A bargaining approach to the consistent value for NTU games with coalition structure

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Abstract

The mechanism by Hart and Mas-Colell (1996) for NTU games is generalized so that a coalition structure among players is taken into account. The new mechanism yields the Owen value for TU games with coalition structure as well as the consistent value (Maschler and Owen 1989, 1992) for NTU games with trivial coalition structure.

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1 Introduction

Hart and Mas-Colell (1996) develop a bargaining mechanism which yields the consistent value (Maschler and Owen 1989, 1992) for NTU games. First, a player is randomly chosen in order to propose a payoff. In case this proposal not be accepted by all other players, the mechanism is played again under the same conditions with probability $\delta \in [0, 1)$. With probability $1 - \delta$, the proposer leaves the game and the mechanism is repeated with the rest of the players. Hart and Mas-Colell consider that the consistent value is a very appropriate generalization for the Shapley (1953) value (used in TU games) to NTU games.

Other non-cooperative mechanisms which implement the Shapley value are, for example, Gul (1989), Hart and Moore (1990), Winter (1994), Evans (1996), Dasgupta and Chiu (1998), Pérez-Castrillo and Wettstein (2001) and Mutuswami, Pérez-Castrillo and Wettstein (2002). Navarro and Perea (2001) design a mechanism which implements the Myerson (1977) value, which is an extension of the Shapley value to graph-restricted games.

Sometimes, however, players are associated in a priori coalitions. Owen (1977) studies them in TU games. He proposes a value, known as the Owen

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value, which generalizes the Shapley value for games with a coalition structure. Later, Winter (1991) proposes a value, called the Game Coalition Structure value, which is a generalization of the Harsanyi (1963) value NTU games and the Owen value for TU games with a coalition structure.

A non-cooperative mechanism which implements the Owen value in the TU case is given by Bergantiños and Vidal-Puga (2002b).

In this paper, we develop a non-cooperative mechanism that takes into account the coalition structure and implements both the consistent value for NTU games, and the Owen value for TU games.

The mechanism is as follows: First, a player is randomly chosen out of each coalition and proposes a payoff. Then, each proposal is voted by the rest of the members of its own coalition. If one of them rejects the proposed payoff, the mechanism is either played again under the same conditions (probability ρ), or the proposer leaves the game and the mechanism is repeated with the rest of the players (probability $1-\rho$). If there is no rejection, the proposal of one of the coalitions is randomly chosen. If this proposal is not accepted by all other coalitions, the mechanism is played again under the same conditions (probability ρ), or the entire proposing coalition leaves the game and the mechanism is repeated with the rest of the players (probability $1-\rho$).

When the coalition structure is trivial (i.e., either there is a single grand coalition or all the coalitions are singletons), this mechanism coincides with Hart and Mas-Colell's. Thus, the consistent value arises in equilibrium. Furthermore, when the mechanism is applied to a transferable utility (TU) game with coalition structure, the Owen value is implemented.

As for general NTU games with coalition structure, the arising equilibrium payoff is a recently studied solution concept: the consistent coalitional value (Vidal-Puga and Bergantiños, 2002a).

Assume we change the mechanism so that, before any proposal is set, all the players know who is bound to be the proposer. This new mechanism also coincides with Hart and Mas-Colell's when the coalition structure is trivial. However, for general NTU games with coalition structure, a new coalitional value arises. We study this value in Section 4.

The structure of this paper is as follows: In Section 2 we give the definitions and results used in the paper. In Section 3 we describe the coalitional mechanism and give the main results: Theorem 11 deals with the existence of equilibria. Theorem 12 proves the result for hyperplane games. Theorem 14 gives the general convergence result. In Section 4, we present a slight modification in the coalitional mechanism. Finally, the proofs are located in the Appendix.

2 Definitions and previous results

Mainly, we follow the notation in Hart and Mas-Colell (1996). Let $N = \{1, 2, ..., n\}$ and $2^N = \{S : S \subset N\}$. Given $x, y \in \mathbb{R}^N$, we say $y \leq x$ when $y^i \leq x^i$ for every $i \in N$. We denote by $x \cdot y$ the scalar product $\sum_{i \in N} x^i y^i$. We denote $\mathbb{R}^N_+ = \left\{x \in \mathbb{R}^N : x^i \geq 0, \forall i\right\}$, and $\mathbb{R}^N_{++} = \left\{x \in \mathbb{R}^N : x^i > 0, \forall i\right\}$.

A *non transferable utility game*, or *NTU game*, is a pair (N, V) where N is the set of *players* and V is a correspondence which assigns to each *coalition* $S \subset N$, $S \neq \emptyset$ a subset $V(S) \subset \mathbb{R}^S$ representing all the possible payoffs that the members of S can obtain for themselves when play cooperatively. For $S \subset N$, we maintain the notation V when refer to the application V restricted to S as player set. For simplicity, we denote $V(i)$ instead of $V(\{i\})$, $S \cup i$ instead of $S \cup \{i\}$ and $N \backslash i$ instead of $N \backslash \{i\}$.

We impose the next conditions on the function V:

(A.1) For each $S \subset N$, the set $V(S)$ is closed, convex, *comprehensive* (i.e., if $x \in V(S)$ and $y \in \mathbb{R}^S$ with $y \leq x$, then $y \in V(S)$) and *upper bounded* (i.e., for each $x \in \mathbb{R}^S$, the set $\{y \in V(S) : y \geq x\}$ is bounded).

(A.2) For each $S \subset N$, the boundary of $V(S)$, which we denote by $\partial V(S)$, is *smooth* (this means that on each point of the boundary there exists an unique outward ortonormal vector) and *nonlevel* (this means that the outward vector on each point of $\partial V(S)$ has its coordinates positive).

(A.3) *Monotonicity*: For each $T \subset S$, $V(T) \times \{0^{S \backslash T}\} \subset V(S)$.

(A.4) *Normalization*: For each $S \subset N$, 0^S belongs to $V(S)$.

For each $i \in N$, let $r^i := \max\{x : x \in V(i)\}$ (notice that, by (A.4), $r^i \geq 0$). When

$$V(S) = \left\{x \in \mathbb{R}^S : \sum_{i \in S} x^i \leq v(S)\right\}$$

for some $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, we say that (N, V) is a *transfer utility game* (or *TU game*) and it is represented by (N, v). We denote by $TU(N)$ the set of TU games over N.

When

$$V(S) = \left\{x \in \mathbb{R}^S : \lambda_S \cdot x \leq v(S)\right\} \tag{1}$$

for some $\lambda_S \in \mathbb{R}^S_{++}$ and $v : 2^N \to \mathbb{R}$, we say that (N, V) is a *hyperplane game*.

Notice that every TU game is a hyperplane game with $\lambda^i_S = 1$ for every $S \subset N$ and $i \in S$.

If $r^S \in \partial V(S)$ for all $S \square N$ and $r^N \in V(N)$, we say that (N, V) is a *pure bargaining game*.

We say that an NTU game is *totally essential* if $r^S \in V(S)$ for all $S \subset N$. We say that an NTU game is *zero-monotonic* if $V(i) \times V(S \backslash i) \subset V(S)$ for all $i \in S \subset N$.

Given N, we call *coalition structure* over N a partition of the player set, i.e., $\mathcal{C} = \{C_1, C_2, ..., C_p\} \subset 2^N$ is a coalition structure if it satisfies $\bigcup_{C_q \in \mathcal{C}} C_q = N$ and $C_q \cap C_r = \emptyset$ when $q \neq r$.

We denote by $(N, V, \mathcal{C})$ an NTU game (N, V) with coalition structure $\mathcal{C}$ over N. We denote by $CNTU(N)$ the set of NTU games with coalition structure over N. For coalitions $S \subset N$, we denote by $\mathcal{C}_S$ the restriction of $\mathcal{C}$ to the players in S (notice that this implies that $\mathcal{C}_S$ may have less or the same number of coalitions as $\mathcal{C}$). We also denote $\mathcal{C}_{-i} := \mathcal{C}_{N \backslash i}$.

Given G a subset of $NTU(N)$ or $CNTU(N)$, a *value* on G is a correspondence which assigns to each element in G a subset of $\mathbb{R}^N$. When these subsets are singletons we call the value a *single value*. A well known single value for TU games is the *Shapley value* (Shapley, 1953). We denote by $\varphi_N \in \mathbb{R}^N$ the Shapley value of the TU game (N, v). For TU games with coalition structure, Owen (1977) proposes a single value based on Shapley's which takes into account the coalition structure $\mathcal{C}$. We call this value the *Owen coalitional value*, or simply the *Owen value*. We denote by $\phi_N \in \mathbb{R}^N$ the Owen value of the TU game with coalition structure $(N, v, \mathcal{C})$.

The *consistent value* for NTU games is introduced by Maschler and Owen (1989, 1992). Let (N, V) be a hyperplane game defined as in (1). Given $i \in N$, the consistent value Ψ is defined recursively as follows

$$\Psi^i_{\{i\}} = r^i.$$

Assume we know Ψ^j_S for all $S \subsetneq N$ and $j \in S$. Then,

$$\Psi^i_N = \frac{1}{|N| \lambda^i_N} \left(\sum_{j \in N \backslash i} \lambda^i_N \Psi^i_{N \backslash j} - \sum_{j \in N \backslash i} \lambda^j_N \Psi^j_{N \backslash i} + v(N) \right).$$

For a general NTU game (N, V), Maschler and Owen (1992) take for each coalition $S \subset N$ a vector λ_S normal to the boundary of $V(S)$. Let (N, V') be the resulting hyperplane game, i.e. $V'(S) = \left\{ x \in \mathbb{R}^S : \lambda_S \cdot x \leq v(S, \lambda_S) \right\} \supset V(S)$, with

$$v(S, \lambda_S) := \max \{ \lambda_S \cdot x : x \in V(S) \}.$$

Let $\Psi = (\Psi_S)_{S \subset N}$ with Ψ_S the (only) consistent value for (S, V'). If Ψ is a feasible payoff in (N, V) (i.e., $\Psi_S \in V(S), \forall S \subset N$) then Ψ_N is a *consistent value* for V.

The consistent value coincides with the Shapley value for TU games. Maschler and Owen (1992) also show that the consistent value exists (it is not always unique though) for any NTU game.

Let $(N, V, \mathcal{C})$ be a hyperplane game with coalition structure. Vidal-Puga and Bergantiños (2002a) define recursively the *consistent coalitional value* as follows. Given $i \in C_q \in \mathcal{C}$:

$$\Phi^i_{\{i\}} = r^i$$

Assume we know Φ_S^j for all $S \square N$ and $j \in S$. Then,

$$\left.\begin{aligned}\Phi_N^i = & \frac{1}{|\mathcal{C}|\,|C_q|\,\lambda_N^i}\left(\sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_q}\lambda_N^j\Phi_{N\backslash C_r}^j-\sum_{j\in C_r}\lambda_N^j\Phi_{N\backslash C_q}^j\right)\right)\\ & +\frac{1}{|C_q|\,\lambda_N^i}\left(\sum_{j\in C_q\backslash i}\lambda_N^i\Phi_{N\backslash j}^i-\sum_{j\in C_q\backslash i}\lambda_N^j\Phi_{N\backslash i}^j\right)\\ & +\frac{1}{|\mathcal{C}|\,|C_q|\,\lambda_N^i}v\left(N\right).\end{aligned}\right\}(2)$$

Following the usual practice, we consider a *payoff configuration* as a set of payoffs $x = (x_S)_{S\subset N}$ with $x_S \in \mathbb{R}^S$ for all $S \subset N$.

The generalization of Φ to NTU games (not necessarily hyperplane games) is done analogously to the consistent value. For an NTU game with coalition structure $(N, V, \mathcal{C})$, we take for each coalition $S \subset N$ a normal vector λ_S to the boundary of $V(S)$. Let $(N, V', \mathcal{C})$ be the resulting hyperplane game. Let $\Phi := (\Phi_S)_{S\subset N}$ for all $S \subset N$ be the (unique) consistent coalitional payoff configuration for V'. If Φ is a feasible payoff configuration for $(N, V, \mathcal{C})$, then Φ is a consistent coalitional payoff configuration for V.

Vidal-Puga and Bergantiños (2002a) prove that the consistent coalitional value exists for any NTU game (although it is not necessarily unique) and give the following characterization. Given $S \subset N$ player set, we denote by $C'_q := C_q \cap S$ (when different from $\emptyset$) the restriction of C_q in $\mathcal{C}_S$. The set $\Phi = (\Phi_S)_{S\subset N}$ is a consistent coalitional payoff configuration for $(N, V, \mathcal{C})$ if and only if for each $S \subset N$ there exists a vector $\lambda_S \in \mathbb{R}^S_{++}$, orthogonal to $V(S)$, such that:

(B.1) $\Phi_S \in \partial V(S)$;

(B.2) $\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\left[\sum_{i\in C'_q}\lambda_S^i(\Phi_S^i-\Phi_{S\backslash C'_r}^i)\right] = \sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\left[\sum_{i\in C'_r}\lambda_S^i(\Phi_S^i-\Phi_{S\backslash C'_q}^i)\right]$ for every $C'_q \in \mathcal{C}_S$;

(B.3) $\sum_{j\in C'_q\backslash i}\lambda_S^i(\Phi_S^i-\Phi_{S\backslash j}^i) = \sum_{j\in C'_q\backslash i}\lambda_S^j(\Phi_S^j-\Phi_{S\backslash i}^j)$ for every $i \in C'_q \in \mathcal{C}_S$.

Thus, (B.1), (B.2) and (B.3) generalize the characterization of balanced contributions of the Owen value (Calvo, Lasaga and Winter, 1996) and the consistent value (Hart and Mas-Colell, 1996).

3 The coalitional mechanism

In this section we describe the coalitional mechanism. This mechanism is a modification of the bargaining mechanism presented by Hart and Mas-Colell (1996).

In order to characterize the equilibria, we need to restrict the class of games. This restriction is given by property (A.5) below. We claim that this property is not too restrictive by showing that a significative class of games (including TU zero-monotonic games and pure bargaining games) satisfies it. Then, we characterize the equilibria and show that there exits at least an equilibrium. Finally, we prove that the equilibria yield the consistent coalitional value.

Given an NTU game (N, V) and $\rho \in [0, 1)$, Hart and Mas-Colell (1996) define the following bargaining mechanism (associated to (N, V) and ρ):

> "In each round there is a set of *active* players, and a *proposer* $i \in S$. In the first round $S = N$. The proposer is chosen at random out of S, with all players in S being equally likely to be selected. The proposer makes a proposal which is feasible, i.e. a payoff vector in $V(S)$. If all the members of S accept it – they are asked in some prespecified order – then the game ends with these payoffs. If it is rejected by even one member of S, then we move to the next round where, with probability ρ, the set of active players is again S and, with probability $1-\rho$, the proposer i *drops out* and the set of active players becomes $S \backslash i$. In the latter case the dropped i gets a final payoff of 0."

Hart and Mas-Colell (1996) prove that, for each hyperplane game, and for each $\rho \in [0, 1)$, the bargaining mechanism implements the consistent value for subgame perfect equilibria.

Furthermore, for a general NTU game (N, V), if for each $S \subset N$, $a_S(\rho)$ is the payoff of a subgame perfect equilibrium for $\rho \in [0, 1)$ and a_S is a limit point of $a_S(\rho)$ as $\rho \to 1$, then $(a_S)_{S \subset N}$ is a consistent payoff configuration of the NTU game (N, V).

Now we describe the coalitional bargaining mechanism formally. For each $S \subset N$, we denote by Γ_S the set of applications $\gamma : \mathcal{C}_S \to S$ satisfying $\gamma(C'_q) \in C'_q$ for each $C'_q \in \mathcal{C}_S$. For simplicity, we denote $\Gamma = \Gamma_N$.

The coalitional bargaining mechanism associated to $(N, V, \mathcal{C})$ and ρ is defined as follows:

> In each round there is a set $S \subset N$ of active players. At first round, $S = N$. Each round has two stages. On the first stage, a *proposer* is randomly chosen out of each coalition. Namely, a function $\gamma \in \Gamma_S$ is randomly chosen, being each γ equally likely to be chosen. Players in $C'_q \in \mathcal{C}_S$ are aware of the identity of proposer $\gamma\left(C'_q\right)$, but not of the proposers in other coalitions. The coalitions play sequentially (say, for example, in the order $\left(C'_1, C'_2, ..., C'_p\right)$) on the following way: Proposer $\gamma\left(C'_1\right)$ proposes a feasible payoff, i.e. a vector in $V(S)$. The members of $C'_1 \backslash \gamma\left(C'_1\right)$ are then asked in some prespecified order. If one of them rejects the proposal, then we move to next round where the set of active players is S with probability ρ and $S \backslash \gamma\left(C'_1\right)$ with probability $1-\rho$. In the latter case, player $\gamma\left(C'_1\right)$

gets a final payoff of 0. If all of them accept the proposal, the game moves to next coalition C_2'. Then, players of C_2', unaware of $\gamma\left(C_1'\right)$'s identity and his proposal, proceed to repeat the process under the same conditions, and so on. If all the proposals are accepted in each coalition, the proposers are called *representatives.* We denote by $a\left(S, \gamma\left(C_q'\right)\right) \in V(S)$ the proposal of $\gamma\left(C_q'\right)$.

On the second stage, a proposal $a\left(S, \gamma\left(C_q'\right)\right)$ is randomly chosen, being each proposal equally likely to be chosen. We call $\gamma\left(C_q'\right)$ the *representative-proposer*, or simply *r.p.* If all the members of $S \backslash C_q'$ accept $a\left(S, \gamma\left(C_q'\right)\right)$– they are asked in some prespecified order – then the game ends with these payoffs. If it is rejected by at least one member of $S \backslash C_q'$, then we move to the next round where, with probability ρ, the set of active players is again S and, with probability $1-\rho$, the entire coalition C_q' *drops out* and the set of active players becomes $S \backslash C_q'$. In the latter case each member of the dropped coalition C_q' gets a final payoff of 0.

Clearly, given any set of strategies, this mechanism finishes in a finite number of rounds with probability 1.

Also note that the proposed payoff of $\gamma(C_q')$ is independent on who are the proposers in other coalitions.

Remark 1 *The normalization given by property (A.4) does not affect our results, although the bargaining mechanism must be changed as follows: The player $i \in N$ who drops out, receives an amount $x^i \in \mathbb{R}$ such that $x^i \in V(i)$. This x^i can be considered as a "penalty payoff". Also, the monotonic property must be changed to $V(T) \times \left(x^{S \backslash T}\right) \subset V(S)$ for each $T \subset S$.*

The coalitional bargaining mechanism may be interpreted as the mechanism by Hart and Mas-Colell played on two stages, one of them by the coalitions and another by the players inside the same coalition. On the second stage, the coalitions play Hart and Mas-Colell's mechanism. This means that a coalition is randomly chosen to propose a payoff. The disagreement to this payoff by at least one of the other coalitions puts the whole proposing coalition in jeopardy. In order to decide the proposals, the members of each coalition play Hart and Mas-Colell's mechanism on a first stage. Thus, a player is randomly chosen inside each coalition and proposes a feasible payoff. Only if all the rest of the members of his coalition agree to this payoff, the proposal goes on to the second stage. Otherwise the proposer is in jeopardy. However, once the proposal is presented on the second stage, it is backed by the whole proposing coalition, so that its rejection may imply the whole coalition leaves the game.

In our study, as in Hart and Mas-Colell's, we consider stationary subgame perfect equilibria. In this context, an equilibrium is stationary if the players strategies depend only on the set S of active players. It does not depend, however, on the previous history nor the number of played rounds.

We also assume, as Hart and Mas-Colell, that players break ties in favor of quick termination of the game. We must note that this assumption is not needed in Hart and Mas-Colell's model. However, Example 15 shows that we cannot avoid it in our coalitional mechanism.

From now on, when we say equilibrium, we mean stationary subgame perfect equilibrium satisfying this tie-breaking rule.

Given a set of stationary strategies, let S denote the set of active players.

We denote by $a(S,i) \in V(S)$ the payoff proposed by $i \in C'_q \in \mathcal{C}_S$ when the set of proposers is determined by some $\gamma \in \Gamma_S$ with $\gamma(C'_q) = i$. We also define, for a given $\gamma \in \Gamma_S$:

$$a(S)_\gamma := \frac{1}{|\mathcal{C}_S|} \sum_{C'_q \in \mathcal{C}_S} a\left(S, \gamma(C'_q)\right).$$

Since $V(S)$ is a convex set and each $a(S,\gamma(C'_q))$ belongs to $V(S)$, their average also belongs to $V(S)$. When all the proposals are accepted, $a(S)_\gamma$ is the expected final payoff when γ determines the set of proposers (or representatives).

Given $i \in C'_q \in \mathcal{C}_S$, let $\Gamma_{S,i}$ ($\Gamma_i = \Gamma_{N,i}$) be the subset of functions $\gamma \in \Gamma_S$ such that $\gamma(C'_q) = i$. Notice that $|\Gamma_S| = |\Gamma_{S,i}| \left|C'_q\right|$ for all $i \in C'_q \in \mathcal{C}_S$. Then,

$$a\left(S|_i\right) := \frac{1}{|\mathcal{C}_S|} \sum_{C'_r \in \mathcal{C}_S} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a\left(S, \gamma\left(C'_r\right)\right) \in V\left(S\right)$$

is the expected final payoff when all the proposals are accepted and player i is the proposer (and representative) of his coalition.

We denote:

$$a(S) := \frac{1}{|\Gamma_S|} \sum_{\gamma \in \Gamma_S} a\left(S\right)_\gamma \in V(S)$$

as the expected final payoff when all the proposals are accepted. Given $C'_q \in \mathcal{C}_S$, it is straightforward to prove that $a(S)$ may also be expressed as:

$$a(S) = \frac{1}{|C'_q|} \sum_{i \in C'_q} a\left(S|_i\right).$$

It is also straightforward to prove:

$$a\left(S\right) = \frac{1}{|\mathcal{C}_S|} \sum_{C'_q \in \mathcal{C}_S} \frac{1}{|C'_q|} \sum_{i \in C'_q} a\left(S, i\right). \tag{3}$$

Proposition 1 in Hart and Mas-Colell (1996) characterizes the proposals corresponding to an equilibrium by (1) $a\left(S,i\right) \in \partial V\left(S\right)$ and (2) $a\left(S,i\right)^j = \delta a\left(S\right)^j + \left(1-\rho\right) a\left(S \backslash i\right)^j$.

We now introduce some properties which generalize (1) and (2) in Hart and Mas-Colell (1996) to games with coalition structure.

We consider the following properties:

(C.1) $a(S,i) \in \partial V(S)$ for every $i \in N$;

(C.2) $a(S|_i)^j = \rho a(S)^j + (1-\rho)a(S\backslash i)^j$ for every $i,j \in C'_q \in \mathcal{C}_S$ with $j \neq i$;

(C.2') $a(S,i)^j = \rho a(S)^j + (1-\rho)\left[|\mathcal{C}_S| a(S\backslash i)^j - \sum_{C'_r \in \mathcal{C}_S \backslash C'_q} a(S\backslash C'_r)^j\right]$ for every $i,j \in C'_q \in \mathcal{C}_S$;

(C.3) $a(S,i)^j = \rho a(S)^j + (1-\rho)a(S\backslash C'_q)^j$ for every $i \in C'_q \in \mathcal{C}_S$, $j \notin C'_q$.

Of course (C.1) coincides with Property (1) of Proposition 1 in Hart and Mas-Colell (1996). Property (2) is split in two properties: (C.2) or (C.2'), and (C.3) following usual practice in the literature on games with coalition structure.

Proposition 2 *If (C.3) holds, then (C.2) is equivalent to (C.2').*

The proof of Proposition 2 is in the Appendix.

Proposition 3 *If (C.3) holds, then* $a(S)^j = a(S|_i)^j$ *for every* $i \in C'_q \in \mathcal{C}_S$, $j \notin C'_q$.

The proof of Proposition 3 is in the Appendix.

Proposition 4 *Let* $(N,V,\mathcal{C})$ *be a hyperplane game with coalition structure. Assume a set of strategies* $\left(a(S,i)_{i\in S}\right)_{S\subset N}$ *satisfies (C.1), (C.2) and (C.3). Then,* $(a(S))_{S\subset N}$ *is the consistent coalitional value for the game* $(N,V,\mathcal{C})$.

The proof of Proposition 4 is located in the Appendix.

By Proposition 2, Proposition 4 also holds if we replace (C.2) by (C.2').

However, in order to characterize the equilibria, properties (C.1), (C.2) and (C.3) are not enough in general. Thus, we impose an additional condition to the NTU games considered.

Given $\left(a(S,i)_{i\in S}\right)_{S\subset N}$ set of proposals, we define the vector $c(S,i) \in \mathbb{R}^S$ with $S \subset N$ and $i \in S$ as follows:

$$\left.\begin{array}{ll} c(S,i)^i = -\sum_{C'_r \in \mathcal{C}_S \backslash C'_q} a(S\backslash C'_r)^i & \\ c(S,i)^j = |\mathcal{C}_S|\, a(S\backslash i)^j - \sum_{C'_r \in \mathcal{C}_S \backslash C'_q} a(S\backslash C'_r)^j & \text{for all } j \in C'_q \backslash i \\ c(S,i)^j = a\left(S\backslash C'_q\right)^j & \text{for all } j \in S\backslash C'_q. \end{array}\right\} \quad (4)$$

We consider the following property:

(A.5) For any $(a(S,i))_{S\subset N, i\in S}$ set of proposals satisfying (C.1), (C.2) and (C.3), we have that, for every $S \subset N$, $i \in C_q^{'} \in \mathcal{C}_S$, the vector $c(S,i)$ belongs to $V(S)$.

This property is not satisfied by general NTU games, as next example shows:

Example 5 *Let* $(N, V, \mathcal{C})$ *be such that* $N = \{1,2,3\}$, $\mathcal{C} = \{\{1,2\},\{3\}\}$ *and* V *be defined as follows,*

$V(i) = 0 - \mathbb{R}_+$, $i = 1,2,3$;
$V(\{1,2\}) = V(\{1,3\}) = (0,0) - \mathbb{R}_+^2$;
$V(\{2,3\}) = \left\{(x_2,x_3) : \frac{4}{5}x_2 + 2x_3 \le 1, x_2 + x_3 \le \frac{7}{8}\right\}$ *and*
$V(N) = \{(x_1,x_2,x_3) : x_1 + x_2 + x_3 \le 1\}$.

It can be easily checked that this game is superadditive[1]*. Furthermore, if we take* $S = N$, $C_q = \{1,2\}$ *and* $i = 1$*, it can be checked that, given* $\rho \in \left(\frac{25-\sqrt{145}}{24}, 1\right)$ *and a set of proposals satisfying (C.1), (C.2) and (C.3), we have*

$$\left(-a(\{1,2\})^1, 2a(\{2,3\})^2 - a(\{1,2\})^2, a(\{3\})^3\right) = \left(0, \frac{70-55\rho}{12\rho^2 - 80\rho + 80}, 0\right) \notin V(N).$$

Nevertheless, next proposition shows that several interesting subclasses of NTU games satisfy (A.5).

Proposition 6 *Property (A.5) is satisfied by the next class of games,*

- *zero-monotonic TU games;*
- *totally essential three-player hyperplane*[2] *games; and*
- *pure bargaining games.*

The proof of Proposition 6 is located in the Appendix.

Proposition 7 *The proposals in any equilibrium of an NTU game satisfying (A.5) are characterized by (C.1), (C.2) and (C.3). Moreover, all the proposals are accepted and* $a(S) \ge 0^S$ *for all* $S \subset N$.

The proof of Proposition 7 is located in the Appendix.

Remark 8 *There is a subtle difference between the result given by Proposition 7 and Proposition 1 in Hart and Mas-Colell (1996). In Hart and Mas-Colell's model, the proposals* $a(S,i)$ *are nonnegative. In our model, the proposals do not need to be nonnegative, as it can be checked in Example 16. However, their (weighted) average* $a(S)$ *is always nonnegative in equilibrium.*

[1] It is not smooth, but we can make it smooth by a small modification which does not change our result.
[2] It is enough that the game coincides with a hyperplane game in $V(S) \cap \mathbb{R}_+^S$ for all $S \subset N$.

Now, we see two important corollaries of Proposition 7.

Corollary 9 *Let $(N, V, \mathcal{C})$ be an NTU game with coalition structure satisfying (A.5). Then, a player's expected payoff in equilibrium is independent on who is the proposer in other coalitions. Namely:*

$$a(S)^j = a(S|_i)^j \qquad \forall i \in C'_q \in \mathcal{C}_S ; j \notin C'_q.$$

The proof of Corollary 9 is immediate from Proposition 3 and Proposition 7.

Hart and Mas-Colell say: "if ρ is close to 1– i.e., the 'cost of delay' is low – then there is little dispersion among individual proposals: all the $a(N,i)$ constitute[3] small deviations of $a(N)$. This implies, first, that $a(N)$ *is almost Pareto optimal* (since the $a(N,i)$ are Pareto optimal). And second, that *there is no substantial advantage or disadvantage to being the proposer*; the 'first-mover' effect vanishes."

Next corollary states that the coalitional bargaining mechanism behaves in the same way.

Corollary 10 *There exists $M \in \mathbb{R}$ such that $\left|a(N,i)^j - a(N)^j\right| < M(1-\rho)$ for all $i, j \in N$.*

The proof of Corollary 10 is in the Appendix.

In next theorem we prove the existence of equilibria.

Theorem 11 *Let $(N, V, \mathcal{C})$ an NTU game with coalition structure satisfying (A.5). Then, for each $\rho \in [0, 1)$, there exists an equilibrium.*

The proof of Theorem 11 is located in the Appendix.

Next results characterize the equilibrium payoffs.

Theorem 12 *Let $(N, V, \mathcal{C})$ be a hyperplane game with coalition structure satisfying (A.5). Then, for each $\rho \in [0, 1)$, there exists a unique equilibrium. Furthermore, the equilibrium payoff configuration equals the unique consistent coalitional payoff configuration of $(N, V, \mathcal{C})$.*

Theorem 12 is an immediate consequence of Proposition 4, Proposition 7 and Theorem 11.

Corollary 13 *The coalitional mechanism, when applied to zero-monotonic TU games, implements the Owen value.*

Since the consistent coalitional value coincides with the Owen value in TU games with coalition structure, Corollary 13 is an immediate consequence of Proposition 6 and Theorem 12.

Notice that the coalitional bargaining mechanism implements the Shapley value for zero-monotonic games because the Shapley value coincides with the Owen value when the coalition structure is trivial.

[3] Hart and Mas-Colell denote $a(N,i)$ and $a(N)$ as $a_{N,i}$ and a_N, respectively.

Theorem 14 *Let $(N, V, \mathcal{C})$ be an NTU game with coalition structure satisfying (A.5). If $a_\rho := (a_\rho(S))_{S \subset N}$ is an equilibrium payoff configuration for each ρ and a is the limit of a_ρ when $\rho \to 1$, then a is a consistent coalitional payoff configuration of $(N, V, \mathcal{C})$.*

The proof of Theorem 14 is located in the Appendix.

If we do not assume the tie-breaking rule, the consistent coalitional value is still an equilibrium payoff. However, there can be other equilibria which do not yield the consistent coalitional value, as next example shows.

Example 15 *Consider $(N, v, \mathcal{C})$, where $N = \{1, 2, 3, 4\}$, $\mathcal{C} = \{C_1, C_2\}$, $C_1 = \{1, 2\}$, $C_2 = \{3, 4\}$. Moreover, v is the characteristic function associated to the weighted majority game where the quota is 3 and the weights are 1, 1, 1, and 2 respectively. This means that $v(S) = 1$ if and only if S contains some of the following subsets: $\{1, 2, 3\}$, $\{1, 4\}$, $\{2, 4\}$, or $\{3, 4\}$.*

It is straightforward to prove that

$$\begin{aligned} \Phi_N &= \left(0, 0, \frac{1}{2}, \frac{1}{2}\right) \\ \Phi_{N\backslash 1} &= \left(-, 0, \frac{1}{4}, \frac{3}{4}\right) \\ \Phi_{N\backslash 2} &= \left(0, -, \frac{1}{4}, \frac{3}{4}\right) \\ \Phi_{N\backslash 3} &= \left(\frac{1}{4}, \frac{1}{4}, -, \frac{1}{2}\right) \\ \Phi_{N\backslash 4} &= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, -\right). \end{aligned}$$

We now define an equilibrium whose payoff outcome is $\left(0, 0, \frac{1}{4}, \frac{3}{4}\right)$.

First, we describe the strategies of players 1 and 2. When one of them is chosen as proposer, his proposal is $a(N, \gamma(C_1)) = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$. Moreover, players 1 and 2 accept an offer if and only if it offers them something positive. In the subgame obtained after $\gamma(C_1)$ drops out of the game the strategy of player j coincides with the strategy with $\Phi_{N\backslash\gamma(C_1)}$ as payoff outcome. In the subgame obtained after C_2 drops out of the game the strategy of players 1 and 2 coincides with the strategy with $\Phi_{N\backslash C_2}$ as payoff outcome.

We now describe the strategies of players 3 and 4. In the subgame obtained after the offer of $\gamma(C_1)$ is accepted, the strategies of players 3 and 4 coincide with the strategies with Φ_N as payoff outcome. In the subgame obtained after $\gamma(C_1)$ drops out the game, the strategies of players 3 and 4 coincide with the strategies with $\Phi_{N\backslash\gamma(C_1)}$ as payoff outcome. In the subgame obtained after C_1 drops out the game, the strategies of players 3 and 4 coincide with the strategies with $\Phi_{N\backslash C_1}$ as payoff outcome.

It is not difficult to check that these strategies are an equilibrium.

According to these strategies, the offer of player $\gamma(C_1)$ is rejected, which means that player $\gamma(C_1)$ obtains a final payoff of 0*. Then, players of* $N \setminus \gamma(C_1)$ *obtain* $\Phi_{N\setminus\gamma(C_1)}$ *as final payoff. This means that the final payoff induced by these strategies is* $\left(0, 0, \frac{1}{4}, \frac{3}{4}\right)$.

4 A modification in the model

In this section we present a slight modification of the coalitional bargaining mechanism defined previously. The new mechanism is simpler. Unfortunately, when we restrict it to TU games with coalition structure, the payoffs of the equilibria can be different from the Owen value.

We assume that a single proposer is chosen, and his proposal is voted first by the members of his own coalition and then by the members of the other coalitions.

Formally,

> In each round there is a set $S \subset N$ of active players. At first round, $S = N$. First, a coalition C'_q out of $\mathcal{C}_S$ is randomly chosen, being each coalition equally likely to be chosen. Then, a *proposer* is randomly chosen out of C'_q, being each player equally likely to be chosen. We denote by q^* this proposer. Player q^* proposes a feasible payoff, i.e. a vector in $V(S)$. The members of $S\backslash q^*$ are then asked in some prespecified order, but beginning with the members of $C'_q\backslash q^*$. If one of the members of $C'_q\backslash q^*$ rejects the proposal, then we move to the next round where the set of active players is S with probability ρ and $S\backslash q^*$ with probability $1-\rho$. In the latter case, player q^* gets a final payoff of 0. If the offer is accepted by all the members of $C'_q\backslash q^*$ and rejected by at least one member of $S\backslash C'_q$, then we move to the next round where, with probability ρ, the set of active players is again S and, with probability $1-\rho$, the entire coalition C'_q *drops out* and the set of active players becomes $S\backslash C'_q$. In the latter case each member of the dropped coalition C'_q gets a final payoff of 0. If all the members of $S\backslash q^*$ accept the proposal, then the game ends with these payoffs.

This mechanism also generalizes Hart and Mas-Colell's bargaining mechanism.

The main difference between the bargaining coalitional mechanism and this new mechanism is that, in the latter, when the players of a coalition accept the proposal of one of their members, they know that this proposal is due to be voted by the other coalitions. In the first mechanism, however, players only know this proposal would have a chance to be voted by the other coalitions.

This slight difference is not innocuous and affects in an important way to the behavior of agents, as we can see in the following example:

Example 16 *Consider $(N, v, \mathcal{C})$, where $N = \{1, 2, 3\}$, $\mathcal{C} = \{C_1, C_2\}$, $C_1 = \{1, 2\}$, $C_2 = \{3\}$. Moreover, v is the characteristic function associated to the weighted majority game where the quota is 3 and the weights are 2, 1, and 1 respectively. This means that $v(S) = 1$ if and only if S contains $\{1, 2\}$ or $\{1, 3\}$. Otherwise, $v(S) = 0$.*

The Owen value for this game is $\left(\frac{3}{4}, \frac{1}{4}, 0\right)$.

Assume they play the bargaining coalitional mechanism with $\rho = 0$. Player 3 would propose $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, since this is the payoff players in C_1 would get in absence of him. Player 2 would propose $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ for a similar reason. Player 1, however, would propose $\left(\frac{3}{2}, -\frac{1}{2}, 0\right)$, and player 2 accepts! Notice that, by rejecting, player 2 gets 0, and by accepting, his final payoff is $\frac{1}{2}$ if the r.p. is player 3, and $-\frac{1}{2}$ if the r.p. is player 1. In expected terms, player 2 gets 0.

The expected final payoff is the Owen value:

$$\frac{1}{4}\left(\frac{3}{2}, -\frac{1}{2}, 0\right) + \frac{1}{4}\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \left(\frac{3}{4}, \frac{1}{4}, 0\right).$$

Assume now they play the new mechanism. Again, both player 3 and 2 are due to propose $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ should any of them be the proposer. In this case, however, player 1 cannot expect player 2 to accept a negative payoff. Player 1 proposes $(1, 0, 0)$, which seems more standard. However, the final expected payoff is $\left(\frac{5}{8}, \frac{3}{8}, 0\right)$, i.e. different from the Owen value.

We proceed now to characterize the equilibria in this new mechanism. To do so, for each $S \subset N$, we keep the notation $a(S, i)$ for the proposal made by player i if he is chosen as proposer.

Given a hyperplane game $(N, V, \mathcal{C})$, we inductively define the following solution concept. For all $i \in C_q \in \mathcal{C}$:

$$\chi^i_{\{i\}} = r^i$$

Assume we know χ^j_S for all $S \subsetneq N$ and $j \in S$. Then,

$$\begin{aligned} \chi^i_N &= \frac{1}{|\mathcal{C}|\,|C_q|\,\lambda^i_N} \sum_{C_r \in \mathcal{C} \setminus C_q} \left(|C_q|\,\lambda^i_N \chi^i_{N \setminus C_r} - \sum_{j \in C_r} \lambda^j_N \chi^j_{N \setminus C_q} \right) \\ &\quad + \frac{1}{|\mathcal{C}|\,|C_q|\,\lambda^i_N} \left(\sum_{j \in C_q \setminus i} \lambda^i_N \chi^i_{N \setminus j} - \sum_{j \in C_q \setminus i} \lambda^j_N \chi^j_{N \setminus i} \right) \\ &\quad + \frac{1}{|\mathcal{C}|\,|C_q|\,\lambda^i_N} v(N) \end{aligned}$$

It is straightforward to prove that $\chi_N \in \partial V(N)$.

We can also generalize χ to any NTU game analogously to Ψ and Φ – i.e. by means of supporting hyperplanes.

Let $a(S)$ be defined as in (3). We consider next property:

(C.4) $a(S,i)^j = \rho a(S)^j + (1-\rho) a(S\backslash i)^j$ for every $i, j \in C'_q \in \mathcal{C}_S$ with $j \neq i$.

Proposition 17 *Let $(N, V, \mathcal{C})$ be a hyperplane game with coalition structure. Assume a set of strategies $\left(a(S,i)_{i\in S}\right)_{S\subset N}$ for the new mechanism satisfies (C.1), (C.3), and (C.4). Then, $(a(S))_{S\subset N} = (\chi_S)_{S\subset N}$.*

The proof of Proposition 17 is in the Appendix.

Given $(a(S,i))_{S\subset N, i\in S}$ set of proposals, we define the vector $d(S,i) \in R^S$ with $S \subset N$ and $i \in S$ as follows:

$$\begin{array}{ll} d(S,i)^i = 0 & \\ d(S,i)^j = a(S\backslash i)^j & \text{for all } j \in C'_q\backslash i \\ d(S,i)^j = a\left(S\backslash C'_q\right)^j & \text{for all } j \in S\backslash C'_q \end{array}$$

Again, we consider a new property:

(A.6) For any $(a(S,i))_{S\subset N, i\in S}$ set of proposals satisfying (C.1), (C.3), and (C.4), we have that, for every $S \subset N$, $i \in C'_q \in \mathcal{C}_S$, the vector $d(S,i)$ belongs to $V(S)$.

The proofs of Proposition 18, Theorem 19, Theorem 20 and Theorem 21 are analogous to those of Proposition 7, Theorem 11, Theorem 12 and Theorem 14, respectively, and we omit them.

Proposition 18 *The proposals in any equilibrium of the new mechanism of an NTU game satisfying (A.6) are characterized by (C.1), (C.3), and (C.4). Moreover, all the proposals are accepted and $a(S) \geq 0^S$ for all $S \subset N$.*

Notice the differences between the characterizations in both models. In both mechanisms, the proposals are Pareto efficient (property (C.1)) and satisfy (C.3). However, in the new mechanism, property (C.2) is replaced by (C.4). Now, the members of the proposer's coalition know that the proposal would also be proposed to the other coalitions should they accept it.

Theorem 19 *Let $(N, V, \mathcal{C})$ be an NTU game satisfying (A.6). Then, for each $\rho \in [0,1)$, there exists an equilibrium.*

Theorem 20 *Let $(N, V, \mathcal{C})$ be a hyperplane game satisfying (A.6). Then, for each $\rho \in [0,1)$, there exists a unique equilibrium. Furthermore, the equilibrium payoff configuration equals $(\chi_S)_{S\subset N}$.*

Theorem 21 *Let $(N, V, \mathcal{C})$ be an NTU game with coalition structure satisfying (A.6). If $a_\rho := (a_\rho(S))_{S\subset N}$ is an equilibrium payoff configuration for the new mechanism for each ρ and a is the limit of a_ρ when $\rho \to 1$, then $a = (\chi_S)_{S\subset N}$.*

We must note that, however χ does not generalize the Owen value for TU games, it does generalize the consistent value for NTU games with trivial coalition structure.

5 Appendix

5.1 *Proof* of Proposition 2

Fix $i, j \in C'_q \in \mathcal{C}_S$ with $j \neq i$. By definition of $a(S|_i)$:

$$\begin{aligned}
a(S|_i)^j &= \frac{1}{|\mathcal{C}_S|} \sum_{C'_r \in \mathcal{C}_S} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a(S, \gamma(C'_r))^j \\
&= \frac{1}{|\mathcal{C}_S|} \sum_{C'_r \in \mathcal{C}_S \setminus C'_q} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a\left(S, \gamma(C'_r)\right)^j + \frac{1}{|\mathcal{C}_S|} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a(S, \gamma(C'_q))^j \\
&= \frac{1}{|\mathcal{C}_S|} \sum_{C'_r \in \mathcal{C}_S \setminus C'_q} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a(S, \gamma(C'_r))^j + \frac{1}{|\mathcal{C}_S|} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a(S, i)^j \\
&= \frac{1}{|\mathcal{C}_S|} \sum_{C'_r \in \mathcal{C}_S \setminus C'_q} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a(S, \gamma(C'_r))^j + \frac{1}{|\mathcal{C}_S|} a(S, i)^j.
\end{aligned}$$

So,

$$a(S, i)^j = |\mathcal{C}_S| a(S|_i)^j - \sum_{C'_r \in \mathcal{C}_S \setminus C'_q} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a(S, \gamma(C'_r))^j. \tag{5}$$

We first prove that, under (C.3), (C.2) implies (C.2').

By (C.3), we know $a(S, \gamma(C'_r))^j = \rho a(S)^j + (1-\rho) a(S \setminus C'_r)^j$ for any $C'_r \neq C'_q$, so:

$$\begin{aligned}
a(S, i)^j &= |\mathcal{C}_S| a(S|_i)^j - \sum_{C'_r \in \mathcal{C}_S \setminus C'_q} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} \left[\rho a(S)^j + (1-\rho) a(S \setminus C'_r)^j\right] \\
&= |\mathcal{C}_S| a(S|_i)^j - \sum_{C'_r \in \mathcal{C}_S \setminus C'_q} \left[\rho a(S)^j + (1-\rho) a(S \setminus C'_r)^j\right]
\end{aligned}$$

by (C.2), we know $a(S|_i)^j = \rho a(S)^j + (1-\rho) a(S \setminus i)^j$, so:

$$\begin{aligned}
&= |\mathcal{C}_S| \left[\rho a(S)^j + (1-\rho) a(S \setminus i)^j\right] - \sum_{C'_r \in \mathcal{C}_S \setminus C'_q} \left[\rho a(S)^j + (1-\rho) a(S \setminus C'_r)^j\right] \\
&= \rho a(S)^j + (1-\rho) \left[|\mathcal{C}_S| a(S \setminus i)^j - \sum_{C'_r \in \mathcal{C}_S \setminus C'_q} a(S \setminus C'_r)^j\right].
\end{aligned}$$

which is condition (C.2').

We now prove the reciprocal. By (5):

$$a(S|_i)^j = \frac{1}{|\mathcal{C}_S|} \sum_{C'_r \in \mathcal{C}_S \setminus C'_q} \frac{1}{|\Gamma_{S,i}|} \sum_{\gamma \in \Gamma_{S,i}} a(S, \gamma(C'_r))^j + \frac{1}{|\mathcal{C}_S|} a(S, i)^j$$

by (C.3) and (C.2'),

$$
\begin{aligned}
&= \frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}\left[\rho a(S)^j+(1-\rho)a(S\backslash C'_r)^j\right] \\
&\quad +\frac{1}{|\mathcal{C}_S|}\left\{\rho a(S)^j+(1-\rho)\left[|\mathcal{C}_S|a(S\backslash i)^j-\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}a(S\backslash C'_r)^j\right]\right\} \\
&= \rho\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S)^j \\
&\quad +(1-\rho)\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S\backslash C'_r)^j \\
&\quad +\rho\frac{1}{|\mathcal{C}_S|}a(S)^j+(1-\rho)\frac{1}{|\mathcal{C}_S|}|\mathcal{C}_S|a(S\backslash i)^j \\
&\quad -(1-\rho)\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}a(S\backslash C'_r)^j \\
&= \rho\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}a(S)^j+(1-\rho)\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}a(S\backslash C'_r)^j \\
&\quad +\rho\frac{1}{|\mathcal{C}_S|}a(S)^j+(1-\rho)a(S\backslash i)^j-(1-\rho)\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}a(S\backslash C'_r)^j \\
&= \rho\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}a(S)^j+\rho\frac{1}{|\mathcal{C}_S|}a(S)^j+(1-\rho)a(S\backslash i)^j \\
&= \rho\frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S}a(S)^j+(1-\rho)a(S\backslash i)^j \\
&= \rho a(S)^j+(1-\rho)a(S\backslash i)^j.
\end{aligned}
$$

which is condition (C.2).□

5.2 *Proof* of Proposition 3

We first prove that $a(S|_i)^j = a(S|_k)^j, \forall i,k\in C'_q\in\mathcal{C}_S; j\notin C'_q$:

$$
\begin{aligned}
a(S|_i)^j &= \frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S,\gamma(C'_r))^j+\frac{1}{|\mathcal{C}_S|}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S,\gamma(C'_q))^j \\
&= \frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,i}|}\sum_{\gamma\in\Gamma_{S,i}}a(S,\gamma(C'_r))^j+\frac{1}{|\mathcal{C}_S|}a(S,i)^j \\
&= \frac{1}{|\mathcal{C}_S|}\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\frac{1}{|\Gamma_{S,k}|}\sum_{\gamma\in\Gamma_{S,k}}a(S,\gamma(C'_r))^j+\frac{1}{|\mathcal{C}_S|}a(S,i)^j
\end{aligned}
$$

by (C.3), $a(S,i)^j = \rho a(S)^j + (1-\rho)a(S\backslash C_q')^j = a(S,k)^j$, so:

$$= \frac{1}{|\mathcal{C}_S|} \sum_{C_r' \in \mathcal{C}_S \backslash C_q'} \frac{1}{|\Gamma_{S,k}|} \sum_{\gamma \in \Gamma_{S,k}} a(S,\gamma(C_r'))^j + \frac{1}{|\mathcal{C}_S|} a(S,k)^j = a(S|_k)^j.$$

Now, we have:

$$a(S)^j = \frac{1}{|C_q'|} \sum_{k \in C_q'} a(S|_k)^j = \frac{1}{|C_q'|} \sum_{k \in C_q'} a(S|_i)^j = a(S|_i)^j.$$

□

5.3 *Proof* of Proposition 4

We proceed by induction. The case of one player is trivial. Assume the result is true for hyperplane games with less than n players. Assume $V(N) = \left\{x \in \mathbb{R}^N : \lambda \cdot x \leq v(N)\right\}$ for some $\lambda \in \mathbb{R}^N_{++}$.

By Bergantiños and Vidal-Puga (2002), it is enough to prove that $a(N)$ satisfies (B.1), (B.2), and (B.3).

We know that $a(N) = \frac{1}{|\mathcal{C}|} \sum_{C_q \in \mathcal{C}} \frac{1}{|C_q|} \sum_{i \in C_q} a(N,i)$. Moreover, $\lambda \cdot a(N,i) = v(N)$ for each $i \in N$ because $a(N,i) \in \partial V(N)$ by (C.1). Then, $\lambda \cdot a(N) = v(N)$ and hence $a(N)$ satisfies (B.1).

We now prove that $a(N)$ satisfies (B.2). For each $\gamma \in \Gamma$ with $\gamma(C_q) = i \in C_q \in \mathcal{C}$:

$$\begin{aligned}
|\mathcal{C}| \sum_{j \in C_q} \lambda^j a(N)^j_\gamma &= \sum_{j \in C_q} \left(\sum_{C_r \in \mathcal{C}} \lambda^j a(N,\gamma(C_r))^j \right) \\
&= \sum_{j \in C_q} \lambda^j a(N,i)^j + \sum_{j \in C_q} \left(\sum_{C_r \in \mathcal{C} \backslash C_q} \lambda^j a(N,\gamma(C_r))^j \right) \\
&= \lambda^i a(N,i)^i + \sum_{j \in C_q \backslash i} \lambda^j a(N,i)^j + \sum_{C_r \in \mathcal{C} \backslash C_q} \left(\sum_{j \in C_q} \lambda^j a(N,\gamma(C_r))^j \right)
\end{aligned}$$

by (C.1), $\lambda \cdot a(N,i) = v(N)$ and then:

$$\begin{aligned}
&= v(N) - \sum_{j \in N \backslash i} \lambda^j a(N,i)^j + \sum_{j \in C_q \backslash i} \lambda^j a(N,i)^j + \sum_{C_r \in \mathcal{C} \backslash C_q} \left(\sum_{j \in C_q} \lambda^j a(N,\gamma(C_r))^j \right) \\
&= v(N) - \sum_{C_r \in \mathcal{C} \backslash C_q} \left(\sum_{j \in C_r} \lambda^j a(N,i)^j \right) + \sum_{C_r \in \mathcal{C} \backslash C_q} \left(\sum_{j \in C_q} \lambda^j a(N,\gamma(C_r))^j \right)
\end{aligned}$$

by (C.3),

$$= \quad v\left(N\right) - \sum_{C_r \in \mathcal{C} \backslash C_q} \left(\sum_{j \in C_r} \lambda^j \left[\rho a(N)^j + (1-\rho) a(N \backslash C_q)^j \right] \right)$$
$$+ \sum_{C_r \in \mathcal{C} \backslash C_q} \left(\sum_{j \in C_q} \lambda^j \left[\rho a(N)^j + (1-\rho) a(N \backslash C_r)^j \right] \right).$$

This amount is independent of γ. Then:

$$\begin{aligned}
|\mathcal{C}| \sum_{j \in C_q} \lambda^j a(N)^j &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |\mathcal{C}| \sum_{j \in C_q} \lambda^j a\left(N\right)^j_\gamma \\
&= v\left(N\right) - \sum_{C_r \in \mathcal{C} \backslash C_q} \left(\sum_{j \in C_r} \lambda^j \left[\rho a(N)^j + (1-\rho) a(N \backslash C_q)^j \right] \right) \\
&\quad + \sum_{C_r \in \mathcal{C} \backslash C_q} \left(\sum_{j \in C_q} \lambda^j \left[\rho a(N)^j + (1-\rho) a(N \backslash C_r)^j \right] \right).
\end{aligned}$$

Since $a\left(N\right)$ satisfies (B.1):

$$v\left(N\right) = \sum_{j \in N} \lambda^j a(N)^j = \sum_{j \in C_q} \lambda^j a(N)^j + \sum_{C_r \in \mathcal{C} \backslash C_q} \left(\sum_{j \in C_r} \lambda^j a(N)^j \right).$$

Hence,

$$\begin{aligned}
|\mathcal{C}| \sum_{j\in C_q} \lambda^j a\left(N\right)^j &= \sum_{j\in C_q} \lambda^j a(N)^j + \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \lambda^j a(N)^j \right) \\
&\quad - \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \lambda^j \left[\rho a(N)^j + (1-\rho) a(N\setminus C_q)^j\right] \right) \\
&\quad + \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_q} \lambda^j \left[\rho a(N)^j + (1-\rho) a(N\setminus C_r)^j\right] \right) \\
&= \sum_{j\in C_q} \lambda^j a(N)^j \\
&\quad + \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \lambda^j \left[(1-\rho) a(N)^j - (1-\rho) a(N\setminus C_q)^j\right] \right) \\
&\quad - \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_q} \lambda^j \left[(1-\rho-1) a(N)^j - (1-\rho) a(N\setminus C_r)^j\right] \right) \\
&= \sum_{j\in C_q} \lambda^j a(N)^j \\
&\quad +(1-\rho) \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \lambda^j \left[a(N)^j - a(N\setminus C_q)^j\right] \right) \\
&\quad -(1-\rho) \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_q} \lambda^j \left[a(N)^j - a(N\setminus C_r)^j\right] \right) \\
&\quad + \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_q} \lambda^j a(N)^j \right). \\
&= (1-\rho) \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_r} \lambda^j \left[a(N)^j - a(N\setminus C_q)^j\right] \right) \\
&\quad - (1-\rho) \sum_{C_r\in\mathcal{C}\setminus C_q} \left(\sum_{j\in C_q} \lambda^j \left[a(N)^j - a(N\setminus C_r)^j\right] \right) \\
&\quad |\mathcal{C}| \sum_{j\in C_q} \lambda^j a\left(N\right)^j .
\end{aligned}$$

From where we get (dividing by $(1-\rho)$):

$$\sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_r}\lambda^j\left[a(N)^j-a(N\backslash C_q)^j\right]\right)=\sum_{C_r\in\mathcal{C}\backslash C_q}\left(\sum_{j\in C_q}\lambda^j\left[a(N)^j-a(N\backslash C_r)^j\right]\right).$$

Which is precisely property (B.2) when $S=N$ and $\lambda_N=\lambda$.

We now prove that $(a(N))$ satisfies (B.3). Given $i\in C_q$, we know that:

$$|C_q|\lambda^i a(N)^i=\sum_{j\in C_q}\lambda^i a(N|_j)^i=\lambda^i a(N|_i)^i+\sum_{j\in C_q\backslash i}\lambda^i a(N|_j)^i.$$

Since $a\left(N|_i\right)=\frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}}\frac{1}{|\Gamma_i|}\sum_{\gamma\in\Gamma_i}a\left(S,\gamma\left(C_r\right)\right)$ and $a\left(N,\gamma\left(C_r\right)\right)\in\partial V\left(N\right)$ for each $\gamma\in\Gamma_i$ and $C_r\in\mathcal{C}$, we conclude that $\sum_{j\in N}\lambda^j a\left(N|_i\right)^j=v\left(N\right)$. Then,

$$|C_q|\lambda^i a(N)^i=v\left(N\right)-\sum_{j\in N\backslash C_q}\lambda^j a(N|_i)^j-\sum_{j\in C_q\backslash i}\lambda^j a(N|_i)^j+\sum_{j\in C_q\backslash i}\lambda^i a(N|_j)^i$$

we proved before that $a\left(N\right)\in\partial V\left(N\right)$ and hence,

$$\begin{aligned}=\;&\lambda^i a(N)^i+\sum_{j\in C_q\backslash i}\lambda^j a(N)^j+\underbrace{\sum_{j\in N\backslash C_q}\lambda^j a(N)^j}\\&-\underbrace{\sum_{j\in N\backslash C_q}\lambda^j a(N|_i)^j}-\sum_{j\in C_q\backslash i}\lambda^j a(N|_i)^j+\sum_{j\in C_q\backslash i}\lambda^i a(N|_j)^i\end{aligned}$$

the terms over the brackets are equal because $a(N)^j=a(N|_i)^j$ for all $j\in N\backslash C_q$ (Proposition 3)

$$=\lambda^i a(N)^i+\sum_{j\in C_q\backslash i}\lambda^j a(N)^j-\sum_{j\in C_q\backslash i}\lambda^j a(N|_i)^j+\sum_{j\in C_q\backslash i}\lambda^i a(N|_j)^i$$

by (C.2):

$$\begin{aligned}=\;&\lambda^i a(N)^i+\sum_{j\in C_q\backslash i}\lambda^j a(N)^j\\&-\sum_{j\in C_q\backslash i}\lambda^j\left[\rho a(N)^j+(1-\rho)a(N\backslash i)^j\right]+\sum_{j\in C_q\backslash i}\lambda^i\left[\rho a(N)^i+(1-\rho)a(N\backslash j)^i\right]\end{aligned}$$

we add and subtract $\sum_{j\in C_q\backslash i}\lambda^i a(N)^i$ and gather terms to obtain:

$$\begin{aligned}=\;&\sum_{j\in C_q}\lambda^i a(N)^i+(1-\rho)\sum_{j\in C_q\backslash i}\lambda^j a(N)^j\\&-(1-\rho)\sum_{j\in C_q\backslash i}\lambda^j a(N\backslash i)^j-(1-\rho)\sum_{j\in C_q\backslash i}\lambda^i a(N)^i+(1-\rho)\sum_{j\in C_q\backslash i}\lambda^i a(N\backslash j)^i.\end{aligned}$$

This first term is $|C_q|\lambda^i a(N)^i$. So, the rest of terms must equal zero. Dividing by $(1-\rho)$:

$$\sum_{j\in C_q\backslash i}\lambda^j a(N)^j - \sum_{j\in C_q\backslash i}\lambda^j a(N\backslash i)^j - \sum_{j\in C_q\backslash i}\lambda^i a(N)^i + \sum_{j\in C_q\backslash i}\lambda^i a(N\backslash j)^i = 0.$$

Or:

$$\sum_{j\in C_q\backslash i}\lambda^j\left[a(N)^j - a(N\backslash i)^j\right] = \sum_{j\in C_q\backslash i}\lambda^i\left[a(N)^i - a(N\backslash j)^i\right].$$

Which is property (B.3) when $S=N$ and $\lambda_N=\lambda$.□

5.4 *Proof* of Proposition 6

5.4.1 Zero-monotonic TU games

By Proposition 4, we know that $a(S)=\phi_S$ for all $S\subset N$.

We use next result, proved by Bergantiños and Vidal-Puga (2001). Given a triple $(N,v,\mathcal{C})$ such that (N,v) is a zero-monotonic TU game, $S\subset N$ and $i\in C'_q\in\mathcal{C}_S$, then

$$\sum_{j\in C'_q}\phi_S^j \geq \sum_{j\in C'_q\backslash i}\phi_{S\backslash i}^j + v(i).$$

By normalization, $v(i)\geq 0$ and thus

$$\sum_{j\in C'_q}\phi_S^j \geq \sum_{j\in C'_q\backslash i}\phi_{S\backslash i}^j. \tag{6}$$

Now, we have:

$$\begin{aligned}\sum_{j\in S}c(S,i)^j &= -\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\phi_{S\backslash C'_r}^i + \sum_{j\in C'_q\backslash i}\left(|\mathcal{C}_S|\,\phi_{S\backslash i}^j - \sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\phi_{S\backslash C'_r}^j\right) + \sum_{j\in S\backslash C'_q}\phi_{S\backslash C'_q}^j\\ &= \sum_{j\in C'_q\backslash i}|\mathcal{C}_S|\,\phi_{S\backslash i}^j + \sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\left(\sum_{j\in C'_r}\phi_{S\backslash C'_q}^j - \sum_{j\in C'_q}\phi_{S\backslash C'_r}^j\right)\end{aligned}$$

by (B.2), $\sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\left(\sum_{j\in C'_r}\phi_{S\backslash C'_q}^j - \sum_{j\in C'_q}\phi_{S\backslash C'_r}^j\right) = \sum_{C'_r\in\mathcal{C}_S\backslash C'_q}\left(\sum_{j\in C'_r}\phi_S^j - \sum_{j\in C'_q}\phi_S^j\right)$

and thus,

$$\begin{aligned}
&= \sum_{j\in C'_q\setminus i} |\mathcal{C}_S|\, \phi^j_{S\setminus i} + \sum_{C'_r\in\mathcal{C}_S\setminus C'_q} \left(\sum_{j\in C'_r} \phi^j_S - \sum_{j\in C'_q} \phi^j_S \right) \\
&= \sum_{j\in C'_q\setminus i} |\mathcal{C}_S|\, \phi^j_{S\setminus i} + \sum_{j\in S\setminus C'_q} \phi^j_S - (|\mathcal{C}_S| - 1) \sum_{j\in C'_q} \phi^j_S \\
&= |\mathcal{C}_S| \left(\sum_{j\in C'_q\setminus i} \phi^j_{S\setminus i} - \sum_{j\in C'_q} \phi^j_S \right) + \sum_{j\in S} \phi^j_S
\end{aligned}$$

by (6):

$$\leq \sum_{j\in S} \phi^j_S = v(S).$$

which means that (A.5) holds for $(N, v, \mathcal{C})$.□

5.4.2 Essential three-player hyperplane games

By Proposition 4, we know that $a(S) = \Phi_S$ for all $S \subset N$.

We consider $i = 1$ and the coalition structure $\mathcal{C} = \{\{1,2\},\{3\}\}$. The other possibilities are equivalent or trivial.

Let $(\lambda_S)_{S\subset N}$ be the coefficients of a hyperplane game $(N, V, \mathcal{C})$. We want to prove that $-\lambda^1_N \Phi^1_{12} + 2\lambda^2_N \Phi^2_{23} - \lambda^2_N \Phi^2_{12} + \lambda^3_N \Phi^3_3 \leq v(N)$.

By applying the inductive formula given by (2), we have,

$$\begin{aligned}
-\lambda^1_N \Phi^1_{12} &= -\frac{\lambda^1_N v(12)}{2\lambda^1_{12}} - \frac{\lambda^1_N r^1}{2} + \frac{\lambda^2_{12}\lambda^1_N r^2}{2\lambda^1_{12}} \\
2\lambda^2_N \Phi^2_{23} &= \frac{\lambda^2_N v(23)}{\lambda^2_{23}} + \lambda^2_N r^2 - \frac{\lambda^3_{23}\lambda^2_N r^3}{\lambda^2_{23}} \\
-\lambda^2_N \Phi^2_{12} &= -\frac{\lambda^2_N v(12)}{2\lambda^2_{12}} - \frac{\lambda^2_N r^2}{2} + \frac{\lambda^1_{12}\lambda^2_N r^1}{2\lambda^2_{12}} \\
\lambda^3_N \Phi^3_3 &= \lambda^3_N r^3.
\end{aligned}$$

So, their sum is

$$\begin{aligned}
&-\frac{\lambda^1_N v(12)}{2\lambda^1_{12}} - \frac{\lambda^1_N r^1}{2} + \frac{\lambda^2_{12}\lambda^1_N r^2}{2\lambda^1_{12}} + \frac{\lambda^2_N v(23)}{\lambda^2_{23}} + \lambda^2_N r^2 \\
&-\frac{\lambda^3_{23}\lambda^2_N r^3}{\lambda^2_{23}} - \frac{\lambda^2_N v(12)}{2\lambda^2_{12}} - \frac{\lambda^2_N r^2}{2} + \frac{\lambda^1_{12}\lambda^2_N r^1}{2\lambda^2_{12}} + \lambda^3_N r^3.
\end{aligned} \tag{7}$$

Let $\left(\frac{v(23) - \lambda^3_{23} r^3}{\lambda^2_{23}}, r^3\right) \in V(23)$. By monotonicity,

$$\left(0, \frac{v(23) - \lambda^3_{23} r^3}{\lambda^2_{23}}, r^3\right) \in V(N).$$

Thus,

$$\frac{\lambda_N^2 v(23)}{\lambda_{23}^2} - \frac{\lambda_N^2 \lambda_{23}^3 r^3}{\lambda_{23}^2} + \lambda_N^3 r^3 \leq v(N).$$

So, the amount given in (7) is not larger than

$$\begin{aligned} &-\frac{\lambda_N^1 v(12)}{2\lambda_{12}^1} - \frac{\lambda_N^1 r^1}{2} + \frac{\lambda_{12}^2 \lambda_N^1 r^2}{2\lambda_{12}^1} + \lambda_N^2 r^2 \\ &-\frac{\lambda_N^2 v(12)}{2\lambda_{12}^2} - \frac{\lambda_N^2 r^2}{2} + \frac{\lambda_{12}^1 \lambda_N^2 r^1}{2\lambda_{12}^2} + v(N). \end{aligned} \tag{8}$$

By essentiality, $\left(r^1, r^2\right) \in V(12)$. So, $\lambda_{12}^1 r^1 + \lambda_{12}^2 r^2 \leq v(12)$. Thus, the expression given by (8) is not more than

$$-\lambda_N^1 r^1 + v(N) \leq v(N).$$

□

5.4.3 Pure bargaining games

We first prove by induction that, for $S \subseteq N$, $a(S) = r^S$. By (C.1), the result is trivial for $n = 1$. Assume that $a(T) = r^T$ for all $T \subset S$. Then, given $i \in C_q' \in \mathcal{C}_S$:

By (C.2'), $a(S,i)^j = \rho a(S)^j + (1-\rho)\left(|\mathcal{C}_S| r^j - \sum_{C_r' \in \mathcal{C}_S \setminus C_q'} r^j\right) = \rho a(S)^j + (1-\rho) r^j$ for all $j \in C_q' \setminus i$.

By (C.3), $a(S,i)^j = \rho a(S)^j + (1-\rho) r^j$ for all $j \in S \setminus C_q'$.

Thus, $a(S,i)$ coincide with $\rho a(S) + (1-\rho) r$ in all coordinates but (at most) the ith. Moreover, both $a(S)$ and r belong to $V(S)$, and so $\rho a(S) + (1-\rho) r$ does. Thus, by (C.1), $a(S,i)^i \geq \rho a(S)^i + (1-\rho) r^i$. By averaging over i, we have $a(S)^i \geq \rho a(S)^i + (1-\rho) r^i$ and thus $a(S)^i \geq r^i$. We have then $a(S) \geq r^S$. Since $r^S \in \partial V(S)$ and $a(S) \in V(S)$, we conclude that $a(S) = r^S$.

Now, we have:

$$\begin{aligned} c(S,i) &= \left(-\sum_{C_r \in \mathcal{C}_S \setminus C_q} r^i, \left(|\mathcal{C}_S| r^j - \sum_{C_r \in \mathcal{C}_S \setminus C_q} r^j\right)_{j \in C_q' \setminus i}, \left(r^j\right)_{j \in S \setminus C_q'}\right) \\ &= \left(-\left(|\mathcal{C}| - 1\right) r^i, \left(r^j\right)_{j \in C_q' \setminus i}, \left(r^j\right)_{j \in S \setminus C_q'}\right) \\ &= \left(-\left(|\mathcal{C}| - 1\right) r^i, r^{S \setminus i}\right). \end{aligned}$$

By (A.4), $r^i \geq 0$ and thus $c(S,i) \leq \left(0, r^{S \setminus i}\right)$. By (A.3), $\left(0, r^{S \setminus i}\right) \in V(S)$. By comprehensiveness, $c(S,i) \in V(S)$.□

5.5 *Proof* of Proposition 7

We proceed by induction. The result holds trivially when $n = 1$. Assume that it is true when there are at most $n - 1$ players.

Assume we are in an equilibrium. By induction hypothesis, the expected payoff for the players in $S \subsetneq N$ in any equilibrium with S as set of active players is $a(S)$. Let $b_N \in \mathbb{R}^N$ be the expected payoff when N is the set of active players. We must prove that (C.1), (C.2), and (C.3) hold for $S = N$.

We proceed by a series of Claims:

> *Claim (A)*: Given $C_q \in \mathcal{C}$ on the second stage, assume the proposers are determined by $\gamma \in \Gamma$ and the r.p. is $\gamma(C_q)$. Then, all players in $N \backslash C_q$ accept $\gamma(C_q)$'s proposal if $a(N, \gamma(C_q))^i \geq \rho b_N^i + (1-\rho)\, a(N \backslash C_q)^i$ for every $i \in N \backslash C_q$. Otherwise, the proposal is rejected.

Notice that, in the case of rejection on the second stage, the expected payoff of a player $i \in N \backslash C_q$ is, by induction hypothesis, $\rho b_N^i + (1-\rho) a(N \backslash C_q)^i$.

Suppose we reach the second stage. We assume without loss of generality that $C_q = \{1, 2, ..., c_q\}$ and $(c_q + 1, ..., n)$ is the order in which the players in $N \backslash C_q$ are asked.

If the game reaches player n, i.e., there has been no previous rejection, his optimal strategy involves accepting the proposal if $a(N, \gamma(C_q))^n$ is equal (by the tie-breaking rule) or higher than $\rho b_N^n + (1-\rho) a(N \backslash C_q)^n$ and rejecting it if it is lower than $\rho b_N^n + (1-\rho) a(N \backslash C_q)^n$. Player $n - 1 \in N \backslash C_q$, anticipates reaction of player n. Hence, if $a(N, \gamma(C_q))^n \geq \rho b_N^n + (1-\rho) a(N \backslash C_q)^n$ and $a(N, \gamma(C_q))^{n-1} \geq \rho b_N^{n-1} + (1-\rho) a(N \backslash C_q)^{n-1}$, and the game reaches player $n - 1$, he accepts the proposal. If $a(N, 1)^n < \rho b_N^n + (1-\rho) a(N \backslash C_q)^n$, player $n - 1$ is indifferent between accepting or rejecting the proposal, since he knows player n is bound to reject the proposal should the game reach him. In any case, the proposal is rejected. By going backwards, we prove the result for all players in $N \backslash C_q$ on the second stage.

> *Claim (B)*: Let $\gamma \in \Gamma$ be the correspondence which determines the set of proposers on the first stage. Given any $C_q \in \mathcal{C}$, assume all the coalitions which choose representative after C_q are bound to choose their proposer as representative should the game reach them. Given $i \in C_q$, let $b_{N,i}$ be the expected final payoff in equilibrium restricted to i be a representative. Then, all players in $C_q \backslash \gamma(C_q)$ accept $\gamma(C_q)$'s proposal if $b_{N,\gamma(C_q)}^j \geq \rho b_N^j + (1-\rho)\, a(N \backslash \gamma(C_q))^j$ for every $j \in C_q \backslash \gamma(C_q)$. Otherwise, the proposal is rejected.

Notice that, under our hypothesis, in the case of rejection of $\gamma(C_q)$'s proposal on the first stage, the expected payoff to a player $j \in C_q \backslash \gamma(C_q)$ is $\rho b_N^j + (1 - \rho) a(N \backslash \gamma(C_q))^j$.

We assume without loss of generality that $C_q = \{1, ..., c_q\}$, $\gamma(C_q) = 1$ and players in $C_q \backslash 1$ are asked in the order $(2, ..., c_q)$.

If the game reaches player c_q, i.e., there has been no previous rejection, his optimal strategy involves accepting any proposal from 1 satisfying $b_{N,1}^{c_q} \geq \rho b_N^{c_q} + (1-\rho)a(N\backslash 1)^{c_q}$ and rejecting any proposal such that $b_{N,1}^{c_q} < \rho b_N^{c_q} + (1-\rho)a(N\backslash 1)^{c_q}$. Player $c_q - 1 \in C_q$, anticipates reaction of player c_q. Hence, if $b_{N,1}^{c_q} \geq \rho b_N^{c_q} + (1-\rho)a(N\backslash 1)^{c_q}$ and $b_{N,1}^{c_q-1} \geq \rho b_N^{c_q-1} + (1-\rho)a(N\backslash 1)^{c_q-1}$, and the game reaches player $c_q - 1$, he accepts the proposal. If $b_{N,1}^{c_q} < \rho b_N^{c_q} + (1-\rho)a(N\backslash 1)^{c_q}$, player $c_q - 1$ is indifferent between accepting or rejecting the proposal, since he knows player c_q is bound to reject the proposal should the game reach him. In any case, the proposal is rejected. By going backwards, we prove the result for all players in $C_q\backslash 1$ on the first stage.

Claim (C): All the offers on the first stage are accepted.

Assume coalitions play the first stage in the order $(C_1, C_2, ..., C_p)$ and that the mechanism reaches coalition C_p; i.e. there has been no previous rejection. Assume the proposal of $\gamma(C_p)$ is rejected. This means the final payoff for player $\gamma(C_p)$ is $\rho b_N^{\gamma(C_p)}$.

We can assume without loss of generality that $C_p = \{1, 2, ..., c_p\}$, $\gamma(C_p) = 1$ and players are asked in the order $(2, ..., c_p)$.

We define a new proposal $a(N,1)$ for player 1 as follows. Let $c(N,1)$ be defined as in (4). By (A.5) and induction hypothesis, $c(N,1) \in V(N)$. By convexity, $\rho b_N + (1-\rho)c(N,1) \in V(N)$. Let $a(N,1) = \rho b_N + (1-\rho)c(N,1)$.

Assume the mechanism reaches c_p; i.e. has not been previous rejection. Then, by rejecting $a(N,1)$, the expected final payoff for player c_p is $\rho b_N^{c_p} + (1-\rho)a(N\backslash 1)^{c_p}$.

If c_p accepts $a(N,1)$ and the proposal chosen in the second stage is from $C_r \neq C_q$, then c_p can obtain $\rho b_N^{c_p} + (1-\rho)a(N\backslash C_r)^{c_p}$ by rejecting it. If the proposal chosen in the second stage is from C_q, then it is accepted (by *Claim (A)*).

Thus, if c_p accepts $a(N,1)$, his expected final payoff is at least:

$$\frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}\backslash C_p}\left[\rho b_N^{c_p} + (1-\rho)a(N\backslash C_r)^{c_p}\right] + \frac{1}{|\mathcal{C}|}a(N,1)^{c_p}$$

$$\begin{aligned}
= \;& \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}\backslash C_p}\rho b_N^{c_p} + \frac{1}{|\mathcal{C}|}(1-\rho)\sum_{C_r\in\mathcal{C}\backslash C_p}a(N\backslash C_r)^{c_p} \\
& + \frac{1}{|\mathcal{C}|}\rho b_N^{c_p} + (1-\rho)\left(a(N\backslash 1)^{c_p} - \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}\backslash C_p}a(N\backslash C_r)^{c_p}\right) \\
= \;& \rho b_N^{c_p} + (1-\rho)a(N\backslash 1)^{c_p}.
\end{aligned}$$

Thus, by the tie-breaking rule, it is optimal for c_p to accept $a(N,1)$. By going backwards, we can prove that it is optimal for $c_p - 1, c_p - 2, ..., 2$ to accept

$a(N,1)$. Furthermore, the expected final payoff for player 1 is not less than:

$$\begin{aligned} & \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}\backslash C_p}\left[\rho b_N^1+(1-\rho)\,a\,(N\backslash C_r)^1\right]+\frac{1}{|\mathcal{C}|}a\,(N,1)^1 \\ = \; & \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}\backslash C_p}\left[\rho b_N^1+(1-\rho)\,a\,(N\backslash C_r)^1\right] \\ & +\frac{1}{|\mathcal{C}|}\left[\rho b_N^1-(1-\rho)\sum_{C_r\in\mathcal{C}\backslash C_p}a\,(N\backslash C_r)^1\right] \\ = \; & \rho b_N^1. \end{aligned}$$

So, by the tie-breaking rule, it is optimal for 1 to change his proposal to $a(N,1)$. This contradiction proves that there are not proposals rejected on the first stage in C_p. By going backwards, we prove that no proposal is rejected on the first stage in $C_{p-1},...,C_1$.

Claim (D): All the offers on the second stage are accepted.

Suppose the proposal of $\gamma(C_q)$ is rejected on the second stage. Then, the final payoff for the members of C_q (including $\gamma(C_q)$) is 0 with probability $\frac{1}{|\mathcal{C}|}>0$. By *Claim (B)*, we know that $b^j_{N,\gamma(C_q)}\geq\rho b_N^j+(1-\rho)\,a\,(N\backslash\gamma(C_q))^j$ for all $j\in C_q\backslash\gamma(C_q)$. Assume that $\gamma(C_q)$ change his strategy and proposes

$$a\,(N,\gamma(C_q))=\left(0^{C_q},\rho b_N^{N\backslash C_q}+(1-\rho)\,a\,(N\backslash C_q)^{N\backslash C_q}\right).$$

By convexity and monotonicity, $a(N,\gamma(C_q))\in V(N)$. By *Claim (A)*, this proposal is bound to be accepted should $\gamma(C_q)$ be the r.p. on the second stage. However, $b_{N,\gamma(C_q)}$ remains unaltered. So, by *Claim (B)*, $a(N,\gamma(C_q))$ is also accepted on the first stage. Moreover, the expected final payoff for $\gamma(C_q)$ also remains the same. By the tie-breaking rule, we are not in an equilibrium. This contradiction proves that the proposals on the second stage are always accepted.

Since all the proposals are accepted, we can assure that $b_N=a(N)$ and $b_{N,i}=a(N|_i)$ for all $i\in N$.

We show now (C.1), (C.2), and (C.3) hold.

Suppose (C.1) does not hold, i.e., there exists a player $i\in C_q$ such that $a(N,i)$ is not Pareto optimal. Thus, $a(N,i)$ belongs to the interior of $V(N)$; so, there exists $\varepsilon>0$ such that $d:=a(N,i)+\left(\varepsilon,0^{N\backslash i}\right)\in V(N)$.

Notice that, since the proposal $a(N,i)$ of player i is accepted, by *Claim (B)* – together with *Claim (C)* and *Claim (D)* – we know that $a(N|_i)^j\geq\rho b_N^j+(1-\rho)a(N\backslash i)^j$ for every $j\in C_q\backslash i$ and, by *Claim (A)*, $a(N,i)^j\geq\rho b_N^j+(1-\rho)a(N\backslash C_q)^j$ for every $j\in N\backslash C_q$. So $d^j\geq\rho b_N^j+(1-\rho)a(N\backslash i)^j$ for every $j\in C_q\backslash i$ and $d^j\geq\rho b_N^j+(1-\rho)a(N\backslash C_q)^j$ for every $j\in N\backslash C_q$. By *Claim (A)* and *Claim (B)*, if player i changes his proposal to d, it is bound to be accepted and his expected final payoff improves by $\frac{\varepsilon}{|\mathcal{C}|\,|C_q|} > 0$. This contradiction proves (C.1).

Suppose (C.2) does not hold. Let $j_0 \in C_q \backslash i$ be a player such that $a(N|_i)^{j_0} = \rho a\,(N)^{j_0} + (1-\rho)a(N\backslash i)^{j_0} + \alpha$ with $\alpha \neq 0$. By *Claim (B)*, $\alpha > 0$.

By comprehensiveness and nonlevelness, we have $a(N,i) - \left(|\mathcal{C}|\,\alpha, 0^{N\backslash j_0}\right)$ belongs to the interior of $V(N)$. Thus, there exists an $\varepsilon > 0$ such that $\widehat{a}(N,i) := a(N,i) - \left(|\mathcal{C}|\,\alpha, 0^{N\backslash j_0}\right) + (\varepsilon, 0^{N\backslash i})$ belongs to $V(N)$. Suppose player i changes his proposal to $\widehat{a}(N,i)$. The new value $\widehat{a}(N|_i)$ satisfies:

$\widehat{a}(N|_i)^i = a(N|_i)^i + \frac{\varepsilon}{|\mathcal{C}|}$;

$\widehat{a}(N|_i)^{j_0} = a(N|_i)^{j_0} - \alpha = \rho b_N^{j_0} + (1-\rho)a(N\backslash i)^{j_0}$;

$\widehat{a}(N|_i)^j = a(N|_i)^j \geq \rho b_N^j + (1-\rho)a(N\backslash i)^j$ for all $j \in C_q \backslash \{i, j_0\}$;

$\widehat{a}(N,i)^j = a(N,i)^j \geq \rho b_N^j + (1-\rho)a(N\backslash C_q)^j$ for all $j \in N\backslash C_q$.

So, by *Claim (A)* and *Claim (B)*, the new proposal of player i is due to be accepted. Also, player i improves his expected payoff by $\frac{\varepsilon}{|\mathcal{C}|\,|C_q|} > 0$. This contradiction proves (C.2).

The reasoning for (C.3) is similar to (C.2) and we omit it.

It remains to show that $a\,(N) \geq 0$. Notice that player $i \in N$ can guarantee a payoff of at least 0 by proposing always 0^N and accepting only proposals which give him a nonnegative expected payoff. Thus, $a\,(N) \geq 0$.

Conversely, we show that proposals $(a(S,i))_{i\in S\subset N}$ satisfying (C.1), (C.2) and (C.3) can be supported as an equilibrium.

First, we prove that $a\,(S) \geq 0$ for all $S \subset N$. By induction hypothesis, this is true for any S □ N. Given $i \in C_q \in \mathcal{C}$, by (A.5), we have $c\,(N,i) \in V\,(N)$. By convexity, $\widetilde{c}\,(N,i) := \rho a\,(N) + (1-\rho)\,c\,(N,i) \in V\,(N)$.

Since $a\,(N,i)$ satisfies (C.2) and (C.3), by Proposition 2, $a\,(N,i)$ also satisfies (C.2'). Then, $a\,(N,i)^{N\backslash i} = \widetilde{c}\,(N,i)^{N\backslash i}$. We now conclude that $a\,(N,i) \geq \widetilde{c}\,(N,i)$ because $a\,(N,i) \in \partial V\,(N)$ and $\widetilde{c}\,(N,i) \in V\,(N)$. Hence,

$$a\,(N,i)^i \geq \widetilde{c}\,(N,i)^i = \rho a\,(N)^i - (1-\rho) \sum_{C_r \in \mathcal{C}\backslash C_q} a\,(N\backslash C_r)^i\,.$$

So,

$$\begin{aligned} a\,(N|_i)^i &= \frac{1}{|\mathcal{C}|} \sum_{C_r \in \mathcal{C}} \frac{1}{|\Gamma_i|} \sum_{\gamma \in \Gamma_i} a\,(N, \gamma\,(C_r))^i \\ &= \frac{1}{|\mathcal{C}|} \sum_{C_r \in \mathcal{C}\backslash C_q} \frac{1}{|\Gamma_i|} \sum_{\gamma \in \Gamma_i} a\,(N, \gamma\,(C_r))^i + \frac{1}{|\mathcal{C}|} a\,(N,i)^i \end{aligned}$$

by (C.3),

$$
\begin{aligned}
&= \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}\backslash C_q}\left[\rho a\left(N\right)^i+(1-\rho)\,a\left(N\backslash C_r\right)^i\right]+\frac{1}{|\mathcal{C}|}a\left(N,i\right)^i \\
&\geq \frac{1}{|\mathcal{C}|}\sum_{C_r\in\mathcal{C}\backslash C_q}\left[\rho a\left(N\right)^i+(1-\rho)\,a\left(N\backslash C_r\right)^i\right] \\
&\quad +\frac{1}{|\mathcal{C}|}\left[\rho a\left(N\right)^i-(1-\rho)\sum_{C_r\in\mathcal{C}\backslash C_q}a\left(N\backslash C_r\right)^i\right] \\
&= \rho a\left(N\right)^i .
\end{aligned}
$$

Furthermore, by (C.2) and $a\left(N\backslash j\right)\geq 0$, we have $a\left(N|_j\right)^i\geq \rho a\left(N\right)^i$ for all $j\in C_q\backslash i$. Thus,

$$
a\left(N\right)^i=\frac{1}{|C_q|}\sum_{j\in C_q}a\left(N|_j\right)^i\geq\frac{1}{|C_q|}\sum_{j\in C_q}\rho a\left(N\right)^i=\rho a\left(N\right)^i
$$

and so $a\left(N\right)^i\geq 0$. Moreover,

$$
a\left(N|_i\right)^i\geq\rho a\left(N\right)^i\geq 0.
$$

We now follow the same reasoning by Hart and Mas-Colell to verify that the strategies corresponding to these proposals form an equilibrium. By the induction hypothesis, this is so in any subgame after a player (or coalition) has dropped out. Fix a player $i\in C_q$. If he rejects the proposal from a proposer $j\in C_q\backslash i$, his expected final payoff is $\rho a(N)^i+(1-\rho)a(N\backslash j)^i$. If he rejects the proposal from a r.p. $j\in C_r\neq C_q$, his expected final payoff is $\rho a(N)^i+(1-\rho)a(N\backslash C_r)^i$. In any case, his expected final payoff is the same as that the other player is offering. Since the rest of the players accept the proposal, he does not improve his expected final payoff by rejecting it. If the proposer is player i himself, the strategies of the other players do not allow him to decrease his proposal to any of them (since it would be rejected by *Claim (A)* and *Claim* (*B*)). Moreover, increasing one or more of his offers to the other players keeping the rest unaltered implies his own payment decreases (by (C.1) and nonlevelness). Finally, by offering an unacceptable proposal, he may be dropped out and his expected final payment becomes 0, which does not improve his final payoff (because $a\left(N|_i\right)^i\geq 0$). Thus, the proposals do form an equilibrium.□

5.6 *Proof* of Corollary 10

Fix $i\in C_q\in\mathcal{C}$. Given $j\in N\backslash C_q$, by (C.3):

$$
\left|a(N,i)^j-a(N)^j\right|=\left|\rho a(N)^j+(1-\rho)a(N\backslash C_q)^j-a(N)^j\right|=(1-\rho)\left|a(N\backslash C_q)^j-a(N)^j\right|.
$$

So, we take $M_1 \in \mathbb{R}$ as the maximum of the set:

$$\left\{ \left| a(N\backslash C_q)^j - a(N)^j \right| : C_q \in \mathcal{C}, j \in N\backslash C_q \right\}.$$

This maximum exists because $a(S) \geq 0$ for all $S \subset N$.[4]
We have then $\left| a(N,i)^j - a(N)^j \right| \leq M_1(1-\rho)$ for all $j \in N\backslash C_q$.
Given $j \in C_q\backslash i$, by (C.2'):

$$\begin{aligned}
& \left| a(N,i)^j - a(N)^j \right| \\
= & \left| \rho a(N)^j + (1-\rho)\left[|\mathcal{C}|\, a(N\backslash i)^j - \sum_{C_r \in \mathcal{C}\backslash C_q} a(N\backslash C_r)^j \right] - a(N)^j \right| \\
= & (1-\rho)\left| |\mathcal{C}|\, a(N\backslash i)^j - \sum_{C_r \in \mathcal{C}\backslash C_q} a(N\backslash C_r)^j - a(N)^j \right|
\end{aligned}$$

Thus, we take $M_2 \in \mathbb{R}$ as the maximum of the set:

$$\left\{ \left| |\mathcal{C}|\, a(N\backslash i)^j - \sum_{C_r \in \mathcal{C}\backslash C_q} a(N\backslash C_r)^j - a(N)^j \right| : i, j \in C_q, j \neq i \right\}$$

This maximum exists because $a(S) \geq 0$ for all $S \subset N$.
We have then $\left| a(N,i)^j - a(N)^j \right| \leq M_2(1-\rho)$.
We now study $\left| a(N,i)^i - a(N)^i \right|$. We know:

$$a(N)^i = \sum_{C_r \in \mathcal{C}} \left[\sum_{j \in C_r} \frac{1}{|\mathcal{C}|\, |C_r|} a(N,j)^i \right].$$

Then,

$$a(N,i)^i = |\mathcal{C}|\, |C_q| \left[a(N)^i - \sum_{C_r \in \mathcal{C}\backslash C_q} \left(\sum_{j \in C_r} \frac{1}{|\mathcal{C}|\, |C_r|} a(N,j)^i \right) - \sum_{j \in C_q\backslash i} \frac{1}{|\mathcal{C}|\, |C_q|} a(N,j)^i \right].$$

[4]This set may be no finite, because we are considering the proposals for any ρ.

So:

$$
\begin{aligned}
& \left|a(N,i)^i - a(N)^i\right| \\
= \quad & |\mathcal{C}|\,|C_q| \left| a(N)^i - \sum_{C_r \in \mathcal{C}\backslash C_q} \left(\sum_{j \in C_r} \frac{1}{|\mathcal{C}|\,|C_r|} a(N,j)^i \right) - \sum_{j \in C_q \backslash i} \frac{1}{|\mathcal{C}|\,|C_q|} a\left(N,j\right)^i - \frac{1}{|\mathcal{C}|\,|C_q|} a(N)^i \right| \\
= \quad & |\mathcal{C}|\,|C_q| \left| \sum_{C_r \in \mathcal{C}\backslash C_q} \left(\sum_{j \in C_r} \frac{1}{|\mathcal{C}|\,|C_r|} \left[a(N)^i - a(N,j)^i\right] \right) - \sum_{j \in C_q \backslash i} \frac{1}{|\mathcal{C}|\,|C_q|} \left[a(N)^i - a(N,j)^i\right] \right| \\
\leq \quad & \sum_{C_r \in \mathcal{C}\backslash C_q} \left(\sum_{j \in C_r} \frac{|C_q|}{|C_r|} \left|a(N)^i - a(N,j)^i\right| \right) + \sum_{j \in C_q \backslash i} \left|a(N)^i - a(N,j)^i\right| \\
\leq \quad & \sum_{C_r \in \mathcal{C}\backslash C_q} \left(\sum_{j \in C_r} \frac{|C_q|}{|C_r|} M_1 \right) + \sum_{j \in C_q \backslash i} M_2 = |C_q|\left(|\mathcal{C}| - 1\right) M_1 + \left(|C_q| - 1\right) M_2.
\end{aligned}
$$

So, we take $M = \max\left\{|C_q|(|\mathcal{C}| - 1)M_1 + (|C_q| - 1)M_2 : C_q \in \mathcal{C}\right\}$.$\square$

5.7 *Proof* of Theorem 11

By Proposition 7, we only need to prove that there exist proposals satisfying (C.1), (C.2), and (C.3). We proceed by induction on the number of players. Clearly, the result is true for $n = 1$. Assume now that we have $a\left(S,i\right)$ for each $S \subset N$ and each $i \in S$ satisfying (C.1), (C.2) and (C.3) when $S \square N$. By Proposition 7, $a\left(S\right) \geq 0$ for all $S \square N$.

For each $i \in C_q \in \mathcal{C}$, by property (A.5), the vector $c\left(N,i\right)$ belongs to $V\left(N\right)$. By property (A.2), there exists an unique $z_i \in \partial V(N)$ such that $z_i^j = c\left(N,i\right)^j$ for all $j \in N\backslash i$. We define:

$$
\begin{aligned}
\beta_1 &= \min\{a^i\left(S\right) : i \in S \square N\} \in \mathbb{R} \\
\beta_2 &= \min\left\{z_j^i : i,j \in N\right\} \in \mathbb{R} \\
\beta &= \min\left(\beta_1,\beta_2\right) \in \mathbb{R} \\
K &= \left\{x \in V(N) : x \geq \left(\beta,...,\beta\right)\right\}.
\end{aligned}
$$

This set K is nonempty ($z_i \in K$ for all $i \in N$), closed (because $V(N)$ is closed) and bounded (by (A.1)). Thus, K is a compact set. Furthermore, K is convex (because $V(N)$ is convex).

We define n functions $\alpha_i : K \to K$ as follows. Given $i \in C_q \in \mathcal{C}$, $\alpha_i^j(x) := \rho x^j + (1-\rho)c\left(N,i\right)^j$ for each $j \in N\backslash i$ and $\alpha_i^i\left(x\right)$ is defined in such a way that $\alpha_i\left(x\right) \in \partial V\left(N\right)$.

These functions are well defined, because $y_i := \rho x + (1-\rho)z_i$ belongs to K (by convexity) and $\alpha_i(x)$ equals y_i in all coordinates but i's, which we increase until reaching the boundary of $V(N)$.

Also, because of the smoothness of property (A.2) the functions α_i are continuous. By the convexity of the domain, $\frac{1}{|\mathcal{C}|}\sum_{C_q\in\mathcal{C}}\frac{1}{|C_q|}\sum_{i\in C_q}\alpha_i(x)\in K$ for each $x\in K$. By a standard fix point theorem, there exists a vector $a(N)\in K$ satisfying $a(N)=\frac{1}{|\mathcal{C}|}\sum_{C_q\in\mathcal{C}}\frac{1}{|C_q|}\sum_{i\in C_q}\alpha_i(a(N))$.

We define $a(N,i)=\alpha_i(a(N))$ for each $i\in N$. It is trivial to see that $(a(N,i))_{i\in N}$ satisfies (C.1), (C.2'), and (C.3). By Proposition 2, $(a(N,i))_{i\in N}$ also satisfies (C.2).□

5.8 *Proof* of Theorem 14

By Bergantiños and Vidal-Puga (2002), it is enough to prove that $a=(a(S))_{S\subset N}$ satisfies (B.1), (B.2), and (B.3). By Corollary 10, $a_\rho(S,i)\to a(S)$ for any $i\in S\subset N$. Since $a_\rho(S,i)\in\partial V(S)$ for every $S\subset N$ and every $\rho\in[0,1)$, and $\partial V(S)$ is closed, we conclude that $a(S)\in\partial V(S)$ for every $S\subset N$. Thus, a satisfies property (B.1) of the characterization of the consistent coalitional payoff configuration.

Let λ_S be the unit length normal to $\partial V(S)$ at $a(S)$ for each $S\subset N$. We associate to each ρ a hyperplane game with coalition structure $(N,V_\rho,\mathcal{C})$ as follows:

Given $\rho\in[0,1)$ and $S\subset N$ with $|S|$ elements, there exists at least one hyperplane on $\mathbb{R}^S$ containing the $|S|$ points $\{a_\rho(S,i):i\in S\}$. If there are more than one hyperplane, we take the one whose unit length outward orthogonal vector $\lambda_S(\rho)$ is closest to λ_S.

We define

$$V_\rho(S):=\left\{x\in\mathbb{R}^S:\lambda_S(\rho)\cdot x\leq\lambda_S(\rho)\cdot a_\rho(S,i),i\in S\right\}.$$

The half-space $V_\rho(S)$ is well defined because $\lambda_S(\rho)\cdot a_\rho(S,i)=\lambda_S(\rho)\cdot a_\rho(S,j)$ for all $i,j\in S$.

By Corollary 10, $a_\rho(S,i)\to a(S)$. By the smoothness of $\partial V(S)$, $\lambda_S(\rho)\to\lambda_S$. Therefore,

$$V_\rho(S)\to V'(S):=\left\{x\in\mathbb{R}^S:\lambda_S\cdot x\leq\lambda_S\cdot a(S)\right\}.$$

By Proposition 7, the proposals $\{a_\rho(S,i):S\subset N,i\in S\}$ satisfy (C.1), (C.2), and (C.3) for $(N,V,\mathcal{C})$. But these properties are the same for $(N,V_\rho,\mathcal{C})$. Thus, by Proposition 7, a_ρ is an equilibrium payoff configuration for $(N,V_\rho,\mathcal{C})$. By Theorem 12, this implies that a_ρ is the only consistent coalitional payoff configuration for $(N,V_\rho,\mathcal{C})$.

Hence, each a_ρ satisfies properties (B.1), (B.2), and (B.3) for vectors $(\lambda_S(\rho))_{S\subset N}$. Given $S\subset N$, $x:=(x_S)_{S\subset N}\in\mathbb{R}^{2^S}$ (with $x_\emptyset=0$), $\mu:=(\mu_S)_{S\subset N}\in\mathbb{R}^{2^S}$ (with

$\mu_\emptyset = 0$), and $i \in C'_q \in \mathcal{C}_S$, we define the following functions:

$$
\begin{aligned}
F_1(S,x,\mu) &= \sum_{C'_r \in C_S \setminus C'_q} \left[\sum_{j \in C'_q} \mu_S^j \left(x_S^j - x_{S \setminus C'_r}^j \right) \right] - \sum_{C'_r \in C_S \setminus C'_q} \left[\sum_{j \in C'_r} \mu_S^j \left(x_S^j - x_{S \setminus C'_q}^j \right) \right] \\
F_2(S,i,x,\mu) &= \sum_{j \in C'_q \setminus i} \mu_S^i \left(x_S^i - x_{S \setminus j}^i \right) - \sum_{j \in C'_q \setminus i} \mu_S^j \left(x_S^j - x_{S \setminus i}^j \right) \\
F(x,\mu) &= \sum_{S \subset N} \left[\sum_{C'_q \in C_S} \left(F_1(S,x,\mu)^2 + \sum_{i \in C'_q} F_2(S,i,x,\mu)^2 \right) \right].
\end{aligned}
$$

These functions are continuous. Thus, $F^{-1}(0) := \{(x,\mu) : F(x,\mu) = 0\}$ is a closed set. Since (a_ρ, λ_ρ) satisfies (B.2) and (B.3) we conclude that $F_1(S, a_\rho, \lambda_\rho) = 0$ and $F_2(S, i, a_\rho, \lambda_\rho) = 0$ for all $S \subset N$ and $i \in C'_q \in \mathcal{C}_S$. Then, $(a_\rho, \lambda_\rho) \in F^{-1}(0)$. We know that $F^{-1}(0)$ is a closed set and $(a_\rho, \lambda_\rho) \to (a, \lambda)$. Then, $(a, \lambda) \in F^{-1}(0)$. So, a satisfies (B.2) and (B.3).

Since a satisfies (B.1), (B.2), and (B.3), we conclude that this vector is the consistent coalitional value of the hyperplane game $(N, V', \mathcal{C})$. Now it is easy to conclude (by the definition of Φ) that a is a consistent coalitional payoff configuration of $(N, V, \mathcal{C})$. Q.E.D.

5.9 *Proof* of Proposition 17

We proceed by induction on n. For $n = 1$, the result is trivial, because $\chi_{\{i\}}^i = r^i$. Assume the result is true for at most $n-1$ players.

By induction hypothesis, $a(S) = \chi_S$ for all $S \square N$. By (C.1), given $i \in C_q \in \mathcal{C}$,

$$
\lambda_N^i a(N,i)^i = v(N) - \sum_{C_r \in \mathcal{C} \setminus C_q} \left[\sum_{j \in C_r} \lambda_N^j a(N,i)^j \right] - \sum_{j \in C_q \setminus i} \lambda_N^j a(N,i)^j .
$$

Thus,

$$\begin{aligned}
\left|\mathcal{C}\right| \lambda_N^i a(N)^i &= \sum_{C_r \in \mathcal{C}} \frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i a(N,j)^i \\
&= \sum_{C_r \in \mathcal{C}\setminus C_q} \frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i a(N,j)^i + \frac{1}{|C_q|} \sum_{j \in C_q \setminus i} \lambda_N^i a(N,j)^i \\
&\quad + \frac{1}{|C_q|} v(N) - \sum_{C_r \in \mathcal{C}\setminus C_q} \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j a(N,i)^j - \frac{1}{|C_q|} \sum_{j \in C_q \setminus i} \lambda_N^j a(N,i)^j \\
&= \sum_{C_r \in \mathcal{C}\setminus C_q} \left[\frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i a(N,j)^i - \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j a(N,i)^j \right] \\
&\quad + \frac{1}{|C_q|} \left[\sum_{j \in C_q \setminus i} \lambda_N^i a(N,j)^i - \sum_{j \in C_q \setminus i} \lambda_N^j a(N,i)^j \right] + \frac{1}{|C_q|} v(N)
\end{aligned}$$

by (C.3), (C.4), and induction hypothesis

$$\begin{aligned}
&= \sum_{C_r \in \mathcal{C}\setminus C_q} \left[\frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i \left(\rho a(N)^i + (1-\rho) \chi_{N\setminus C_r}^i \right) - \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j \left(\rho a(N)^j + (1-\rho) \chi_{N\setminus C_q}^j \right) \right] \\
&\quad + \frac{1}{|C_q|} \left[\sum_{j \in C_q \setminus i} \lambda_N^i \left(\rho a(N)^i + (1-\rho) \chi_{N\setminus j}^i \right) - \sum_{j \in C_q \setminus i} \lambda_N^j \left(\rho a(N)^j + (1-\rho) \chi_{N\setminus i}^j \right) \right] + \frac{1}{|C_q|} v(N) \\
&= \rho \sum_{C_r \in \mathcal{C}\setminus C_q} \frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i a(N)^i - \rho \sum_{C_r \in \mathcal{C}\setminus C_q} \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j a(N)^j \\
&\quad + \rho \frac{1}{|C_q|} \sum_{j \in C_q \setminus i} \lambda_N^i a(N)^i - \rho \frac{1}{|C_q|} \sum_{j \in C_q \setminus i} \lambda_N^j a(N)^j \\
&\quad + (1-\rho) \sum_{C_r \in \mathcal{C}\setminus C_q} \left[\frac{1}{|C_r|} \sum_{j \in C_r} \lambda_N^i \chi_{N\setminus C_r}^i - \frac{1}{|C_q|} \sum_{j \in C_r} \lambda_N^j \chi_{N\setminus C_q}^j \right] \\
&\quad + (1-\rho) \frac{1}{|C_q|} \left[\sum_{j \in C_q \setminus i} \lambda_N^i \chi_{N\setminus j}^i - \sum_{j \in C_q \setminus i} \lambda_N^j \chi_{N\setminus i}^j \right] + \frac{1}{|C_q|} v(N).
\end{aligned}$$

By (C.1), $v(N) = \sum_{j \in N} \lambda_N^j a(N,i)^j$ for all $i \in N$. By (3) and averaging over i we conclude that $v(N) = \sum_{j \in N} \lambda_N^j a(N)^j = \sum_{C_r \in \mathcal{C}\setminus C_q} \sum_{j \in C_r} \lambda_N^j a(N)^j +$

$\sum_{j\in C_q\setminus i} \lambda_N^j a(N)^j + \lambda_N^i a(N)^i$ and so,

$$\begin{aligned}
& |\mathcal{C}|\,\lambda_N^i a(N)^i \\
= \; & \rho\left[\sum_{C_r\in\mathcal{C}\setminus C_q}\frac{1}{|C_r|}\sum_{j\in C_r}\lambda_N^i a(N)^i + \frac{1}{|C_q|}\sum_{j\in C_q\setminus i}\lambda_N^i a(N)^i + \frac{1}{|C_q|}\lambda_N^i a(N)^i - \frac{1}{|C_q|}v(N)\right] \\
& + (1-\rho)\sum_{C_r\in\mathcal{C}\setminus C_q}\left[\frac{1}{|C_r|}\sum_{j\in C_r}\lambda_N^i\chi_{N\setminus C_r}^i - \frac{1}{|C_q|}\sum_{j\in C_r}\lambda_N^j\chi_{N\setminus C_q}^j\right] \\
& + (1-\rho)\frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus i}\lambda_N^i\chi_{N\setminus j}^i - \sum_{j\in C_q\setminus i}\lambda_N^j\chi_{N\setminus i}^j\right] + \frac{1}{|C_q|}v(N) \\
= \; & \rho\left[\sum_{C_r\in\mathcal{C}}\frac{1}{|C_r|}\sum_{j\in C_r}\lambda_N^i a(N)^i - \frac{1}{|C_q|}v(N)\right] \\
& + (1-\rho)\sum_{C_r\in\mathcal{C}\setminus C_q}\left[\lambda_N^i\chi_{N\setminus C_r}^i - \frac{1}{|C_q|}\sum_{j\in C_r}\lambda_N^j\chi_{N\setminus C_q}^j\right] \\
& + (1-\rho)\frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus i}\lambda_N^i\chi_{N\setminus j}^i - \sum_{j\in C_q\setminus i}\lambda_N^j\chi_{N\setminus i}^j\right] + \frac{1}{|C_q|}v(N) \\
= \; & \rho\,|\mathcal{C}|\,\lambda_N^i a(N)^i - \frac{\rho}{|C_q|}v(N) \\
& + (1-\rho)\sum_{C_r\in\mathcal{C}\setminus C_q}\left[\lambda_N^i\chi_{N\setminus C_r}^i - \frac{1}{|C_q|}\sum_{j\in C_r}\lambda_N^j\chi_{N\setminus C_q}^j\right] \\
& + (1-\rho)\frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus i}\lambda_N^i\chi_{N\setminus j}^i - \sum_{j\in C_q\setminus i}\lambda_N^j\chi_{N\setminus i}^j\right] + \frac{1}{|C_q|}v(N).
\end{aligned}$$

By gathering terms and dividing by $(1-\rho)$ we have

$$\begin{aligned}
|\mathcal{C}|\,\lambda_N^i a(N)^i \;=\; & \sum_{C_r\in\mathcal{C}\setminus C_q}\left[\lambda_N^i\chi_{N\setminus C_r}^i - \frac{1}{|C_q|}\sum_{j\in C_r}\lambda_N^j\chi_{N\setminus C_q}^j\right] \\
& + \frac{1}{|C_q|}\left[\sum_{j\in C_q\setminus i}\lambda_N^i\chi_{N\setminus j}^i - \sum_{j\in C_q\setminus i}\lambda_N^j\chi_{N\setminus i}^j\right] + \frac{v(N)}{|C_q|}.
\end{aligned}$$

And so,

$$
\begin{aligned}
a(N)^i &= \frac{1}{|\mathcal{C}|\,|C_q|\,\lambda_N^i} \sum_{C_r \in \mathcal{C} \setminus C_q} \left(|C_q|\,\lambda_N^i \chi_{N \setminus C_r}^i - \sum_{j \in C_r} \lambda_N^j \chi_{N \setminus C_q}^j \right) \\
&\quad + \frac{1}{|\mathcal{C}|\,|C_q|\,\lambda_N^i} \sum_{j \in C_q \setminus i} \left(\lambda_N^i \chi_{N \setminus j}^i - \lambda_N^j \chi_{N \setminus i}^j \right) \\
&\quad + \frac{1}{|\mathcal{C}|\,|C_q|\,\lambda_N^i} v(N) \\
&= \chi_N^i .
\end{aligned}
$$

□

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