qx_q.\end{aligned}$$

Thus, $\mathbf{cav}u(q) \geq qx_q$ and therefore $\mathbf{cav}u(q) = qx_q$, as required.

As for the inverse direction, assume that for every $q \in \Delta$, $\mathbf{cav}u(q) = qx_q$. We will use Proposition 3 to show that v is convex. Let $T \subseteq S$ be two coalitions, $0 \leq \beta \leq 1$, and denote $q = \beta C_T + (1 - \beta)C_S$. Assume that the equation $\sum_R \alpha_R C_R = q$ holds with $\alpha_R \geq 0$ and $\sum_R \alpha_R = 1$. By Lemma 1, $\sum_R \alpha_R u(C_R) \leq \mathbf{cav}u(q)$.
The vector q has the following formation:

$$q_i = \begin{cases} \frac{\beta}{|T|} + \frac{1-\beta}{|S|} & i \in T \\ \frac{1-\beta}{|S|} & i \in S \setminus T \\ 0 & i \notin S \end{cases}$$

Consider the permutation of q according to π_q. The first $|T|$ coordinates of this vector will equal $\frac{\beta}{|T|} + \frac{1-\beta}{|S|}$, the next $|S| - |T|$ coordinates will equal $\frac{1-\beta}{|S|}$, and the last $n - |S|$ will equal 0. Moreover, $R_q^{|T|} = T$, and $R_q^{|S|} = S$. Therefore, by assumption we obtain,

$$\begin{aligned}
\mathbf{cav}u(q) &= \sum_{i=1}^{|T|} \left(\frac{\beta}{|T|} + \frac{1-\beta}{|S|} \right) x_q^i + \sum_{i=|T|+1}^{|S|} \frac{1-\beta}{|S|} x_q^i \\
&= \left(\frac{\beta}{|T|} + \frac{1-\beta}{|S|} \right) \sum_{i=1}^{|T|} x_q^i + \frac{1-\beta}{|S|} \sum_{i=|T|+1}^{|S|} x_q^i \\
&= \left(\frac{\beta}{|T|} + \frac{1-\beta}{|S|} \right) v(T) + \frac{1-\beta}{|S|} \left(v(S) - v(T) \right) \\
&= \beta \frac{v(T)}{|T|} + (1-\beta) \frac{v(S)}{|S|} = \beta u(C_T) + (1-\beta) u(C_S).
\end{aligned}$$

Therefore, $\sum_R \alpha_R u(C_R) \leq \mathbf{cav}u(q) = \beta u(C_T) + (1-\beta) u(C_S)$. By Proposition 3 v is convex. ■

Proposition 4 states that when a game is convex, $\mathbf{cav}u(q)$ is the weighted average (according to q) of the marginal contributions of the players when ordered according to the order of the coordinates of q.

Proposition 5. *v is convex iff there is a family $\{f_\ell\}_\ell$ of linear functions such that whenever $R \subseteq S$, $u(C_R) = \min\{f_\ell(C_R);\ f_\ell(C_S) = u(C_S)\}$.*

Proof. If the above condition holds, then due to Proposition 2 (b), v and each of its sub-games are exact. Therefore, by Proposition 3, v is convex.

To show the converse, assume that v is convex. Consider an order π of the set of players and the vector of marginal contributions with respect to π, x_π. Let $f_\pi(q) = x_\pi q$. We show that the collection of linear functions $\{f_\pi\}$ over all permutations π satisfies the above condition.

First, since x_π is in the core of v for every π, we have for every R, $f_\pi(C_R) = x_\pi C_R \geq \frac{v(R)}{|R|} = u(C_R)$. In particular, $u(C_R) \leq \min\{f_\pi(C_R);\ f_\pi(C_S) = u(C_S)\}$ for every $R \subseteq S$.

For two particular coalitions $R \subseteq S$ choose any order π with the property that the first $|R|$ players in π are the coalition R and the first $|S|$ players are the coalition S. Then obviously, $u(C_R) = f_\pi(C_R)$ and $u(C_S) = f_\pi(C_S)$. It follows that $u(C_R) \geq \min\{f_\pi(C_R);\ f_\pi(C_S) = u(C_S)\}$ and the proposition follows. ∎

Proposition 5 is a kind of consistency property. Consider the set N as the set of states and v as a non-additive probability over N. Suppose that non-additive probability (Schmeidler, 1989) v is obtained by taking the minimum over a set of additive probabilities (see Gilboa and Schmeidler, 1989). v is convex if this set of additive probabilities owes the following property.

Upon receiving the information that the event R occurred, the non-additive probability is updated and becomes v_R. However, v_R itself is also obtained as a minimum of additive probabilities: of those according to which the probability of S is precisely $v(S)$.

6. Refernces

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