

Dynamic Behavior in Minimum Effort Coordination Games

Some Theory of Group Size and Inter-Group Competition as
Coordination Devices

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Abstract

This paper presents a model of individual behavior in minimum effort coordination games, focusing primarily on the effects of the number of players and the introduction of inter-group competition. It is shown that independent of the number of players and the number of competing groups, the most inefficient equilibrium is always the stochastically stable one. Yet, it turns out that the ‘security’ of more efficient equilibria increases with a decrease of the number of players and with an increase of the number of competing groups.

key words Minimum Effort Coordination, Group Competition, Stochastic Stability, Dynamic Games

JEL classifications C72, C92

1 Introduction

This paper develops a model of individual behavior in repeated minimum effort coordination games. The model focuses on two variants of the game, the ‘basic’ game following Van Huyck et al. (1990), and a variant involving inter–group competition for a bonus described by Bornstein et al. (2002). The model is motivated by the apparent lack of a theoretical benchmark case, the experimental observations in this game can be contrasted to. Thus, the primary objective of this paper is to suggest a theoretical foundation of individual behavior in repeated minimum effort coordination situations that abstracts from non–mainstream influences like ‘abnormal’ behavior or strategic uncertainty.

The problem of coordination failure, i.e. inefficient coordination of individual behavior, is prevalent in many fields of economics, ranging from Post–Walrasian macroeconomics (see, for example, Cooper and John 1988; Bryant 1996) to questions of providing work incentives in industrial organization (Nalbantian and Schotter, 1997). The problem is stripped down to its very core in the minimum effort coordination game initially proposed by Bryant (1983) and introduced to experimental literature by Van Huyck et al. (1987, 1990). The outstanding feature of this game is the fact that it gives rise to multiple Nash equilibria which are Pareto ranked. At the same time, the game is an exemplary for the type of games showing a tension between efficiency and security (Foster and Young, 1990; Young, 1993; Kandori et al., 1993). The ‘safest’ equilibrium is the most inefficient one, while the efficient equilibrium is the most ‘insecure’ outcome (Straub, 1995; Crawford, 1991; Goeree and Holt, 1998; Riechmann, 2002). Moreover, and different from the closely related class of discrete public good games (see, for example, Palfrey and Rosenthal 1984), the play of an inefficient but secure equilibrium cannot be rationally caused by the intention to free–ride on the actions of others.

In the basic setting of the game (Van Huyck et al., 1990), players approach the play of the inefficient equilibrium over time, thus displaying the problem of coordination failure. Still, there is a number of notably different settings where people play the efficient equilibrium instead.¹ Apparently, these settings represent ways of solving the problem of coordination failure. The present paper is primarily interested in cases of efficient coordination in settings relying on a very small number of players or on the introduction of inter–group competition.

We know from the experiments by Van Huyck et al. (1990, 2001) that small groups coordinate more efficiently than large ones. We do not know, however, why this is the case. There is no theoretical model explaining the working of

¹See e.g. Berninghaus and Erhart (1998, 2001); Blume and Ortmann (2000); Cachon and Camerer (1996); Cooper et al. (1992, 1994); Van Huyck et al. (1993).

the coordination device ‘small groups’. Experiments by Bornstein et al. (2002) and Riechmann and Weimann (2004) show that players coordinate efficiently, if they belong to several groups that compete against each other. Again, there is no theoretical model on inter–group competition in these games.

This paper will be devoted to setting up a dynamic model of minimum effort coordination incorporating the influence of group size and inter–group competition. The first part of the paper will derive the location of the stochastically stable equilibrium, thus employing a rather classical concept of equilibrium selection. Roughly speaking, the stochastically stable equilibrium is the ‘most secure’ equilibrium in the long run. In the presence of rare mistakes of others, players can be expected to approach the play of this equilibrium over time. It will be shown that independent of the group size and in both settings, i.e. in the basic and in the inter–group competition setting, the most inefficient equilibrium is the stochastically stable one. In a way, this could be the end of the analysis. The model would suggest that, independent of the setting and the number of players involved, theory always predicts the play of the most inefficient equilibrium.

The paper takes a further step. It aims at quantifying the security or reliability of all equilibria of the game. Security will be measured in terms of expected sojourn time of states of the dynamics underlying the repeated game. This measure of reliability, subsequently called ‘stickiness’ throughout this paper, will be derived and analyzed with respect to its sensitivity towards changes in group size and the number of competing groups. It will be shown that, even though the most inefficient equilibrium remains the most secure one, the security of efficient equilibria increases with a decrease in group size and with an increase in the number of competing groups. This finding might give rise to the development of more refined models of individual behavior in this type of games.

The paper proceeds as follows. After the formal introduction of the basic and the repeated game, the stochastically stable equilibrium of the one–group game is derived and shown to be the most inefficient one, a task that at the same time serves as a preparation of a more broader characterization of the reliability of every single equilibrium of the game. After that, the inter–group competition variant of the game is introduced and analyzed, again including the derivation of the stochastically stable equilibrium, which is shown to still be equal to the most inefficient equilibrium. After that, the idea of ‘stickiness’ of equilibria is introduced as a measure of their security. It is shown that stickiness of efficient equilibria increases with an increasing number of competing groups and with decreasing group size. Finally, some numerical computations for a number of well–known experiments are conducted, indicating that indeed, stickiness of the most efficient equilibrium is a good though not sufficient predictor of individual behavior in the minimum effort coordination game.

2 The Basic Game

The basic game in focus is the minimum effort coordination game inspired by Bryant (1983) and introduced to experimental economic literature by Van Huyck et al. (1990). It involves a group consisting of n players indexed i , each choosing his action e_i (because of a common framing of this game as a game of ‘team production’ (Bryant, 1996) often called ‘effort level’) from the set of individual actions: $e_i \in \mathbb{E} = \{e^1 = 1, e^2 = 2, e^3 = 3, \dots, e^m = m\}$, \mathbb{E} is a discrete set of m elements.² Hence, action profiles are given by $(e_1, e_2, \dots, e_n) \in \mathbb{E}^n$.

Payoffs to agent i , π_i , are essentially determined by the minimum effort level in the group, $\underline{e}_n = \min_{j \in \{1, 2, \dots, n\}} \{e_j\}$, and the agent’s own effort level, e_i :

$$\pi_i = a \underline{e}_n - b e_i \quad (1)$$

with $a > b > 0$.

Is it straightforward to conclude that best responses are given by

$$e_i^* = \min_{j \neq i} \{e_j\}, \quad (2)$$

and thus, all uniform action profiles, i.e. profiles with all agents choosing the same effort level, constitute the Nash equilibria. The Nash equilibria are Pareto-ranked: The equilibrium with all agents choosing the maximum effort level is the Pareto efficient one, lower common effort levels mean lower levels of efficiency.

3 The Repeated One-Group Game

In the following, the dynamics of the repeated game are introduced and analyzed. As the first aim of the paper is the derivation of the stochastically stable equilibrium of the game, the paper follows the ‘cookbook recipe’ for finding stochastically stable equilibria by Fudenberg and Levine (1998). It leaves the track of this recipe, when it comes to using the Freidlin–Wenzel method, and proceeds by application of a basic Markov chain framework instead.

3.1 Unperturbed Best Response Dynamics

The dynamic model is based on a simple assumption of individual play, namely myopic best response play.³ People do not know the current (period t) choices

²It is straightforward to extend the analysis to the case of action spaces containing actions with a common distance of δ , i.e. $\mathbb{E} = \{e^1, e^2, e^3, \dots, e^m\}$ with $e^i - e^{i-1} = \delta \forall i$. As this yields no further results, but adds significant notational inconvenience, we will restrict ourselves to the ‘simpler case’ of unit distances.

³Besides the fact that this is a standard form of dynamics in theoretical analyses, it can be found from various experimental results that this indeed the rationale that is used by the majority

of the other players, but they know the group minimum from the previous period ($t - 1$). In period t , every agent is assumed to play the best response to the action profile from the previous period.⁴

$$\forall i: e_i(t) = \underline{e}_n(t - 1) \quad (3)$$

The resulting dynamics of the repeated game can be described as a degenerate Markov process⁵ modeling the transitions from one Nash equilibrium to another.⁶ Thus, every Nash equilibrium represents a state of the Markov process. Accordingly, there are m different states, each identified by the common individual action played in it: $1, 2, \dots, m$. With all agents playing best responses, all Nash equilibria are absorbing states of the Markov process. The transition matrix is equal to the $m \times m$ identity matrix.

3.2 Perturbed Best Response Dynamics

As a second step of analysis, the focus is put on (slightly) perturbed dynamics. Assume that every player almost surely plays a best response but very rarely, i.e. with a very small probability ε , does not do so, but chooses his action by chance. ε is assumed to be the same for every player and constant over time. Moreover, ε is common knowledge. The act of randomly choosing a new action will be called a ‘mutation’, thus adopting the term from evolutionary theory. Now, due to the fact that single choices of individual effort levels e_i might occur at random, the group minimum effort level \underline{e}_n becomes a random variable.

of participants in the baseline settings of the game.

⁴In fact, agent i should play a best response to previous period’s action profile of *everybody else but himself*, i.e. $\min_{j, j \neq i} \{e_j(t - 1)\}$. This, probably more correct specification yields one difference in behavior to the specification in (3): If agent i was the only agent playing the minimum effort level in $t - 1$, he should in fact *increase* his effort level in t , only to find that in the same period, everybody else *decreased* his effort to what i played in $t - 1$. Thus, in the course of at most two time steps, the specification of this footnote and the one given in (3) lead to the same dynamics. Moreover, as (a) this paper concentrates on transitions between equilibria and (b) the situation leading to the difference between the two types of dynamics in focus can only occur once, namely in the very first period of the game, the specification in (3) should be sufficiently precise.

⁵The Markov process in the unperturbed model is degenerate, because the process itself is deterministic. Still, a formulation of the model in terms of a Markov process is in order, since it serves as a preparation of the upcoming analysis of truly stochastic dynamics.

⁶Because agents are assumed to play best responses, there is no need to explicitly focus on transitions to and from states which are not Nash equilibria, which remarkably facilitates the analysis.

3.2.1 Simultaneous Mutations

The probability of an agent's mutation, ε , will be called his 'individual mutation probability'. Let the individual mutation probabilities be the same for every agent and let agents' mutations take place independent of each other. Then, the probability of μ mutations happening in the same period, P_μ , is given by the following proposition.

PROPOSITION 1 *If the individual mutation probability is ε , the probability of μ out of n agents mutating at the same time is given by*

$$P_\mu = B(\mu, n, \varepsilon) = \binom{n}{\mu} \varepsilon^\mu (1 - \varepsilon)^{n-\mu}. \quad (4)$$

PROOF: Every individual mutation is a Bernoulli trial with ε giving the probability of a 'success'. The probability of μ mutations in n independent such trials is Binomially distributed with parameters μ , n , and ε . \square

3.2.2 Probability of Minima

If an agent mutates, he chooses his new strategy randomly. Assume that every action in the action space \mathbb{E} is chosen with positive probability $P(e^k) > 0 \forall e^k \in \mathbb{E}$. In statistical terms, because of the act of mutation, player i 's action choice e_i becomes a random variable. Let E^μ denote the *minimum* of μ i.i.d. random variables $e_i \in \mathbb{E}$. Then, the probability that the minimum of μ i.i.d. action choices (by mutation) equals e^k is given by

PROPOSITION 2

$$P(E^\mu = e^k) = \left[1 - P(e_i \leq e^k - 1)\right]^\mu - \left[1 - P(e_i \leq e^k)\right]^\mu \quad (5)$$

The important aspect of Proposition 2 is the fact that the probabilities of minima exist, are well defined, and can easily be computed. The proof is delayed to Appendix A.

In order to provide a sensible behavioral model, it will be assumed that the play of low group minima has a weakly higher probability than that of higher ones. ASSUMPTION

$$P(E^\mu = e^k) \leq P(E^\mu = e^k - 1). \quad (6)$$

3.2.3 Transition Probabilities

Transitions from one period to the next in the perturbed repeated game can be seen as processes involving two steps: First, (possibly,) mutations take place. Then, secondly, every agent, no matter if having mutated or not, plays the best response to the resulting — possibly perturbed — tentative action profile. This order of steps makes sure that at the beginning of every new period, a Nash equilibrium is played. The only question of course is, which one of the m possible equilibria this will be. The answer to this question is seemingly simple: Is is the equilibrium that is characterized by the minimum effort level of both mutants and non-mutants.

Perturbed best response dynamics can, again, be viewed as a Markov process with m states corresponding to the m equilibria of the underlying game. The random event that characterizes a (non trivial) transition in the process is a change in group minimum play, i.e. a change of \underline{e}_n . Transition probabilities from state u to state v in the perturbed minimum effort coordination game depend on two factors only: (a) the direction of the transition, which can be upwards ($u < v$), downwards ($u > v$), or trivial ($u = v$), and (b) the destination of the transition, v . Probabilities of upwards transitions to state v are denoted as $\bar{p}(v)$, downwards transition probabilities to v read $\underline{p}(v)$, probabilities of trivial transitions from u to v are denoted $\tilde{p}(v)$. Probabilities of transitions into the same direction are independent of their particular starting state u . A downwards transition to state $v = 3$, for example, that started in $u = 4$ has the same probability as one that started in, say, $u = 6$. The only important aspect is the fact that the transition must really be *downwards*, i.e. the starting state u must higher than v .

Let $p_{u,v}$ denote the probability of a direct transition from state u to state v in the Markov process: $p_{u,v} = P(\underline{e}_n(t) = v | \underline{e}_n(t-1) = u)$. Then, transition probabilities are given in the proposition.

PROPOSITION 3 *The probabilities of a direct transition from state u to state v under the regime of perturbed best response dynamics are given by*

$$p_{u,v} = \begin{cases} \bar{p}(v) &= P_n P(E^n = v) & \text{for } u < v \\ \underline{p}(v) &= \sum_{\mu=1}^n P_\mu P(E^\mu = v) & \text{for } u > v \\ \tilde{p}(v) &= P_0 + \underline{p}(v) + \sum_{\mu=1}^{n-1} P_\mu (1 - P(E^\mu \leq v)) & \text{for } u = v \end{cases} \quad (7)$$

PROOF: For an upwards transition, $u < v$, all n agents must mutate *and* the minimum outcome of their mutations must equal v . The respective probability is

$$\bar{p}(v) = P_n P(E^n = v).$$

In order to transit from the present minimum effort level of u downwards to a lower minimum effort level of v , there must be one mutation resulting in v , or

two mutations, the minimum of which is v , or three mutations with a minimum result of v , or \dots , or n mutations with a minimum outcome of v . The aggregate probability of these random events is

$$\underline{p}(v) = \sum_{\mu=1}^n P_{\mu} P(E^{\mu} = v).$$

Finally, there are several random events that let the process remain in the present state. First, the process remains in state u ($u = v$) if there is no mutation. Secondly, the process remains in u , if there are between one and n mutations with a minimum outcome of u . As this is the same event as the one leading to a downwards transition *to* state u , the probability for this event is $\underline{p}(u)$. Last, the process will not leave u if there are between one and $n - 1$ mutations, the minima of which are greater than u . (In these cases, the remaining non-mutants' effort level of u represents the minimum effort level.) The joint probability of all of these events is

$$\tilde{p}(v) = P_0 + \underline{p}(v) + \sum_{\mu=1}^{n-1} P_{\mu} (1 - P(E^{\mu} \leq v)).$$

□

From (7), the transition matrix of the Markov process can be assembled:⁷

$$P = \begin{pmatrix} \tilde{p}(1) & \bar{p}(2) & \bar{p}(3) & \cdots & \bar{p}(m) \\ \underline{p}(1) & \tilde{p}(2) & \bar{p}(3) & \cdots & \bar{p}(m) \\ \vdots & & & & \\ \underline{p}(1) & \underline{p}(2) & \underline{p}(3) & \cdots & \tilde{p}(m) \end{pmatrix} \quad (8)$$

3.2.4 Stochastic Stability

As long as the individual mutation probability ε is strictly positive, $\varepsilon > 0$, every entry of matrix (8) is strictly positive. This means that the underlying Markov process is regular, i.e. irreducible and aperiodic (Goodman, 1988, pp. 164). From this, it follows that the unique long term distribution of states (the limit distribution) can be computed, which is described by the stochastic row vector α with

$$\alpha P = \alpha \quad \text{and} \quad \sum_{k=1}^m \alpha_k = 1, \quad (9)$$

with α_k giving the long term frequency of play of state k .

⁷Following common use of transition matrices (see e.g. Feller 1968), the starting states of transition are denoted in the rows and the destinations in the columns.

As the structure of P is comparably simple, so are the elements of α . In search of the limit distribution of states for vanishing individual mutation probability, i.e. $\lim_{\varepsilon \rightarrow 0} \alpha$, first, α_1 , the long term frequency of play of the lowest possible effort level, $e^1 = 1$, must be computed. Solving the system of linear equations given in (9) results in (among others) α_1 :

$$\alpha_1 = \alpha_1 \tilde{p}(1) + (1 - \alpha_1) \underline{p}(1) = \frac{\underline{p}(1)}{1 - \tilde{p}(1) + \underline{p}(1)} \quad (10)$$

From this, it can be concluded that a (unique) stochastically stable equilibrium exists and is equal to the one characterized by common play of an effort level of 1.

PROPOSITION 4 *Under the regime of perturbed best response dynamics, the stochastically stable equilibrium of the one-group game is the one characterized by common play of the lowest effort level:*

$$\lim_{\varepsilon \rightarrow 0} \alpha_1 = 1. \quad (11)$$

4 Inter-Group Competition

In the inter-group competition setting, the minimum coordination game is simultaneously played by γ different groups, each consisting of n members. These groups are competing with each other. Every member i of group g gets a higher payoff, i.e. a bonus of $c > 0$, if his group's minimum, \underline{e}_{n_g} , is the (weakly) highest of all groups' minima. If the highest minimum effort level is played by several groups, members of all of these groups get the higher payoff.⁸ With $\overline{e}^\gamma = \max_{h \in \{1, 2, \dots, \gamma\}} \{\underline{e}_{n_h}\}$ denoting the maximum group minimum effort level of all γ competing groups, individual payoffs result as

$$\pi_{i,g} = \begin{cases} a \underline{e}_{n_g} - b e_i & \text{if } \underline{e}_{n_g} < \overline{e}^\gamma \\ a \underline{e}_{n_g} - b e_i + c & \text{if } \underline{e}_{n_g} = \overline{e}^\gamma \end{cases} \quad (12)$$

All action profiles with a uniform action played *within* groups but not necessarily *across* groups are Nash equilibria. (The simplest way of seeing that this is true is to verify that no single player can increase his payoff by unilaterally deviating from his group's common effort level, independent of whatever the common

⁸This is the setting from Riechmann and Weimann (2004). In the experiments by Bornstein et al. (2002), involving two groups in competition, in case of a tie, both (winning) groups share the bonus, i.e. get an extra pay of $c/2$.

effort level of other, competing groups may be.) Thus, in a game with γ groups and m different effort levels, there are m^γ different Nash equilibria.

Individual myopic best responses are straightforward: No matter if your group is the best or a worse one, play *your group's* minimum effort level. (This rationale results from the specific static expectations imposed by the definition of myopic best response dynamics.) Consequence is the same kind of dynamics as the one in the basic one-group game: As the location of the Nash equilibria is untouched, the resulting myopic best response dynamics are nothing else than γ many independent processes described for the one-group setting. This means, among others, that under the regime of perturbed best response dynamics, the location of the stochastically stable equilibrium does not change from the one-group to the inter-group competition setting.

4.1 Competitive Response Dynamics

Experimental evidence (Bornstein et al., 2002; Riechmann and Weimann, 2004) strongly suggests that individual behavior in inter-group competition situations significantly differs from behavior in the one-group setting. Involved in group competition, people do *not* stick to the simple myopic best response behavior as defined by (3). Instead, group competition seems to induce the wish to make one's own group the better one. This notion is strongly supported by results from the field of social psychology, where it is found that — given certain conditions — inter-group competition enhances individual motivation and effort. (See Bornstein and Erev 1994, and, for a survey, Tajfel 1982.)

One important result of the experiments by Bornstein et al. (2002) as well as by Riechmann and Weimann (2004) is the observation that, in addition to the minimum effort level of the own group, the minimum effort levels of the other, competing groups strongly guide individuals' behavior. It turns out that the minimum effort level of the best group is the most prominent attractor of individual behavior. Experimental evidence suggests that individual behavior is *not* independent of what players in other groups do. Consequently, simple myopic best response dynamics can be found to be entirely inappropriate as a model of individual behavior in the inter-group competition setting.

Based on the experimental findings mentioned above, in this section, a type of dynamics will be defined that take account of both, the own group's minimum and the best groups' minimum choices. For this means, it is assumed that every agent follows the same (lexicographic) behavioral rules, for convenience called 'competitive response' behavior:

- first priority: Try to make your group the best group. (This concerns inter-group relations.)

- second priority: Try not to waste any effort. (This concerns intra group relations.)

These two rules translate into a simple rationale requiring every member i of every group g to try to play the maximum of all groups' minima,

$$e_{i,g}^* = \overline{e^\gamma}. \quad (13)$$

All groups exerting the highest group minimum effort level receive the higher payoff. Consequently, all equilibria produced by competitive response dynamics are states with every agent in every group playing the same action. Thus, again, the number of different equilibria in the dynamic γ -group competition game is the same as the number of possible individual actions, m .

Proceeding in accordance to the method used for the one-group game, the unperturbed competitive dynamics are viewed first. The basic assumption is that in every period t , every individual plays the competitive response to the previous period's strategy profile, which is best represented by the set of group minimum effort levels. Thus, unperturbed competitive dynamics are completely described by a lagged version of (13).

$$\forall i, g : e_{i,g}(t) = \overline{e^\gamma}(t-1). \quad (14)$$

Like in the one-group game, these dynamics can be seen as a degenerate Markov-process describing the transition between the equilibria of the game. Again, all equilibria represent absorbing states of the process, such that, under the regime of competitive responses (13), the transition matrix is equal to the $m \times m$ identity matrix.

4.2 Perturbed Competitive Dynamics

Let perturbed competitive dynamics be a form of competitive dynamics which is occasionally disturbed by (rare) individual mutations. Mutations are (again) i.i.d., such that the probability of simultaneous mutations is again described by Proposition 1.

4.2.1 Probability of Maxima

The maximum of the minimum effort levels of γ competing groups, $\overline{e^\gamma}$, is a random variable. This results from the notion that every group minimum effort level e_n is a random variable with probabilities (densities) given in Proposition 2. Group minima are i.i.d. across groups. As $\overline{e^\gamma}$ is simply the maximum of γ many random variables which are i.i.d. in \mathbb{E} , the probabilities for $\overline{e^\gamma}$ can be computed.

PROPOSITION 5 *The probability that $\overline{e^\gamma}$, the maximum of γ many i.i.d. group minima \underline{e}_n equals e^k is given by*

$$P\left(\overline{e^\gamma} = e^k\right) = P\left(\underline{e}_n \leq e^k\right)^\gamma - P\left(\underline{e}_n \leq e^k - 1\right)^\gamma \quad (15)$$

4.2.2 Transition Probabilities

For the process of perturbed competitive response dynamics in the γ -group competition game, a transition matrix must be derived. Now, transitions are direct moves from one maximum of group minima to another one. Consequently, transition probabilities for the γ -group competition setting can be computed as probabilities of maxima: Let $p_{u,v}(\gamma)$ denote the probability of a direct transition from state u to state v in the γ -group setting, $p_{u,v}(\gamma) = P\left(\overline{e^\gamma}(t) = v \mid \overline{e^\gamma}(t-1) = u\right)$. A transition from state u to state v takes place if there is at least one group transiting to v and if v is the highest state played by any of the γ groups. In other words, the destination state v must be the maximum of all single groups' transition destinations. Consequently, the transition probability can be computed from Prop. 5 and the single group transition probabilities in (7):⁹

$$p_{u,v}(\gamma) = \left(\sum_{k=1}^v p_{u,k}\right)^\gamma - \left(\sum_{k=1}^{v-1} p_{u,k}\right)^\gamma \quad (16)$$

Again, there is a difference between downwards transitions, trivial transitions, and upwards transitions, such that the respective proposition reads:

PROPOSITION 6 *In the perturbed inter-group competition game with γ groups, the probability of a direct transition from state u to state v is given by*

$$p_{u,v}(\gamma) = \begin{cases} \overline{p}_\gamma(v) = \left(\sum_{k=1}^{u-1} \underline{p}(k) + \tilde{p}(u) + \sum_{k=u+1}^v \overline{p}(k)\right)^\gamma \\ \quad - \left(\sum_{k=1}^{u-1} \underline{p}(k) + \tilde{p}(u) + \sum_{k=u+1}^{v-1} \overline{p}(k)\right)^\gamma & \text{for } u < v \\ \underline{p}_\gamma(v) = \left(\sum_{k=1}^v \underline{p}(k)\right)^\gamma - \left(\sum_{k=1}^{v-1} \underline{p}(k)\right)^\gamma & \text{for } u > v \\ \tilde{p}_\gamma(v) = \left(\sum_{k=1}^{v-1} \underline{p}(k) + \tilde{p}(v)\right)^\gamma - \left(\sum_{k=1}^{v-1} \underline{p}(k)\right)^\gamma & \text{for } i = j \end{cases} \quad (17)$$

PROOF

By putting the respective one-group transition probabilities from Proposition 3 into (16). \square

⁹It is easily verified that, of course, $p_{u,v}(1) = p_{u,v}$ as given in (7), clarifying that the one-group game from the previous section is merely a special case of the general γ -group competition game.

From Proposition 6, the transition matrix of the γ -group game results as

$$P = \begin{pmatrix} \tilde{p}_\gamma(1) & \bar{p}_\gamma(2) & \bar{p}_\gamma(3) & \cdots & \bar{p}_\gamma(m) \\ \underline{p}_\gamma(1) & \tilde{p}_\gamma(2) & \bar{p}_\gamma(3) & \cdots & \bar{p}_\gamma(m) \\ \vdots & & & & \\ \underline{p}_\gamma(1) & \underline{p}_\gamma(2) & \underline{p}_\gamma(3) & \cdots & \tilde{p}_\gamma(m) \end{pmatrix} \quad (18)$$

4.2.3 Stochastic Stability

From the transition matrix (18), the unique long run distribution of states can be computed. For the long run frequency of 1-play, this gives

$$\alpha_1 = \frac{\underline{p}_\gamma(1)}{1 - \tilde{p}_\gamma(1) + \underline{p}_\gamma(1)} \quad (19)$$

Letting the individual mutation probability ε approach zero, it can be found that the only effort level used in the long run is the lowest one. This leads to the proposition:

PROPOSITION 7 *In the minimum effort coordination game with γ competing groups, the equilibrium characterized by the lowest possible effort level is the stochastically stable one.*

$$\lim_{\varepsilon \rightarrow 0} \alpha_1 = 1. \quad (20)$$

This result shows that not only in the one group minimum effort coordination game, but in fact for any number of competing groups, the equilibrium characterized by the lowest possible effort level is the stochastically stable equilibrium.

5 Stickiness as a Measure of Security of Equilibria

While the notion of stochastic stability is a notion of long run behavior in the game, it is certainly not enough to characterize (objective) uncertainty of the game. There is still a need for a measure of ‘objective riskiness’ or ‘objective security’ of the different equilibria of the game. The basic question that cannot be answered by stochastic stability alone is: ‘How risky is it to (try to) play the 7-equilibrium, how risky is the 6-equilibrium etc.?’ For means of measuring this type of ‘security’ of equilibria, a clear measure is needed. Consequently, in this section, a notion of ‘stickiness’ of equilibria is introduced. (This phrase is not intended to parallel any notions of economic rigidities sometimes labeled by this term.) An equilibrium is the stickier, the higher the probability is that it, once reached, will be played

again in the next period. This notion is of course closely related to the idea of the expected sojourn time of the respective state of the underlying Markov process (Goodman, 1988, p. 177). Expected sojourn time gives the expected number of periods the dynamic process will remain in a given state of the Markov chain, i.e. in the respective equilibrium of the game. In the case of this paper (i.e. for regular Markov chains), the expected sojourn time for equilibrium v in a game with γ groups, $s_\gamma(v)$, is given by

$$s_\gamma(v) = \frac{1}{1 - \tilde{p}_\gamma(v)}. \quad (21)$$

In order to keep things simple, the paper will not make direct use of $s_\gamma(v)$, but will rely on $\tilde{p}_\gamma(v)$ as a measure of stickiness of equilibrium v .

It can be found that there are two major factors influencing $\tilde{p}_\gamma(v)$, which are the number of agents per group, n , and the number of competing groups, γ . For both factors, the direction of influence is roughly characterized as follows.

- More efficient ('higher') equilibria become stickier with decreasing number of agents per group and increasing number of competing groups.
- Less efficient ('low') equilibria become less sticky with decreasing number of agents per group and increasing number of competing groups.

Consequently, there are two alternative ways of increasing stickiness, thus decreasing strategic uncertainty and by that increasing group performance, the increase of the number of competing groups and the decrease of the size of groups. Many small groups will perform much more efficient than few large ones.

5.1 Equilibrium Level

First, the relation between stickiness and the equilibrium effort level v is analyzed. This relation is characterized by the expression

$$\Delta_v \tilde{p}_\gamma(v) = \tilde{p}_\gamma(v) - \tilde{p}_\gamma(v-1) \quad (22)$$

PROPOSITION 8 *For low numbers of competing groups, low-level equilibria (small v) are stickier than high-level equilibria (large v). For sufficiently high numbers of competing groups, this relation turns to the opposite.*

$$\Delta_v \tilde{p}_\gamma(v) \begin{cases} < 0 & \text{for small } \gamma \\ > 0 & \text{for sufficiently high } \gamma \end{cases} \quad (23)$$

5.2 Number of Groups

In order to find the direction of influence of the number of competing groups, γ , on the stickiness of equilibria, $\tilde{p}_\gamma(v)$, the sign of following expression is of primary interest:

$$\Delta_\gamma \tilde{p}_\gamma(v) = \tilde{p}_\gamma(v) - \tilde{p}_{\gamma-1}(v) \quad (24)$$

PROPOSITION 9 *The stickiness of low equilibria (small v) decreases in the number of competing groups. The stickiness of high equilibria (large v) increases in the number of competing groups.*

$$\Delta_\gamma \tilde{p}_\gamma(v) \begin{cases} < 0 & \text{for low } v \\ > 0 & \text{for high } v \end{cases} \quad (25)$$

5.3 Number of Group Members

Up to this point in the paper, the number of members per group has not been explicitly taken account of. Now, while deriving the influence of group size on the stickiness of equilibria, the notation needs to be enhanced. Thus, for example, let $\tilde{p}_\gamma(v, n)$ denote the probability of staying in equilibrium v with γ competing groups, each consisting of n members. Other notations parallel this one.

The influence of the number of group members on stickiness of equilibria will be analyzed by means of the following measure:

$$\Delta_n \tilde{p}_\gamma(v) = \tilde{p}_\gamma(v, n) - \tilde{p}_\gamma(v, n-1) \quad (26)$$

PROPOSITION 10 *The stickiness of low equilibria (small v) increases in n . The stickiness of high equilibria (high v) decreases in n .*

$$\Delta_n \tilde{p}_\gamma(v) \begin{cases} > 0 & \text{for low } v \\ < 0 & \text{for high } v \end{cases} \quad (27)$$

5.4 Numerical Results

For well-known experimental settings following the set-up described in this paper, numerical computations of the stickiness of the equilibria have been carried out. These computations are based on the assumption that individual mutation probability is $\varepsilon = 0.1$ and that individual mutations are uniformly distributed in $[1, \dots, 7]$. Table 5.4 gives the results of the computations (in the lines ‘stickiness of’) in form of $\tilde{p}_\gamma(v)$ and in form of expected sojourn time (in brackets). Moreover, the table lists some benchmark results of the respective experiments. It is easy to recognize from the table that indeed those experiments

source treatment	vHBB90 ^a A	vHBB90 A	vHBB90 A	RW04 ^b base1	BGN02 ^c NC	vHBB90 C	RW04 comp2a
<i>Parameters</i>							
γ	1	1	1	1	1	1	2
n	16	15	14	7	7	2	7
no. of indep. observations	4	2	1	6	6	14	4
minimum ^d	1.00	1.00	1.00	2.17	2.17	6.07	6.38
mean ^e	1.59	1.27	1.79	2.60	2.62	6.21	6.64
<i>stickiness of^f</i>							
1	1.00 (∞)	1.00 (∞)	1.00 (∞)	1.00 (∞)	0.99 (100)	1.00 (∞)	
2	0.79 (4.8)	0.81 (5.3)	0.82 (5.6)	0.90 (10)	0.97 (33.3)	0.99 (100)	
3	0.63 (2.7)	0.65 (2.9)	0.67 (3.0)	0.81 (5.3)	0.94 (16.7)	0.96 (25)	
4	0.49 (2.0)	0.52 (2.1)	0.54 (2.2)	0.74 (3.9)	0.91 (11.1)	0.93 (14.3)	
5	0.39 (1.7)	0.41 (1.7)	0.44 (1.8)	0.66 (2.9)	0.89 (9.1)	0.89 (9.1)	
6	0.31 (1.4)	0.32 (1.5)	0.35 (1.5)	0.60 (2.5)	0.86 (7.1)	0.84 (6.2)	
7	0.24 (1.3)	0.26 (1.4)	0.29 (1.4)	0.53 (2.1)	0.84 (6.2)	0.78 (4.5)	

^aVan Huyck et al. (1990)

^bRiechmann and Weimann (2004)

^cBornstein et al. (2002)

^daverage group minimum effort level in second-to-last period

^eaverage group mean effort level in second-to-last period

^f $\tilde{p}_\gamma(v)$, expected sojourn time in brackets

Table 1: Numerical Computations. $\tilde{p}_\gamma(v)$ and $s_\gamma(v)$ as functions of n and γ . $\varepsilon = 0.1$, $m = 7$, individual mutations uniformly distributed

lead to a relatively high stickiness of the efficient ('7') equilibrium are the ones with experimental outcomes close to 7. The latter experiments are the one by Van Huyck et al. (1990) with a very low number of members per group and the one by Riechmann and Weimann (2004) with two competing groups.

This finding might lead to simple behavioral hypotheses like for example the idea that players might do something like 'security satisficing': Instead of just maximizing payoffs or minimizing risk, they might tend to play the equilibrium that is 'sufficiently save'. If for example the aspiration level in terms of stickiness that players wish to achieve is about six periods expected sojourn time, they will chose the '1'-equilibrium in all basic settings, but the '6'- or even the '7'-equilibrium in the setting with one group of two players and in the setting with two competing groups of seven players each. Still, the formulation of behavioral hypotheses is not the main aim of this paper, and, moreover, that aim requires much more experimental work than has currently been undertaken.

6 Summary

This paper presents a model of dynamic behavior in two different variants of the minimum effort coordination game. The derivation of stochastically stable equilibria shows that in the basic game as well as in an inter-group competition variant of the game, the most inefficient equilibrium is the stochastically stable one. In order to gain more insight into the structure of uncertainty underlying the equilibria of the game, a notion of ‘stickiness’ of the equilibria is derived. The paper shows that generally, more efficient equilibria are less sticky than less efficient ones. Yet, stickiness of more efficient equilibria can be notably enhanced by either decreasing the group size or increasing the number of competing groups.

Some numerical computations indicate that objective probability might be sufficient to develop some simple behavioral hypotheses of individual behavior. This, nevertheless, is not the aim of the present paper. The development of a *behavioral* theory on individual behavior in the minimum effort coordination game should be the next step of research, a step that requires both, theoretical work and more and refined experimental observation.

A The Repeated One-Group Game

PROPOSITION 2

$$P(E^\mu = e^k) = \left[1 - P(e_i \leq e^k - 1)\right]^\mu - \left[1 - P(e_i \leq e^k)\right]^\mu$$

PROOF

First note that for the discrete distribution on $\mathbb{E} = \{1, 2, \dots, m\}$, whose elements have a distance of 1 between each adjacent pair of actions, it is true that

$$P(e_i < e^k + 1) = P(e_i \leq e^k). \quad (28)$$

Now, interpret the (random) events described by the following probabilities:

- $P(e_i \leq e^k)$: In a random trial, the outcome e_i is less or equal e^k .
- $1 - P(e_i \leq e^k)$: In a random trial, the outcome e_i greater than e^k .
- $[1 - P(e_i \leq e^k)]^n$: In n i.i.d. random trials, every outcome e_i is greater than e^k .
- $1 - [1 - P(e_i \leq e^k)]^n$: In n i.i.d. random trials, there is at least one outcome e_i with $e_i \leq e^k$.

Denote this last probability by $P(E^n \leq e^k)$:

$$P(E^n \leq e^k) = P(E^n < e^k + 1) = 1 - \left[1 - P(e_i \leq e^k)\right]^n. \quad (29)$$

Analogously, the probability

$$P(E^n < e^k) = 1 - \left[1 - P(e_i \leq e^k - 1)\right]^n \quad (30)$$

gives the probability of the event that in n trials, there is at least one outcome e_i strictly less than e^k .

Now, consider the difference between the probabilities from (29) and (30): The expression

$$P(E^n = e^k) = P(E^n \leq e^k) - P(E^n < e^k) \quad (31)$$

gives the probability of the event that in n trials, there is at least one outcome $e_i \leq e^k$, but no outcome $e_i < e^k$. This is of course the same event as having at least one $e_i = e^k$, but no outcome $e_i < e^k$. This, finally, is the same as the event that $e_i = e^k$ is the minimum outcome of the n random trials.

It remains to be shown that the difference stated in (31) is indeed equal to the one given in the proposition, which is straightforward from a look at (28)

$$\begin{aligned} P(E^n \leq e^k) - P(E^n < e^k) &= 1 - \left[1 - P(e_i \leq e^k)\right]^n - \left\{1 - \left[1 - P(e_i \leq e^k - 1)\right]^n\right\} \\ &= \left[1 - P(e_i \leq e^k - 1)\right]^n - \left[1 - P(e_i \leq e^k)\right]^n \end{aligned}$$

□

B Stochastic Stability in the One-Group Game

LEMMA 1 *For individual mutation probability ε approaching zero, the probability of a total of zero mutations approaches unity.*

$$\lim_{\varepsilon \rightarrow 0} P_\mu = \begin{cases} 1 & \text{for } \mu = 0 \\ 0 & \text{for } \mu > 0 \end{cases} \quad (32)$$

PROOF Compute the respective limits of (4).

LEMMA 2 *For individual mutation probability ε approaching zero, the limit of the first derivative of P_μ with respect to ε is given by*

$$\lim_{\varepsilon \rightarrow 0} \frac{dP_\mu}{d\varepsilon} = \begin{cases} -n & \text{for } \mu = 0 \\ n & \text{for } \mu = 1 \\ 0 & \text{for } \mu > 1 \end{cases} . \quad (33)$$

PROOF

Compute the first derivative of (4) and let ε approach zero:

$$\begin{aligned} \frac{dP_\mu}{d\varepsilon} &= \binom{n}{\mu} [\mu \varepsilon^{\mu-1} (1-\varepsilon)^{n-\mu} - (n-\mu) \varepsilon^\mu (1-\varepsilon)^{n-\mu-1}] \\ &= \begin{cases} -n(1-\varepsilon)^{n-1} & \text{for } \mu = 0 \\ n[(1-\varepsilon)^{n-1} - (n-1)\varepsilon(1-\varepsilon)^{n-2}] & \text{for } \mu = 1 \\ \binom{n}{\mu} [\mu \varepsilon^{\mu-1} (1-\varepsilon)^{n-\mu} - (n-\mu) \varepsilon^\mu (1-\varepsilon)^{n-\mu-1}] & \text{for } \mu > 1 \end{cases} \end{aligned}$$

The limits of these first derivatives for vanishing ε are the ones given in Lemma 2. \square

LEMMA 3

(a) *The probability of reaching the lowest state, state 1, from a higher state $u > 1$, $\underline{p}(1)$, approaches zero for individual mutation probability ε approaching zero,*

$$\lim_{\varepsilon \rightarrow 0} \underline{p}(1) = 0, \quad (34)$$

(b) *and the first derivative of this probability with respect to zero goes to $nP(E^1 = 1)$ for vanishing individual mutation probability,*

$$\lim_{\varepsilon \rightarrow 0} \frac{d\underline{p}(1)}{d\varepsilon} = nP(E^1 = 1). \quad (35)$$

PROOF

- (a) $\underline{p}(1) = \sum_{\mu=1}^n P_{\mu} P(E^{\mu} = 1) \Rightarrow \lim_{\varepsilon \rightarrow 0} \underline{p}(1) = 0$
because of $\lim_{\varepsilon \rightarrow 0} P_{\mu} = 0$ for $\mu > 0$ (Lemma 1).

(b)

$$\begin{aligned} \frac{d\underline{p}(1)}{d\varepsilon} &= \sum_{\mu=1}^n \frac{dP_{\mu}}{d\varepsilon} P(E^{\mu} = 1) \\ \Rightarrow \lim_{\varepsilon \rightarrow 0} \frac{d\underline{p}(1)}{d\varepsilon} &= \lim_{\varepsilon \rightarrow 0} \sum_{\mu=1}^n \frac{dP_{\mu}}{d\varepsilon} P(E^{\mu} = 1) \end{aligned}$$

and thus, because of Lemma 1,

$$= \lim_{\varepsilon \rightarrow 0} \frac{dP_1}{d\varepsilon} P(E^1 = 1) = nP(E^1 = 1)$$

□

LEMMA 4

- (a) *The probability of staying in the lowest possible state approaches unity if the individual mutation probability vanishes,*

$$\lim_{\varepsilon \rightarrow 0} \tilde{p}(1) = 1, \quad (36)$$

- (b) *and the first derivative of this probability vanishes if the individual mutation probability vanishes,*

$$\lim_{\varepsilon \rightarrow 0} \frac{d\tilde{p}(1)}{d\varepsilon} = 0. \quad (37)$$

PROOF

- (a) First, recognize that $\tilde{p}(1)$ can be simplified:

$$\tilde{p}(1) = P_0 + \underline{p}(1) + \sum_{\mu=1}^{n-1} P_{\mu} [1 - P(E^{\mu} \leq 1)].$$

Then, because of $P(E^{\mu} < 1) = 0$,

$$\begin{aligned} \tilde{p}(1) &= P_0 + \sum_{\mu=1}^n P_{\mu} P(E^{\mu} = 1) + \sum_{\mu=1}^{n-1} P_{\mu} [1 - P(E^{\mu} = 1)] \\ &= P_0 + \sum_{\mu=1}^n P_{\mu} P(E^{\mu} = 1) + \sum_{\mu=1}^{n-1} P_{\mu} - \sum_{\mu=1}^{n-1} P_{\mu} P(E^{\mu} = 1) \\ &= P_0 + P_n P(E^n = 1) + \sum_{\mu=1}^{n-1} P_{\mu} \\ &= P_0 + P_n P(E^n = 1) + (1 - P_0 - P_n) \\ &= P_0 + P_n [P(E^n = 1) - 1] - P_0 + 1 \\ &= 1 - P_n [1 - P(E^n = 1)] \end{aligned} \quad (38)$$

The last line of the above allows for a simple interpretation: The probability of staying in the state of 1 equals one minus the probability of having all n agents mutate but not reach a minimum of 1.

Letting $\varepsilon \rightarrow 0$, P_n becomes zero, which proves (36).

(b) The first derivative is

$$\frac{d\tilde{p}(1)}{d\varepsilon} = -\frac{dP_n}{d\varepsilon} [1 - P(E^n = 1)]$$

and consequently

$$\lim_{\varepsilon \rightarrow 0} \frac{d\tilde{p}(1)}{d\varepsilon} = 0$$

because of $\lim_{\varepsilon \rightarrow 0} \frac{dP_n}{d\varepsilon} = 0$ for $n > 1$ (Lemma 2).

□

PROPOSITION 4

$$\lim_{\varepsilon \rightarrow 0} \alpha_1 = 1.$$

PROOF

It is known that

$$\alpha_1 = \frac{\underline{p}(1)}{1 - \tilde{p}(1) + \underline{p}(1)}. \quad (10)$$

In search of $\lim_{\varepsilon \rightarrow 0} \alpha_1$, it is found that, because of the first parts of Lemmata 3 and 4, the denominator in (10) becomes zero. Consequently, make use of L'Hôpital's rule:

$$\lim_{\varepsilon \rightarrow 0} \alpha_1 = \frac{\lim_{\varepsilon \rightarrow 0} \frac{d\underline{p}(1)}{d\varepsilon}}{\lim_{\varepsilon \rightarrow 0} \frac{d}{d\varepsilon} (1 - \tilde{p}(1) + \underline{p}(1))}$$

which, because of the second parts of Lemmata 3 and 4, becomes

$$\lim_{\varepsilon \rightarrow 0} \alpha_1 = \frac{\lim_{\varepsilon \rightarrow 0} \frac{d\underline{p}(1)}{d\varepsilon}}{\lim_{\varepsilon \rightarrow 0} \frac{d\underline{p}(1)}{d\varepsilon}} = \frac{nP(E^1 = 1)}{nP(E^1 = 1)} = 1$$

□

C Inter-Group Competition

PROPOSITION 5

The probability that \overline{e}^γ , the maximum of γ many i.i.d. group minima \underline{e}_n in \mathbb{E} equals e^k is given by

$$P(\overline{e}^\gamma = e^k) = P(\underline{e}_n \leq e^k)^\gamma - P(\underline{e}_n \leq e^k - 1)^\gamma$$

PROOF

First note that

$$P\left(\max_{g \in \{1, 2, \dots, \gamma\}} \{e_{n_g}\} \leq e^k\right) = P(e_{n_1} \leq e^k) P(e_{n_2} \leq e^k) \dots P(e_{n_\gamma} \leq e^k). \quad (39)$$

As all e_{n_i} are i.i.d., this can be simplified:

$$P(\overline{e}^\gamma \leq e^k) = P(e_n \leq e^k)^\gamma \quad (40)$$

Now note that in order to make e^k the maximum outcome, the maximum choice must be less or equal e^k but at the same time not less than e^k , i.e. not less or equal $e^k - 1$. Consequently, Proposition 5 holds. \square

In order to show that $e^k = 1$ characterizes the stochastically stable state for every number γ of competing groups, analyze the expression given in (19), thereby making use of the following lemmata.

LEMMA 5 For $\varepsilon \rightarrow 0$, the γ group transition probability $\underline{p}_\gamma(1)$ approaches zero.

$$\lim_{\varepsilon \rightarrow 0} \underline{p}_\gamma(1) = 0 \quad (41)$$

Proof

First notice that the γ group transition probability $\underline{p}_\gamma(1)$ can be denoted in terms of the one group transition probability $\underline{p}(1)$:

$$\begin{aligned} \underline{p}_\gamma(1) &= \left(\sum_{k=1}^j \underline{p}(k)\right)^\gamma - \left(\sum_{k=1}^{j-1} \underline{p}(k)\right)^\gamma \\ &= \left(\sum_{k=1}^1 \underline{p}(k)\right)^\gamma - \left(\sum_{k=1}^0 \underline{p}(k)\right)^\gamma \\ &= \underline{p}(1)^\gamma \end{aligned}$$

According to the first part of Lemma 3, $\underline{p}(1)$ vanishes for vanishing ε and consequently, this is what $\underline{p}_\gamma(1)$ does. \square

In addition to Lemma 4, which describes the behavior of $\tilde{p}(1)$ and its first derivative for vanishing ε , we find that every higher derivative of $\tilde{p}(1)$ with respect to ε vanishes for vanishing ε :

LEMMA 6 For vanishing individual mutation probability ε , all higher derivatives of $\tilde{p}(1)$ with respect to ε approach zero:

$$\lim_{\varepsilon \rightarrow 0} \frac{d^x \tilde{p}(1)}{d\varepsilon^x} = 0 \quad \forall x > 0 \quad (42)$$

Proof:

A simplified expression for $\tilde{p}(1)$ is given by (38). Higher derivatives of $\tilde{p}(1)$ with respect to ε are given by

$$\frac{d^x \tilde{p}(1)}{d\varepsilon^x} = -\frac{d^x P_n}{d\varepsilon^x} [1 - P(E^n = 1)]$$

From Lemma 2, we can find that $\lim_{\varepsilon \rightarrow 0} \frac{d^x P_n}{d\varepsilon^x} = 0$ for $n > 1$, such that Lemma 6 holds true. \square

From this, we can conclude that for $\varepsilon \rightarrow 0$, $\tilde{p}_\gamma(1)$ approaches unity and all derivatives of $\tilde{p}_\gamma(1)$ with respect to ε vanish.

LEMMA 7 *For vanishing individual mutation probability,*

(a) $\tilde{p}_\gamma(1)$ approaches unity,

$$\lim_{\varepsilon \rightarrow 0} \tilde{p}_\gamma(1) = 1. \quad (43)$$

and

(b) every derivative of the γ -group transition probability $\tilde{p}_\gamma(1)$ with respect to ε vanishes,

$$\lim_{\varepsilon \rightarrow 0} \frac{d^x \tilde{p}_\gamma(1)}{d\varepsilon^x} = 0 \quad \forall x > 0. \quad (44)$$

Proof

First recognize that the γ -group transition probability $\tilde{p}_\gamma(1)$ can be written in terms of the one-group transition probability $\tilde{p}(1)$:

$$\begin{aligned} \tilde{p}_\gamma(1) &= \left(\sum_{k=1}^{j-1} \underline{p}(k) + \tilde{p}(1) \right)^\gamma + \left(\sum_{k=1}^{j-1} \underline{p}(k) \right)^\gamma \\ &= \left(\sum_{k=1}^0 \underline{p}(k) + \tilde{p}(1) \right)^\gamma + \left(\sum_{k=1}^0 \underline{p}(k) \right)^\gamma \\ &= \tilde{p}(1)^\gamma \end{aligned} \quad (45)$$

Thus, $\lim_{\varepsilon \rightarrow 0} \tilde{p}_\gamma(1) = (\lim_{\varepsilon \rightarrow 0} \tilde{p}(1))^\gamma$, which, according to Lemma 4, gives $\lim_{\varepsilon \rightarrow 0} \tilde{p}_\gamma(1) = 1$. This proves the first part of the Lemma.

From (45), the first derivative is easily computed.

$$\frac{d \tilde{p}_\gamma(1)}{d\varepsilon} = \gamma \tilde{p}(1)^{\gamma-1} \frac{d \tilde{p}(1)}{d\varepsilon}.$$

From Lemma 6, we know that $\lim_{\varepsilon \rightarrow 0} \frac{d \tilde{p}(1)}{d\varepsilon} = 0$, such that

$$\lim_{\varepsilon \rightarrow 0} \frac{d \tilde{p}_\gamma(1)}{d\varepsilon} = 0.$$

For higher derivatives, the same holds true.

$$\frac{d^2 \tilde{p}_\gamma(1)}{d\varepsilon^2} = \gamma \left[(\gamma-1) \tilde{p}(1)^{\gamma-2} \left(\frac{d \tilde{p}(1)}{d\varepsilon} \right)^2 + \tilde{p}(1)^{\gamma-1} \frac{d^2 \tilde{p}(1)}{d\varepsilon^2} \right].$$

It is obvious that every higher derivative of $\tilde{p}_\gamma(1)$ contains a derivative of $\tilde{p}(1)$ in every additive term. As we know from Lemma 6 that in the limit for $\varepsilon \rightarrow 0$, every derivative of $\tilde{p}(1)$ vanishes, we can conclude that Lemma 7 holds. \square

From the above, we can state a condition for state 1 being stochastically stable.

LEMMA 8 *A stochastically stable state exists and is equal to the state characterized by common play of $e^k = 1$, if there is a natural number $x \in \mathbb{N}$, such that the x th derivative of $\tilde{p}_\gamma(1)$ with respect to ε is different from zero:*

$$\lim_{\varepsilon \rightarrow 0} \alpha_1 = 1 \quad \text{if} \quad \exists x \in \mathbb{N} \quad \text{such that} \quad \frac{d^x \underline{p}_\gamma(1)}{d\varepsilon^x} \neq 0 \quad (46)$$

Proof

From Lemma 5 and the first part of Lemma 7, we know that the denominator of (19) approaches zero in the limit for $\varepsilon \rightarrow 0$. Consequently, the limit of α_1 must be computed by means of a finite repetition of L'Hôpital's rule:

$$\lim_{\varepsilon \rightarrow 0} \alpha_1 = \frac{\lim_{\varepsilon \rightarrow 0} \frac{d^x \underline{p}_\gamma(1)}{d\varepsilon^x}}{\lim_{\varepsilon \rightarrow 0} \frac{d^x 1}{d\varepsilon^x} - \lim_{\varepsilon \rightarrow 0} \frac{d^x \tilde{p}_\gamma(1)}{d\varepsilon^x} + \lim_{\varepsilon \rightarrow 0} \frac{d^x \underline{p}_\gamma(1)}{d\varepsilon^x}}$$

As we know that $\frac{d^x 1}{d\varepsilon^x} = 0$ and $\frac{d^x \tilde{p}_\gamma(1)}{d\varepsilon^x} = 0$ (Lemma 7), we can conclude that

$$\lim_{\varepsilon \rightarrow 0} \alpha_1 = \frac{\lim_{\varepsilon \rightarrow 0} \frac{d^x \underline{p}_\gamma(1)}{d\varepsilon^x}}{\lim_{\varepsilon \rightarrow 0} \frac{d^x \underline{p}_\gamma(1)}{d\varepsilon^x}}$$

This means that $\lim_{\varepsilon \rightarrow 0} \alpha_1 = 1$, if there exists an x such that the x th derivative $\frac{d^x \underline{p}_\gamma(1)}{d\varepsilon^x}$ is different from zero (in order to have a denominator different from zero). \square

Thus, it remains to show that such an x does indeed exist:

LEMMA 9 *A natural number x meeting the condition given in Lemma 8 exists and is equal to γ .*

Proof

The first derivative of $\underline{p}_\gamma(1)$ is

$$\frac{d \underline{p}_\gamma(1)}{d\varepsilon} = \gamma \underline{p}(1)^{\gamma-1} \frac{d \underline{p}(1)}{d\varepsilon}$$

Because of $\lim_{\varepsilon \rightarrow 0} \underline{p}(1) = 0$ (Lemma 3, part (a)), the limit of this for vanishing ε approaches zero

$$\lim_{\varepsilon \rightarrow 0} \frac{d \underline{p}_\gamma(1)}{d\varepsilon} = 0$$

Starting a process of complete induction, computation of the second derivative yields

$$\frac{d^2 \underline{p}_\gamma(1)}{d\varepsilon^2} = \gamma \left[(\gamma-1) \underline{p}(1)^{\gamma-2} \left(\frac{d \underline{p}(1)}{d\varepsilon} \right)^2 + \underline{p}(1)^{\gamma-1} \frac{d^2 \underline{p}(1)}{d\varepsilon^2} \right]$$

which in the limit for vanishing ε again approaches zero.

$$\lim_{\varepsilon \rightarrow 0} \frac{d^2 \underline{p}_\gamma(1)}{d\varepsilon^2} = 0$$

Again, this is due to the fact that $\lim_{\varepsilon \rightarrow 0} \underline{p}(1) = 0$ (Lemma 3, part (a)).

Continuing the process of induction, it can be found that the third derivative contains a term $y(\gamma - 1)(\gamma - 2)\underline{p}(1)^{\gamma-3} \left(\frac{d\underline{p}(1)}{d\varepsilon}\right)^3$. All terms additively contained in the third derivative approach zero in the limit for $\varepsilon \rightarrow 0$. The critical ingredient to the equation is indeed the fact that every summand contains a power of $\underline{p}(1)$ which makes the entire summand vanish in the limit for vanishing ε .

Moreover, the γ th derivative can be found to include the term

$$\gamma! \underline{p}(1)^{\gamma-\gamma} \left(\frac{d\underline{p}(1)}{d\varepsilon}\right)^\gamma$$

which is additively connected to all the other terms. In this case (and only in this case), the critical term $\underline{p}(1)$ disappears, because it is raised to the power of zero. Consequently, as the limit of $\underline{p}(1)^{\gamma-\gamma} = \underline{p}(1)^0 = 1$ is positive, the limit of this entire term is positive and so is the limit of $\frac{d^\gamma \underline{p}(1)}{d\varepsilon^\gamma}$:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{d^\gamma \underline{p}(1)}{d\varepsilon^\gamma} &= \lim_{\varepsilon \rightarrow 0} \gamma! \underline{p}(1)^0 \left(\frac{d\underline{p}(1)}{d\varepsilon}\right)^\gamma \\ &= \lim_{\varepsilon \rightarrow 0} \gamma! \left(\frac{d\underline{p}(1)}{d\varepsilon}\right)^\gamma \\ &= \gamma! (nP(E^1 = 1))^\gamma \\ &\neq 0 \end{aligned}$$

Thus, a number x is found, which makes the limit of a higher derivative of $\underline{p}_\gamma(1)$ unequal to zero. This x is $x = \gamma$. \square

PROPOSITION 7

In the minimum effort coordination game with γ competing groups, the equilibrium characterized by the lowest possible effort level is the stochastically stable one.

$$\lim_{\varepsilon \rightarrow 0} \alpha_1 = 1.$$

PROOF

According to Lemmata 5 and 7, $\lim_{\varepsilon \rightarrow 0} \alpha_1$ must be computed via L'Hôpital's rule. Lemma 8 in connection with Lemma 9 states that the limit does exist and is equal to one. This shows that the long-run frequency of play of the lowest effort level approaches unity if the individual mutation probability vanishes. Thus, it completes the proof of Proposition 7. \square

D Stickiness

D.1 Equilibrium Level

LEMMA 10

$$0 \leq \tilde{p}(v) \leq 1 \quad \forall v \tag{47}$$

Proof

$\tilde{p}(v)$ is a probability. Consequently, Lemma 10 holds by definition of probabilities. \square

LEMMA 11

$$\sum_{k=1}^{v-1} \underline{p}(k) + \tilde{p}(v) \leq 1 \quad \forall v \in \mathbb{E} \quad (48)$$

Proof

The expression in (48) reaches its maximum possible value at $v = m$. In this case, $\sum_{k=1}^{v-1} \underline{p}(k) + \tilde{p}(v)$ gives the sum of all elements of the last row of transition matrix (8). Transition matrices are stochastic matrices, i.e. their rows sum up to one. Consequently, Lemma 11 holds. \square

LEMMA 12 $\sum_{k=1}^{v-1} \underline{p}(k) + \tilde{p}(v)$ weakly increases in v .

$$\left[\sum_{k=1}^{v-1} \underline{p}(k) + \tilde{p}(v) \right] > \left[\sum_{k=1}^{v-2} \underline{p}(k) + \tilde{p}(v-1) \right]. \quad (49)$$

PROOF

$$\begin{aligned} & \left[\sum_{k=1}^{v-1} \underline{p}(k) + \tilde{p}(v) \right] - \left[\sum_{k=1}^{v-2} \underline{p}(k) + \tilde{p}(v-1) \right] \\ &= \underline{p}(v-1) + \tilde{p}(v) - \tilde{p}(v-1) \\ &= \underline{p}(v-1) + P_0 + \underline{p}(v) + \sum_{\mu=1}^{n-1} P_\mu [1 - P(E^\mu \leq v)] - P_0 - \underline{p}(v-1) - \sum_{\mu=1}^{n-1} P_\mu [1 - P(E^\mu \leq v-1)] \\ &= \underline{p}(v) + \sum_{\mu=1}^{n-1} P_\mu [1 - P(E^\mu \leq v-1) - P(E^\mu = v)] - \sum_{\mu=1}^{n-1} P_\mu [1 - P(E^\mu \leq v-1)] \\ &= \underline{p}(v) - \sum_{\mu=1}^{n-1} P_\mu [1 - P(E^\mu = v)] \\ &= \sum_{\mu=1}^n P_\mu P(E^\mu = v) - \sum_{\mu=1}^{n-1} P_\mu P(E^\mu = v) \\ &= P_n P(E^n = v) \\ &\geq 0 \end{aligned}$$

\square

LEMMA 13 *In the one-group case, low equilibria are weakly stickier than high ones,*

$$\Delta_v \tilde{p}(v) = \tilde{p}(v) - \tilde{p}(v-1) \leq 0. \quad (50)$$

PROOF

$$\begin{aligned} \Delta_v \tilde{p}(v) &= \tilde{p}(v) - \tilde{p}(v-1) \\ &= P_0 + \underline{p}(v) + \sum_{\mu=1}^{n-1} P_\mu [1 - P(E^\mu \leq v)] - P_0 - \underline{p}(v-1) - \sum_{\mu=1}^{n-1} P_\mu [1 - P(E^\mu \leq v-1)] \\ &= \underline{p}(v) - \underline{p}(v-1) + \sum_{\mu=1}^{n-1} P_\mu [1 - P(E^\mu \leq v-1) - P(E^\mu = v)] - \sum_{\mu=1}^{n-1} P_\mu [1 - P(E^\mu \leq v-1)] \end{aligned}$$

$$\begin{aligned}
&= \underline{p}(v) - \underline{p}(v-1) - \sum_{\mu=1}^{n-1} P_{\mu} [1 - P(E^{\mu} = v)] \\
&= \sum_{\mu=1}^n P_{\mu} P(E^{\mu} = v) - \sum_{\mu=1}^n P_{\mu} P(E^{\mu} = v-1) - \sum_{\mu=1}^{n-1} P_{\mu} [1 - P(E^{\mu} = v)] \\
&= P_n P(E^n = v) - \sum_{\mu=1}^n P_{\mu} P(E^{\mu} = v-1) \\
&= P_n [P(E^n = v) - P(E^n = v-1)] - \sum_{\mu=1}^{n-1} P_{\mu} P(E^{\mu} = v-1)
\end{aligned}$$

As $P(E^n = v) \leq P(E^n = v-1)$ holds by assumption (6), it can be concluded that $\Delta_v \tilde{p}(v) \leq 0$ which proves Lemma 13. \square

PROPOSITION 8 For low numbers of competing groups, low-level equilibria (small v) are stickier than high-level equilibria (large v). For sufficiently high numbers of competing groups, this relation turns to the opposite.

$$\Delta_v \tilde{p}_{\gamma}(v) \begin{cases} < 0 & \text{for small } \gamma \\ > 0 & \text{for sufficiently high } \gamma \end{cases} \quad (23)$$

PROOF

Note that $\Delta_v \tilde{p}_{\gamma}(v)$ can be written as

$$\begin{aligned}
\Delta_v \tilde{p}_{\gamma}(v) &= \tilde{p}_{\gamma}(v) - \tilde{p}_{\gamma}(v-1) \\
&= \left[\left(\sum_{k=1}^{v-1} \underline{p}(k) + \tilde{p}(v) \right)^{\gamma} - \left(\sum_{k=1}^{v-1} \underline{p}(k) \right)^{\gamma} \right] - \left[\left(\sum_{k=1}^{v-2} \underline{p}(k) + \tilde{p}(v-1) \right)^{\gamma} - \left(\sum_{k=1}^{v-2} \underline{p}(k) \right)^{\gamma} \right]
\end{aligned} \quad (51)$$

Substituting $a := \sum_{k=1}^{v-1} \underline{p}(k)$, $b := \tilde{p}(v)$, $c := \sum_{k=1}^{v-2} \underline{p}(k)$, and $d := \tilde{p}(v-1)$, (51) can be written as

$$\Delta_v \tilde{p}_{\gamma}(v) = [(a+b)^{\gamma} - a^{\gamma}] - [(c+d)^{\gamma} - c^{\gamma}]$$

Lemma 10 shows that $b \geq 0$, $d \geq 0$. From Lemma 11, it is known that $a+b \leq 1$, $c+d \leq 1$. $a > c$ holds by simple observation. Lemma 12 shows that $a+b \geq c+d$. Finally, due to Lemma 13, $d \geq b$.

Lemma 13 shows that in the one-group case, i.e. for $\gamma = 1$, lower equilibria are stickier than higher ones. Generally, depending on the value of γ , higher or lower equilibria may be stickier. The following sketches (Fig. 1) make clear that for low numbers of competing groups, $[(c+d)^{\gamma} - c^{\gamma}] > [(a+b)^{\gamma} - a^{\gamma}]$, i.e. that lower equilibria are stickier than higher ones (Figure 1(a)). Sketch 1(b) shows, that for sufficiently high γ (i.e. a sufficiently convex curvature of the function x^{γ}), the opposite will be the case. \square

D.2 Number of Groups

LEMMA 14 In the interval $0 \leq x \leq 1$ the function $D(x) = x^{\gamma-1} - x^{\gamma}$ has a maximum at $x = \frac{\gamma-1}{\gamma}$ for $\gamma \in \mathbb{N}$.

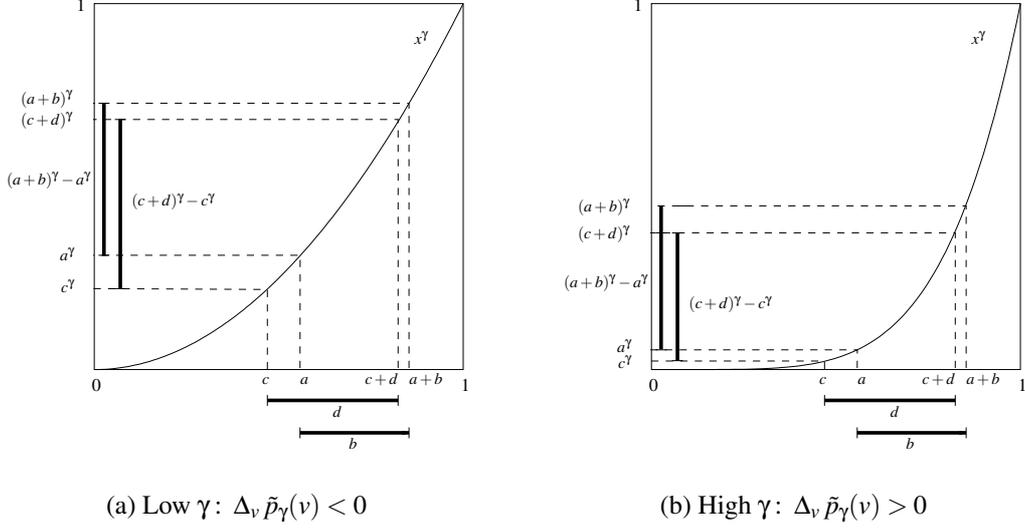


Figure 1: Stickiness and Level of Equilibrium

PROOF

Compute the first derivative of $D(x) = x^{\gamma-1} - x^\gamma$ and set it equal to zero:

$$\frac{dD(x)}{dx} = (\gamma-1)x^{\gamma-2} - \gamma x^{\gamma-1} \stackrel{!}{=} 0$$

which results in $x = \frac{\gamma-1}{\gamma}$. The second derivative at this point is negative,

$$\left. \frac{d^2 D(x)}{dx^2} \right|_{x=\frac{\gamma-1}{\gamma}} < 0,$$

which shows that the optimum is indeed a maximum. □

PROPOSITION 9

The stickiness of low equilibria (small ν) decreases in the number of competing groups. The stickiness of high equilibria (large ν) increases in the number of competing groups.

$$\Delta_\nu \tilde{p}_\gamma(\nu) \begin{cases} < 0 & \text{for low } \nu \\ > 0 & \text{for high } \nu \end{cases} \quad (25)$$

PROOF

Re-write the expression in (25):

$$\begin{aligned} \Delta_\nu \tilde{p}_\gamma(\nu) &= \tilde{p}_\gamma(\nu) - \tilde{p}_{\gamma-1}(\nu) \\ &= \left[\left(\sum_{k=1}^{\nu-1} p(k) + \tilde{p}(\nu) \right)^\gamma - \left(\sum_{k=1}^{\nu-1} p(k) \right)^\gamma \right] - \left[\left(\sum_{k=1}^{\nu-1} p(k) + \tilde{p}(\nu) \right)^{\gamma-1} - \left(\sum_{k=1}^{\nu-1} p(k) \right)^{\gamma-1} \right] \end{aligned} \quad (52)$$

Substitute $a := \sum_{k=1}^{v-1} \underline{p}(k)$, $b := \tilde{p}(v)$, such that (52) becomes

$$\Delta_\gamma \tilde{p}_\gamma(v) = (a+b)^\gamma - a^\gamma - (a+b)^{\gamma-1} + a^{\gamma-1}$$

If it holds that $(a+b)^{\gamma-1} - (a+b)^\gamma > a^{\gamma-1} - a^\gamma$, then $\Delta_\gamma \tilde{p}_\gamma(v) < 0$. Making use of Lemmata 10 and 11, Figures 2(a) and 2(b) are constructed. Figure 2(a) shows that $\Delta_\gamma \tilde{p}_\gamma(v) < 0$ must hold whenever $a+b < \frac{\gamma-1}{\gamma}$, where $\frac{\gamma-1}{\gamma}$ gives the location of the maximum distance between $x^{\gamma-1}$ and x^γ for $0 \leq x \leq 1$ (see Lemma 14). (This is of course a sufficient, but not a necessary condition.) According to Lemma 12, $a+b$ will be small for low equilibria (small v). Consequently, the case of $\Delta_\gamma \tilde{p}_\gamma(v) < 0$, i.e. stickiness of equilibria falling with an increase in the number of competing groups, is the relevant case for low equilibria and high γ .

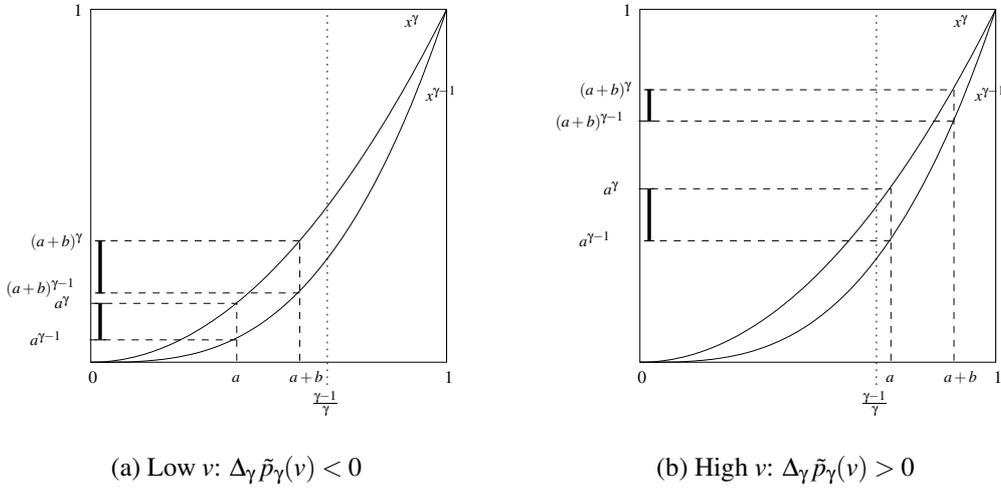


Figure 2: Stickiness and Number of Groups

On the other hand, if $(a+b)^{\gamma-1} - (a+b)^\gamma < a^{\gamma-1} - a^\gamma$, it turns out that $\Delta_\gamma \tilde{p}_\gamma(v) > 0$. Figure 2(b) shows that this is generally the case for $a > \frac{\gamma-1}{\gamma}$ (sufficient, but not necessary condition!). a is higher, the higher v is. This finally means that for high equilibria, the stickiness increases with an increase in the number of competing groups. \square

D.3 Number of Group Members

LEMMA 15

$$\Delta_n P_\mu = P_{\mu,n} - P_{\mu,n-1} = P_{\mu,n} \frac{\mu - n\varepsilon}{n(1-\varepsilon)} \quad (53)$$

PROOF

$$\Delta_n P_\mu = P_{\mu,n} - P_{\mu,n-1}$$

due to Proposition 1

$$\begin{aligned}
&= \binom{n}{\mu} \varepsilon^\mu (1-\varepsilon)^{n-\mu} - \binom{n-1}{\mu} \varepsilon^\mu (1-\varepsilon)^{n-\mu-1} \\
&= \frac{\varepsilon^\mu}{\mu!} \left[\frac{n!}{(n-\mu)!} (1-\varepsilon)^{n-\mu} - \frac{(n-1)!}{(n-\mu-1)!} (1-\varepsilon)^{n-\mu-1} \right] \\
&= \frac{\varepsilon^\mu (n-1)!}{\mu! (n-\mu)!} [n(1-\varepsilon)^{n-\mu} - (n-\mu)(1-\varepsilon)^{n-\mu-1}] \\
&= \frac{\varepsilon^\mu (n-1)!}{\mu! (n-\mu)!} (1-\varepsilon)^{n-\mu} [n - (n-\mu)(1-\varepsilon)^{-1}] \\
&= \frac{(n-1)!n}{(n-\mu)! \mu!} \varepsilon^\mu (1-\varepsilon)^{n-\mu} \frac{1}{n} [n - (n-\mu)(1-\varepsilon)^{-1}] \\
&= \frac{n!}{(n-\mu)! \mu!} \varepsilon^\mu (1-\varepsilon)^{n-\mu} \left[1 - \frac{n-\mu}{n} (1-\varepsilon)^{-1} \right]
\end{aligned}$$

and, because of $\frac{n!}{(n-\mu)! \mu!} \varepsilon^\mu (1-\varepsilon)^{n-\mu} = P_{\mu,n}$ (Proposition 1):

$$= P_{\mu,n} \frac{\mu - n\varepsilon}{n(1-\varepsilon)}$$

□

There is an important special case to Lemma 15:

COROLLARY 1

$$\Delta_n P_n = P_{n,n} \tag{54}$$

PROOF:

By making use of Lemma 15:

$$\Delta_n P_n = P_{n,n} \frac{n - n\varepsilon}{n(1-\varepsilon)} = P_{n,n}$$

□

LEMMA 16

$$\Delta_n \underline{p}(v) = \underline{p}(v, n) - \underline{p}(v, n-1) = \sum_{\mu=1}^n \Delta_n P_\mu \times P(E^\mu = v) \tag{55}$$

PROOF

$$\begin{aligned}
\Delta_n \underline{p}(v) &= \underline{p}(v, n) - \underline{p}(v, n-1) \\
&= \sum_{\mu=1}^n P_{\mu,n} P(E^\mu = v) - \sum_{\mu=1}^{n-1} P_{\mu,n-1} P(E^\mu = v) \\
&= \sum_{\mu=1}^{n-1} \Delta_n P_\mu P(E^\mu = v) + P_{n,n} P(E^n = v)
\end{aligned}$$

Using Corollary 1, this finally becomes

$$\Delta_n \underline{p}(v) = \sum_{\mu=1}^n \Delta_n P_\mu \times P(E^\mu = v)$$

□

LEMMA 17

$$\Delta_n \underline{p}(v) \begin{cases} > 0 & \text{for small } n \text{ (and/or small } \varepsilon) \\ < 0 & \text{for high } n \text{ (and/or high } \varepsilon) \end{cases} \quad (56)$$

PROOF

Note that every summand of the right-hand side of (55) reads $\Delta_n P_\mu \times P(E^\mu = v)$, which, according to Lemma 15, is equal to $P_{\mu,n} \frac{\mu - n\varepsilon}{n(1-\varepsilon)} P(E^\mu = v)$. As it is known that $P_{\mu,n} \geq 0$, $n(1-\varepsilon) \geq 0$, and $P(E^\mu) \geq 0$, the sign of each summand is the sign of the expression $\mu - n\varepsilon$. The sign of this expression depends on the number of mutants on the one hand, and the number of group members and the individual mutation probability on the other. There is one special case, which is well determined: If all group members mutate, i.e. if $\mu = n$, and if $\varepsilon < 1$ (which should be straightforward), the sign is positive. (This can be shown by using Corollary 1.)

$$\Delta_n P_n \times P(E^n = v) = P_{n,n} \frac{n - n\varepsilon}{n(1-\varepsilon)} P(E^n = v) > 0 \quad \text{if } \varepsilon < 1.$$

It is easy to conclude that for the general case, there are the more positive summands, the lower n (and/or ε) is. Consequently, for small group sizes n , $\Delta_n \underline{p}(v)$ is likely to be positive, while for large n (and high individual mutation probability ε), $\Delta_n \underline{p}(v)$ is likely to be of negative sign. □

LEMMA 18

$$\Delta_n \tilde{p}(v) = \tilde{p}(v, n) - \tilde{p}(v, n-1)$$

can be of either sign. It is negative for high n and not-too-small ε , and it is positive for low n and small ε .

PROOF

From Proposition 7, it is known that

$$\tilde{p}(v, n) = P_{0,n} + \underline{p}(v, n) + \sum_{\mu=1}^{n-1} P_\mu [1 - P(E^\mu \leq v)].$$

Consequently, the difference is given by

$$\begin{aligned} \Delta_n \tilde{p}(v) &= \tilde{p}(v, n) - \tilde{p}(v, n-1) \\ &= \Delta_n P_0 + \Delta_n \underline{p}(v) + \sum_{\mu=1}^{n-2} \Delta_n P_\mu [1 - P(E^\mu \leq v)] + P_{n-1,n} [1 - P(E^{n-1} \leq v)] \end{aligned}$$

$\Delta_n P_0$ is negative. $\Delta_n \underline{p}(v)$ and $\sum_{\mu=1}^{n-2} \Delta_n P_\mu [1 - P(E^\mu \leq v)]$ will both be negative for high n , but positive for low n and/or low ε . $P_{n-1,n} [1 - P(E^{n-1} \leq v)]$ will almost always be positive. Altogether, it can be concluded, that $\Delta_n \tilde{p}(v)$ is negative for sufficiently high n and not-too-small ε , whereas for low n and/or small ε , it is positive. □

PROPOSITION 10 The stickiness of low equilibria (small v) increases in n . The stickiness of high equilibria (high v) decreases in n .

$$\Delta_n \tilde{p}_\gamma(v) \begin{cases} > 0 & \text{for low } v \\ < 0 & \text{for high } v \end{cases} \quad (27)$$

PROOF

Re-write $\Delta_n \tilde{p}_\gamma(v)$:

$$\begin{aligned} \Delta_n \tilde{p}_\gamma(v) &= \tilde{p}(v, n) - \tilde{p}(v, n-1) \\ &= \left(\sum_{k=1}^{v-1} \underline{p}(k, n) + \tilde{p}(v, n) \right)^\gamma - \left(\sum_{k=1}^{v-1} \underline{p}(k, n) \right)^\gamma \\ &\quad - \left(\sum_{k=1}^{v-1} \underline{p}(k, n-1) + \tilde{p}(v, n-1) \right)^\gamma + \left(\sum_{k=1}^{v-1} \underline{p}(k, n-1) \right)^\gamma \end{aligned} \quad (57)$$

In order to find the sign of $\Delta_n \tilde{p}_\gamma(v)$, the notation is simplified by substituting $a = \sum_{k=1}^{v-1} \underline{p}(k, n)$, $b = \tilde{p}(v, n)$, $c = \sum_{k=1}^{v-1} \underline{p}(k, n-1)$, and $d = \tilde{p}(v, n-1)$, such that (57) reads

$$\Delta_n \tilde{p}_\gamma(v) = (a+b)^\gamma - a^\gamma - (c+d)^\gamma + c^\gamma$$

For the case of low n and not-too-large ϵ , it is known from Lemma 17 that $a > c$ and from Lemma 18 that $b > d$, which in combination yields $a+b > c+d$. Figure 3(a) shows that this results in $(a+b)^\gamma + a^\gamma > (c+d)^\gamma - c^\gamma$ and thus that $\Delta_n \tilde{p}_\gamma(v) > 0$. On the other hand, for sufficiently large n (and not-too-small ϵ), we can conclude from Lemmata 17 and 18 and Figure 3(b) that $\Delta_n \tilde{p}_\gamma(v) < 0$.

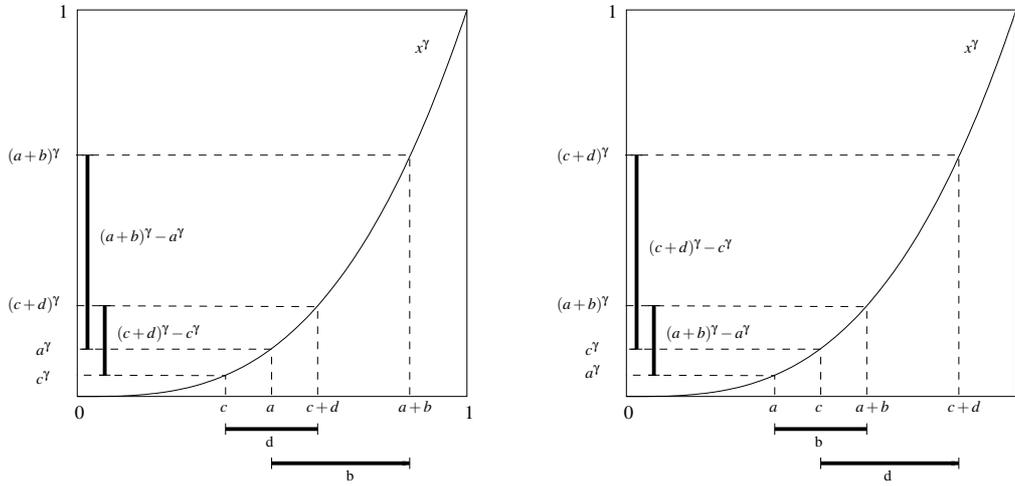


Figure 3: Stickiness and Number of Group Members

It can be seen that $\Delta_n \tilde{p}_\gamma(v) < 0$, if $d = \tilde{p}(v, n-1)$ is relatively large, which according to Lemma 13, is the case if v is relatively small. \square

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