

# Demand commitment in legislative bargaining\*

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## Abstract

Morelli (*American Political Science Review*, 1999) provides a majoritarian bargaining model in which the parties make payoff demands and the order of moves is chosen by the leading party. Morelli's main proposition states that the ex post distribution of payoffs inside the coalition that forms is proportional to the homogeneous representation of the game. We provide a counterexample and prove a weaker result: proportional payoffs hold if the rules are modified so that the parties must move in decreasing order of weight.

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# 1 Introduction

In a parliamentary democracy, many important decisions including government formation are the outcome of bargaining between the parties in Parliament. The most influential model of legislative bargaining is the closed rule model of Baron and Ferejohn (1989). In this model, a party is randomly recognized to propose a complete distribution of ministerial payoffs and the remaining parties can accept or reject the proposal. This model has led to many applications and extensions.<sup>1</sup> It has some properties that may be perceived as drawbacks: the proposer has a large advantage<sup>2</sup> (he receives more than half of the total payoff under simple majority), and there is a multiplicity of subgame perfect equilibria. In order to single out a unique prediction, the stationary equilibrium is selected. Stationary strategies are simple but by no means uncontroversial: a stationary strategy requires a party to always make the same proposal regardless of the history of the negotiations so far. Moreover, Norman (2002) shows that sharp predictions using stationarity are only possible in the infinite horizon version of the model: in the finite horizon version there is a continuum of equilibria, all of them with history-independent strategies.

An alternative model of legislative bargaining by Morelli (1999) is based not on complete proposals but on demands. Parties make individual demands for ministerial payoffs and a coalition emerges between parties making compatible demands. The Head of State chooses the first mover, and the latter chooses the order in which the parties formulate demands. Morelli's

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<sup>1</sup>For example, McKelvey and Riezman (1992) analyze seniority in legislatures. Other papers incorporate policy preferences (see e.g. Baron, 1991; Jackson and Moselle, 2002), different risk attitudes or discount factors (Harrington 1990, Eraslan 2002), different voting rules (Winter 1996, Snyder et al. 2005, Montero 2006) or arbitrary recognition probabilities (Kalandrakis, 2006). Banks and Duggan (2000) show existence of stationary equilibrium under very general conditions.

<sup>2</sup>This feature is also present in other proposal-making models (Austen-Smith and Banks (1988), Bloch and Rottier (2002)).

main result (Proposition 2) is that the ex post distribution of payoffs inside the coalition that forms is proportional to the homogeneous representation of the game regardless of which party is chosen to be the first mover. This is an attractive result in at least two respects: the first mover has no disproportionate advantage, and a sharp result can be obtained without resorting to controversial equilibrium refinements.<sup>3</sup>

In this paper we provide a counterexample to Morelli's Proposition 2. In our counterexample, the first mover, even if it is the smallest party in parliament, can exploit the competition between the other parties and capture *the entire surplus* if it is allowed to choose the order of moves. We then go on to prove a weaker result: proportional payoffs hold if the rules are modified so that the parties must move in decreasing order of weight. This procedure mirrors the assumption of Austen-Smith and Banks (1988) that parties are asked to try to form a government in decreasing order of weight. Parties making demands in decreasing order of weight is not a completely unnatural assumption. In some countries the largest party must be asked to form the government first; in the absence of this rule the largest party is still selected quite often.<sup>4</sup> Moving first in the demand bargaining procedure is not equivalent to being formateur (the rules of the game allow other par-

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<sup>3</sup>The empirical evidence lies somewhere in between. A large body of empirical studies (see e.g. Browne and Franklin 1973, Schofield and Laver 1985, Warwick and Druckman 2001) find little or no advantage to being formateur: a party's share of cabinet posts is nearly proportional to its share of legislative seats in the governing coalition. Ansolabehere et al. (2005) find a formateur advantage using voting weights instead of seat shares, though this advantage is well below the value predicted by the Baron-Ferejohn model. Interestingly, experiments on majority games do not find as sharp a behavioral difference between both types of procedures as the theory predicts (see Fréchet et al., 2005).

<sup>4</sup>The Greek constitution prescribes that the leader of the largest party must be chosen as the first formateur; if he fails, the leader of the second largest party is selected, to be followed by the leader of the third largest party if he too fails. Even if the constitution is silent on this matter, a convention may emerge (Laver and Schofield, 1990, p. 210). For a quantitative analysis of formateur selection see table 1 in Warwick (1996), table 3 in Diermeier and Merlo (2004), and table 3 in Ansolabehere et al. (2005).

ties to exclude the largest party), though in equilibrium the largest party is always included in the government.

There have been other demand bargaining models in the literature. Binmore (1985) presents a three-player “market model” where demands are carried over to the next round and infinite plays are possible. Selten (1992) presents a general but relatively complicated model, including random draws and costs of both formulating a demand and forming a coalition. Bennett and van Damme (1991) study a simpler version in which each player selects the next one to move, and show that there may be a multiplicity of subgame perfect equilibria. Using a refinement, they select the proportional payoff division for apex games. Winter (1994), Dasgupta and Chiu (1998), and Vidal-Puga (2004) use various demand commitment procedures to implement the Shapley value in convex games.

## 2 Preliminaries

### 2.1 Weighted majority games

Consider a legislature in which  $n$  parties are represented. We denote these as  $N = \{1, 2, \dots, n\}$ . There is a budget of size 1 to be divided by majority rule. Each party  $i$  has  $\omega_i$  votes, and a quota of  $q$  is needed for a majority. The pair  $[q; (w_i)_{i \in N}]$  is a *weighted majority game*. Notice that the game is not affected if weights and quota are multiplied by the same positive constant.

Given a vector  $x \in \mathbb{R}^N$  and a coalition  $S \subset N$ , we denote as  $x_S$  the sum of the coordinates of the members of  $S$ ,  $x_S := \sum_{i \in S} x_i$ .

A coalition  $S \subset N$  is *winning* if  $\omega_S \geq q$ ; it is *minimal winning* if it is winning and no  $T \subsetneq S$  is winning. We denote as  $\Omega(\omega)$  the set of all winning coalitions, and as  $\Omega^m(\omega)$  the set of all minimal winning coalitions. A *dummy player* is a player who does not belong to any minimal winning coalition.

A weighted majority game is *constant-sum* if  $S \in \Omega(\omega) \iff N \setminus S \notin \Omega(\omega)$ .

$\Omega(\omega)$  for all  $S$ . It admits an equivalent *homogeneous representation* if there exists a vector of votes  $(\omega_1^h, \dots, \omega_n^h)$  and a quota  $q^h$  such that  $\Omega^m(\omega) = \Omega^m(\omega^h) = \{S \subset N : \omega_S^h = q^h\}$ . A weighted majority game that admits an equivalent homogeneous representation is called a *homogeneous game*.

Homogeneous representations do not always exist and when they exist they may not be unique. For example,  $[5; 3, 2, 2, 1]$  and  $[7; 4, 3, 3, 1]$  are two homogeneous representations of the same game. Peleg (1968) shows that constant-sum homogeneous games have a unique homogeneous representation (up to multiplication by a positive constant and to the weight that is assigned to dummies, which may be 0 or a sufficiently small number).

## 2.2 Morelli's bargaining procedure

There are  $n$  parties, 1 unit of private benefits to be distributed between them, and a policy to be chosen from the one-dimensional policy space  $[0, 1]$ . Decisions are taken by weighted majority.

Party  $i$  has utility function  $u_i = x_i + 1 - \beta|\theta - \theta_i^*|$ , where  $x_i$  denotes the share of private benefits accruing to  $i$ ,  $\theta$  is the policy implemented and  $\theta_i^*$  is party  $i$ 's ideal policy.

Bargaining proceeds as follows: First, the Head of State chooses a party  $i$ . Second,  $i$  chooses an order of play  $\rho : N \rightarrow \{1, 2, \dots, n\}$  so that  $i$  is the first one in the order, *i.e.*  $\rho(i) = 1$ . Third, each party  $j$  demands a pair  $(d_j, \theta_j)$  following the order of play, where  $d_j \in [0, 1]$  is the share of the private benefit  $j$  claims and  $\theta_j \in [0, 1]$  is a policy. If, after party  $j$  makes its demand, there exists a winning coalition  $S \subset \{k : \rho(k) \leq \rho(j)\}$  such that  $d_S \leq 1$  and  $\theta_k = \theta_l$  for every  $k, l \in S$ , then  $j$  has the additional choice of forming  $S$ , in which case the policy is implemented and the demands of parties in  $S$  are granted. In case of more than one possible coalition, party  $j$  decides which one is formed. If all parties have moved and no winning coalition has been formed, the Head of State chooses a first mover again. If after  $T$  rounds no agreement is reached, no private benefits are distributed and the policy

outcome is the one preferred by the median voter.

**Example 1** (*cf. theorem 1 in Bennett and van Damme 1991 and Morelli 1999*) *There are three parties, with 1 vote each. The quota is 2. Parties do not care about policy, that is,  $u_i = x_i$ . Let  $T = 1$ . In the unique subgame perfect equilibrium parties 1 and 2 form a coalition and get  $\frac{1}{2}$  each.*

**Proof.** Since the parties are symmetric the identity of the party chosen by the Head of State is irrelevant, and the order of moves this party chooses is irrelevant as well. Without loss of generality suppose party 1 is chosen to move first, and it chooses party 2 to move second.

Proceeding by backwards induction, suppose it is party 3's turn to move. If  $\min(d_1, d_2) < 1$ , party 3 will form a coalition with the party with the lowest demand; if  $d_1 = d_2$ , party 3 is indifferent between 1 and 2 and any choice is part of an equilibrium at this subgame.

Now consider the situation faced by party 2 after observing  $d_1$ . If 2 decides to formulate a demand and let the game go on, setting  $d_2 > d_1$  leads to a payoff of 0; setting  $d_2 < d_1$  leads to  $d_2$ . The payoff of setting  $d_2 = d_1$  depends on party 3's tie-breaking rule. As we will see below, party 3 must break ties in favor of 2 in order for 2 to have a best response at all subgames.

Party 2 then compares the payoff of forming a coalition with party 1 ( $1 - d_1$ ) and the payoff of matching party 1's demand and inducing a coalition with party 3 ( $d_1$ ). Party 2 sets  $\theta_2 = \theta_1$  and forms a coalition if  $1 - d_1 > d_1$  (i.e. if  $d_1 < \frac{1}{2}$ ), and makes a demand if  $d_1 > \frac{1}{2}$ . If 3 would break ties in favor of 1, 2 would not have a best response after observing  $d_1 > \frac{1}{2}$ : it would want to set a demand as close as possible to  $d_1$ , but not  $d_1$ .

Party 1 obtains  $d_1$  if it sets  $d_1 < \frac{1}{2}$  and 0 if  $d_1 > \frac{1}{2}$ . Again, 1 does not have a best response unless 2 breaks ties in its favor for  $d_1 = \frac{1}{2}$ . Party 1 is then chooses  $d_1 = \frac{1}{2}$  and an arbitrary value of  $\theta_1$ . ■

The example illustrates an important property of this type of bargaining procedure: parties will typically face choices among which they (but not others) are indifferent, and they must break ties in a particular way in order

for other parties to have a best response earlier on. We will use this property repeatedly in the rest of the paper.

Proposition 2 in Morelli (1999) states that, when parties only care about private benefits and there exists a unique equivalent homogeneous representation  $(\omega, q)$ , then there is a unique equilibrium payoff distribution, where a winning coalition  $S^*(\rho)$  is formed and each party  $i \in S^*(\rho)$  receives a payoff  $\frac{\omega_i}{q}$  proportional to its number of votes in the homogeneous representation.

This is indeed the case for some games including symmetric and apex games, but not in general as the following counterexample illustrates.

**Proposition 1** *There are five parties, with 3, 2, 2, 1 and 1 votes respectively. The quota is 5. This is a game with a unique homogeneous representation. Let  $T = 1$ . The party chosen as the first mover can always find an order of play that allows it to get the whole surplus.*

**Proof.** See appendix A. ■

The intuition for this result is as follows. If party 1 (3 votes) is chosen to be proposer, it can choose the order [14523], associated to votes [31122], and get all the surplus. To see this, suppose party 1 demands the whole surplus. Party 4 (with 1 vote) can either go along with party 1 and demand 0, or make a positive demand and try to form an alternative coalition with parties 2 and 3. However, given the order of moves, any positive demand can be undercut by party 5 and will result in coalition  $\{2, 3, 5\}$ . Thus, party 4 may as well demand 0 after observing a demand of 1 by party 1 (indeed, it *must* demand 0 in order for party 1 to have a best response). Given that party 4 demands 0, party 5 is helpless as well: a positive demand would result in parties 2 and 3 forming a coalition with 4.

If party 2 (2 votes) is chosen to be proposer, it can choose the order [23451], associated to votes [22113], and get all the surplus. Suppose the first mover demands the whole surplus. This prevents the second mover from getting a positive payoff in any coalition that includes the first mover. The only other alternative, a coalition with the last mover, would always be

sabotaged by the two small parties. Thus, the second mover may as well demand 0 after observing a demand of 1 by the first mover. But this in turn prevents party 4 from getting a positive payoff in a coalition with party 1: party 1 will always prefer to form a coalition with party 3. Thus, party 4 may as well form a coalition and get 0.

If party 4 (1 vote) is chosen to be proposer, it can choose the order [42315], associated to votes [12231], and get all the surplus. Suppose party 4 demands the whole surplus. Then party 2 may as well demand 0: any positive demand can be undercut by party 3 and would lead to a coalition of parties 1 and 3. On the other hand, if party 2 demands 0, party 3 cannot get a positive payoff: a positive demand would result in party 1 forming a coalition with party 2.

By committing itself to a demand and sequencing the order of moves of the other parties in a suitable way, the first mover exploits the demand competition between the other parties in its favor. This is the case even though the first mover has no monopoly proposal power and the rules of the game allow the first mover to be excluded from the government.

Morelli's argument for proportionality was that a higher than proportional demand would trigger the reaction of an alternative minimal winning coalition that can divide payoffs proportionally: any party that deviates can be replaced without changing the payoff shares for the others (see Morelli 1999 p. 818). Indeed, such a minimal winning coalition always exists, but by choosing the order of moves the first mover can ensure that the members of the coalition cannot coordinate on forming an alternative government. For example, in the order [14523], after party 1 demands the whole surplus there exists one minimal winning coalition that could exclude 1 and divide payoffs proportionally: coalition  $\{2, 3, 4\}$ . However, the members of this coalition do not move consecutively and party 4 knows that any attempt to induce coalition  $\{2, 3, 4\}$  will be sabotaged by party 5. There is an alternative minimal winning coalition,  $\{2, 3, 5\}$ , whose members move consecutively, but



they do not move immediately after party 1: any attempt of party 5 to form  $\{2, 3, 5\}$  will be forestalled by party 4 setting a sufficiently low demand.

However, in the next section we restate Morelli's result as follows: when the voting game is homogeneous and constant-sum, and the order of moves is exogenously determined to be by decreasing weight, then there is a unique equilibrium payoff distribution, where a minimal winning coalition  $S^*$  is formed and each party  $i \in S^*$  receives a payoff  $\frac{\omega_i}{q}$  proportional to its number of votes in the homogeneous representation.

### 3 The model

Let  $[q; (w_i)_{i \in N}]$  be a constant-sum homogeneous weighted majority game. There is a budget of size 1 to divide. Party  $i$ 's utility function is  $u_i = x_i$ , where  $x_i$  is  $i$ 's share of the budget. Bargaining proceeds as follows. Parties move in decreasing order of weight. We label the parties in this order, so that party 1 moves first, followed by party 2, etc.

Each party  $i$  makes a demand  $d_i$ , following the order of play, where  $d_i \in [0, 1]$  is the share of the private benefit party  $i$  claims. If, after party  $i$  makes its demand, there exists a winning coalition  $S \subset \{j : j \leq i\}$  such that  $d_S \leq 1$ , party  $i$  has the additional choice of forming coalition  $S$ , in which case the private benefits are distributed according to the demands made. If there is more than one possible  $S$ , party  $i$  decides which one is formed. If party  $n$  forms no coalition, the game ends with each party getting zero.

Given  $i \in N$ , we denote as  $P_i$  the set of predecessors of  $i$ . Namely:

$$P_i := \{j \in N : j < i\}.$$

Since parties only care about private benefits, we will assume that they do not demand a policy outcome. We also assume  $T = 1$ , i.e., there is only one round of bargaining.<sup>5</sup>

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<sup>5</sup> Any finite  $T$  would lead to the same equilibrium outcome, regardless of whether parties discount future payoffs.

As it will become clear from the analysis, dummy players must get 0 in equilibrium, so for simplicity we assume there are no dummy players. We will use the homogeneous representation with  $\omega_n = 1$ ; i.e. the weakest party has exactly 1 vote. Under these circumstances, every party in a constant-sum homogeneous game has a positive integer number of votes. Furthermore:

**Lemma 1** *Let  $[q; (w_i)_{i \in N}]$  be a constant-sum homogeneous game. Then,  $\omega_N = 2q - 1$ .*

**Proof.** Because  $n$  is not a dummy player, there exists  $S \in \Omega^m(\omega)$  such that  $n \in S$ . Homogeneity implies  $\omega_S = q$ . Because  $S \in \Omega^m(\omega)$ ,  $S \setminus \{n\}$  must be losing. Since the game is constant-sum,  $(N \setminus S) \cup \{n\} \in \Omega(\omega)$ . Moreover, by deleting the weakest party (i.e. party  $n$ ) we obtain a losing coalition  $N \setminus S$ . Thus,  $(N \setminus S) \cup \{n\} \in \Omega^m(\omega)$ . So,  $\omega_{(N \setminus S) \cup \{n\}} = q$  and  $\omega_{N \setminus S} = q - 1$ . Hence

$$\omega_N = \omega_S + \omega_{N \setminus S} = q + q - 1 = 2q - 1.$$

■

**Corollary 1** *Let  $[q; (w_i)_{i \in N}]$  be a constant-sum homogeneous game. Then,  $S$  is maximal losing (i.e.  $N \setminus S \in \Omega^m(\omega)$ ) iff  $\omega_S = q - 1$ .*

**Proof.** Since  $(N, v)$  is constant-sum and homogeneous,  $S$  is maximal losing iff  $N \setminus S \in \Omega^m(\omega)$ , which means  $\omega_{N \setminus S} = q$  and thus, under Lemma 1,

$$\omega_S = \omega_N - \omega_{N \setminus S} = 2q - 1 - q = q - 1.$$

■

**Lemma 2** *Let  $[q; (w_i)_{i \in N}]$  be a weighted majority game. Then, there is a party  $i$  such that  $P_{i+1} \in \Omega^m(\omega)$ .*

**Proof.** Suppose this was not the case. Consider the smallest index  $i$  such that  $S = \{1, \dots, i\}$  is a winning coalition. There is a minimal winning coalition  $S' \subset S$ , and  $S'$  is obtained from  $S$  by deleting at least one party  $j < i$ .

However this is impossible because by assumption  $\{1, \dots, i-1\}$  is a losing coalition, and, since  $w_j \geq w_i$  for all  $j < i$ , this coalition has at least as many votes as  $S'$ . ■

Lemma 2 does not hold for arbitrary orders of the parties. For example, if we take the game  $[3; 2, 1, 1, 1]$  and order the parties in such a way that the party with 2 votes is in the third place, no set of parties  $\{1, \dots, i\}$  is a minimal winning coalition. If the parties play the game in this order, the party that moves first cannot get a positive payoff for any demand, and this leads to a continuum of subgame perfect equilibria.

**Theorem 1** *Let  $[q; (w_i)_{i \in N}]$  be a constant-sum homogeneous game. Suppose parties play a demand commitment game in decreasing order of weight. Then in any subgame perfect equilibrium the minimal winning coalition of Lemma 2 forms with each party  $i$  demanding  $\frac{\omega_i}{q}$ .*

**Proof.** See Appendix B. ■

The equilibrium strategies are roughly as follows (for a formal description see Appendix B). Given the demands of the parties that have moved so far, party  $i$  determines two things: the optimal coalition to be (eventually) formed and the optimal demand to make.

In general, the optimal coalition  $S$  will control exactly  $q$  votes. This coalition will generally include some parties that have moved before  $i$ , as well as some parties moving after  $i$ . Since  $T = S \cap P_i$  is a group of parties that have already formulated a demand,  $1 - d_T$  is the benefit from buying the votes of the parties in  $T$ ; this benefit will be shared by the parties in  $S \setminus T$ . Buying less votes leads to a higher benefit, but more votes from parties moving after  $i$  will be needed to complete a winning coalition. The coalition  $S$  is chosen such that the *average benefit per vote*,  $\frac{1-d_T}{q-\omega_T}$ , is maximized.

The optimal demand for party  $i$  will normally be  $d_i = \omega_i \frac{1-d_T}{q-\omega_T}$ , that is, party  $i$  will claim a share of the benefit proportional to its number of votes. Only in some subgames outside the equilibrium path can party  $i$  demand more than a proportional share.

Below we present a worked out example.

**Example 2** *There are five parties, with 3, 2, 2, 1 and 1 votes respectively. The quota is 5. If the parties play a demand commitment game in decreasing order there is a unique subgame perfect equilibrium, in which coalition  $\{1, 2\}$  forms with  $d_1 = \frac{3}{5}$  and  $d_2 = \frac{2}{5}$ .*

**Proof.** We proceed by backward induction.

At stage 5, party 5 faces a vector of demands  $(d_1, d_2, d_3, d_4)$ . It has three choices:

- a) Form coalition  $\{1, 4, 5\}$  and get  $1 - d_1 - d_4$ .
- b) Form coalition  $\{2, 3, 5\}$  and get  $1 - d_2 - d_3$ .
- c) Form no coalition and get 0.

Suppose forming some coalition is optimal. Then party 5 will form coalition  $\{1, 4, 5\}$  if  $1 - d_1 - d_4 \geq 1 - d_2 - d_3$ , or  $d_4 \leq d_2 + d_3 - d_1$ . Ties are broken in favor of forming the coalition that includes party 4, to guarantee that party 4 has a best response in the previous stage. Hence the maximum demand 4 can make and still get into a coalition with 5 is  $d_4 = d_2 + d_3 - d_1$ .

At stage 4, party 4 faces a vector of demands  $(d_1, d_2, d_3)$ . It can form coalition  $\{2, 3, 4\}$  or make a demand that will lead to  $\{1, 4, 5\}$ . It forms  $\{2, 3, 4\}$  if  $1 - d_2 - d_3 \geq d_2 + d_3 - d_1$ , or

$$1 - d_2 - d_3 \geq \frac{1 - d_1}{2}.$$

Thus, party 4 is effectively comparing the average benefit associated to buying the votes of 2 and 3 (in which case 1 vote is enough to complete a winning coalition) or the votes of 1 (in which case 2 votes are needed to complete a winning coalition and 4 must share the benefit with 5).

From the inequality above, the maximum demand party 3 can make at the previous stage and still induce  $\{2, 3, 4\}$  is

$$d_3 = \frac{1 - 2d_2 + d_1}{2}.$$

At stage 3, party 3 faces a vector of demands  $(d_1, d_2)$ . It can form coalition  $\{1, 3\}$  or make a demand that will induce  $\{2, 3, 4\}$ . It makes a demand if  $\frac{1-2d_2+d_1}{2} \geq 1 - d_1$  or

$$\frac{1 - d_2}{3} \geq \frac{1 - d_1}{2}.$$

Again, party 3 may buy the votes of party 1 (in which case 2 votes are required to complete a winning coalition), or the votes of party 2 (in which case 3 votes are required to complete a winning coalition). It chooses the alternative with the highest average benefit.

The maximum demand party 2 can make in the previous stage and still induce coalition  $\{2, 3, 4\}$  is

$$d_2 = \frac{3d_1 - 1}{2}.$$

At stage 2, party 2 compares  $1 - d_1$  and  $\frac{3d_1-1}{2}$ . It forms  $\{1, 2\}$  if  $\frac{3d_1-1}{2} \leq 1 - d_1$ , or  $d_1 \leq \frac{3}{5}$ . This inequality can be rewritten as  $\frac{1-d_1}{2} \geq \frac{1}{5}$  (where  $\frac{1}{5}$  is the average benefit of buying no votes).

Anticipating this, party 1 sets  $d_1 = \frac{3}{5}$ . Party 2 will then set  $d_2 = \frac{2}{5}$  and coalition  $\{1, 2\}$  is formed. ■

## 4 Discussion

We have shown that Morelli's proportionality result still holds if parties must move in decreasing order of weight. But if parties move in decreasing order of weight, the coalition that forms is the minimal winning coalition with the smallest number of parties, as hypothesized by Leiserson (1968, p. 775). If parties are asymmetric, the smaller parties are never part of the government. One may ask whether the Head of State can achieve proportional payoffs for an arbitrary minimal winning coalition by choosing the order of moves appropriately. The answer is negative: for the game  $[5; 3, 2, 2, 1, 1]$ , there is no order of moves for which coalition  $\{1, 4, 5\}$  forms with a proportional

payoff division. There are three types of possible orders for which the parties in this coalition move first: [31122], [13122] and [11322]. It can be shown that the first mover gets the whole budget in order [31122], whereas in the other two orders the first mover gets half of the budget.

If the game is not constant-sum and homogeneous, proportionality may break down. In some cases, this is due to the presence of a party that can be "held hostage" by others, as pointed by Morelli (1999).

**Example 3** *There are four parties, with 3, 2, 2 and 1 votes respectively. The quota is 5. If the parties play a demand commitment game in decreasing order the unique subgame perfect equilibrium results in coalition  $\{1, 2\}$  with  $d_1 = \frac{1}{2}$  and  $d_2 = \frac{1}{2}$ .*

Party 4 is helpless because there is only one minimal winning coalition it can form. Knowing this, party 3 will either form a coalition with 1 and get  $1 - d_1$ , or set  $d_3 = 1 - d_2$ . Party 2 can then form a coalition with 1 (obtaining  $1 - d_1$ ) or set  $d_2 = d_1$  and induce coalition  $\{2, 3, 4\}$ . Anticipating this, party 1 sets  $d_1 = \frac{1}{2}$ . The game  $[5; 3, 2, 2, 1]$  has many homogeneous representations, but in none of them do parties 1 and 2 have the same number of votes.

Proportionality can break down even if no party can be held hostage by others, as the following example illustrates.

**Example 4** *Consider the game  $[7; 4, 3, 2, 2, 1, 1]$ . If the parties play a demand commitment game in decreasing order, the unique subgame perfect equilibrium results in coalition  $\{1, 2\}$  with  $d_1 = d_2 = \frac{1}{2}$ .*

The game above is constant-sum but not homogeneous. None of the parties can be held hostage in the sense of Morelli: given any two parties, each of them can form a minimal winning coalition that does not include the other. Moreover, coalition  $\{1, 2\}$  has exactly 7 votes. Nevertheless, proportionality fails because  $\{1, 3, 4\}$  and  $\{2, 3, 4\}$  are both minimal winning coalitions. From the point of view of parties 3 and 4, parties 1 and 2 are

equally valuable even though they have a different number of votes. If the turn reaches party 3, which of the two coalitions forms will depend on whether  $d_1$  is higher or lower than  $d_2$ . Anticipating this, party 2 has two options: it can form a coalition with 1 and get  $1 - d_1$ , or set  $d_2 = d_1$  and induce coalition  $\{2, 3, 4\}$ . Party 2 will form a coalition if  $1 - d_1 \geq d_1$ , or  $d_1 \geq \frac{1}{2}$ .

## 5 Conclusion

We have presented a demand bargaining model that makes sharp predictions regarding coalition formation and payoff division. As in the original model of Morelli, we need no controversial refinements of subgame perfect equilibrium. The model can be extended to any finite horizon, and its predictions are independent of the discount factors and the risk attitudes of the parties. Moreover, the first mover has no disproportionate advantage. Because only the homogeneous representation matters, Gamson's predictions (see Gamson, 1961) hold only approximately, unless the actual seat shares coincide with the homogeneous weights.

From a normative point of view, proportional payoffs are intuitive in the absence of policy preferences. They are also predicted by many solution concepts like von Neumann-Morgenstern's (1944) main simple solution, the set of balanced aspirations (Cross, 1967), the competitive solution (McKelvey et al., 1978) and the demand bargaining set (Morelli and Montero, 2003). The empirical evidence is consistent with the proportional payoff prediction, at least for parties other than the formateur (see Ansolabehere et al. (2005) and the references therein). We have shown that Morelli's appealing results regarding proportionality in demand bargaining do not hold generally. However, one should keep in mind that they hold for some important types of games, including symmetric and apex games.

## 6 Appendix A: Proof of Proposition 1

We will denote  $\min(a, b)$  by  $a \wedge b$  and  $\max(a, b)$  by  $a \vee b$ . There are three types of parties that can be chosen to be first mover: the party with 3 votes, one of the parties with 2 votes, and one of the parties with 1 vote. We will examine each case in turn. Given the order chosen by the first mover, we divide the game in stages (each stage corresponding to one party moving) and construct an equilibrium starting by the last party to move. We then show that there is no other equilibrium.

**CASE 1: Party 1 (3 votes) is the first mover.** It can choose the order [14523], associated to votes [31122], and get the whole surplus.

**Stage 5.** Party 3 (2 votes) faces a vector of demands  $(d_1, d_2, d_4, d_5)$  and a vector of policies  $(\theta_1, \theta_2, \theta_4, \theta_5)$ . It has four choices<sup>6</sup>:

- a) Form coalition  $\{1, 3\}$  and get  $1 - d_1$ .
- b) If  $\theta_2 = \theta_4$ , it can also form  $\{2, 3, 4\}$  and get  $1 - d_2 - d_4$ .
- c) If  $\theta_2 = \theta_5$ , it can also form  $\{2, 3, 5\}$  and get  $1 - d_2 - d_5$ .
- d) Form no coalition and get 0.

Parties 4 and 5 are interchangeable and no minimal winning coalition includes  $\{3, 4, 5\}$ , so 3 will either include the cheapest of the two parties in the coalition or none of them. Denote this party by  $m$  (formally,  $m \in \arg \min_{i \in \{4, 5\}} d_i$ ). Suppose  $\theta_2 = \theta_m$  and forming a coalition is optimal. Then 3 will form  $\{1, 3\}$  if  $1 - d_1 > 1 - d_2 - d_m$ , and  $\{2, 3, m\}$  in the reverse case. Ties are solved in favor of a coalition with 2, and, if  $d_4 = d_5$ , of  $\{2, 3, 5\}$ .

**Stage 4.** Party 2 has two options: to form coalition  $\{1, 2\}$ , or to set  $\theta_2 = \theta_m$  and make a demand that will induce party 3 to form  $\{2, 3, m\}$ . The maximum demand 2 can make and still induce  $\{2, 3, m\}$  is  $d_2 = d_1 - d_m$ . If  $d_1 - d_m \geq 1 - d_1$ , party 2 sets  $\theta_2 = \theta_m$  and makes this demand; otherwise it

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<sup>6</sup>In fact, it may have more choices (e.g. forming coalition  $\{2, 3, 4, 5\}$ ) but all of them are dominated by at least one of these four. Without loss of generality we will not consider dominated choices. We will also exclude some situations that do not arise in equilibrium (e.g. demands so high that all coalitions are unfeasible).



sets  $\theta_2 = \theta_1$  and forms  $\{1, 2\}$ . Ties are solved in favor of inducing  $\{2, 3, m\}$ , and, if  $d_4 = d_5$ , of inducing  $\{2, 3, 5\}$ .

If party 5 wants to induce  $\{2, 3, 5\}$  it must set  $d_5 \leq d_4$ , so that  $m = 5$ . If  $d_1 - d_4 \geq 1 - d_1$  setting  $d_5 = d_4$  will do; otherwise  $d_5 = 2d_1 - 1$ . Thus, the maximum  $d_5$  that induces  $\{2, 3, 5\}$  is  $d_5 = d_4 \wedge (2d_1 - 1)$ . Note that, if  $d_1 < \frac{1}{2}$ , party 5 cannot induce  $\{2, 3, 5\}$ : 2 will form  $\{1, 2\}$  for any  $d_5 \geq 0$ .

**Stage 3.** Party 5 faces  $(d_1, d_4)$  and  $(\theta_1, \theta_4)$ . If  $\theta_1 = \theta_4$ , party 5 compares  $1 - d_1 - d_4$  and  $d_4 \wedge (2d_1 - 1)$ . Then party 5 forms  $\{1, 4, 5\}$  if  $1 - d_1 - d_4 \geq d_4 \wedge (2d_1 - 1)$ . If  $\theta_4 \neq \theta_1$ , the only possible coalition for party 5 is  $\{2, 3, 5\}$ , or no coalition if  $\{2, 3, 5\}$  cannot be induced by any  $d_5 \geq 0$ . In either case party 4 is excluded, so there is no reason for 4 to set  $\theta_4 \neq \theta_1$ .

The maximum value of  $d_4$  that still induces  $\{1, 4, 5\}$  depends on the size of  $d_1$ . For a relatively large  $d_1$  ( $d_1 \geq \frac{3}{5}$ ) the critical value is  $d_4 = \frac{1-d_1}{2}$ .

**Stage 2.** The only alternative for party 4 is to induce coalition  $\{1, 4, 5\}$ . If  $\frac{3}{5} < d_1 < 1$ , party 4's best response is to set  $\theta_4 = \theta_1$  and  $d_4 = \frac{1-d_1}{2}$ ; a higher demand would result in party 5 inducing coalition  $\{2, 3, 5\}$ . If  $d_1 = 1$  any demand is optimal, and ties are solved in favor of  $\theta_4 = \theta_1$  and  $d_4 = \frac{1-d_1}{2}$ .

**Stage 1.** Party 1 sets  $d_1 = 1$  together with an arbitrary  $\theta_1$ .

We now show the uniqueness of equilibrium payoffs. Essentially we will show that ties *must* be solved in favor of coalition  $\{1, 4, 5\}$  when  $d_1 = 1$  in order for an equilibrium to exist. It is enough to show that  $d_1 = 1 - \epsilon$  must lead to coalition  $\{1, 4, 5\}$  for any small  $\epsilon > 0$  in a subgame perfect equilibrium. Having established this, it follows that parties 4 and 5 must solve ties in favor of party 1 if  $d_1 = 1$ ; otherwise party 1 would have no best response.

Let  $d_1 = 1 - \epsilon$ . If party 4 sets  $d_4 < \frac{1-d_1}{2}$ , party 5's unique best response is to form coalition  $\{1, 4, 5\}$  regardless of the tie-breaking rules used by 2 and 3. It follows that 5 must form  $\{1, 4, 5\}$  for  $d_4 = \frac{1-d_1}{2}$  as well: otherwise 4 would not have a best response after observing  $d_1 = 1 - \epsilon$ .

Since  $\{1, 4, 5\}$  must form for  $d_1 = 1 - \epsilon$ , it must also form for  $d_1 = 1$ .

Notice that in this case any value of  $d_4$  is optimal regardless of 5's tie-breaking rule; however, 4 and 5 must solve ties in favor of 1.

**CASE 2: Party 2 (2 votes) is the first mover.** It can choose the order [23451], associated to votes [22113], and get all the surplus.

**Stage 5.** Party 1 (3 votes) faces a vector of demands  $(d_2, d_3, d_4, d_5)$  and a vector of policies  $(\theta_2, \theta_3, \theta_4, \theta_5)$ . It has four choices:

- a) Form coalition  $\{1, 2\}$  and get  $1 - d_2$ .
- b) Form coalition  $\{1, 3\}$  and get  $1 - d_3$ .
- c) If  $\theta_4 = \theta_5$ , it can also form  $\{1, 4, 5\}$  and get  $1 - d_4 - d_5$ .
- d) Form no coalition and get 0.

Suppose  $\theta_4 = \theta_5$  and forming some coalition is optimal. Then party 1 forms  $\{1, 4, 5\}$  if  $1 - d_4 - d_5 \geq 1 - (d_2 \wedge d_3)$ . Thus, the maximum demand party 5 can make and still induce coalition  $\{1, 4, 5\}$  is  $d_5 = (d_2 \wedge d_3) - d_4$ .

**Stage 4.** If  $\theta_2 = \theta_3$ , party 5 can form  $\{2, 3, 5\}$  and get  $1 - d_2 - d_3$ . Alternatively, it can induce  $\{1, 4, 5\}$  by setting  $\theta_5 = \theta_4$  and  $d_5 = (d_2 \wedge d_3) - d_4$ . It will do so if  $(d_2 \wedge d_3) - d_4 \geq 1 - d_2 - d_3$ , or  $d_4 \leq d_2 + d_3 + (d_2 \wedge d_3) - 1$ .

**Stage 3.** Party 4 can induce coalition  $\{1, 4, 5\}$  by setting  $d_4 = d_2 + d_3 + (d_2 \wedge d_3) - 1$ . If  $\theta_2 = \theta_3$ , it can also form coalition  $\{2, 3, 4\}$ . It forms  $\{2, 3, 4\}$  if  $1 - d_2 - d_3 \geq d_2 + d_3 + (d_2 \wedge d_3) - 1$ .

**Stage 2.** If  $d_2 \geq \frac{2}{3}$ , party 3 can induce coalition  $\{2, 3, 4\}$  by setting  $\theta_3 = \theta_2$  and  $d_3 = \frac{2-2d_2}{3}$ . A larger demand or/and setting  $\theta_3 \neq \theta_2$  would result in party 4 inducing  $\{1, 4, 5\}$ .

**Stage 1.** Party 2 sets  $d_2 = 1$  together with an arbitrary value of  $\theta_2$ .

To see that equilibrium payoffs are unique, let  $d_2 = 1 - \epsilon$ . If  $d_3 < \frac{2-2d_2}{3}$ , party 4 strictly prefers to form  $\{2, 3, 4\}$ . In order for 3 to have a best response, 4 must solve ties in favor of  $\{2, 3, 4\}$  for  $d_3 = \frac{2-2d_2}{3}$ . But then both 3 and 4 must solve ties in favor of party 2 for  $d_2 = 1$ .

**CASE 3: Party 4 (1 vote) is the first mover.** It can choose the order [42315], associated to votes [12231], and get all the surplus.

**Stage 5.** If  $\theta_1 = \theta_4$ , party 5 can form  $\{1, 4, 5\}$  and get  $1 - d_1 - d_4$ .

If  $\theta_2 = \theta_3$ , it can form  $\{2, 3, 5\}$  and get  $1 - d_2 - d_3$ . It can also form no coalition and get 0.

Suppose  $\theta_1 = \theta_4$  and  $\theta_2 = \theta_3$ . Party 5 forms  $\{1, 4, 5\}$  if  $1 - d_1 - d_4 \geq (1 - d_2 - d_3) \vee 0$ . If  $\theta_2 \neq \theta_3$ , the relevant condition is  $1 - d_1 - d_4 \geq 0$ . Thus the critical value of  $d_1$  is (weakly) higher if  $\theta_2 \neq \theta_3$ : because party 5 cannot form  $\{2, 3, 5\}$ , party 1 can get a better deal in coalition  $\{1, 4, 5\}$ .

**Stage 4.** Party 1 can form the cheapest of coalitions  $\{1, 2\}$  and  $\{1, 3\}$  and get  $1 - (d_2 \wedge d_3)$ , or induce  $\{1, 4, 5\}$  by setting  $\theta_1 = \theta_4$  and  $d_1 = 1 - d_4$  (if  $d_2 + d_3 > 1$  or  $\theta_2 \neq \theta_3$ ) or  $d_1 = d_2 + d_3 - d_4$  (if  $d_2 + d_3 \leq 1$  and  $\theta_2 = \theta_3$ ). Thus for  $\theta_2 = \theta_3$ , party 1 forms a two-party coalition if  $1 - (d_2 \wedge d_3) \geq (d_2 + d_3 - d_4) \wedge (1 - d_4)$ .

**Stage 3.** Party 3 can form  $\{2, 3, 4\}$  (provided  $\theta_2 = \theta_4$ ) or induce  $\{1, 3\}$ . In order to induce  $\{1, 3\}$ , party 3 must set  $d_3 \leq d_2$ , and can do no better than setting  $\theta_2 = \theta_3$ . What is the highest value of  $d_3$  that still induces coalition  $\{1, 3\}$ ? If  $d_4 > \frac{1}{2}$  there are two possible cases:

If  $d_2 \leq \frac{1}{2}$ ,  $d_3 = d_2$  will induce  $\{1, 3\}$ . If party 3 sets  $d_3 = d_2$ , we have  $d_3 + d_2 \leq 1$ , thus the relevant inequality for party 1 is  $1 - d_2 \geq 2d_2 - d_4$ , or  $d_2 \leq \frac{1+d_4}{3}$ . This is satisfied for  $d_4 > \frac{1}{2}$ . Since party 3 can induce  $\{1, 3\}$  by setting  $d_3 = d_2$ , party 3 will form  $\{2, 3, 4\}$  if  $1 - d_2 - d_4 \geq d_2$ , or  $d_2 \leq \frac{1-d_4}{2}$ .

If  $d_2 > \frac{1}{2}$ ,  $\{2, 3, 4\}$  leads to a negative payoff. Thus, party 3 always induces  $\{1, 3\}$ . This is achieved by setting  $d_3 = d_2 \wedge d_4$ .

**Stage 2.** If  $\frac{1}{2} < d_4 \leq 1$ , it is a best response for party 2 to set  $\theta_2 = \theta_4$  (setting  $\theta_2 \neq \theta_4$  would result in coalition  $\{1, 3\}$ ) and  $d_2 = \frac{1-d_4}{2}$ .

**Stage 1.** Party 4 sets  $d_4 = 1$  and an arbitrary  $\theta_4$ .

Uniqueness can be established analogously to the previous cases.

## 7 Appendix B: Proof of Theorem 1

The result trivially follows if there is a veto player. In constant-sum games, a veto player must be a dictator, thus  $\omega_1 = q$ , and  $d_1 = 1$  would be the equilibrium outcome. We will assume from now on that  $\omega_i < q$  for all  $i$ .

We denote as  $\mathbb{B}(d, i)$  with  $i \in N$  and  $d \in \mathbb{R}^{P_i}$  the subgame which begins when it is party  $i$ 's turn, facing a vector  $d$  of demands. At subgame  $\mathbb{B}(d, i)$ , party  $i$  will determine the optimal winning coalition  $S \ni i$  to be formed, and will formulate a demand  $d_i$  that will lead to  $S$  being formed. We will show how party  $i$  determines which coalition is optimal as well as how the optimal coalition can be induced by the choice of  $d_i$ .

Suppose we are in  $\mathbb{B}(d, i)$ , and party  $i$  plans to make a demand in the belief that a coalition  $S \in \Omega(\omega)$  with  $i \in S$  will be formed. This coalition should include some parties from  $N \setminus P_i$  (party  $i$  and possibly parties that move after it) and may also include some predecessors from  $P_i$ . Let  $\alpha$  be the number of votes controlled by parties in  $S \cap (N \setminus P_i)$ . Then, the parties in  $S \cap P_i$  should control at least  $q - \alpha$  votes. We denote as  $b(i, \alpha)$  the maximum benefit that can be achieved by buying these  $q - \alpha$  votes from parties in  $P_i$ .

$$b(i, \alpha) := \max \{1 - d_T : T \subset P_i, \omega_T \geq q - \alpha\}.$$

Party  $i$  can calculate  $b(i, \alpha)$  for every feasible value of  $\alpha$ . Notice that not all integers between 0 and  $q$  are feasible for every player. First,  $\alpha$  cannot be so small that even the votes of all the parties in  $P_i$  would not suffice. Let

$$\gamma_0^i := q - \omega_{P_i}.$$

In order for  $b(i, \alpha)$  to exist we need  $\alpha \geq \gamma_0^i$ .

Since party  $i$  must be in  $S$ , it seems reasonable to require  $\alpha \geq \omega_i$  as well. The next lemma shows that this is unnecessary: there is no positive benefit from buying more than  $q - \omega_i$  votes.

**Lemma 3** *Let  $\gamma_0^i \leq \alpha < \omega_i$  and assume no party  $j < i$  has made a strictly dominated choice of  $d_j$ . Then,  $b(i, \alpha) \leq 0$ . Moreover,  $b(i, \alpha) = 0$  implies  $b(i, \omega_i) \geq 0$ .*

**Proof.** Let  $T \subset P_i$  such that  $\omega_T \geq q - \alpha$ . Since  $\alpha < \omega_i$ , we have

$$\omega_{T \cup \{i\}} = \omega_T + \omega_i > \omega_T + \alpha \geq q.$$

Hence, since the game is homogeneous,  $T \cup \{i\}$  cannot be a minimal winning coalition. Moreover, party  $i$  is the party with less votes in  $T \cup \{i\}$ , thus coalition  $T$  should be winning. This means that either  $d_T \geq 1$  (implying  $b(i, \alpha) \leq 0$ ) or  $d_T < 1$ , in which case the smallest party in  $T$  (party  $j$ ) would have been strictly better-off by setting a higher demand and forming a coalition, regardless of the actions of the parties moving after  $j$ .

Moreover, when  $b(i, \alpha) = 0$ ,  $b(i, \omega_i) \geq 0$  follows from the fact that  $b(i, \cdot)$  is nondecreasing in the second variable. ■

We will eliminate strictly dominated strategies, thus in all the subgames we study it will be the case that  $b(i, \alpha) \leq 0$  for  $\gamma_0^i \leq \alpha < \omega_i$ . Otherwise the turn would never have reached party  $i$ .

Since there is no positive benefit from buying more than  $q - \omega_i$  votes, and (given that there is no benefit left to be divided) the particular value of  $\alpha$  is irrelevant if  $b(i, \alpha) = 0$ , any lower bound between 0 and  $\omega_i$  can be equivalently used by party  $i$ . We take  $\alpha$  to be greater or equal to:<sup>7</sup>

$$\gamma^i := \max \{1, \gamma_0^i\}.$$

Moreover, party  $i$  is constrained by the number of votes owned by parties in  $N \setminus P_i$ . Thus,  $\alpha$  must be smaller or equal to

$$\delta^i := \omega_{N \setminus P_i}.$$

Notice that  $\delta^{i+1} = \delta^i - \omega_i$  for all  $i < n$ . Also,  $\omega_i < q$  implies  $\gamma_0^i \leq \delta^{i+1}$ . It follows from lemma 4.9 in Ostmann (1987) that  $\omega_i \leq \delta^{i+1}$  for all  $i < n$ , thus  $\gamma^i \leq \delta^{i+1}$  for all  $i < n$ .

For party 1 only  $\alpha \geq q$  is feasible and  $b(1, \alpha) = 1$  for all  $\alpha \geq q$ . For party  $n$ , only  $\alpha = 1$  is feasible and  $b(n, 1)$  is simply  $n$ 's payoff from buying the votes of one of the cheapest coalitions controlling at least  $q - 1$  votes.

The following lemma shows how  $b(i + 1, \alpha)$  is determined from  $b(i, \cdot)$  and  $d_i$ . It may be the case that, having  $\alpha$  votes in its pocket, party  $i + 1$  cannot

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<sup>7</sup>A lower bound of 1 has the advantage of being independent of  $i$  and allowing division by all values of  $\alpha$ , but the proof can be adapted to any other choice.

form a winning coalition without party  $i$ . Then  $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$  irrespective of  $d_i$ . Otherwise party  $i+1$  will compare the best coalition that includes  $i$  with the best coalition that does not include  $i$ . Given that  $i$  is included in the coalition,  $i+1$  needs to buy the remaining votes ( $q - (\alpha + \omega_i)$ ) from  $P_i$ , and the best way to do this leads to a benefit of  $b(i, \alpha + \omega_i)$ ; after paying  $d_i$ , there would be  $b(i, \alpha + \omega_i) - d_i$  left. Without party  $i$ , the maximum benefit from buying  $q - \alpha$  votes without buying  $i$ 's votes is precisely  $b(i, \alpha)$ . Party  $i$  will then be included if  $d_i$  is sufficiently low.

Whether  $d_i$  is sufficiently low depends on the demands of the parties in  $P_i$ . Because parties may be complements, in some cases no positive demand by  $i$  would be low enough, as the following example illustrates.

Consider the game  $[10; 7, 3, 3, 3, 1, 1, 1]$ . Let  $i = 3$ ,  $i+1 = 4$ . We have  $b(3, 7) = \max(1 - d_1, 1 - d_2)$  and  $b(3, 4) = 1 - d_1$ . Having 7 votes in its pocket, party 3 may buy the votes of either party 1 (with a benefit of  $1 - d_1$ ) or party 2 (with a benefit of  $1 - d_2$ ). On the other hand, having only 4 votes, party 3 must buy the votes of party 1, with a benefit of  $1 - d_1$ .

If party 4 wants to compute  $b(4, 4)$  it compares  $1 - d_1$  and  $1 - d_2 - d_3$ . Thus in this particular case parties 2 and 3 are complements. If  $d_3$  is high, then  $b(4, 4) = 1 - d_1$ , which is precisely  $b(3, 4)$ . If *both*  $d_3$  and  $d_2$  are sufficiently low, then  $b(4, 4) = 1 - d_2 - d_3$  and  $b(3, 7) = 1 - d_2$ , hence  $b(4, 4) = b(3, 4 + \omega_3) - d_3$ . If  $d_2 > d_1$ , no positive value of  $d_3$  is sufficiently low.

**Lemma 4** *Assume we are in  $\mathbb{B}(d, i+1)$ . Let  $\alpha$  such that  $\gamma_0^{i+1} \leq \alpha \leq \delta^{i+1}$ . Then  $\gamma_0^i \leq \alpha + \omega_i \leq \delta^i$  and furthermore:*

- a) if  $\alpha < \gamma_0^i$ , then  $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$ ;
- b) if  $\alpha \geq \gamma_0^i$ , then  $b(i, \alpha)$  exists and

$$b(i+1, \alpha) = \max\{b(i, \alpha), b(i, \alpha + \omega_i) - d_i\}.$$

**Proof.** We have to prove that  $\gamma_0^i \leq \alpha + \omega_i \leq \delta^i$ . It is straightforward:

$$\alpha \leq \delta^{i+1} \implies \alpha + \omega_i \leq \delta^{i+1} + \omega_i = \delta^i.$$

$$\alpha \geq \gamma_0^{i+1} = q - \omega_{P_{i+1}} \implies \alpha + \omega_i \geq q - \omega_{P_{i+1}} + \omega_i = q - \omega_{P_i} = \gamma_0^i.$$

a) If  $\alpha < \gamma_0^i$ , every  $T \subset P_{i+1}$  with  $\omega_T \geq q - \alpha$  satisfies  $i \in T$ . Then:

$$\begin{aligned} b(i+1, \alpha) &= \max_{T \subset P_{i+1}: \omega_T \geq q - \alpha} (1 - d_T) = \max_{T \subset P_{i+1}: i \in T, \omega_T \geq q - \alpha} (1 - d_T) \\ &= \max_{T \subset P_i: \omega_T \geq q - \alpha - \omega_i} (1 - d_T) - d_i = b(i, \alpha + \omega_i) - d_i. \end{aligned}$$

b) If  $\alpha \geq \gamma_0^i$ ,  $b(i, \alpha)$  is well defined and

$$\begin{aligned} b(i+1, \alpha) &= \max_{T \subset P_{i+1}: \omega_T \geq q - \alpha} (1 - d_T) \\ &= \max \left\{ \max_{T \subset P_{i+1}: i \notin T, \omega_T \geq q - \alpha} (1 - d_T), \max_{T \subset P_{i+1}: i \in T, \omega_T \geq q - \alpha} (1 - d_T) \right\} \\ &= \max \left\{ \max_{T \subset P_i: \omega_T \geq q - \alpha} (1 - d_T), \max_{T \subset P_i: \omega_T \geq q - \alpha - \omega_i} (1 - d_T) - d_i \right\} \\ &= \max \{ b(i, \alpha), b(i, \alpha + \omega_i) - d_i \}. \end{aligned}$$

■

We have defined the best way to form a coalition that contains  $\alpha$  votes from  $N \setminus P_i$  and at least  $q - \alpha$  votes from  $P_i$ . It remains to choose the optimal value of  $\alpha$ , and the optimal demand  $d_i$ .

We denote as  $\Sigma^i$  the set of values between  $\gamma^i$  and  $\delta^i$  that maximize  $b(i, \alpha) / \alpha$ . Thus:

$$\Sigma^i := \arg \max_{\gamma^i \leq \alpha \leq \delta^i} \frac{b(i, \alpha)}{\alpha}$$

The next lemma shows that the only interesting bargaining occurs when  $b(i, \sigma^i) \geq 0$  for some/all<sup>8</sup>  $\sigma^i \in \Sigma^i$ .

**Lemma 5** *Assume we are in a subgame perfect equilibrium (SPE) of  $\mathbb{B}(d, i)$ . If  $b(i, \sigma^i) < 0$  for some/all  $\sigma^i \in \Sigma^i$ , then every party gets zero.*

**Proof.** Since  $b(i, \sigma^i) / \sigma^i$  is maximum, we deduce that  $b(i, \alpha) < 0$  for every  $\alpha \geq \gamma^i$ . The same occurs for  $\alpha = 0$  since  $b(i, \alpha)$  is nondecreasing in  $\alpha$ . This means that no winning coalition can be formed. ■

<sup>8</sup>Of course,  $b(i, \sigma) \geq 0$  for some  $\sigma \in \Sigma^i$  implies  $b(i, \sigma) \geq 0$  for all  $\sigma \in \Sigma^i$ .

Thus, if  $b(i, \sigma^i) < 0$  for some/all  $\sigma^i \in \Sigma^i$ , party  $i$  formulates an arbitrary demand and the game eventually ends with no coalition being formed.

From now on, we will assume that  $b(i, \sigma^i) \geq 0$  for all  $\sigma^i \in \Sigma^i$ . We will show that in equilibrium party  $i$  always chooses some  $\alpha \in \Sigma^i$ .

The following lemma shows that all values of  $\alpha$  between  $\delta^{i+1} + 1$  and  $\delta^{i+1} + \omega_i = \delta^i$  lead to the same  $b(i, \alpha)$ . The extra votes are not valuable because they are not enough to replace any party from  $P_i$ .

For example, in the game  $[10; 7, 3, 3, 3, 1, 1, 1]$ ,  $\delta^4 = 6$  and  $\delta^5 = 3$ . Consider the situation of party 4. If it takes  $\alpha = 4$ , there are two ways to form a winning coalition: buying the votes of party 1, or buying the votes of parties 2 and 3. Thus,  $b(4, 4) = \max(1 - d_1, 1 - d_2 - d_3)$ . If instead it takes  $\alpha = 5$  or  $\alpha = 6$ , exactly the same parties are needed: none of party 4's predecessors can be dispensed with despite the extra votes.

**Lemma 6** *Assume we are in the subgame  $\mathbb{B}(d, i)$ . Then*

$$\{T \subset P_i : \omega_T \geq q - (\delta^{i+1} + \alpha)\} = \{T \subset P_i : \omega_T \geq q - \delta^i\}$$

for all  $\alpha = 1, 2, \dots, \omega_i$ .

**Proof.** “ $\subset$ ” Let  $T \subset P_i$  such that  $\omega_T \geq q - (\omega_{N \setminus P_{i+1}} + \alpha)$ . Then

$$\omega_T \geq q - (\omega_{N \setminus P_i} - \omega_i + \alpha) = q - \omega_{N \setminus P_i} + (\omega_i - \alpha) \geq q - \omega_{N \setminus P_i}.$$

“ $\supset$ ” Let  $T \subset P_i$  such that  $\omega_T \geq q - \omega_{N \setminus P_i}$ . Then,  $T \cup (N \setminus P_i)$  is winning and contains party  $i$ . We study two cases:

- $T \cup (N \setminus P_i) \setminus \{i\} = T \cup (N \setminus P_{i+1})$  is also winning. Then,  $\omega_T \geq q - \omega_{N \setminus P_{i+1}}$  and the result is proved.
- $T \cup (N \setminus P_{i+1})$  is losing. Then, since the game is constant-sum, we conclude that its complement,  $(N \setminus T) \cap P_{i+1}$ , is winning and contains party  $i$  as the weakest member. By taking out party  $i$ , we obtain the coalition  $(N \setminus T) \cap P_i$  which is losing (since its complementary  $T \cup$



$(N \setminus P_i)$  is winning). Thus,  $(N \setminus T) \cap P_{i+1}$  is minimal winning and  $T \cup (N \setminus P_{i+1})$  is maximal losing. Hence, under Corollary 1:

$$\omega_T = q - \omega_{N \setminus P_{i+1}} - 1 \geq q - \omega_{N \setminus P_{i+1}} - \alpha.$$

■

**Corollary 2** *In  $\mathbb{B}(d, i)$ , we have  $b(i, \delta^{i+1} + \alpha) = b(i, \delta^i)$  for all  $\alpha = 1, 2, \dots, \omega_i$ . Moreover, for all  $\sigma^i \in \Sigma^i$ , if  $b(i, \sigma^i) > 0$ ,*

$$\sigma^i > \delta^{i+1} \implies \sigma^i = \delta^{i+1} + 1.$$

**Proof.** Under Lemma 6, it is clear that  $b(i, \delta^{i+1} + \alpha) = b(i, \delta^i)$  for all  $\alpha = 1, 2, \dots, \omega_i$ , since they minimize  $d_T$  on the same coalitions  $T$ . Hence, if  $b(i, \delta^i) > 0$ ,

$$\frac{b(i, \delta^{i+1} + \alpha)}{\delta^{i+1} + \alpha} < \frac{b(i, \delta^{i+1} + 1)}{\delta^{i+1} + 1}$$

for all  $\alpha = 2, \dots, \omega_i$  and thus the maximum is  $b(i, \delta^{i+1} + 1) / (\delta^{i+1} + 1)$ . Let  $\sigma^i \in \Sigma^i$  such that  $\sigma^i > \delta^{i+1}$ . Since  $\sigma^i = \delta^{i+1} + \alpha$  for some  $\alpha = 1, 2, \dots, \omega_i$ , we conclude the result. ■

Now we define the maximum demand party  $i$  can make at  $\mathbb{B}(d, i)$ . This depends on what party  $i + 1$  can achieve without party  $i$ . If party  $i + 1$  decides to exclude party  $i$ , it is in a similar situation to party  $i$  except that it has less feasible values for  $\alpha$ . It will be choosing an  $\alpha$  between  $\gamma^i$  and  $\delta^{i+1}$ , and the maximum benefit from buying  $q - \alpha$  votes without party  $i$  is precisely  $b(i, \alpha)$ . We define  $T^i$  as the set of values between  $\gamma^i$  and  $\delta^{i+1}$  that maximize  $b(i, \alpha) / \alpha$  (recall that  $\gamma^i \leq \delta^{i+1}$ , so the interval is nonempty).

$$T^i := \arg \max_{\gamma^i \leq \alpha \leq \delta^{i+1}} \frac{b(i, \alpha)}{\alpha}.$$

Let  $\tau^i \in T^i$ . Because  $\gamma^i \leq \tau^i$  and  $\tau^i \leq \delta^{i+1} < \delta^i$ ,  $\frac{b(i, \tau^i)}{\tau^i} \leq \frac{b(i, \sigma^i)}{\sigma^i}$  for all  $\sigma^i \in \Sigma^i$ .

For any values of  $\sigma^i \in \Sigma^i$  and  $\tau^i \in T^i$ , we define

$$d_i^* := \left\{ \begin{array}{ll} \frac{\omega_i b(i, \sigma^i)}{\sigma^i} & \text{if } \sigma^i \leq \delta^{i+1} \\ b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i) & \text{if } \sigma^i > \delta^{i+1} \text{ and } b(i, \tau^i) \geq 0 \\ b(i, \sigma^i) & \text{if } \sigma^i > \delta^{i+1} \text{ and } b(i, \tau^i) < 0. \end{array} \right\} \quad (1)$$

It is easy to prove that  $d_i^*$  is independent of the particular choice of  $\sigma^i$  and  $\tau^i$ . By definition,  $\frac{b(i, \tau^i)}{\tau^i}$  and  $\frac{b(i, \sigma^i)}{\sigma^i}$  are independent of the  $\tau^i$  and  $\sigma^i$  chosen. Also,  $b(i, \tau^i) \geq 0$  for some  $\tau^i \in T^i$  if and only if  $b(i, \tau^i) \geq 0$  for all  $\tau^i \in T^i$ . If  $b(i, \sigma^i) = 0$  for some  $\sigma^i$ , then  $b(i, \sigma^i) = 0$  for all  $\sigma^i$  and  $b(i, \tau^i) \leq 0$  for all  $\tau^i \in T^i$ . Thus,  $d_i^* = 0$  regardless of the choice of  $\sigma^i$  and  $\tau^i$ . If  $b(i, \sigma^i) > 0$ ,  $d_i^*$  is the same for all  $\sigma^i \leq \delta^{i+1}$ . If  $\sigma^i > \delta^{i+1}$ ,  $\sigma^i = \delta^{i+1} + 1$ . If  $\Sigma^i$  contains some  $\sigma^i \leq \delta^{i+1}$  as well as  $\sigma^i = \delta^{i+1} + 1$ ,  $d_i^*$  will still be independent of the choice of  $\sigma^i$  because in this case  $T^i = \Sigma^i \setminus \{\delta^{i+1} + 1\}$ , thus  $\frac{b(i, \tau^i)}{\tau^i} = \frac{b(i, \sigma^i)}{\sigma^i}$ .

In order to prove that  $d_i^*$  is the equilibrium demand of party  $i$ , the following lemmas will be useful. Notice that  $\sigma^i > \omega_i$  for some  $\sigma^i \in \Sigma^i$  implies  $i < n$ , because  $\omega_n = 1$  and  $\gamma^n = \delta^n = 1$ .

**Lemma 7** *Assume we are in  $\mathbb{B}(d, i)$ . If  $\sigma^i > \omega_i$  for some  $\sigma^i \in \Sigma^i$  and party  $i$  demands  $d_i \leq d_i^*$ , then*

$$\gamma^{i+1} \leq \sigma^i - \omega_i \leq \delta^{i+1} \quad (2)$$

$$b(i+1, \sigma^i - \omega_i) = b(i, \sigma^i) - d_i \quad (3)$$

and, given any  $\tau^i \in T^i$ ,

$$\frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i} \geq \left\{ \begin{array}{ll} \frac{b(i, \sigma^i)}{\sigma^i} & \text{if } \sigma^i \leq \delta^{i+1} \\ \frac{b(i, \tau^i)}{\tau^i} & \text{if } \sigma^i > \delta^{i+1} \text{ and } b(i, \tau^i) \geq 0 \\ 0 & \text{if } \sigma^i > \delta^{i+1} \text{ and } b(i, \tau^i) < 0. \end{array} \right\} \quad (4)$$

Furthermore, inequality in (4) is strict iff  $d_i < d_i^*$ .

**Proof.** Let  $\sigma^i \in \Sigma^i$  such that  $\sigma^i > \omega_i$ . We first prove (2):

$$\sigma^i \leq \delta^i \implies \sigma^i - \omega_i \leq \delta^i - \omega_i = \delta^{i+1}.$$

$$\sigma^i \geq \gamma^i \geq \gamma_0^i = q - \omega_{P_i} \implies \sigma^i - \omega_i \geq q - \omega_{P_i} - \omega_i = q - \omega_{P_{i+1}} = \gamma_0^{i+1}.$$

$$\sigma^i > \omega_i \implies \sigma^i - \omega_i > 0 \implies \sigma^i - \omega_i \geq 1.$$

We have just proven that  $\sigma^i - \omega_i$  is a feasible value of  $\alpha$  for party  $i + 1$ . Notice that for  $b(i, \sigma^i) > 0$ , homogeneity implies  $\sigma^i - \omega_i \geq \omega_{i+1}$ .

We prove now (3) and (4). Under Lemma 4a), (3) is true when  $\sigma^i - \omega_i < \gamma_0^i$ . Then (4) follows immediately by replacing  $\frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}$  by  $\frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i}$  and then using  $d_i \leq d_i^*$ .<sup>9</sup> Assume then  $\sigma^i - \omega_i \geq \gamma_0^i$ . We have two cases:

1. If  $\sigma^i \leq \delta^{i+1}$ , then  $d_i^* = \frac{\omega_i b(i, \sigma^i)}{\sigma^i}$ . Since  $\sigma^i \in \Sigma^i$ , re-arranging terms,

$$\begin{aligned} \frac{b(i, \sigma^i - \omega_i)}{\sigma^i - \omega_i} &\leq \frac{b(i, \sigma^i)}{\sigma^i} \implies b(i, \sigma^i - \omega_i) \leq b(i, \sigma^i) - \frac{\omega_i b(i, \sigma^i)}{\sigma^i} \\ &\implies b(i, \sigma^i - \omega_i) \leq b(i, \sigma^i) - d_i. \end{aligned}$$

Hence, (3) follows under lemma 4b). Moreover

$$\begin{aligned} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i} &= \frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i} \geq \frac{b(i, \sigma^i) - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}}{\sigma^i - \omega_i} \\ &= \frac{(\sigma^i - \omega_i) b(i, \sigma^i)}{\sigma^i (\sigma^i - \omega_i)} = \frac{b(i, \sigma^i)}{\sigma^i} \end{aligned}$$

with strict inequality iff  $d_i < d_i^*$ .

2. If  $\sigma^i > \delta^{i+1}$ , recall that  $\gamma^i \leq \sigma^i - \omega_i \leq \delta^{i+1}$ . Then for any  $\tau^i \in T^i$

$$\frac{b(i, \tau^i)}{\tau^i} \geq \frac{b(i, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.$$

Re-arranging terms,

$$\begin{aligned} \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i) &\geq b(i, \sigma^i - \omega_i) \\ &\implies b(i, \sigma^i - \omega_i) + b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i) \leq b(i, \sigma^i) \\ &\implies b(i, \sigma^i - \omega_i) + d_i \leq b(i, \sigma^i). \end{aligned}$$

Hence, (3) follows under lemma 4b).

To show (4), we distinguish two subcases:

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<sup>9</sup> Actually,  $\sigma^i - \omega_i < \gamma_0^i$  implies  $\sigma^i \leq \delta^{i+1}$ , so two of the three cases are void.

(a) If  $b(i, \tau^i) \geq 0$ , then  $d_i^* = b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)$

$$\begin{aligned} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i} &= \frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i} \\ &\geq \frac{b(i, \sigma^i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\sigma^i - \omega_i} \\ &= \frac{b(i, \tau^i)}{\tau^i} \end{aligned}$$

with strict inequality iff  $d_i < d_i^*$ .

(b) If  $b(i, \tau^i) < 0$ , then  $d_i^* = b(i, \sigma^i)$  and thus

$$\frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i} = \frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i} \geq 0$$

with strict inequality iff  $d_i < d_i^*$ .

■

**Lemma 8** Assume we are in  $\mathbb{B}(d, i+1)$  and  $\sigma^i > \omega_i$  for some  $\sigma^i \in \Sigma^i$ .

a) If  $d_i < d_i^*$ , then  $i \in S$  for all  $S \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$  and all  $\sigma^{i+1} \in \Sigma^{i+1}$ .

b) If  $d_i = d_i^*$ , then  $\sigma^i - \omega_i \in \Sigma^{i+1}$ . Moreover,  $S \in \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$  implies  $S \cup \{i\} \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - (\sigma^i - \omega_i)} (1 - d_T)$ .

c) If  $d_i = d_i^*$ , given  $\sigma^{i+1} \in \Sigma^{i+1}$  and  $S \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$ ,  $i \in S$  implies  $S \cap P_i \in \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$  for some  $\sigma^i \in \Sigma^i$ .

**Proof.** a) Let  $\sigma^{i+1} \in \Sigma^{i+1}$ . Suppose there exists  $S \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$

such that  $i \notin S$ . Then,  $b(i+1, \sigma^{i+1}) = b(i, \sigma^{i+1})$ . We see three cases:

1. If  $\sigma^i \leq \delta^{i+1}$ ,

$$\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^{i+1})}{\sigma^{i+1}} \leq \frac{b(i, \sigma^i)}{\sigma^i} \stackrel{\text{(Lemma 7)}}{<} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}$$

which contradicts that  $\sigma^{i+1} \in \Sigma^{i+1}$ .

2. If  $\sigma^i > \delta^{i+1}$ ,

$$\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^{i+1})}{\sigma^{i+1}} \leq \frac{b(i, \tau^i)}{\tau^i} \stackrel{\text{(Lemma 7)}}{<} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}$$

which contradicts that  $\sigma^{i+1} \in \Sigma^{i+1}$ .

b) Let  $\alpha$  such that  $\gamma^{i+1} \leq \alpha \leq \delta^{i+1}$ . Under Lemma 4, either  $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$  or  $b(i+1, \alpha) = b(i, \alpha)$ . We have to prove that

$$\frac{b(i+1, \alpha)}{\alpha} \leq \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.$$

If  $b(i+1, \alpha) = b(i, \alpha)$ , we proceed like in case a).

If  $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$ , we have three cases:

1. If  $\sigma^i \leq \delta^{i+1}$ , then

$$\begin{aligned} \frac{b(i+1, \alpha)}{\alpha} &= \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} = \frac{b(i, \alpha + \omega_i) - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}}{\alpha} \\ &\leq \frac{\frac{b(i, \sigma^i)}{\sigma^i}(\alpha + \omega_i) - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}}{\alpha} \\ &= \frac{b(i, \sigma^i)}{\sigma^i} \stackrel{\text{(Lemma 7)}}{\leq} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}. \end{aligned}$$

2. If  $\sigma^i > \delta^{i+1}$  and  $b(i, \tau^i) \geq 0$  for some/all  $\tau^i \in T^i$ , then either  $\frac{b(i, \alpha + \omega_i)}{\alpha + \omega_i} \leq \frac{b(i, \tau^i)}{\tau^i}$  (if  $\alpha + \omega_i \leq \delta^{i+1}$ ) or  $b(i, \alpha + \omega_i) = b(i, \sigma^i)$  (if  $\alpha + \omega_i > \delta^{i+1}$ , by Corollary 2).

If  $\alpha + \omega_i \leq \delta^{i+1}$ ,

$$\begin{aligned} \frac{b(i+1, \alpha)}{\alpha} &= \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} = \frac{b(i, \alpha + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\alpha} \\ &\leq \frac{\frac{\alpha + \omega_i}{\tau^i} b(i, \tau^i) - \frac{\sigma^i}{\tau^i} b(i, \tau^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\alpha} \\ &= \frac{b(i, \tau^i)}{\tau^i} \stackrel{\text{(Lemma 7)}}{\leq} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}. \end{aligned}$$

If  $\alpha + \omega_i > \delta^{i+1}$ ,

$$\begin{aligned} \frac{b(i+1, \alpha)}{\alpha} &= \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} = \frac{b(i, \alpha + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\alpha} \\ &= \frac{\sigma^i - \omega_i}{\alpha} \frac{b(i, \tau^i)}{\tau^i}. \end{aligned}$$

If  $b(i, \sigma^i) > 0$ , corollary 2 implies  $\sigma^i = \delta^{i+1} + 1$ . Then  $\alpha - \omega_i > \delta^{i+1}$  implies  $\alpha + \omega_i \geq \delta^{i+1} + 1 = \sigma^i$ , or  $(\sigma^i - \omega_i) / \alpha \leq 1$ . If  $b(i, \sigma^i) = 0$ ,  $b(i, \tau^i) = 0$ , implying  $\frac{\sigma^i - \omega_i}{\alpha} \frac{b(i, \tau^i)}{\tau^i} = \frac{b(i, \tau^i)}{\tau^i}$ . In either case,

$$\frac{b(i+1, \alpha)}{\alpha} \leq \frac{b(i, \tau^i)}{\tau^i} \stackrel{\text{(Lemma 7)}}{\leq} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.$$

3. If  $\sigma^i > \delta^{i+1}$  and  $b(i, \tau^i) < 0$  for some/all  $\tau^i \in T^i$ , then either  $b(i, \alpha + \omega_i) < 0$  (if  $\alpha + \omega_i \leq \delta^{i+1}$ ) or  $b(i, \alpha + \omega_i) = b(i, \sigma^i)$  (if  $\alpha + \omega_i > \delta^{i+1}$ , by Corollary 2).

If  $b(i, \alpha + \omega_i) < 0$ ,

$$\frac{b(i+1, \alpha)}{\alpha} = \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} < -\frac{d_i}{\alpha} \leq 0 \leq \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.$$

If  $b(i, \alpha + \omega_i) = b(i, \sigma^i)$ ,

$$\frac{b(i+1, \alpha)}{\alpha} = \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} = 0 \leq \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.$$

We now prove the second statement. Let  $S \in \arg \max_{T \subset P_i; \omega_T \geq q - \sigma^i} (1 - d_T)$ . We have to prove  $b(i+1, \sigma^i - \omega_i) = 1 - d_{S \cup \{i\}}$ . Using (3),

$$b(i+1, \sigma^i - \omega_i) = b(i, \sigma^i) - d_i = 1 - d_S - d_i = 1 - d_{S \cup \{i\}}.$$

- c) Since  $i \in S$ ,  $b(i+1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i$ , or

$$b(i, \sigma^{i+1} + \omega_i) = b(i+1, \sigma^{i+1}) + d_i. \quad (5)$$

Let  $\sigma^i > \omega_i$ . We have shown that  $\sigma^i - \omega_i \in \Sigma^{i+1}$ , thus

$$\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}. \quad (6)$$

1. If  $\sigma^i \leq \delta^{i+1}$  for some  $\sigma^i \in \Sigma^i$ , it follows from (6) and (3) that

$$\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^i)}{\sigma^i}. \quad (7)$$

Then

$$\begin{aligned} \frac{b(i, \sigma^{i+1} + \omega_i)}{\sigma^{i+1} + \omega_i} &\stackrel{(5)}{=} \frac{b(i+1, \sigma^{i+1}) + d_i}{\sigma^{i+1} + \omega_i} = \frac{b(i+1, \sigma^{i+1}) + \frac{\omega_i}{\sigma^i} b(i, \sigma^i)}{\sigma^{i+1} + \omega_i} \\ &\stackrel{(7)}{=} \frac{\frac{\sigma^{i+1}}{\sigma^i} b(i, \sigma^i) + \frac{\omega_i}{\sigma^i} b(i, \sigma^i)}{\sigma^{i+1} + \omega_i} = \frac{b(i, \sigma^i)}{\sigma^i}. \end{aligned}$$

Hence  $\sigma^{i+1} + \omega_i \in \Sigma^i$  and  $b(i, \sigma^{i+1} + \omega_i) = b(i+1, \sigma^{i+1}) + d_i = 1 - d_{S \cap P_i}$ .

2. If  $\sigma^i > \delta^{i+1}$  for all  $\sigma^i \in \Sigma^i$ ,  $\delta^{i+1} + 1$  always belongs to  $\Sigma^i$ .

Suppose  $S \cap P_i \notin \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$  for all  $\sigma^i \in \Sigma^i$ . Then it must be the case that for any  $\sigma^i$  either  $\omega_{S \cap P_i} < q - \sigma^i$ , or  $\omega_{S \cap P_i} \geq q - \sigma^i$  but  $1 - d_{S \cap P_i}$  is not maximal.

Suppose  $\omega_{S \cap P_i} < q - \sigma^i$  for all  $\sigma^i \in \Sigma^i$ . Since  $\delta^{i+1} + 1 \in \Sigma^i$ , it follows from Lemma 6 that  $\omega_{S \cap P_i} < q - \delta^i$ . But then  $\omega_{S \cap P_i} + \omega_{N \setminus P_i} = \omega_{S \cap P_{i+1}} + \omega_{N \setminus P_{i+1}} < q$ , contradicting the assumption that  $\omega_{S \cap P_{i+1}} \geq q - \sigma^{i+1}$ .

Suppose  $\omega_{S \cap P_i} \geq q - \sigma^i$  but  $1 - d_{S \cap P_i} < 1 - d_T$  for some  $\sigma^i \in \Sigma^i$  and  $T \subset P_i$  with  $\omega_T \geq q - \sigma^i$ .

If  $\sigma^{i+1} + \omega_i > \delta^{i+1}$ ,

$$b(i, \sigma^{i+1} + \omega_i) \stackrel{(6)}{=} b(i, \sigma^i) > 1 - d_{S \cap P_i} = b(i+1, \sigma^{i+1}) + d_i$$

contradicting (5).

If  $\sigma^{i+1} + \omega_i \leq \delta^{i+1}$ ,  $\frac{b(i, \sigma^{i+1} + \omega_i)}{\sigma^{i+1} + \omega_i} \leq \frac{b(i, \tau^i)}{\tau^i}$ . There are two possibilities.

- If  $b(i, \tau^i) \geq 0$ , it follows from (6) and (3) that  $\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \tau^i)}{\tau^i}$ .

Then

$$\begin{aligned} \frac{b(i, \tau^i)}{\tau^i}(\sigma^{i+1} + \omega_i) &\geq b(i, \sigma^{i+1} + \omega_i) \stackrel{(5)}{=} b(i+1, \sigma^{i+1}) + d_i = \\ &= \frac{b(i, \tau^i)}{\tau^i} \sigma^{i+1} + b(i, \sigma^i) - (\sigma^i - \omega_i) \frac{b(i, \tau^i)}{\tau^i} \end{aligned}$$

implying  $\frac{b(i, \tau^i)}{\tau^i} \geq \frac{b(i, \sigma^i)}{\sigma^i}$ , thus  $\frac{b(i, \tau^i)}{\tau^i} = \frac{b(i, \sigma^i)}{\sigma^i}$ . Then  $\frac{b(i, \sigma^{i+1} + \omega_i)}{\sigma^{i+1} + \omega_i} = \frac{b(i, \sigma^i)}{\sigma^i}$ . Hence  $\sigma^{i+1} + \omega_i \in \Sigma^i$  and the result follows.

- If  $b(i, \tau^i) < 0$ , it follows from (6) and (3) that  $\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = 0$ .

Then

$$b(i, \sigma^{i+1} + \omega_i) = b(i+1, \sigma^{i+1}) + d_i = b(i, \sigma^i).$$

Hence  $b(i, \sigma^i) = 1 - d_{S \cap P_i}$  and the result follows.

■

**Lemma 9** Assume we are in  $\mathbb{B}(d, i+1)$  and  $d_i > d_i^*$ .

a) If  $b(i, \tau^i) \geq 0$  for some/all  $\tau^i \in T^i$ , then

$i \notin S$  for all  $S \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$  and all  $\sigma^{i+1} \in \Sigma^{i+1}$ .

b) If  $b(i, \tau^i) < 0$  for some/all  $\tau^i \in T^i$ , then every party obtains zero.

**Proof.** a) Let  $\sigma^{i+1} \in \Sigma^{i+1}$  and  $\tau^i \in T^i$ . We need to prove that  $b(i, \sigma^{i+1})$  exists and  $b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i$ . This will be due to party  $i+1$  having the option of setting  $\alpha = \sigma^i$  (if  $\sigma^i \leq \delta^{i+1}$ ) or  $\alpha = \tau^i$  (if  $\sigma^i > \delta^{i+1}$ ).

We examine each case in turn:

1. If  $\sigma^i \leq \delta^{i+1}$ , then  $d_i > \frac{\omega_i b(i, \sigma^i)}{\sigma^i}$ .

Since  $\sigma^i \leq \delta^{i+1}$ ,  $b(i+1, \sigma^i)$  exists. Moreover, lemma 4b) implies

$$b(i+1, \sigma^i) \geq b(i, \sigma^i). \quad (8)$$

In principle, there are three possibilities for  $\sigma^{i+1}$ : either  $\sigma^{i+1} < \gamma_0^i$ , or  $\sigma^{i+1} \geq \gamma_0^i$  and  $b(i, \sigma^{i+1}) \leq b(i, \sigma^{i+1} + \omega_i) - d_i$ , or  $\sigma^{i+1} \geq \gamma_0^i$  and



$b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i$ . We will show that the first two possibilities lead to a contradiction. In both cases, Lemma 4 implies

$$b(i+1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i. \quad (9)$$

From (9) we can deduce:

$$\begin{aligned} \frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} &= \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} \\ &< \frac{b(i, \sigma^{i+1} + \omega_i) - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}}{\sigma^{i+1}} \\ &\leq \frac{\frac{(\sigma^{i+1} + \omega_i) b(i, \sigma^i)}{\sigma^i} - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}}{\sigma^{i+1}} \\ &= \frac{b(i, \sigma^i)}{\sigma^i} \stackrel{(8)}{\leq} \frac{b(i+1, \sigma^i)}{\sigma^i}. \end{aligned}$$

which contradicts that  $\sigma^{i+1} \in \Sigma^{i+1}$ . Thus,  $\sigma^{i+1} \geq \gamma_0^i$  (i.e.  $b(i, \sigma^{i+1})$  does exist) and  $b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i$ . We conclude then that  $i \notin S$  for all  $S \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$ .

2. If  $\sigma^i > \delta^{i+1}$ , then  $d_i > b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)$ .

Under Lemma 4b):

$$b(i+1, \tau^i) = \max \{b(i, \tau^i), b(i, \tau^i + \omega_i) - d_i\} \geq b(i, \tau^i). \quad (10a)$$

Suppose  $b(i, \sigma^{i+1})$  does not exist (i.e.  $\sigma^{i+1} < \gamma_0^i$ ), or  $b(i, \sigma^{i+1})$  exists and  $b(i, \sigma^{i+1}) \leq b(i, \sigma^{i+1} + \omega_i) - d_i$ . In both cases, under Lemma 4,

$$b(i+1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i. \quad (11)$$

We will prove that (11) leads to a contradiction, so that  $b(i, \sigma^{i+1})$  exists and  $b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i$ , which implies  $i \notin S$  for all  $S \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$  as desired.

We have two cases:

- If  $\sigma^{i+1} + \omega_i \leq \delta^{i+1}$ . Then  $\frac{b(i, \sigma^{i+1} + \omega_i)}{\sigma^{i+1} + \omega_i} \leq \frac{b(i, \tau^i)}{\tau^i}$  and

$$\begin{aligned}
\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} &= \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} \\
&< \frac{b(i, \sigma^{i+1} + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\sigma^{i+1}} \\
&\leq \frac{\frac{\sigma^{i+1} + \omega_i}{\tau^i} b(i, \tau^i) - \frac{\sigma^i}{\tau^i} b(i, \tau^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\sigma^{i+1}} \\
&= \frac{b(i, \tau^i)}{\tau^i} \stackrel{(10a)}{\leq} \frac{b(i+1, \tau^i)}{\tau^i}
\end{aligned}$$

which is a contradiction.

- If  $\sigma^{i+1} + \omega_i > \delta^{i+1}$ , then under Corollary 2,  $b(i, \sigma^{i+1} + \omega_i) = b(i, \sigma^i)$ . If  $b(i, \sigma^i) > 0$ ,  $\sigma^i = \delta^{i+1} + 1$  and  $\sigma^{i+1} + \omega_i \geq \sigma^i$ , which implies  $(\sigma^i - \omega_i) / \sigma^{i+1} \leq 1$ . If  $b(i, \sigma^i) = 0$ ,  $b(i, \tau^i) = 0$ . Hence:

$$\begin{aligned}
\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} &= \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} \\
&< \frac{b(i, \sigma^{i+1} + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\sigma^{i+1}} \\
&= \frac{\sigma^i - \omega_i}{\sigma^{i+1}} \frac{b(i, \tau^i)}{\tau^i} \stackrel{(10a)}{\leq} \frac{b(i, \tau^i)}{\tau^i} \leq \frac{b(i+1, \tau^i)}{\tau^i}
\end{aligned}$$

which is a contradiction.

b) Recall that we assumed  $b(i, \sigma^i) \geq 0$  for all  $\sigma^i \in \Sigma^i$ . Thus,  $b(i, \tau^i) < 0$  for some  $\tau^i \in T^i$  implies  $\sigma^i > \delta^{i+1}$ . Under Corollary 2, this means  $b(i, \sigma^i) = b(i, \delta^i)$ . Let  $\alpha$  be such that  $\gamma^{i+1} \leq \alpha \leq \delta^{i+1}$ . Under Lemma 4, we have two cases:

1.  $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$ . Then

$$b(i+1, \alpha) < b(i, \alpha + \omega_i) - b(i, \delta^i).$$

Since  $\alpha + \omega_i \leq \delta^i$ ,  $b(i, \alpha + \omega_i) \leq b(i, \delta^i)$  and thus  $b(i+1, \alpha) < 0$ .

2.  $b(i+1, \alpha) = b(i, \alpha)$ . Then  $\gamma_0^i \leq \alpha \leq \delta^{i+1}$  and

$$\frac{b(i+1, \alpha)}{\alpha} \leq \frac{b(i, \tau^i)}{\tau^i} < 0$$

and thus  $b(i+1, \alpha) < 0$ .

Since  $b(i+1, \alpha) < 0$  for all  $\alpha$ , we conclude  $b(i+1, \sigma^{i+1}) < 0$  for all  $\sigma^{i+1} \in \Sigma^{i+1}$  and thus by Lemma 5 all the parties get zero. ■

Let us consider the following strategy profile for the parties. In  $\mathbb{B}(d, n)$ , party  $n$  forms a coalition  $S \cup \{n\}$  with  $S \in \arg \max_{T \subset P_n: \omega_T \geq q - \omega_n} (1 - d_T)$  after demanding  $d_n = 1 - d_S$ . If there is more than one possible choice of  $S$ , party  $n$  uses the following **tie-breaking rule**: First, select only the coalitions that contain the party with the highest index (party  $n-1$ , or, if party  $n-1$  is in none of the coalitions, party  $n-2$  etc.). If there are several coalitions containing this party, select the ones that contain the party with the second highest index, etc., until only one coalition is left.

Let  $i < n$  and assume we have defined the strategies for parties in  $\mathbb{B}(d, i+1)$ . In  $\mathbb{B}(d, i)$ , party  $i$  proceeds as follows:

1. If  $\sigma^i > \omega_i$  for all  $\sigma^i \in \Sigma^i$ , party  $i$  demands  $d_i = d_i^*$  given as in (1).
2. If  $\Sigma^i = \{\omega_i\}$ , party  $i$  forms coalition  $S \cup \{i\}$  with  $S \in \arg \min_{T \subset P_i: \omega_T \geq q - \omega_i} d_T$ . If there is more than one possible choice of  $S$ , party  $i$  uses the tie-breaking rule: Among all the optimal coalitions  $S \in \arg \min_{T \subset P_i: \omega_T \geq q - \omega_i} d_T$ , party  $i$  selects the ones that contain the party with the highest index ( $i-1$ , or, if party  $i-1$  is in none of the coalitions, party  $i-2$ , etc). If there are several coalitions containing this party, select the ones that contain the party with the second highest index, etc., until only one coalition is left.
3. If  $\{\omega_i\} \subsetneq \Sigma^i$ , party  $i$  can anticipate the coalition  $S^*$  that will be formed should it demand  $d_i^*$  and its followers play the strategies we have defined.

- (a) If  $i \notin S^*$ , party  $i$  forms coalition  $S \cup \{i\}$  with  $S \in \arg \min_{T \subset P_i: \omega_T \geq q - \omega_i} d_T$ .  
If there is more than one possible  $S$ , party  $i$  uses the tie-breaking rule.
- (b) If  $i \in S^*$ , party  $i$  compares the coalitions  $S \in \arg \max_{T \subset P_i: \omega_T \geq q - \omega_i} (1 - d_T)$  and  $S^* \cap P_i$ . Among them, party  $i$  selects a coalition following the tie-breaking rule. If  $S^*$  is chosen, party  $i$  demands  $d_i = d_i^*$  given as in (1). If  $S \neq S^*$  is chosen, then party  $i$  demands  $1 - d_S = b(i, \omega_i) = d_i^*$  and forms coalition  $S \cup \{i\}$ .

The role of the tie-breaking rule is to ensure that parties have a best response at all stages (cf. Example 1).

**Proposition 2** *The above strategies constitute a SPE for any  $\mathbb{B}(d, i)$ .*

**Proof.** We proceed by backwards induction on  $i$ . For  $i = n$ , its strategy is clearly optimal.

Assume now the result is true for  $\mathbb{B}(d, i + 1)$  and moreover assume the following two conditions hold:

**Condition 1** The formed coalition satisfies

$$S \cap P_{i+1} \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$$

for some  $\sigma^{i+1} \in \Sigma^{i+1}$ . (This condition holds trivially for  $i + 1 = n$  because  $\Sigma^n = \{\omega_n\}$ ).

**Condition 2** The above  $S$  and  $\sigma^{i+1}$  are such that  $S \cap P_{i+1}$  is one of the most favorable sets for party  $i$  (*i.e.*  $i \notin S$  implies  $i \notin T$  for all  $T \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$  and all  $\sigma^{i+1} \in \Sigma^{i+1}$ ). Among them, it is one of the most favorable to party  $i - 1$ , etc. (This condition holds for  $i + 1 = n$  because  $\Sigma^n = \{\omega_n\}$  and  $n$  applies the tie-breaking rule).

We check that this remains true for  $\mathbb{B}(d, i)$ . Let  $\tau^i \in T^i$ . We have two cases:

1. If  $\sigma^i > \omega_i$  for all  $\sigma^i \in \Sigma^i$ , then it is straightforward to check that party  $i$  obtains strictly less than  $d_i^*$  by forming coalition. If  $i$  demands  $d_i^*$ ,  $S \cup \{i\} \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$  for  $\sigma^{i+1} = \sigma^i - \omega_i \in \Sigma^{i+1}$ .

The induction hypothesis (Conditions 1 and 2) implies that  $d_i^*$  will be accepted. Assume party  $i$  deviates by demanding  $d_i > d_i^*$ . If  $b(i, \tau^i) \geq 0$ , under Lemma 9a) party  $i$  does not belong to any coalition in  $\arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$  for any  $\sigma^{i+1} \in \Sigma^{i+1}$  and its final payoff is zero under the induction hypothesis (Condition 1). If  $b(i, \tau^i) < 0$ , under Lemma 9b), its final payoff is zero.

Moreover, Conditions 1 and 2 hold for  $i$ . Condition 1 follows from Lemma 8b) and the induction hypothesis applied to Conditions 1 and 2. Condition 2 follows from the tie-breaking rule applied by the party  $j > i$  that eventually forms coalition.

2. If  $\omega_i \in \Sigma^i$ , then  $1 - d_S = b(i, \omega_i) = d_i^*$  for all  $S \in \arg \max_{T \subset P_i: \omega_T \geq q - \omega_i} (1 - d_T)$ .

This means that if party  $i$  forms a winning coalition it obtains a final payoff of  $b(i, \omega_i)$ . Suppose party  $i$  deviates and demands  $d_i > b(i, \omega_i)$ . It is enough to check that  $i \notin S$  for all  $S \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$  and all  $\sigma^{i+1} \in \Sigma^{i+1}$ . Under the induction hypothesis applied to Condition 1, this means that party  $i$  will not be included in any eventual winning coalition, and its final payoff will be zero, while the original strategy yields a nonnegative payoff.

For constant-sum homogeneous games it is always the case that  $\omega_i \leq \delta^{i+1}$ , thus  $b(i+1, \omega_i)$  is well defined. Under Lemma 4b),

$$b(i+1, \omega_i) = \max \{b(i, \omega_i), b(i, 2\omega_i) - d_i\} \geq b(i, \omega_i) \quad (12)$$

Suppose that  $i \in S$  for some  $S \in \arg \min_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} d_T$  and some  $\sigma^{i+1} \in \Sigma^{i+1}$ . This means

$$b(i+1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i$$

and hence

$$\begin{aligned}
\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} &= \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} \\
&< \frac{b(i, \sigma^{i+1} + \omega_i) - b(i, \omega_i)}{\sigma^{i+1}} \\
&\leq \frac{\frac{\sigma^{i+1} + \omega_i}{\omega_i} b(i, \omega_i) - b(i, \omega_i)}{\sigma^{i+1}} \\
&= \frac{(\sigma^{i+1} + \omega_i) b(i, \omega_i) - \omega_i b(i, \omega_i)}{\omega_i \sigma^{i+1}} \\
&= \frac{b(i, \omega_i)}{\omega_i} \stackrel{(12)}{\leq} \frac{b(i+1, \omega_i)}{\omega_i}
\end{aligned}$$

which is a contradiction. This contradiction proves that  $i \notin S$  for all  $S \in \arg \min_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} d_T$ , as desired.

We now check that Conditions 1 and 2 hold for  $i$ . If party  $i$  forms coalition, Condition 1 holds with  $\sigma^i = \omega_i$ , and Condition 2 holds because of the tie-breaking rule. If party  $i$  demands  $d_i^*$  so that  $S^*$  is induced, it must be the case that  $\{\omega_i\} \not\subseteq \Sigma^i$ . Hence, there exists  $\sigma^i \in \Sigma^i$  with  $\sigma^i > \omega_i$ . Then, Condition 1 follows from Lemma 8b) and the induction hypothesis applied to Conditions 1 and 2. Condition 2 follows from the tie-breaking rule applied by the party that eventually forms coalition.

■

The next proposition shows uniqueness of equilibrium *payoffs*. Equilibrium *strategies* are not unique for some subgames. In subgames  $\mathbb{B}(d, i)$  where no coalition can be formed (i.e.,  $b(i, \sigma^i) < 0$ ), any demand vector is part of a SPE and equilibrium payoffs are always 0 for all parties. Multiplicity may also arise in subgames where a coalition can be formed but  $d_i^* = 0$ , as the following example illustrates.

**Example 5** Consider the game  $[5; 3, 2, 2, 1, 1]$  and suppose  $d_1 = d_2 = 1$ . Equilibrium strategies at  $\mathbb{B}(d, 3)$  are not unique, but equilibrium payoffs are.

At  $\mathbb{B}(d, 3)$  we have  $d_3^* = 0$  and  $\Sigma^3 = \{2, 3\}$ . If we look at this subgame in isolation, several equilibrium outcomes are possible: coalition  $\{1, 3\}$  (associated to  $\sigma^3 = 2$ ), coalition  $\{2, 3, 4\}$  or  $\{2, 3, 5\}$  (associated to  $\sigma^3 = 3$ ), coalition  $\{2, 3, 4, 5\}$  (which is not a minimal winning coalition), coalition  $\{1, 4, 5\}$  (which does not include party 3), or even no winning coalition at all. Intuitively, since the parties in  $\{3, 4, 5\}$  cannot get a positive payoff, they are indifferent between all these situations. However, parties that have moved before are *not* indifferent. If we take into account that the strategies must be part of an equilibrium for all the subgames, and in particular for subgame  $\mathbb{B}(d, 2)$ , some of the equilibrium strategies at  $\mathbb{B}(d, 3)$  are not equilibrium strategies for  $\mathbb{B}(d, 2)$  and are discarded (cf. example 1). In particular, a coalition containing party 2 must be formed in order for party 2 to have a best response at  $\mathbb{B}(d, 2)$ . Nevertheless, multiplicity remains: after party 2 sets  $d_2 = 1$ , there are three possible equilibrium coalitions:  $\{2, 3, 4\}$ ,  $\{2, 3, 5\}$  and  $\{2, 3, 4, 5\}$ . Nevertheless, all equilibrium strategies lead to the same payoffs.

**Proposition 3** *Assume we are in a SPE in  $\mathbb{B}(d, i)$ . If  $b(i, \sigma^i) \geq 0$  for some/all  $\sigma^i \in \Sigma^i$ , party  $i$ 's payoff is  $d_i^*$  as defined in (1); otherwise party  $i$ 's payoff is zero.*

**Proof.** We proceed by backwards induction on  $i$ . We prove the following three hypotheses:

1. If  $b(i, \sigma^i) < 0$ , all parties get zero in every SPE of  $\mathbb{B}(d, i)$ .
2. If  $b(i, \sigma^i) > 0$ , party  $i$  receives  $d_i^* > 0$  in every SPE of  $\mathbb{B}(d, i)$  and the coalition that forms satisfies  $S \cap P_i \in \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$  for some  $\sigma^i \in \Sigma^i$ .
3. If  $b(i, \sigma^i) = 0$ ,
  - a) party  $i$  gets  $d_i^* = 0$  in every SPE of  $\mathbb{B}(d, i)$ ;

- b) there is a SPE of  $\mathbb{B}(d, i)$  in which a winning coalition forms;
- c) if a winning coalition  $S$  forms, then  $S \cap P_i \in \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$   
for some  $\sigma^i \in \Sigma^i$ .

The induction hypothesis holds for party  $n$ . Now suppose it holds for party  $i + 1$ . Does it hold for party  $i$ ?

1. If  $b(i, \sigma^i) < 0$ , all parties get zero (Lemma 5).
2. If  $b(i, \sigma^i) > 0$ , party  $i$  cannot get more than  $d_i^*$  by forming coalition. If party  $i$  demands more than  $d_i^*$  and  $b(i, \tau^i) \geq 0$ , we know from Lemma 9a) that  $i \notin \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$  for all  $\sigma^{i+1} \in \Sigma^{i+1}$ . The induction hypothesis implies that party  $i$  gets zero. If party  $i$  demands more than  $d_i^*$  and  $b(i, \tau^i) < 0$ , we know from Lemma 9b) that party  $i$  gets zero.

Now we show that party  $i$  can get at least  $d_i^*$ . This is immediate if  $\omega_i \in \Sigma^i$ . Suppose  $\omega_i \notin \Sigma^i$ . Since  $b(i, \sigma^i) > 0$ , we know  $d_i^* > 0$ . The value of  $d_{i+1}^*$  induced by  $d_i^*$  may be strictly positive or 0. Suppose party  $i$  demands  $d_i < d_i^*$ . Then the corresponding value of  $d_{i+1}^*$  is strictly positive. Under Lemma 8a), party  $i$  belongs to all coalitions associated with some element of  $\Sigma^{i+1}$ , and the induction hypothesis for  $d_{i+1}^* > 0$  implies that party  $i$  gets  $d_i$ . Thus, the perfectness of the equilibrium implies that  $d_i^*$  is accepted (otherwise, party  $i$  would not have a best response).

Moreover, Lemma 8c), the induction hypothesis and the fact that  $d_i^*$  is accepted imply that the coalition that forms satisfies  $S \cap P_i \in \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$  for some  $\sigma^i \in \Sigma^i$ .

3. If  $b(i, \sigma^i) = 0$ , then  $d_i^* = 0$  and, moreover,  $\alpha \in \Sigma^i$  if and only if  $b(i, \alpha) = 0$ .



a) It is trivial that party  $i$  gets  $d_i^* = 0$ . If  $d_i > d_i^*$ , the induction hypothesis implies that no coalition to which party  $i$  belongs will form.

b) There is an equilibrium of the subgame in which a coalition associated with  $\sigma^i \in \Sigma^i$  forms. This is clearly the case for  $\omega_i \in \Sigma^i$ . Otherwise, it is optimal for party  $i$  to demand  $d_i^* = 0$ . Then  $b(i+1, \sigma^{i+1}) = 0$  for all  $\sigma^{i+1} \in \Sigma^{i+1}$  and the induction hypothesis implies that there is a SPE of  $\mathbb{B}(i, d)$  in which a winning coalition is formed.

c) Assume a winning coalition  $S$  is formed with  $S \cap P_i \notin \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$  for all  $\sigma^i \in \Sigma^i$ . This means that, for a given  $\sigma^i \in \Sigma^i$ , either  $\omega_{S \cap P_i} \geq q - \sigma^i$  but  $1 - d_{S \cap P_i}$  is not maximal, or  $\omega_{S \cap P_i} < q - \sigma^i$ .

Assume first there exists  $\sigma^i \in \Sigma^i$  such that  $\omega_{S \cap P_i} \geq q - \sigma^i$  but  $1 - d_{S \cap P_i}$  is not maximal. Since  $b(i, \sigma^i) = 0$ , this means  $d_{S \cap P_i} > 1$  and it cannot be optimal at any subgame to form  $S$ .

Assume now  $\omega_{S \cap P_i} < q - \sigma^i$  for all  $\sigma^i \in \Sigma^i$ . Since  $b(i, \sigma^i) = 0$  and  $b(i, \alpha)$  is nondecreasing in  $\alpha$ ,  $\delta^i \in \Sigma^i$ ; thus  $\omega_{S \cap P_i} < q - \delta^i$ . This means  $\omega_{S \cap P_i} + \omega_{S \cap (N \setminus P_i)} < q$ . Thus,  $S$  is not a winning coalition.

■

**Corollary 3** *In any SPE, the coalition of Lemma 2 forms with each party demanding  $d_i = \frac{\omega_i}{q}$ .*

**Proof.** Denote this coalition by  $S^*$ . Because of lemma 2,  $S^* = P_{l+1}$  for some value of  $l$ . We can show  $d_i = \frac{\omega_i}{q}$  for  $i = 1, \dots, l$  by induction on  $i$ .

Party 1 finds  $\Sigma^1 = \{q\}$  and, since  $q \leq \delta^2$  (due to the absence of veto players and the game being constant-sum) sets a demand  $d_1^* = \frac{\omega_1}{q}$ . Given this demand,  $q - \omega_1 \in \Sigma^2$ .

Assume now  $d_j = \frac{\omega_j}{q}$  for all  $j \in P_i$ , and  $q - \omega_{P_i} \in \Sigma^i$ . Then,

$$d_i^* = \frac{\omega_i b(i, q - \omega_{P_i})}{q - \omega_{P_i}} = \frac{\omega_i \left(1 - \frac{\omega_{P_i}}{q}\right)}{q - \omega_{P_i}} = \frac{\omega_i}{q}.$$

■

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