THE COALITION STRUCTURE CORE IS ACCESSIBLE

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Abstract. For each outcome (i.e. a payoff vector augmented with a coalition structure) of a TU-game with a non-empty coalition structure core there exists a finite sequence of successively dominating outcomes that terminates in the coalition structure core. In order to obtain this result a restrictive dominance relation - which we label outsider independent - is employed.

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1. Introduction

For a TU-game in coalitional form, there are two fundamental and strongly linked problems: (i) what coalitions will form, and (ii) how will the members of these coalitions distribute their total worth. We attempt to answer these questions for a certain class of games. We presuppose some bargaining process and show that the coalition structure core, provided it is non-empty, comes forward as a natural candidate for a solution.

Consider a TU-game and some initial individually rational payoff configuration - i.e. an individual rational payoff vector supported by a coalition structure for which the vector is group rational. In case some coalition D could gain by acting for themselves, it can reject this initial outcome and propose a second outcome. As in Shenoy (1979), Sengupta and Sengupta (1994) and Greenberg (1994, p1326) the improving coalition D becomes a member of the new coalition structure and none of the players in D looses when moving towards the new outcome. We impose an additional condition. The counter-proposal should be *outsider independent*: first, the new coalition structure should contain those coalitions in the initial configuration that do not shelter deviating players; and second, these unaffected coalitions obtain the very same payoffs. Hence, in contrast to Shenoy and Sengupta and Sengupta, the deviating coalition D cannot prescribe the structure and the payoffs of those coalitions that remain unaffected when the players in D separate to form a coalition.[1]

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[1]Diamantoudi (2002) also criticizes this feature of the dominance relation employed by Shenoy (1979) and Sengupta and Sengupta (1994).

Once a counter-proposal is established, another coalition may reject this in favor of a third outcome, and so forth. Apparently, this bargaining process turns the coalition structure core, if non-empty, into an accessible set of outcomes:

> *For each outcome of a TU-game with a non-empty coalition structure core, there exists a finite sequence of successive 'outsider independent' counter-proposals that terminates in the coalition structure core.*

In the search for a dynamic foundation of the core, already Green (1974) made an important contribution. He established a finite process of successive counter-proposals that almost surely reaches the core. Later on, Wu (1977) showed the existence of a bargaining scheme that converges to the core and rephrased this result as the core is globally stable.

Finally, our result is a continuation of the work by Sengupta and Sengupta (1996). Formulated in the language of von Neumann and Morgenstern, they proved the *indirect stability* of the core: no payoff allocation dominates a core outcome, and each outcome is indirectly dominated by a core outcome. We refine this stability property in two dimensions.

First, Sengupta and Sengupta (and also Green and Wu) concentrate on the core. Hence they do not tackle problem (i). They take the coalition structure to be exogenously given and assume that the grand coalition forms. We also take the coalition formation process into account and extend the stability result to the coalition structure core.

Second, we extend the dominance relation employed by Sengupta and Sengupta (1996) to a framework involving coalition structures. And here, as already explained, we motivate a restrictive dominance relation based upon the outsider-independency condition.

The next section collects preliminaries, introduces the coalition structure core, and defines outsider independent domination. Section 3 studies outsider independent dominating chains and proves our result. The coalition structure core is characterized as the smallest set of outcomes that satisfies this accessibility property.

2. Preliminaries

We introduce the notation and define games, outcomes, dominance, and the coalition structure core. As we do not assume that the grand coalition forms, we use outcomes (or individual rational payoff configurations, e.g. (Owen 1982, p236)) instead of imputations.

Let $N = \{1, 2, \ldots, n\}$ be a set of n players. Non-empty subsets of N are called coalitions. A *partition* is a set of pairwise disjoint coalitions so that their union is N and represents the breaking up of the grand coalition N. For a partition $\mathcal{P} = \{C_1, C_2, \ldots, C_m\}$ and a coalition C, the *partners' set* $P(C, \mathcal{P})$ of C in $\mathcal{P}$ is defined as the union of those coalitions in $\mathcal{P}$ that have a non-empty intersection with C:

$$P(C,\mathcal{P}) \;=\; \{i \,|\, i \in C_j \text{ with } j \text{ such that } C_j \cap C \neq \varnothing\} \;=\; \bigcup_{C_j \cap C \neq \emptyset} C_j.$$

A characteristic function $v : 2^N \setminus \{\varnothing\} \longrightarrow \mathbb{R}$ assigns a real value to each coalition. The pair (N, v) is said to be a transferable utility game in characteristic function form, in short, a game. An *outcome* of a game (N, v) is a pair $(x, \mathcal{P})$ with x in $\mathbb{R}^n$ and $\mathcal{P}$ a partition of N. The vector $x = (x_1, x_2, \ldots, x_n)$ lists the payoffs of each player and satisfies

$$\forall i \in N : x_i \geq v(\{i\}) \qquad \text{and} \qquad \forall C \in \mathcal{P} : x(C) = v(C),$$

with $x(C) = \sum_{j \in C} x_j$. The first condition is known as individual rationality: player i will cooperate to form a coalition only if his payoff x_i exceeds the amount he would get on his own. The second condition combines feasibility and the myopic behavior of the players. It states that the payoff vector x is efficient with respect to the coalition structure $\mathcal{P}$: each coalition in the partition $\mathcal{P}$ allocates its value among its members. Outcomes with the same payoff vector are said to be *payoff equivalent*. The set of all outcomes is denoted by $\Omega(N, v)$.

In case the grand coalition forms, then an outcome is a pair $(x, \mathcal{P})$ with $\mathcal{P} = \{N\}$ and x is an *imputation*, that is, a payoff vector, such that $x_i \geq v(\{i\})$, and $x(N) = \sum_{i \in N} x_i = v(N)$. As such, outcomes generalize imputations. Also note that $\Omega(N, v)$ is non-empty: it contains the outcome in which N is split up in singletons.

For a game (N, v), let v^* denote the maximum of $v(\mathcal{Q}) = \sum_{C \in \mathcal{Q}} v(C)$ where $\mathcal{Q}$ runs over all partitions of N. This number v^* is called the value of the game (N, v). The value of a superadditive game is equal to the value of the grand coalition.

Now, we list three dominance relations: domination at the level of payoffs, standard domination, and outsider independent domination. An interpretation and a discussion follows.

Definition 1. *Domination at the level of payoffs.* Let $x, y \in \mathbb{R}^n$ and let C be a coalition. Then, vector x dominates y by C, denoted by $x >_C y$, if

- for each player i in C we have $x_i \geq y_i$, and
- for at least one player i in C we have $x_i > y_i$.

Standard domination. Let (N, v) be a game and let $a = (x, \mathcal{P})$ and $b = (y, \mathcal{Q})$ be two outcomes. Then, outcome a standard dominates b by C if

- the payoff vector x dominates y by C, and
- $\mathcal{P}$ contains C.

Outsider independent domination. Let (N, v) be a game and let $a = (x, \mathcal{P})$ and $b = (y, \mathcal{Q})$ be two outcomes. Then, outcome a outsider independently dominates b by C if

- outcome a standard dominates b by C,
- $\mathcal{P}$ contains all coalitions in $\mathcal{Q}$ that do not intersect C,
- the restrictions of x and y to the set of players outside $P(C, \mathcal{Q})$ coincide.

Furthermore, we hold on the next terminology:

- C is called the deviating coalition, its members are *deviators*.

Outcome a outsider independent dominates b if $\mathcal{P}$ contains a coalition C such that a outsider independent dominates b by C, and we abbreviate this as a o.i.dominates b.

The o.i.dominance relation should be interpreted in a dynamic way. Let $a = (x, \mathcal{P})$ o.i.dominate $b = (y, \mathcal{Q})$ by C. Then, if b is considered as the initial outcome, one can say that coalition C deviates and enforces the new outcome a. Indeed, in order to obtain a higher total payoff, coalition C separates from its partners (and at least one member of C gets strictly better off). The players in $P(C, \mathcal{Q}) \setminus C$ become ex-partners of C. They may reorganize themselves and their payoffs might decrease when moving from b to a. In the worst case, these ex-partners fall apart to singletons. Finally, the outsiders, i.e. the players not in $P(C, \mathcal{Q})$, are left untouched.

The definition clearly indicates that outsider independent domination is more restrictive than standard domination, which was employed by Shenoy (1979) and Sengupta and Sengupta (1994) among others. With respect to standard domination the deviating coalition is allowed to affect the payoffs of all the players and thus to ignore the behavior and the motivation of the outsiders. The use of o.i.domination removes these privileges.

The o.i.dominance relation also models a merger or a breaking up. In the former case, the deviating coalition is the union of some of the coalitions in the initial partition. In the latter case, an initial coalition is split up into two or more subcoalitions; each subcoalition that is better off in the new outcome can be considered as the deviating coalition.

Sengupta and Sengupta (1996) restrict their attention to the core, that is, they assume that the grand coalition forms. As a consequence, they employ the dominance relation at the level of payoff vectors. In contrast, we are also concerned with the coalition formation process. We believe that in the context of coalition formation, the outsider independent dominance relation is a natural and appropriate extension.

Now we repeat the definition of the coalition structure core, and we explain its relation with the different domination relations.

Definition 2. (Greenberg, 1994). Let (N, v) be a game and let $\Omega(N, v)$ be the set of outcomes. The *coalition structure core* $C(N, v)$ is the set of outcomes $(x, \mathcal{P})$ that satisfy coalitional rationality: for each coalition S we have $x(S) \geq v(S)$.

Balancedness conditions in order to check whether or not the coalition structure core is non-empty are well-known (e.g. Greenberg, 1994).

Now, consider a game with value v^* and with a non-empty coalition structure core. Each outcome $(x, \mathcal{P})$ in the coalition structure core satisfies $x(N) = v^*$. On the other hand an outcome $b = (y, \mathcal{Q})$ does not belong to the coalition structure core as soon it can be blocked, i.e. as soon there exists coalition D such that $y(D) < v(D)$. Such a blocking coalition D has an incentive to deviate and is able to propose an outcome $a = (x, \mathcal{P})$ (with D in $\mathcal{P}$) that o.i.dominates and therefore also standard dominates b. The set of outcomes that are not o.i.dominated, the set of outcomes that are not standard dominated, and the coalition structure core all three coincide.

The same arguments hold in case the grand coalition forms and the core is non-empty. An imputation $(y, \{N\})$ does not belong to the core as soon it can be blocked by some coalition D. Again, this blocking coalition D can propose a standard dominating outcome or an outsider independent dominating outcome. In addition, D can propose a new imputation $(x, \{N\})$ such that $x >_D y$. Hence, in this case the conditions of undomination at the payoff level (imputations), standard *un*domination, and o.i.*un*domination all three coincide.

Finally, the coalition structure core might contain payoff equivalent outcomes. In case the grand coalition forms, the coalition structure core includes the core.

3. The coalition structure core is accessible

Consider an initial outcome. If a coalition can obtain a higher payoff, it is allowed to deviate (respecting the outsider independency conditions) and to propose a second outcome, and so forth. This bargaining process gives rise to an o.i.dominating sequence. We show that for each outcome there exists an o.i.dominating sequence that terminates in the coalition structure core. Let (N, v) be a game and let $\Omega = \Omega(N, v)$ be the set of all outcomes.

Definition 3. Let $a, b \in \Omega$. Outcome a is said to be *accessible* from b, and we write $a \leftarrow b$ (or $b \rightarrow a$), if *a sequentially o.i.dominates b*, i.e. there exists a positive integer k and a sequence of outcomes

$$a_0 = b,\ a_1,\ \ldots,\ a_{k-1},\ a_k = a$$

such that a_i o.i.dominates a_{i-1} for $i = 1, 2, \ldots, k$. The integer k is said to be the length of (or the number of steps in) the o.i.dominating sequence.

The relation '$\leftarrow$' describes a possible succession of transitions from one outcome to another. We are interested in the outcomes that appear at the *end* of these sequences.

Definition 4. Let Δ be a set of outcomes. Then, Δ is *accessible* from Ω if for each b in Ω there exists an a in Δ such that $a \leftarrow b$.

Lemma. *Let (N, v) be a game with a non-empty coalition structure core. Then, the coalition structure core is accessible.*

Proof. Let $b_0 = (y_0, \mathcal{Q}_0)$ be an outcome that is not in $C(N, v)$. In case b_0 is o.i.dominated by an outcome in $C(N, v)$, the proof is done. In case no outcome in $C(N, v)$ outsider independent dominates b_0, we construct an o.i.dominating sequence that terminates in the c.s.core.[2] This sequence will be denoted by $b_0 \rightarrow b_1 \rightarrow b_2 \rightarrow \ldots$. As a consequence, coalitions and individual payoffs have a double subscript the first one of which refers to the position in this dominating sequence.

The proof is divided into five steps. In Step 1, we select those players that can be blamed for not being able to go to the c.s.core in one step. We call those players 'overpaid'. In Step 2 we select a deviating coalition and in Step 3 we define an outcome b_1 that o.i.dominates b_0. In Step 4 we repeat Step 2 and Step 3 and we construct an o.i.dominating sequence $b_0, b_1, b_2, \ldots$. In Step 5 we show that this sequence reaches a c.s.core outcomes after a finite number of iterations.

Step 1. Defining the set of overpaid players.

Interpret $b_0 = (y_0, \mathcal{Q}_0)$ as the initial outcome. Let $a = (x, \mathcal{P})$ be a c.s.core outcome. A player i for which $y_{0i} > x_i$ is said to be *overpaid* relative to a. Let $O(b_0, a)$ collect these overpaid players. Since b_0 is not dominated by a, the set $O(b_0, a)$ is non-empty.

Now, we consider the collection of c.s.core outcomes that minimize the number of overpaid

[2]In this proof we use the term 'c.s.core' as a shorthand for 'coalition structure core'.

players. Within this collection, we look for an outcome $a^* = (x^*, \mathcal{P}^*)$ that minimizes the amount overpaid $y_0(O_0) - x^*(O_0)$, where $O_0 = O(b_0, a^*)$. We consider a^* as a c.s.core outcome *close* to b_0. Since a^* belongs to the c.s.core, we have that $x^*(N) = v(\mathcal{P}^*) = v^*$.

Step 2. Selecting a deviating coalition.
Since the outcome $b_0 = (y_0, \mathcal{Q}_0)$ is not in the c.s.core there exists at least one blocking coalition, i.e. a coalition D for which $v(D) > y_0(D)$. We select a deviating coalition D as follows. First, we inspect the coalitions in the partition $\mathcal{P}^*$ and we look for a blocking coalition D among $\mathcal{P}^*$. Next, if the partition $\mathcal{P}^*$ does not contain a blocking coalition, then the outcome b_0 is efficient with respect to $\mathcal{P}^*$ and satisfies $y_0(N) = v^*$. In that case we select a minimal (for inclusion) blocking coalition.

Step 3. Defining a deviating outcome.
In order to define the payoff vector in the deviating outcome $b_1 = (y_1, \mathcal{Q}_1)$ we consider the different types of players separately.

First, we deal with the *deviating players*. Since D blocks b_0 and a^* is a c.s.core outcome, we know that $y_0(D) < v(D) \leq x^*(D)$. Let $i \in D$. The payoff y_{1i} depends upon whether or not D contains overpaid players.

(1) If D does not contain overpaid players, then we define

$$y_{1i} = y_{0i} + \delta_i(D) \leq x_i^*,$$

with $\delta_i(D)$ non-negative and adding up to $\delta(D) = v(D) - y_0(D)$.

(2) If D does contain overpaid players, then we define

$$y_{1i} = \begin{cases} y_{0i} + \frac{1}{|D \cap O_0|}\left[v(D) - y(D)\right] & \text{in case } i \text{ is overpaid,} \\ y_{0i} & \text{in case } i \text{ is not overpaid.} \end{cases}$$

In words, the deviating coalition divides the surplus $\delta(D) = v(D) - y_0(D)$ among its members. The overpaid players are served first and consume the whole surplus. The non-overpaid players experience either a status quo or an improvement.

Secondly, the *ex-partners* of D are assumed to split up into singletons. Hence, each player i in $P(D, \mathcal{Q}_0) \setminus D$ receives his value $v(\{i\})$ as payoff.[3]

Thirdly, the *outsiders* remain untouched: if $i \notin P(D, \mathcal{Q}_0)$, then $y_{1i} = y_{0i}$. As such, we meet the outsider independency conditions.

In conclusion: b_1 o.i.dominates b_0. When moving from b_0 to b_1, the overpaid ex-partners of D become non-overpaid. In case b_1 is either a c.s.core outcome or o.i.dominated by a c.s.core outcome, the proof is complete. Otherwise, execute the next steps.

[3]This assumption can be relaxed. The ex-partners are allowed to reorganize themselves provided none of them is overpaid in the new outcome.

Step 4. An iteration.
We denote the set $O(b_1, a^*)$ of overpaid players in the outcome b_1 by O_1. This set O_1 is a subset of O_0. We repeat Steps 2 and 3 and we generate an o.i.dominating sequence $b_0 \to b_1 \to b_2 \to \ldots$ of outcomes and a corresponding sequence $O_0 \supseteq O_1 \supseteq O_2 \supseteq \ldots$ of sets of overpaid players. The next step shows the finiteness of necessary iterations.

Step 5. The sequence $b_0 \to b_1 \to b_2 \to \ldots$ enters the c.s.core in a finite number of iterations. Along this o.i.dominating sequence the set of overpaid players finds its minimal form, denoted by O, after a finite number of iterations. Let $b_s = (y_s, \mathcal{P}_s)$ be the first outcome in the sequence that satisfies $O_s = O$. In case the partition $\mathcal{P}_s$ differs from $\mathcal{P}^*$, we can execute some more iterations (as described in Step 4). Due to the selection criteria for the deviating coalition (Step 2) we obtain an outcome $b_t = (y_t, \mathcal{P}_t)$ with $t \geq s$, $O_t = O$, which is efficient with respect to $\mathcal{P}^*$.[4] The outcome $b = (y_t, \mathcal{P}^*)$ is payoff equivalent with b_t.

We *claim* that the outcome b is in the c.s.core. Since b and b_t are payoff-equivalent, this immediately implies that b_t is also in the c.s.core.

In case O is empty this claim is obviously true. Hence, assume that O is non-empty and that the iteration is unable to reduce it further. Denote the partners' set of the overpaid players in $\mathcal{P}^*$ by B, i.e. $B = P(O, \mathcal{P}^*)$, and the complement of B by A.

Since A does not contain overpaid players, we have $y_i \leq x_i^*$ for each i in A. Since b belongs to Ω and since A is the union of some of the coalitions in $\mathcal{P}^*$, we have $y(A) = x^*(A)$. Therefore, the payoff vectors y and x^* restricted to A coincide: $y|_A = x^*|_A$.

We complete the proof of the claim (that b belongs to the c.s.core) by contradiction. Assume the existence of a coalition D that blocks the outcome b. Since $y|_A = x^*|_A$, the coalition D is not a subset of A. Hence D intersects B. Let $\bar{D} = P(D, \mathcal{P}^*)$ be the partners' set of D. Since y and x^* are efficient with respect to $\mathcal{P}^*$, we have $y(\bar{D}) = x^*(\bar{D}) = v(\bar{D})$. Since O cannot be reduced, the coalition D contains all the overpaid players in $\bar{D}$. Therefore, $\bar{D} \backslash D$ only contains non-overpaid players and thus satisfies $y(\bar{D} \backslash D) \leq x^*(\bar{D} \backslash D)$. Use the efficiency of y and x^* with respect to $\mathcal{P}^*$ together with the fact that a^* is a c.s.core outcome to conclude that $y(D) \geq x^*(D) \geq v(D)$. Hence, D is not blocking. Contradiction.

Therefore, b and then also b_t belong to the c.s.core and the outsider independent dominating sequence $b_0, b_1, \ldots, b_t$ enters the c.s.core after a finite number of iterations. □

In order to stress the impact of the particular construction in the above proof we give an example of a bargaining scheme that does not enter the coalition structure core.

Example. Consider a three-player game in which each singleton has value 0, each pair has value 2, and the grand coalition has value 6. The core is non-empty. Nevertheless, the next three outcomes generate a cycle of dominating outcomes:

$$((1,1,0), \{1,2\}, \{3\}), \quad ((1,0,1), \{1,3\}, \{2\}), \quad \text{and} \quad ((0,1,1), \{2,3\}, \{1\}).$$

[4] Remember that in looking for a blocking coalition, the coalitions in $\mathcal{P}^*$ are the first candidates to check. Hence, for each outcome that is not efficient with respect to $\mathcal{P}^*$, there will be an efficient outcome later on in the sequence.

We conclude with a characterization of the coalition structure core.

Theorem. *The coalition structure core of a game, if non-empty, is the smallest (for inclusion) set of outcomes that satisfies accessibility.*

Proof. Accessibility follows from the previous lemma. Furthermore, each outcome in the coalition structure core is not outsider independent dominated. This implies minimality. □

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