DISCUSSION PAPER
2004

The Biais-Martimort-Rochet equilibrium with direct mechanisms

Gwenaël PIASER[1]

July 2004

Abstract

In this note we show that the equilibrium characterized by Biais, Martimort, and Rochet (2000) could have been characterized by using direct mechanisms.

Keywords: Common Agency, Revelation Principle, Direct Mechanisms, Nonlinear Prices.
JEL Classification: D82.

[1]Università Ca' Foscari di Venezia.

I would like to thank Andrea Attar and Diego Salzman for their comments and their help.

1 Introduction

Biais, Martimort, and Rochet (2000) (BMR thereafter) consider a multi-principals game to analyze imperfect competition under adverse selection in financial markets. Strategic liquidity suppliers post nonlinear prices (such as limit order schedules) which stand ready to trade with a risk-adverse agent who has private information on the fundamental value of the asset as well as on his hedging needs. They show that there exists an unique equilibrium in convex schedules and they analyze its properties.

In order to do that, they do not use standard mechanism design methods. Usually, in principal-agents games direct mechanisms are sufficient to characterize all equilibria. Peters (2001) and Martimort and Stole (2002) have shown that restricting the attention to direct mechanisms may induce a loss of generality. Some equilibria cannot be characterize by direct mechanisms. Nevertheless, if we consider more general mechanisms, such as menus (or price schedules), one can characterize all equilibria of any common agency game. The drawback of this approach is that menu (or price schedules) are more difficult to handle than direct mechanisms.

BMR showed that using calculus of variations one can characterize equilibria even if we allow principals to use menus. From that point of view BMR is an interesting contribution to the literature as it provides a clear and rigorous methodology.[1] Another methodology would have been to consider only direct mechanisms. If by doing that one cannot characterize all equilibria, Peters (2003) has shown that one characterizes regular equilibria.

In this note, we show that the BMR equilibrium could have been characterized by a much simplified approach as the use of direct mechanisms.

2 The Model

We use exactly the BMR's model. We just quickly present the formal aspects, for a more complete description of the model and its properties please refer to the original article.

[1] It is also a interesting contribution to the financial literature as it provides testable predictions.

There are $(n+1)$ players in the game, n principals and one agent. The principals play first, they offer simultaneously "mechanisms". A "mechanism" is a mapping from a message space ($\mathcal{M}_i$ is the set of all possible message spaces for principal i, $i \in \{1,\ldots,n\}$) to the decision space. Here a principal takes two decisions, a price T and a quantity q, the decision space is $\mathbb{R}^2_+$. Principal i is offering a couple $(M_i, (T_i(.), q_i(.)))$, the agent can either reject or accept the offer. If he accepts then he sends the message $m \in M_i$ (we must have $M_i \in \mathcal{M}_i$), the agent gets from principal i the decision $(T_i(m), q_i(m))$.

In the BMR model the interpretation of $(T_i(m), q_i(m))$ is the following: the agent must trade the quantity $q_i(m)$ at the price $T_i(m)$. If the agent rejects the offer from principal i, he gets $(0,0)$ from him. The agent observe all the offered mechanisms and he/she decides to reject or accept some of them. His preferences are represented by the following utility function.

$$U = \theta \sum q_i - \frac{\gamma\sigma^2}{2}\left(\sum q_i\right)^2 - \sum_i T_i.$$

The variables γ and σ are common knowledge. The variable θ is known only by the agent, principals know only the distribution of that variable over the range of possible values $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$. The density is denoted f. This density is common knowledge.

The principal i's preferences over q_i and T_i are represented by the following utility function:

$$T_i - v(\theta)\, q_i.$$

We consider Perfect Bayesian Equilibria for that game. The problem is quite complex, the set $\mathcal{M}_i$ can be very large (it formalizes all possible communication schemes between a principal and the agent), and it is difficult to characterize the optimal choice of M_i.

If we consider a simple principal-agent game $(n=1)$, the so-called "Revelation Principle" (Myerson 1979, Myerson 1982) states that one can ignore the choice of M_i, and consider that the message space is given and equal to Θ. One can show that the unique principal would have chosen $(\Theta, (T^*(.), q^*(.)))$ even if he would not have been constrainted to play $M = \Theta$ The couple $(T^*(.), q^*(.))$ is called a "direct mechanism"

An immediate consequence of the revelation principle is that we can retrict our attention to direct "revelating" mechanisms. The direct mechanism $(T^*(.), q^*(.))$ is "revelating" if it is such that the agent reveals the actual value of θ. Considering only "direct revelating mechanisms" simplifies a lot the game and the optimal values of $T^*(.)$ and $q^*(.)$ can be then characterized in most of the relevant games.

If we consider a multi-principals game ($n > 1$), the revelation principle does not apply: one cannot impose $M_i = \Theta$ and characterize all equilibria of the game. If we do this we characterize only a subset of the equilibria of the game.[2] If we want to characterize all the equilibria of the game, we can only consider as possible message space all the subset of the decision space, and consider that implement the message receive from the agent (Peters 2001, Martimort and Stole 2002, call that methodology "the Delagation Principle"). In our particular game, rather than considering any element of the abstract set $\mathcal{M}$, we can consider only the subsets of $\mathbb{R}_+^2$ and the mapping $(T_i(.), q_i(.))$ are define by:

$$\forall_{,} \left(\hat{T}, \hat{q}\right) \in Z_i, \ \left(T_i\left(\hat{T}, \hat{q}\right), q_i\left(\hat{T}, \hat{q}\right)\right) = \left(\hat{T}, \hat{q}\right),$$

where $Z_i \subset \mathbb{R}_+^2$. Roughly speaking, the agent gets what he asks from any principal, but he is allow to choose only in a restricted set. These mechanisms are called "menus", or sometimes "catalogs".

Even if this result restricts the possible strategies, it does not simplify a lot the analysis given that we still have problem with the characterization (considering all subset of $\mathbb{R}_+^2$ is out of reach). BMR restrict the communication set by considering only a particular class of subset of $\mathbb{R}_+^2$: they consider that principals are only allow to choose continuous and (almost everywhere) differentiable menus. The message space is $\mathbb{R}_+$, a particular message is $q \in \mathbb{R}_+$, and if the agent sends the message q, he gets $(T(q), q)$, where $T(.)$ is a continuous function, with a finite number of non-differentiable points.

In the following section we will show that BMR equilibrium could attained also using simple direct revelating mechanisms.[3]

[2] See Peters (2003).

[3] Peters (2003) shows that if we modify a the BMR's model, the revelation applies. If q_i is chosen by agent and observable by prinicipal i, there is no restriction to consider direct mechanisms. But as q_i

3 Direct mechanisms equilibria

Principals are using direct mechanisms i.e; mappings $(q_i(.), T_i(.))$ from Θ to $\mathbb{R}_+^2$. If the agent θ reports the vector $\tilde{\theta}$, he gets

$$\begin{aligned} U\left(\tilde{\theta},\theta\right) &= \theta\left(q_i\left(\tilde{\theta}_i\right)+\sum_{-i} q_{-i}\left(\tilde{\theta}_{-i}\right)\right)-\frac{\gamma\sigma^2}{2}\left(q_i\left(\tilde{\theta}_i\right)+\sum_{-i} q_{-i}\left(\tilde{\theta}_{-i}\right)\right)^2 \\ &- T_i\left(\tilde{\theta}_i\right)-\sum_{-i} T_{-i}\left(\tilde{\theta}_{-i}\right). \end{aligned}$$

We concentrate on principal i (the indices $-i$ represents all other principals). He considers others principals strategies $\left(q_j(.), T_j(.)\right)_{j\neq i}$ as given and known. The agent reports truthfully his type to principal i if

$$\theta \dot{q}_i(\theta)-\gamma\sigma^2\left(q_i(\theta)+\sum_{-i} q_{-i}\left(\tilde{\theta}_{-i}\right)\right)\dot{q}_i(\theta)-\dot{T}_i(\theta)=0.$$

The other report $\tilde{\theta}_j$, $j\neq i$ are define by the following first order conditions:

$$\theta \dot{q}_j\left(\tilde{\theta}_j\right)-\gamma\sigma^2\left(\sum_{g\in I} q_g\left(\tilde{\theta}_g\right)\right)\dot{q}_j\left(\tilde{\theta}_j\right)-\dot{T}_j\left(\tilde{\theta}_j\right)=0,$$

where $\tilde{\theta}_i=\theta$.

From that first order conditions, one can define the rent obtained be the agent.

$$\begin{aligned} U(\theta) &= \theta\left(q_i(\theta)+\sum_{-i} q_{-i}\left(\tilde{\theta}_{-i}\right)\right) \\ &- \frac{\gamma\sigma^2}{2}\left(q_i(\theta)+\sum_{-i} q_{-i}\left(\tilde{\theta}_{-i}\right)\right)^2-T_i(\theta)-\sum_{-i} T_{-i}\left(\tilde{\theta}_{-i}\right), \end{aligned}$$

is chosen by the agent, direct mechanisms are quite complex, and the revelation principle is not helpful: for example Laussel and Le Breton (2001) show that even in a complete information setting, observable actions introduce technical difficulties in common agency games. In the folllowing, we keep the BMR's model.

where the $\tilde{\theta}_{-i}$ are chosen optimally and then are implicit functions of θ. Applying the envelop theorem, we get:

$$\dot{U} = q_i + \sum_{-i} q_{-i}.$$

A necessary second order condition can be obtained by using standard methods. [4] For any $\theta \in \Theta$ we have

$$\theta \dot{q}_i(\theta) - \gamma\sigma^2 \left(q_i(\theta) + \sum_{-i} q_{-i}\left(\tilde{\theta}_{-i}\right)\right) \dot{q}_i(\theta) - \dot{T}_i(\theta) \quad = \quad 0,$$

where the $\tilde{\theta}_{-i}$ are chosen optimally. Given the definition of $U(\theta)$, the necessary condition

$$\theta \ddot{q}_i(\theta) - \gamma\sigma^2 \left(q_i(\theta) + \sum_{-i} q_{-i}\left(\tilde{\theta}_{-i}\right)\right) \ddot{q}_i(\theta) - \gamma\sigma^2 \dot{q}_i^2(\theta) - \ddot{T}_i(\theta) \quad < \quad 0,$$

can be written as

$$\theta \dot{q}_i(\theta) > 0.$$

The optimal quantity must be increasing with θ.[5] This condition is standard in mechanism design theory. As the utility of the agent is non monotonic in q_i this condition introduce a technical difficulty. To solve the main problem we will assume in the following that the the strategies of all other principals are such that for low value of θ the utility of the agent is decreasing with q_i and his utility is increasing with q_i when θ is hight enough. We will check that it is the case at equilibrium.

$$U(\theta) = -\int_{\theta}^{\theta_b} \left(q_i + \sum_{-i} q_{-i}\right) dF$$

if $\theta > \theta_b$,

$$U(\theta) = \int_{\theta}^{\theta_a} \left(q_i + \sum_{-i} q_{-i}\right) dF$$

[4] This necessary condition is not u nique, and clearly not sufficient.

[5] We have assumed that the functions $q(.)$ and $T(.)$ are twice differenciable. This asssumption is always made in the literature.

if $\theta < \theta_a$, and

$$U(\theta) = 0.$$

if $\theta \in [\theta_a, \theta_b]$, where $\underline{\theta} < \theta_a \leqslant \theta_b < \overline{\theta}$.

Please note that the function $q(\theta)$ must be continuous around θ_a and θ_b. Otherwise, by applying a simple argument, it would be possible for the principal to improve his profit: when $\theta \in [\theta_a, \theta_b]$, $q(\theta) = 0$, and the marginal profit for the principal i is equal to zero. If $q(\theta)$ does not go to zero when θ goes to θ_a (with $\theta > \theta_a$), then by increasing a little θ_a, the principal i would increase his profit.

Integrating by parts these expressions gives

$$\int_{\underline{\theta}}^{\overline{\theta}} U(\theta)\, dF = \int_{\underline{\theta}}^{\theta_a} \left(q_i + \sum_{-i} q_{-i} \right) \frac{F}{f} dF + \int_{\theta_b}^{\overline{\theta}} \left(q_i + \sum_{-i} q_{-i} \right) \frac{1-F}{f} dF.$$

The profit of principal i can be written as

$$\Pi = \int_{\Theta} \left[T_i(\theta) - v(\theta)\, q_i(\theta) \right] dF(\theta),$$

by using the definition of the utility function we can simplify the former expression:

$$\begin{aligned} \Pi &= \int_{\Theta} \left[\theta \left(q_i + \sum_{-i} q_{-i} \right) - \frac{\gamma\sigma^2}{2} \left(q_i + \sum_{-i} q_{-i} \right)^2 \right. \\ &- \left. U(\theta) + \sum_{-i} T_{-i} - v(\theta)\, q_i(\theta) \right] dF(\theta). \end{aligned}$$

The problem of the principal is equivalent to a point wise maximization problem. The principal maximizes the following expression with respect $q(\theta)$ if $\theta \in \left[\theta_b, \overline{\theta}\right]$.

$$\begin{aligned} \theta \left(q_i + \sum_{-i} q_{-i} \right) &- \frac{\gamma\sigma^2}{2} \left(q_i + \sum_{-i} q_{-i} \right)^2 \\ &- \left(q_i(\theta) + \sum_{-i} q_{-i} \right) \frac{(1-F)}{f} + \sum_{-i} T_{-i} - v(\theta)\, q_i(\theta). \end{aligned}$$

As we look for a symmetric equilibrium, we consider that all other principle play the same strategy, namely $q_j = q$ and $T_j = T$ for all j different from i. It simplifies the expression.

$$\begin{aligned}\theta\left(q_i+(n-1)q\right) &- \frac{\gamma\sigma^2}{2}\left(q_i+(n-1)q\right)^2 \\ &- \left(q_i(\theta)+(n-1)q\right)\frac{(1-F)}{f}+(n-1)T-v(\theta)\,q_i(\theta)\,.\end{aligned}$$

The first order condition is given by

$$\begin{aligned}(\theta-v(\theta))\left(1+(n-1)\frac{\partial q}{\partial q_i}\right)-\gamma\sigma^2 nq\left(1+(n-1)\frac{\partial q}{\partial q_i}\right)& \\ -\left(1+(n-1)\frac{\partial q}{\partial q_i}\right)\frac{(1-F)}{f}+(n-1)\frac{\partial T}{\partial q_i}+(n-1)\,v(\theta)\,\frac{\partial q}{\partial q_i} &= 0.\end{aligned}$$

To characterize the solution we need the expression of $\frac{\partial q}{\partial q_i}$ and $\frac{\partial T}{\partial q_i}$. From the self selection constraint we have derived the expressions:

$$\theta\dot{q}_j\left(\tilde{\theta}_j\right)-\gamma\sigma^2\left(\sum_{g\in I}q_g\left(\tilde{\theta}_g\right)\right)\dot{q}_j\left(\tilde{\theta}_j\right)-\dot{T}_j\left(\tilde{\theta}_j\right) = 0.$$

Without loss of generality, we can rewrite the direct mechanism $\left(q_j(\theta), T_j(\theta)\right)$ as a direct mechanism $\left(q_j(\theta), \tau_j(q(\theta))\right)$, then the former first oder condition becomes:

$$\theta-\gamma\sigma^2\left(q_i+\sum_{-i}q_{-i}\right)=t_j,$$

where $t_j \equiv \tau_j$. Differentiating this equation with respect to $q_i(\theta)$ gives:

$$-\gamma\sigma^2\left(1+\frac{\partial\Sigma_{-i}\,q_{-i}\left(\tilde{\theta}_{-i}\right)}{\partial q_i}\right)=t'\frac{\partial q_j\left(\tilde{\theta}_j\right)}{\partial q_i}.$$

By summing that conditions over $j \neq i$, we get:

$$-(n-1)\,\gamma\sigma^2\left(1+\frac{\partial\Sigma_{-i}\,q_{-i}\left(\tilde{\theta}_{-i}\right)}{\partial q_i}\right)=t'\Sigma_{-i}\frac{\partial q_{-i}\left(\tilde{\theta}_{-i}\right)}{\partial q_i},$$

or equivalently:

$$-\frac{(n-1)\gamma\sigma^2}{t'+(n-1)\gamma\sigma^2}=\sum_{-i}\frac{\partial q_{-i}\left(\tilde{\theta}_{-i}\right)}{\partial q_i}.$$

It follows that:

$$\begin{aligned}
&(\theta-v(\theta))\left(1-\frac{\dot{q}(\theta)}{1-\gamma\sigma^2\dot{q}(\theta)}\gamma\sigma^2(n-1)\right)\\
-\ &\gamma\sigma^2 nq(\theta)\left(1-\frac{\dot{q}(\theta)}{1-\gamma\sigma^2\dot{q}(\theta)}\gamma\sigma^2(n-1)\right)\\
-\ &\left(1-\frac{\dot{q}(\theta)}{1-\gamma\sigma^2\dot{q}(\theta)}\gamma\sigma^2(n-1)\right)\frac{(1-F(\theta))}{f(\theta)}\\
+\ &(n-1)\left[\theta-\gamma\sigma^2 nq(\theta)\right]\frac{\dot{q}(\theta)}{1-\gamma\sigma^2\dot{q}(\theta)}\gamma\sigma^2-(n-1)v(\theta)\frac{\dot{q}(\theta)}{1-\gamma\sigma^2\dot{q}(\theta)}\gamma\sigma^2=0.
\end{aligned}$$

Consequently ,

$$\begin{aligned}
&[q_m(\theta)-nq(\theta)]-\dot{q}(\theta)\gamma\sigma^2[q_m(\theta)-nq(\theta)]\\
-\ &[q_m(\theta)-nq(\theta)]\dot{q}(\theta)\gamma\sigma^2(n-1)+(n-1)[q^*(\theta)-nq(\theta)]\dot{q}(\theta)\gamma\sigma^2=0,
\end{aligned}$$

where $q^*(\theta)=\frac{\theta-v(\theta)}{\gamma\sigma^2}$ and $q_m(\theta)=q^*(\theta)-\frac{1-F(\theta)}{\gamma\sigma^2 f(\theta)}$. Finally we get:

$$\dot{q}(\theta)=\frac{1}{\gamma\sigma^2}\left(1+\frac{(n-1)(q^*(\theta)-q_m(\theta))}{nq(\theta)-q_m(\theta)}\right)^{-1},$$

the expression derived by BMR.[6]

If $\theta\in[\underline{\theta},\theta_a]$, the principal maximizes the following expression with respect to $q(\theta)$:

$$\begin{aligned}
&\theta\left(q_i+\sum_{-i}q_{-i}\right)-\frac{\gamma\sigma^2}{2}\left(q_i+\sum_{-i}q_{-i}\right)^2\\
-\ &\left(q_i(\theta)+\sum_{-i}q_{-i}\right)\frac{F}{f}+\sum_{-i}T_{-i}-v(\theta)q_i(\theta).
\end{aligned}$$

[6]BMR consider aggregate values, we consider individual values. Except that slight difference in the presentation, the formulas are strictly equivalent.

We can derive the same expression for $\dot{q}(\theta)$, except that $q_m(\theta) = q^*(\theta) - \frac{F(\theta)}{\gamma\sigma^2 f(\theta)}$.

Given the expressions of $\dot{q}(\theta)$, θ_a and θ_b must be such that the function q is continuous. As the aggregate supply $nq(.)$ is an increasing function, the form chosen for the utility is justified. Usual conditions on the density f guaranty that q is increasing. [7]

4 Conclusion

The theorem suggests four main remarks:

Direct mechanisms are not able to characterize any equilibria in a common agency game. However, they seem to be quite powerful. It would be very interesting to have a general theorem giving conditions under which an equilibrium cannot be characterized by direct mechanisms.

The BMR methodology remains interesting since we do not have this general theorem. We do not have any hints on the generality of our result.

The BMR equilibrium is the unique equilibrium with convex price schedules. It does not means that is the unique equilibrium of their game. The existence of other equilibria remains an open question. If it exists other equilibria, we do not know if direct mechanisms are able to characterize them.

If we consider common agency games some equilibria can be characterized by direct mechanisms, some that cannot be. It would be interesting to know which kind of equilibria is more likely empirically.

References

BIAIS, B., D. MARTIMORT, AND J.-C. ROCHET (2000): "Competing mechanisms in a common value environment," *Econometrica*, 78(4), 799–837.

LAUSSEL, D., AND M. LE BRETON (2001): "Conflict and Cooperation: the structure of equilibrium payoffs in common agency," *Journal of Economic Theory*, 100, 93–128.

[7] See Miravete (2002) for a discussion of these conditions and their interpretation in the BMR's model.

MARTIMORT, D., AND L. A. STOLE (2002): "The revelation and delegation principles in common agency games," *Econometrica*, 70(4), 1659–1673.

MIRAVETE, E. J. (2002): "Preserving Log-Concavity under convolution: Comment," *Econometrica*, 70(3), 1253–1254.

MYERSON, R. B. (1979): "Incentive compatibility and the bargaining problem," *Econometrica*, 47, 61–73.

——— (1982): "Optimal coordination mechanisms in generalized principal-agent problems," *Journal of Mathematical Economics*, 10, 67–81.

PETERS, M. (2001): "Common Agency and the Revelation Principle," *Econometrica*, 69(5), 1349–1372.

——— (2003): "Negociation and take-it-or-leave-it in common agency," *Journal of Economic Theory*, 111(1), 88–109.