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Some economic applications of Scott domains

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The present paper is structured around two main constructions, fixed points of functors and fibrations and sections of functors. Fixed points of functors are utilized to resolve problems of infinite regress that have recently appeared in economics. Fibrations and sections are utilized to model solution concepts abstractly, so that we can solve equations whose arguments are solution concepts. Most of the objects (games, solution concepts) that we consider can be obtained as some kind of limit of their finite subobjects. Some of the constructions preserve computability. The paper relies heavily on recent work on the semantics of programming languages.

Key words: Game theory; functorial fixed points; infinite regress.

1. Introduction

Recent work in game theory and in applied areas such as industrial organization has considered problems that involve infinite regress. Some authors, such as Mertens and Zamir (1985) and Lipman (1991), have resolved the infinite regress 'by hand', i.e. by working hard to exploit the special features of their models. The message of the present paper is that such resolutions of infinite regress can be thought of as proofs of existence of fixed points of certain 'maps' (called functors) defined on certain 'spaces' (called categories). This change of perspective is useful in several respects. First, there is a general theory of existence, uniqueness, stability and continuity of such fixed points that can be invoked to check whether a particular infinite regress problem can be resolved or not. The theory was developed by computer scientists; Section 2 of the present paper provides an exposition. Secondly, having a general theory allows us to see the already existing examples of resolutions of infinite regress as cases of a single construction. Thirdly, we can resolve infinite regress in cases that had been considered intractable up to now, such

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as the regress reported in Crawford (1985, p. 825) as intractable and resolved in Vassilakis (1990). Fourthly, we can investigate whether each particular regress can be resolved effectively.

In the present paper I have tried to summarize my earlier results, to motivate the constructions with economic examples, and to give a reasonably complete picture of the mathematical fundamentals, with some emphasis on the effectiveness of the main constructions, omitting most proofs but providing detailed references to the sources where proofs can be found. I have also included new results and open problems to indicate the direction this project is taking.

To motivate the introduction of the new mathematical tools, we begin with a bit of history. The economist's workhorse is the competitive model of exchange; it assumes that economic agents determine their demands and supplies by maximizing their objective functions taking prices as given; that there is a price quoted for each commodity; and that all trade takes place at prices that simultaneously clear all markets. Why should agents take prices as given? Who sets prices? How do prices reflect the actions of agents in all markets? These questions (Kreps, 1990, p. 4) have motivated the introduction of game theory into economics. The environment in which individuals have to make decisions is modelled as a game g. A prediction on g is a subset of the set of all possible outcomes in g. A rule that assigns a prediction to each game g is called a solution concept. An outcome predicted by a solution concept is an equilibrium of g. Equilibria might fail to exist, be unique or be efficient. In some games, lack of existence, uniqueness or efficiency is just an artifact of the formalism, but in other games it has been argued that 'institutions will evolve' to restore, to some extent, these properties of equilibria. The examples that follow illustrate the meaning of this argument and show that its consistent application generates an infinite regress.

The first example shows how lack of existence provides incentives for institution formation. Shepsle (1986) considers a game $g = \langle N, X, u \rangle$, where each player $i \in N$ has preferences over a set of alternatives X described by a utility function $u_i : X \to R$. The solution concept is 'majority voting': an alternative y in X is an equilibrium if there is no alternative x preferred to y by a strict majority of the players. Shepsle denotes by $W(y)$ the set of alternatives preferred to y by a strict majority of the players, and describes the 'paradox of voting' as follows:

- For almost all utility functions and for all alternatives y in X, $W(y)$ is not empty (equilibria do not exist).
- For almost all utility functions and for any two alternatives x, y there is a finite sequence $Z_1, \ldots, Z_k$ of alternatives such that

  (i) a strict majority prefers Z_1 to x:

  $$Z_1 \in W(x);$$

  (ii) a strict majority prefers Z_i to Z_{i-1}

  $$Z_i \in W(Z_{i-1}), \quad i = 1, \ldots, K;$$

(iii) a strict majority prefers y to Z_k

$$y \in W(Z_k).$$

In other words, the nonexistence problem is particularly severe: an agenda-setter, namely an individual who can dictate the kind and order of alternatives players vote on, can induce any alternative y as the final outcome of majority voting. This result provides strong support for the position that 'institutions matter', where by institutions we mean the rules that govern the process of voting. A striking example is cited by Shepsle (1986, p. 56): while it takes one line to describe majority rule, the rules that govern the process of voting in the U.S. House of Representatives take 600 pages to describe. Riker (1980, p. 443) states that '... the particular structure of an institution is at least as likely to be predictive of socially enforced values as are the preferences of the citizen body'. Given that the particular structure of institutions is important to players, and given that institutions are just rules under players' control, players will try to change institutions to advance their own interests. This is clearly seen by Riker (1980, p. 445):

> One can expect that losers on a series of decisions under a particular set of rules will attempt (often successfully) to change institutions and hence the kind of decisions produced under them. In that sense rules or institutions are just more alternatives in the policy space and the status quo of one set of rules can be supplanted with another set of rules.

In other words, players will try to influence the outcome of game g by choosing strategies in a larger game $F(g)$: the strategies of $F(g)$ are (proposed) rules of behavior in g. But now, the outcome of game $F(g)$ is at stake, and players will try to influence it by proposing rules of behavior in $F(g)$, i.e. by choosing strategies in $F^2(g)$. To capture all the opportunities of the players to propose rules, we must be able to show that the infinite regress $g, F(g), F^2(g), \ldots$ can be resolved, i.e. that F has a fixed point.

Lack of uniqueness of equilibria in a game g also provides incentives for 'institutions to evolve'. For example, if g is a 'split-the-dollar' bargaining game with two players who announce simultaneously their claims on the dollar, then any division of the dollar is a (Nash) equilibrium. The multiplicity creates a coordination problem; if one player expects the (1/2, 1/2) equilibrium to prevail, he plays 1/2; if the other player expects the (1/3, 2/3) equilibrium to prevail, she plays 2/3. But then (1/2, 2/3) is not an equilibrium and both players get their disagreement payoff (zero). Hence, there is an incentive for each player to propose a particular institution (for example, a variant of Rubinstein's bargaining game) that, to the extent it is accepted, will select an equilibrium and resolve the coordination problem. To capture the ability of players to make such proposals, we construct a larger game $F(g)$ in which these proposals are points in players' strategy spaces. Given that the choice of institution matters (Kreps, 1990, ch. 15), players will try to influence the choice of strategies in $F(g)$ by proposing institutions in $F^2(g)$, etc. Hence the need to resolve infinite regress. I should emphasize at this point that Crawford (1985,

p. 825), and no doubt others, saw this problem but considered the infinite regress intractable.

Lack of efficiency of an equilibrium in a game g will also provide incentives for institution formation. For example, the Cournot equilibrium in an oligopoly game is inefficient from the point of view of the players (firms), and each firm has some incentive to propose an institution that facilitates collusion. Another example, cited by Ordeshook (1980), is a prisoner's dilemma type game that has a unique inefficient equilibrium, and so provides incentives for players to propose institutions (in this case, government) that break the dilemma. As in the previous cases, the proposed institutions will be points in the strategy spaces of a larger game $F(g)$, and the by now familiar argument will generate an infinite regress.

I will now sketch informally the construction that resolves the infinite regress. I start with a game g and a 'map' F that takes games into games, and I construct a game $X(g)$ that satisfies two properties:

(a) $X(g)$ is a fixed point of F.

(b) Each $F^t(g)$, $t=0,1,2,\ldots$, is a 'subgame' of $X(g)$.

$X(g)$ is then the universal game generated by g, and captures all the opportunities available to the players in the environment described by g.

The game $g=\langle A,u\rangle$ will be a normal form, complete information game, where $A=\langle A_1,A_2\rangle$ is a pair of strategy spaces and $u=\langle u_1,u_2\rangle$ a pair of payoff functions. Generalizations to n-player and/or incomplete information games are immediate once the constructions are understood in this simple case. Each A_i is a complete partial order (cpo), namely a poset with the following properties:

- A_i contains a least element $\perp$;
- if $x_1 \leqslant x_2 \leqslant \ldots$ is an increasing sequence in A_i, then the least upper bound $\bigcup_{i=1}^{\infty} x_i$ of the sequence belongs to A_i.

Each payoff function $u_i: A_1 \times A_2 \to R$ is a Scott-continuous function, i.e. it preserves least upper bounds of increasing sequences:

- $$u_j\left(\bigcup_{i=1}^{\infty} x_i\right)=\bigcup_{i=1}^{\infty} u_j(x_i), \quad \text{if } x_1 \leqslant x_2 \leqslant \ldots,$$

where $x_i \in A_1 \times A_2$ and $A_1 \times A_2$ is ordered componentwise.

The meaning of the order relation on each A_i will become clear once we see how to transform any game $\langle B,v\rangle$ into a game $\langle A,u\rangle$ with the required properties. One way would be to define

- $$A_i=B_i \cup \{\perp\}, \text{ where } \perp \notin B_i;$$
- $$a_i \leqslant a_i' \text{ iff } a_i=\perp \text{ or } a_i=a_i';$$
- $$u_i(a)=\begin{cases} -\infty, & \text{if } a_i=\perp, \text{ some } i, \\ v_i(a), & \text{otherwise.} \end{cases}$$

That is, to add an element $\perp$ to each B_i and define the order on each A_i to be the least restrictive one that has $\perp$ as its least element. The payoff functions are then defined to agree with v_i on $B_1 \times B_2$ and to ensure that $\perp$ is never played. Another way would be to define

- A_i = all closed subsets of B_i;
- $a_i \leqslant a_i'$ iff a_i' is a subset of a_i;
- u_i = the unique extension of v_i that is continuous with respect to the Hausforff topology on A_i.

One could also define A_i as a subfamily of the closed sets of B_i. In all cases the elements of A_i are properties of strategies and the order relation is a precision, or information-content, relation: $a_i \leqslant a_i'$ means that a_i is a less precise property than a_i'. The least element is the totally non-informative property. Scott continuity means that the utility of a property of strategies equals the least upper bound of utilities of properties that approximate it. How restrictive is Scott continuity? This depends on the order on A_i. Under the first definition of $\leqslant$, Scott continuity is so weak it is almost vacuous. Under the second definition, though, it is quite strong. Intermediate degrees of strength can be achieved if one restricts the subsets of B_i that can be compared by the order relation.

The fact that we call the functions that preserve least upper bounds of increasing sequences Scott-continuous suggests there is a topology on each A_i such that the continuous functions with respect to this topology are precisely the Scott-continuous functions. The open sets U of this topology are defined by two properties:

- U is an upper set: $x \in U$, $x \leqslant y$ implies $y \in U$.
- U is inaccessible by least upper bounds of increasing sequences: if $x_1 \leqslant x_2 \leqslant \dots$ and $\bigcup_{i=1}^{\infty} x_i \in U$, then, for some i, $x_i \in U$.

For example, if $A_i = [0, 1]$, then the Scott open sets are the open half-says $(t, 1]$, $0 \leqslant t \leqslant 1$.

The crucial step in the construction is the definition of the map F that assigns to each game g the game Fg whose strategy spaces include the institutions that players can propose to coordinate actions in g. By the revelation principle (Kreps, 1990, Ch. 18) institutions can be modelled, without loss of generality, as direct mechanisms on g.

A (direct) mechanism on g is a probability measure p on the aggregate strategy space $A_1 \times A_2$ of g;$p(E)$ is the probability that mechanism p will recommend to players an action in $E \subseteq A_1 \times A_2$. Let $\Delta(A_1 \times A_2)$ be the set of all Borel probability measures on $A_1 \times A_2$ with respect to the Scott topology; it can be ordered as follows: $p \leqslant q$ iff $p(U) \leqslant q(U)$ for all Scott-opens U. With this order, $\Delta(A_1 \times A_2)$ is a cpo; its least element is the measure that assigns probability 1 to the least element $(\perp, \perp)$ of $A_1 \times A_2$. The space $\Delta(A_1 \times A_2)$ is interpreted as the set of all institutions that each agent can choose from to coordinate his actions in g.

Given that all agents can propose a mechanism, each agent i will receive two recommended actions, i.e. a point in A_i^2. The response of i to these recommendations is determined by a Scott-continuous function $\delta_i : A_i^2 \to A_i$, that is chosen by agent i. For example, if agent 1 chooses to obey the recommendations of agent 2, then 1 chooses a function $\delta_1 : A_1^2 \to A_1$ defined by $\delta_1(a_1, a_1') = a_1'$ for all (a_1, a_1'). The fact that agents can choose such functions is the formal expression of the fact that agents do not have to obey recommendations unless it is in their interest to do so. The space $[A_i^2 \to A_i]$ of Scott-continuous functions, when ordered pointwise, is a cpo.

We can now define $F(g)$ as a game $\langle B, \upsilon\rangle$; its strategy spaces are given by

- $B_i = \Delta(A_1 \times A_2) \times [A_i^2 \to A_i]$.

In other words, in $F(g)$ each agent i chooses the mechanism $p_i \in \Delta(A_1 \times A_2)$ he proposes and the 'deviation function' $\delta_i \in [A_i^2 \to A_i]$ that he uses to respond to recommendations. What happens if each agent i chooses a particular strategy $(p_i \delta_i)$? Agent j will receive recommendations from agent i with probability law p_{ij}, namely the marginal of p_i on A_j. Hence, agent j will receive a pair of recommendations with probability law $p_{1j} \times p_{2j}$. He is going to respond to these recommendations according to his deviation function δ_j. The probability, therefore, that agent j will take an action in a set $E \subseteq A_j$ equals

$$(p_{ij} \times p_{2j})(\delta_j^{-1}(E)).$$

We can interpret the measure $q_j = (p_{ij} \times p_{2j}) \circ \delta_j^{-1}$ as j's induced mixed strategy on A_j, and the measure $q = q_1 \times q_2$ as the probability distribution on $A_1 \times A_2$ induced by (p_i, δ_i), $i = 1, 2$. Hence, define

- $$\upsilon_i(p_1, \delta_1, p_2, \delta_2) = \int u_i \, \mathrm{d}q.$$

The definition of $F(g)$ is now complete.

We can embed g into $F(g)$ by a pair of Scott-continuous function, $\alpha_i : A_i \to B_i$, defined by $\alpha_i(a_i) = (\perp, \hat{a}_i)$, where $\perp$ is the bottom element of $\Delta(A_1 \times A_2)$ and $\hat{a}_i : A_i^2 \to A_i$ is defined by $\hat{a}_i(b_1, b_2) = a_i$, for all b_1, b_2 in A_i. It is easy to see that

$$\upsilon_i(\alpha_1(a_i), \alpha_2(a_2)) = u_i(a_1, a_2),$$

i.e. that υ_i is an extension of u_i. In the same way, we can embed $F^t(g)$ into $F^{t+1}(g)$ for all $t \geqslant 0$. Denoting $F^t(g) = \langle A(t), u(t)\rangle$, we get a nested sequence of strategy spaces

$$A(1) \to A(2) \to A(3) \to \cdots$$

and a sequence of payoff functions such that $u_i(t+1)$ extends $u_i(t)$, for all i and t (see Fig. 1).

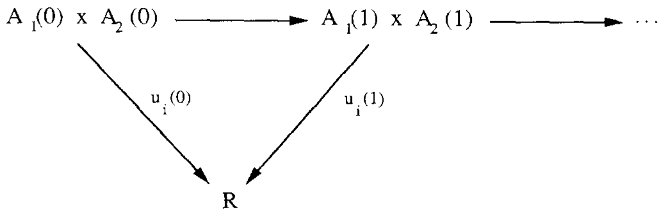

Fig. 1. Payoff functions of higher-order games extend those of lower-order games.

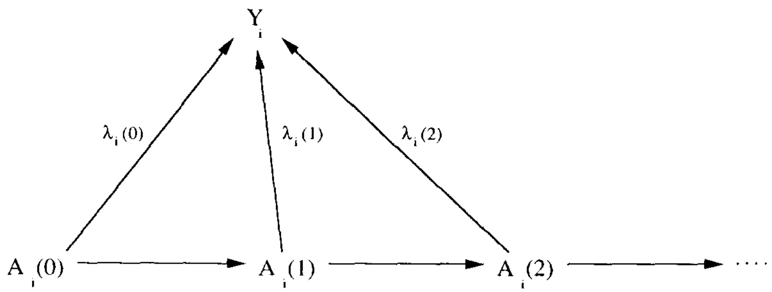

Fig. 2. No junk requirement.

To capture all the opportunities open to the players, we want to construct a game $X(g)=\langle Y, w\rangle$ that is a 'limit' of these sequences and a fixed point of F. To see how Y_i should be defined, recall that every strategy in $A_i(t)$ has to be a strategy in Y_i, otherwise Y_i under-represents the opportunities open to the players. It is tempting to define Y_i as the union $\bigcup_{t=1}^{\infty} A_i(t)$; recall, though, that each $A_i(t)$ is embedded into $A_i(t+1)$, and that this implies that this union contains many different names for the same strategy. We want all these different names of the same strategy to be embedded in Y_i as a single strategy. Hence the

No junk requirement. For each $i=1,2$ and $t=0,1,2,\dots$ there is a Scott-continuous embedding $\lambda_i(t): A_i(t) \to Y_i$ such that Fig. 2 commutes.

A cpo Y_i that satisfies the no junk requirement might be too large, i.e. contain strategies not available to the agents in the environment described by g. To make Y_i the smallest cpo that satisfies no junk we impose the

No exaggeration requirement. If there is another cpo Y_i' and Scott-continuous embeddings $\lambda_i'(t): A_i(t) \to Y_i'$ that also satisfy no junk, then there is a unique Scott-continuous embedding $f: Y_i \to Y_i'$ such that Fig. 3 commutes.

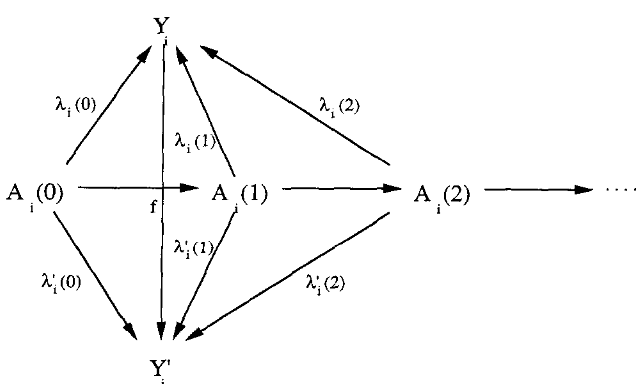

Fig. 3. No exaggeration requirement.

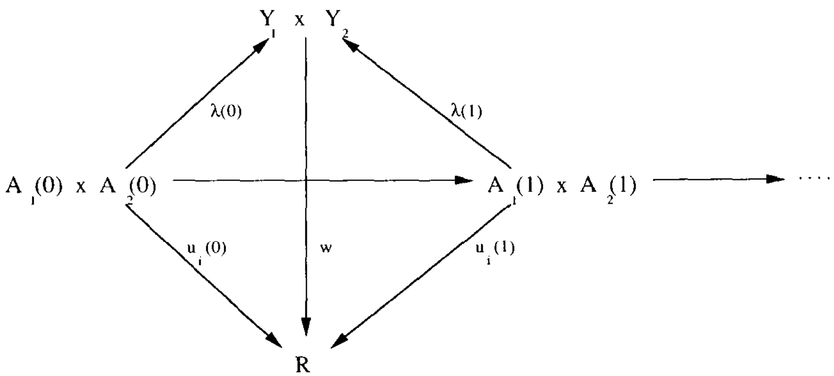

Fig. 4. Payoff function of the universal game.

It turns out that our assumptions are exactly right to guarantee that such Y_i exists and is unique up to isomorphism. We write

$$Y_i = \operatorname{colimit}_t A_i(t).$$

To construct the payoff functions $w_i: Y_i \times Y_2 \to R$, we use the no exaggeration requirement and the fact that each $u_i(t+1)$ extends $u_i(t)$. It then follows that there is a unique Scott-continuous function w_i such that Fig. 4 commutes.

This completes the construction of the universal game $X(g)$. We write concisely

$$X(g) = \operatorname{colimit}_t F^t(g).$$

It turns out that the map F preserves colimits of chains, i.e. $F(\text{colimit}_t\, g_t) = \text{colimit}_t\, F(g_t)$. The universal game $X(g)$, then, is a fixed point of F, because

$$F(X(g)) = F(\operatorname{colimit} F^t(g)) = \operatorname{colimit} F^{t+1}(g) = \operatorname{colimit} F^t(g) = X(g).$$

We now come to the 'computability' of universal games. Clearly, the fact that strategy spaces are not necessarily subsets of the natural numbers means that to define 'computable' strategy spaces we must find a way to extend the definitions of recursive function theory to cpos. At the very least, then, the cpos we single out as 'computable' must have a countable basis, and this basis must be computable in the usual sense.

How exactly should we define a basis of a cpo? We first look at an example. Let D be the set of all subsets of N, ordered by set inclusion, and let E be the set of all finite subsets of N similarly ordered. Clearly, E is countable while D is not. Each X in D can be obtained as the least upper bound (union) of elements of E contained in X:

(a) $X = \bigcup\{Y \in E:\ Y \leqslant X\}$.

Furthermore, the set of elements of E below X is directed, in the sense that if $Y_i \leqslant X$, $Y_i \in E$, then $Y_1 \cup Y_2 \in E$ and $Y_i \leqslant Y_1 \cup Y_2 \leqslant X$. Hence, for each X

(b) $\{Y \in E\colon Y \leqslant X\}$ is directed.

Finally, the basis E itself consists of elements with a 'compactness' property

(c) If $Y \in E$ and if $Y \leqslant \bigcup_{i \in I} X_i$: where $\{X_i\colon i \in I\}$ is a directed set, then there is some i in I such that $Y \leqslant X_i$.

We can now abstract these three properties to define the basis of compact elements of a cpo. An element x of a cpo D is compact if it satisfies

(c) if $x \leqslant \bigcup d_i$, $\langle d_i \rangle$ directed, then $x \leqslant d_i$ for some i.

Let $K(D)$ be the set of compact elements of D, and let $\downarrow d = \{x \in D : x \leqslant d\}$. We say $K(D)$ is a basis of D if for all d in D:

(b) $K(D) \cap \downarrow d$ is directed, and

(a) $d = \bigcup (K(D) \cap \downarrow d)$.

Note that not every cpo will satisfy these properties: for example, the real interval [0, 1] will not. Those cpos that do satisfy them are called algebraic.

The first computability requirement we impose is very natural: $K(D)$ has to be countable, and the order relation restricted to $K(D)$ has to be decidable, i.e. a Turing machine must be able to decide whether or not two basis elements are related by the order relation. Hence

Condition A. There is a surjection $d : N \to K(D)$ such that the set $\{(m, n) \in N^2\colon d_m \leqslant d_n\}$ is effectively decidable.

The second computability requirement is also a natural consequence of the approximation properties *a* and *b*; given that we approximate each element of D by directed subsets of its basis $K(D)$, we want to be able to decide effectively whether a finite subset F of $K(D)$ 'approximates' $K(D)$. To formalize this, we say that a subset F of $K(D)$ is a normal substructure if for each x in $K(D)$, the set $F \cap \downarrow x$ is directed. The computability requirement is

Condition B. For any finite subset T of N, it is decidable whether $\{d_t : t \in T\}$ is a normal substructure of $K(D)$.

We define an effective presented domain as a pair $\langle D, d \rangle$ with properties A and B. If $\langle E, e \rangle$ are effectively presented domains, and if $f : D \to E$ is Scott-continuous we say that

Condition C. f is computable if for each n in N the set $\{(n, m) \in N^2 : e_m \leqslant f(d_n)\}$ is recursively enumerable.

Note that Scott-continuity of f and Conditions A and B imply that knowledge of the values of f on the basis elements d_n suffices to determine the value of f on all elements of D. Hence, roughly speaking, the meaning of Conditions A, B and C is that a domain D contains both 'computable' and 'noncomputable' elements, but the noncomputable ones can be obtained as 'limits' of computable ones; and that the value of a computable function on a computable element d_n can be effectively approximated by computable elements in its range, while the values of such a function on noncomputable elements are limits of its values on computable ones.

Recall that, unless our notion of computability is preserved by the map F, $X(g)$ will not be a 'computable' game even if g is. If D is effectively presentable, $\Delta(D)$ and $[D \to D]$ are not necessarily so, and unfortunately, F is a composite of Δ, $\to$ and x. Can we build on Conditions A and B to find the right notion? We will be guided by the fact that the definition of a computable cpo has to include the real interval $I = [0, 1]$, for otherwise $\Delta(D)$ will not be computable even if D is. For each $t \in TI$, let $Z(t)$ be the smallest integer larger than t.

Define $f_n : I \to I$ by

$$f_n(t) = \frac{Z(10^n t - 1)}{10^n}.$$

The f_n's are Scott-continuous and satisfy:

(i) $f_1 \leqslant f_2 \leqslant \cdots$.

(ii) $f_n(I)$ is a finite set, for all n.

(iii) $t = \bigcup_{n=1}^{\infty} f_n(t)$, for all $t \in [0, 1]$.

Note that $\bigcup_{n=1}^{\infty} f_n(I)$ behaves very much like a countable basis of I. If a cpo D admits a sequence of Scott-continuous functions $f_n : D \to D$ that satisfy properties (i), (ii) and (iii), it is called a finitely continuous cpo. Abusing notation let $K(D) = \bigcup_{n=1}^{\infty} f_n(D)$. If *this* $K(D)$ satisfies Conditions A and B, D is a computable finitely continuous cpo. It turns out that this notion of computability is preserved by Δ, $\to$, x and by the operation of taking colimits. Hence, each strategy space Y_i is a computable finitely continuous cpo if each A_i is. More information can be found in Kanda (1979), Kamimura and Tang (1984, 1986), Graham (1988) and Gunter and Scott (1989). A more recent approach to computability on cpos is developed, under the name of modest sets, in Barr and Wells (1990, p. 333), Rosolini (1990), and Freyd et al. (1990). As of this writing, I do not know whether the payoff functions of each $F^t(g)$ are computable, because their definition involves integration.

2. Categorical preliminaries

The 'map' F is defined on all games g; the collection of all games is not a set but a proper class. Hence the need to consider categories.

Definition 1. A category X consists of

(i) a class of objects = A;B;... ;

(ii) for any two objects A, B, a set of morphisms from A to B: a typical morphism is denoted by $f : A \to B$;

(iii) an associative operation on morphisms called composition: if $f : A \to B$ and

$g: B \to C$, then there is a morphism $gf: A \to C$; for any three morphisms f, g, h for which composition is defined, $(gf)h = g(fh)$;

(iv) for each object A, an identity morphism

$$\mathrm{id}_A: A \to A \text{ that satisfies}$$

$$f\mathrm{id}_A = f$$

$$\mathrm{id}_A g = g$$

for any f and g for which the compositions are defined (see Table 1).

In all cases, the definitions of composition and identity are the obvious ones; some standard references are MacLane (1971), Arbib and Manes (1975), or Adamek et al. (1990).

Maps between categories that preserve composition and identities are called functors.

Definition 2. A function F from category X to category $Y, F: X \to Y$, assigns to each object A in X an object $F(A)$ in Y, and to each morphism $F: A \to B$ in X a morphism $F(f): F(A) \to F(B))$ in Y; in addition,

$$F(gf) = F(g)F(f), \qquad F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

Examples of functors.

The forgetful functor $U: \mathrm{Top} \to \mathrm{Set}$ assigns to each topological space (A, τ) its underlying set A (where τ is the topology on A), and to each continuous function f the function f itself.

The powerset functor $P: \mathrm{Set} \to \mathrm{Set}$ assigns to each set A its powerset $P(A)$ and to each function $f: A \to B$ the 'direct image' function $P(f): P(A) \to P(B)$ defined by $P(f)(E) = f(E)$, $E \subseteq A$.

Table 1
Examples of categories

Category	Objects	Morphisms
Set	all sets	all functions
Top	all topological spaces	all continuous functions
Comphaus	all compact Hausdorff spaces	all continuous functions
Poset	all partially ordered sets	all monotonic functions
ω	all natural numbers	$m \to n$ iff $m \leq n$
ω^{op}	all natural numbers	$m \to n$ iff $m \geq n$
Any set A	elements of A	$a \to b$ iff $a = b$
Any poset A	Elements of A	$a \to b$ iff $a \leq b$

The probability functor Δ : Comp Haus $\to$ Comp Haus assigns to each compact Hausdorff space A the set of all Borel regular probability measures $\Delta(A)$ equipped with the weak-star topology, and to each continuous function $f: A \to B$ the function $\Delta(f): \Delta(A) \to \Delta(B)$ defined by $\Delta(f)(P)(E) = P(f^{-1}(E))$, $p \in \Delta(A)$, E Borel subset of B.

Functors can be composed in the obvious way. Morphisms between functors are called natural transformations.

Definition 3. Let $F, G: X \to Y$ be functors from X to Y. A natural transformation $\lambda: F \to G$ from F to G is a collection of Y-morphisms $\langle \lambda_A : F(A) \to G(A) \rangle_{A \in X}$ such that if $f: A \to B$ is a morphism in X, $G(f)\lambda_A = \lambda_B F(f)$, i.e. Fig. 5 commutes.

Examples of natural transformations.

The sample mean: Let $S \subseteq R$ be the set of all possible outcomes of some experiment; let zero be in S. Define the functors $F, G: \omega \to$ Set as follows:

$$F(n) = S^n; \qquad F(n \to n+1) = f_n,$$

$$G(n) = R; \qquad G(n \to n+1) = g_n,$$

where $f_n: S^n \to S^{n+1}$ is defined by $f_n(x) = (x, 0)$, and $g_n: R \to R$ by

$$g_n(x) = \frac{n}{n+1} x.$$

The sample mean

$$\lambda_n : S^n \to R \text{ is } \lambda_n(x) = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Then, $\lambda: F \to G$ is a natural transformation. In general, a natural transformation λ can be thought of as a rule for transforming elements of $F(A)$ into elements of $G(A)$: the commutativity conditions imply that this rule is the same for all A in the category X; in the example, the rule for forming the sample mean is the same for all sample sizes.

We can now formulate the conditions under which a functor F has a fixed point

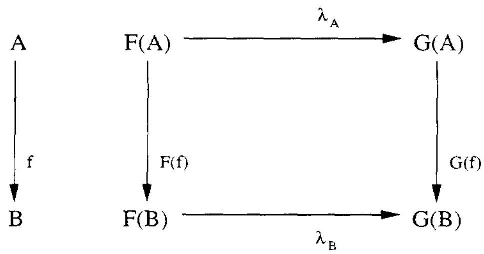

Fig. 5. Naturality square.

$L \simeq F(L)$ such that all the iterations $F^t(A)$, $t \geqslant 0$, are faithfully embedded in L (recall points (a) and (b) on p. 176). These are essentially continuity conditions, i.e. we require that F preserves some sort of 'limit'.

Definition 4. $f: A \to B$ is an isomorphism in the category X if there exists $g: B \to A$ in X such that $gf = \mathrm{id}_A$, $fg = \mathrm{id}_B$.

Definition 5. L is a fixed point of the functor $F: X \to X$ if there is an isomorphism $f: F(L) \to L$. We write $L \simeq F(L)$.

Definition 6. The constant functor at $A, K_A: X \to X$, is defined by

$$K_A(B) = A, \qquad K_A(f) = \mathrm{id}_A.$$

We can now formulate exactly the notion of 'limit' that we need.

Definition 7. Let $T: J \to X$ be a functor. A colimit of T is an object L of X and a natural transformation $\lambda: T \to K_L$ such that if $\lambda': T \to K_{L'}$ is any other natural transformation from T to a functor constant at any other object L', there is a unique X-morphism $f: L \to L'$ such that Fig. 6 commutes for all objects j in J.

We interpret the naturality of λ as 'consistent' embedding of each $T(j)$ into L, i.e. embedding according to some rule. The defining property of L means that L is the smallest object in X into which each $T(j)$ is consistently embedded.

Examples of colimits.

If J is a set, i.e. the only morphisms are the identity morphisms and $T: J \to \mathrm{Set}$, then the colimit L of T is the disjoint union of the $T(j)$'s:

$$L = \bigcup_{j \in J} (T(j) \times (j))$$

and $\lambda_j: T(j) \to L$ is defined by $\lambda_j(x) = (x, j)$, $j \in J$, $x \in T(j)$.

If $J = \omega$ and X is a poset, the colimit L of $T: J \to X$ is the least upper bound of the sequence $T(j)$.

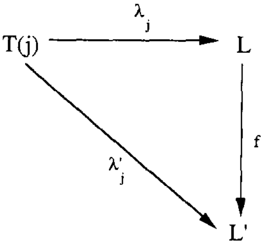

Fig. 6. Definition of limits.

If $J=\omega$ and X is one of Set, Top, Comphaus, then the colimit of $T: J \to X$ is the 'inductive or direct limit' of the $T(j)$'s; its explicit construction is described in Adamek et al. (1990, p. 187). An economic application is in Aliprantis et al. (1984, pp. 239–240).

Theorem 1. *Colimits are unique up to isomorphism* (Adamek et al., 1990, p. 187, Proposition 11.29).

Definition 8. Let $T: J \to X$ be a functor with colimit (L, λ). The functor $F: X \to Y$ preserves the colimit of T if $(F(L), F(\lambda))$ is a colimit of the composite functor FT. We write $F(\text{colimit } T) \simeq \text{colimit}(FT)$.

Definition 9. The functor $F: X \to Y$ is J-continuous if it preserves colimits of all functors $T: J \to X$.

Examples of continuous and discontinuous functors.

The forgetful functor $U: \text{Top} \to \text{Set}$ preserves all colimits.

The powerset functor $P: \text{Set} \to \text{Set}$ does not preserve any colimits.

Constant and identity functors preserve all colimits.

See Adamek et al. (1990, p. 207); more interesting examples will be provided in the next section. To formulate the fixed point theorem, we need the concept of the functor of iterations of a functor F.

Definition 10. Let $F: X \to X$ be a functor, and $f: A \to F(A)$ a morphism in X. The functor of iterations of F with respect to f is $q_f: \omega \to X$, defined by $q_f(n) = F^n(A)$, $q_f(n \to n+1) = F^n(f)$.

Theorem 2. *Let $F: X \to X$ be a functor such that*

(a) *There is a morphism $f: A \to F(A)$ for some A in X.*

(b) *The colimit of the functor of iterations of F w.r.t. f exists (call it (L, λ)) and is preserved by F.*

Then, L is a fixed point of F.

For a proof see Adamek and Koubek (1979, p. 106), Smyth and Plotkin (1982, p. 765), or Manes and Arbib (1986, p. 270). If X is a poset, Theorem 2 reduces to the Kleene fixed point theorem, while if X is a complete lattice, it reduces to Tarski's fixed point theorem. Note that the theorem provides both an existence result and a construction of the fixed point as a colimit; the theorem holds, in particular, for ω-continuous functors F defined on a category X where all ω-colimits exist.

3. Which category?

3.1. Motivation

The categories we are going to consider will have partially ordered sets as objects; hence, the games we can handle in the theory must have strategy spaces that are posets. The next paragraph shows that this is without loss of generality.

Let $g=(A,u)$ be an (ordinary) game, where $A=\langle A_1 \dots A_n\rangle$ is an n-tuple of compact Hausdorff spaces and $u=\langle u_1 \dots u_n\rangle$ an n-tuple of continuous real-valued payoff functions defined on the product $\prod(A)$ of the strategy spaces. Note that there is no order whatsoever on each A_i. Take $\Gamma(A_i)$ to be the set of closed subsets of A_i ordered by inverse set inclusion: if X and Y are two closed subsets of A_i, then $X \leqslant Y$ means that $Y \subseteq X$, i.e. that X and Y are consistent properties of the strategies in A_i and Y refines X, or X approximates Y, or, finally, Y implies X. Furthermore, the map $f: A_i \to \Gamma(A_i)$, $fx=\{x\}$, x in A_i, embeds A_i into $\Gamma(A_i)$. The next two theorems describe some properties of this embedding.

Theorem 3. *If S is compact Hausdorff, then so is $\Gamma(S)$ in the Vietoris topology; furthermore, the 'inclusion' map f is continuous.*

Proof. Gierz (1980, pp. 284–285).

Theorem 4. *If S is compact Hausdorff and $g: S \to R$ is continuous, then there is a continuous extension $\bar{g}: \Gamma(S) \to R$ of g.*

Proof. This is a consequence of Tietze's extension theorem (Kuratowski, 1968, Theorem 1′, p. 161 and Theorem 4, p. 191). If S is a set of alternatives and g a utility function on S, its extension $\bar{g}$ is a utility function on properties of the alternatives in S; the fact that $\bar{g}$ extends g means that the ranking of alternatives does not change when they are embedded into $\Gamma(S)$ as singleton sets.

Theorem 5. *Each ordinary game can be faithfully embedded into a game whose strategy spaces are posets.*

Proof. If $g=(A,u)$ let $g'=(B,\upsilon)$ be defined by $B_i=\Gamma(A_i)$, while $\upsilon_i: \prod(B) \to R$ is a continuous extension of u_i.

It should be emphasized that the interpretation of the partial order on the strategy spaces, namely a precision ordering, is entirely different from the interpretation of orders in games with strategic complementarities, studied by Milgrom and Roberts (1989); in particular a precision ordering can be defined on all games, whether there are complementarities or not. We are now ready to consider the relevant categories.

3.2. Complete partial orders

In this subsection the category that fits our needs is defined, not in one scoop but by successive approximations. To motivate complete partial orders, recall that if S is compact Hausdorff, $A=\Gamma(S)$ is ordered by inverse set inclusion. If $X_1 \supseteq X_2 \supseteq \ldots$ is a nested sequence of closed sets in S, their least upper bound $\bigcap_{i=1}^{\infty} X_i$ is also closed in S: a sequence of 'finer and finer' properties in A has a minimal common refinement in A. Furthermore, A contains S itself, the least informative property in A.

Definition 11. A poset A is a cpo if

(a) every increasing sequence $x_1 \subseteq x_2 \subseteq \ldots$ in A has a least upper bound $\bigsqcup x_i$ in A.

(b) A has a least element $\perp$.

If x, y are in A and $x \subseteq y$, we say that y refines x, or that y implies x, or that x approximates y.

Examples of cpos.

If S is compact Hausdorff, $\Gamma(S)$ is a cpo; if D is *any* subset of $\Gamma(S)$, then $\bigsqcup D=\bigcap\{X: X\in D\}$.

Compact real intervals and the extended real line $R^*=[-\infty,\infty]$ with the usual ordering are cpos; they are still cpos if the usual ordering is inverted.

Open real intervals are not cpos.

We now define a topology on cpos induced by the order; then we define, and interpret, continuous functions in this topology.

Definition 12. A subset U of a cpo A is Scott-open if

(a) U is an upper set: if x is a property in U and y refines x, $x \subseteq y$, then y is also in U.

(b) U is inaccessible by increasing sequences: if $\langle x_i \rangle$ is an increasing sequence and its least upper bound $\bigsqcup x_i$ belongs to U, then some element x_i of the sequence belongs to U.

In other words, (a) means that U is a collection of properties closed under refinement and (b) says that if U contains the least common refinement of a sequence of 'finer and finer' properties, it also contains one of these properties.

Examples of Scott-open sets.

A subset U of the real interval $[a,b]$ is Scott-open iff it is of the form $(x,b]$, where $-\infty \leqslant a$, $b \leqslant \infty$. A subset U of $[a,b]^n$ is Scott-open if it is an upper set open in the ordinary metric topology. If X is compact Hausdorff and V is open in X, $N(V)=\{K\in\Gamma(X): K\subseteq V\}$ is Scott-open in $\Gamma(X)$.

These examples are on p. 100 of Gierz (1980).

Theorem 6. *The collection of all Scott-open sets in a cpo A is a topology.*

Proof. Gierz (1980, p. 100).

Definition 13. A function $f: A \to B$ between cpos is Scott-continuous if it is continuous in the Scott topologies of A and B.

The next theorem characterizes Scott-continuous functions.

Theorem 7. *A function $f: A \to B$ is Scott-continuous if and only if it preserves least upper bounds of increasing sequences: $f(\bigcup x_i) = \bigcup f(x_i)$.*

To see what this means, think of f as a scientific theory (or a computer program) that transforms input data in A into output data in B. Let Tot(A) be the set of total (maximal) elements of A. To fix ideas, let $\text{Tot}(A) \subseteq R^1_+$ be a compact set of initial endowment vectors and Tot(B) be the price simplex in R^1; f is a theory, say Arrow–Debreu, which, given the preferences, assigns to each endowment vector in Tot(A) a price vector in Tot(B). In practice, we can never be sure that we have observed exactly the values of the inputs and the outputs of the theory. This means that, to have a testable theory, we must extend f to a function from A to B, i.e. a function that predicts a property of prices for each property of endowments. The theory is reliable only if, when fed better and better approximations to the true value of initial endowments, it produces better and better approximations to the true value of prices: if $x_1 \subseteq x_2 \subseteq \ldots$ and $\bigcup x_n = x$, then $f(x_1) \subseteq f(x_2) \subseteq \ldots$ and $\bigcup f(x_n) = f(x)$. This interpretation of continuity is due to Scott (1970) for programs and to Laymon (1987) for scientific theories.

Definition 14. cpo is a category with objects all cpos and morphisms all Scott-continuous functions.

Definition 15. cpo^n is a category with objects all n-tuples $\langle A_1, \ldots, A_n \rangle$ of cpos and morphisms all n-tuples of Scott-continuous functions.

Definition 16. If A, B are cpos, then $[A \to B]$ is the set of Scott-continuous functions from A to B, ordered pointwise: $f \subseteq g$ iff $f(x) \subseteq g(x)$ for all x in A. The product $A \times B$ is ordered by $(x, y) \subseteq (x', y')$ if $x \subseteq x'$ and $y \subseteq y'$.

Theorem 8. *If A, B are cpos, then so are $A \times B$ and $[A \to B]$.*

Proof. Gierz (1980, p. 115, Lemma 2.4, and p. 199, Exercise 2.19).

To apply the fixed-point theorem 2, we define the 'map' $\to$ (which assigns to any two cpos A, B their function space $[A \to B]$), on morphisms as well, in such a way that it becomes an ω-continuous functor. This is not possible on cpos (see Manes and Arbib, 1986, p. 307), but it can be done in a subcategory of cpos obtained by restricting the morphisms to be embeddings.

Definition 17. Let A, B be cpos. A Scott-continuous map $f: A \to B$ is an embedding if there is a monotonic map $g: B \to A$ (its adjoint) such that

$$g(f(x)) = x, \quad x \in A,$$

$$f(g(y)) \leqslant y, \quad y \in B.$$

(See Fig. 7.)

Theorem 9.

(a) *Each Scott-continuous f has at most one adjoint.*

(b) *If f has an adjoint g, then f is injective, g is surjective and Scott-continuous, and both f and g preserve least elements.*

(c) *If f is an isomorphism, its adjoint is its inverse.*

(d) *The composition of two embeddings is an embedding.*

Proof. Manes and Arbib (1986, p. 308).

Given this theorem, we denote the adjoint of f by f^*: (f, f^*) is sometimes called an embedding-projection pair. To interpret the meaning of such a pair, let A again be a set of properties of initial endowment vectors and B a set of properties of price vectors, and interpret f as a theory that predicts a property of price vectors for each property of initial endowments. The fact that f is injective means that two different properties of initial endowments will generate two different predictions about prices; the fact that f preserves least elements means that if there is no information about initial endowments the theory will provide no information about prices. The adjoint f^* is the 'best possible' inverse of theory f: given a property of prices, f^* predicts the property of initial endowments that could have generated it. If this property y of prices is in the range of the theory f, say $y = f(x)$, f^* will correctly

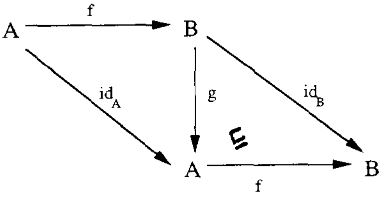

Fig. 7. Adjoint maps.

predict the property of initial endowments x that generated $f(x): f^*(f(x))=x$. And if a property of prices y is not in the range of theory f, $f^*(y)$ is a property of initial endowments that approximately generates y, in the sense that if theory f is fed data $f^*(y)$, it will generate a prediction $f(f^*(y))$ that approximates $y: f(f^*(y)) \subseteq y$.

Definition 18. The category cpo_E has the same objects as cpo, but its morphisms are restricted to be embeddings.

We can now investigate what kind of construction can be done in cpo_E.

Definition 19. An *ω colimit* in the category X is the colimit of a functor $T: \omega \to X$ (see Table 1 for ω).

Theorem 10. *Cpo and* cpo_E *have all ω-colimits. If* $T: \omega \to \text{cpo}_E$, *the colimit* (L, λ) *of T in* cpo_E *is also a colimit in cpo.* (L, λ) *is a colimit of T iff the natural transformation* $\lambda: T \to K_L$ *satisfies:*

(a) $\{\lambda_j \lambda_j^* : j \in \omega\}$ *is an increasing sequence in* $[L \to L]$;

(b) $\bigcup \lambda_j \lambda_j^* = \text{id}_L$.

Proof. Smyth and Plotkin (1982, p. 768, Theorem 2, p. 773, Example 2, and p. 775, the discussion above Fact 1.a); or Gierz (1980, ch. 4.3).

Theorem 11. *Cpo and* cpo_E *all have small products, i.e. products defined on a set.*

Proof. Smyth and Plotkin (1982, p. 774) or Manes and Arbib (1986, p. 297 and p. 311).

Definition 20. The *product functor* $\prod : \text{cpo}_E^n \to \text{cpo}_E$ assigns to each n-tuple $A = \langle A_1, A_2, \ldots, A_n \rangle$ its product $\prod(A)$, and to each n-tuple $f: A \to B$ its product $\prod(f): \prod(A) \to \prod(B)$, defined by $\prod(f)(a) = (f_1(a_1), \ldots, f_n(a_n))$.

Theorem 12. $\prod$ *preserves all ω-colimits.*

Proof. MacLane (1971, p. 115 and p. 69 (products)).

Definition 21. The *function space functor* $\to : \text{cpo}_E \times \text{cpo}_E \to \text{cpo}_E$ is defined as follows:

On objects: $\to(A, B) = [A \to B]$ = all Scott-continuous maps from A to B (*not only* embeddings).

On morphisms: if $f: A \to A'$ and $g: B \to B'$ are two embeddings, then $\to(f, g): [A \to B] \to [A' \to B']$ is an embedding defined by $\to(f, g)(h) = ghf^*$ (see Fig. 8).

Theorem 13. $\to$ *preserves all ω-colimits.*

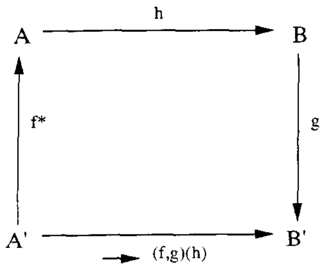

Fig. 8. The function-space functor on morphisms.

Proof. Manes and Arbib (1986, pp. 311–317); Gierz (1980, ch. 4.3); Gunter (1989, p. 84, Theorem 6.5).

3.3. Finitely continuous complete partial orders

The category cpo_E would be adequate for the construction of universal games if it were not for the fact that the Lawson topology on a cpo is not necessarily compact Hausdorff (Section 4). This topology is fine enough (has so many open sets) that requiring payoff functions to be continuous with respect to it is not more restrictive than the continuity conditions imposed as a matter of course in economics (Section 4). Unless the Lawson topology is compact, though, solutions to optimization problems involving Lawson continuous functions will not always exist. We could restrict ourselves to the category of continuous lattices studied in Gierz (1980), but this category is not closed under the probability functor: see the example on p. 225 of Graham (1988). Nonetheless, by imposing a restriction on cpos we can generate a category that satisfies the following properties:

(a) it is closed under all functors involved in the construction of universal games;
(b) its objects are compact Hausdorff in the Lawson topology;
(c) it has all ω colimits;
(d) the functors in (a) preserve all such colimits.

Definition 22. A cpo A is *finitely continuous* if there is a sequence $\langle f_i \rangle$ of Scott-continuous functions $f_i : A \to A$ that satisfies

(a) $f_1 \subseteq f_2 \subseteq \ldots$;
(b) the range of each f_i is finite, for all i;
(c) $\bigcup f_i = \mathrm{id}_A$.

Examples.

Compact real intervals are finitely continuous cpos; see Graham (1988, p. 222, Lemma 2.3).

If S is a separable compact Hausdorff space, $\Gamma(S)$ is a finitely continuous cpo; see Lawson (1989, p. 150).

Definition 23. The *category fcpo* has all finitely continuous cpos as objects and all Scott-continuous functions as morphisms; fcpo_E has the same objects but its morphisms are embeddings.

Theorem 14. *$Fcpo_E$ has finite products, and ω colimits. Hence, if $T: \omega \to fcpo_E$ is a functor, the natural transformation $\lambda: T \to K_L$ is a colimit of T iff conditions* (a) *and* (b) *of Theorem* 10 *are satisfied and L is finitely continuous.*

Proof. This follows from the existence of ω colimits in cpo_E and the definition of a finitely continuous cpo; see Graham (1988, p. 221).

Definition 24. If A is a finitely continuous cpo, $\Delta(A)$ is the set of Borel probability measures on A with respect to the Scott topology.

Definition 25. The *stochastic dominance* order on $\Delta(A)$ is defined by $p \sqsubseteq q$ iff $p(U) \leqslant q(U)$ for all Scott-open sets U.

Example.

If $A = [0, 1]$, then each Scott-open U is of the form $(\alpha, 1]$; hence the name of the order on $\Delta(A)$.

Theorem 15. *If A is a finitely continuous cpo, then so is $\Delta(A)$ with the stochastic dominance order.*

Proof. Graham (1988, p. 224, Theorem 2.4).

Definition 26. The *probability functor* $\Delta: \mathrm{fcpo}_E \to \mathrm{fcpo}_E$ is defined on objects as in Definition 24 and on morphisms $f: A \to B$ as follows: $\Delta(f): \Delta(A) \to \Delta(B)$, $\Delta(f)(p)(E) = p(f^{-1}(E))$, $p \in \Delta(A)$, with E a Borel subset of B.

Theorem 17. Δ preserves all ω colimits.

Proof. Vassilakis (1990, Appendix 2); see also the related Lemmas 3.1 and 3.2 on p. 228 of Graham (1988).

Theorem 18. *The product (see Definition* 20) *and function space (see Definition* 21) *functors map into $fcpo_E$ when restricted on $fcpo_E$.*

Proof. Lawson (1989, p. 150).

Theorem 19. *The product and function space functors on $fcpo_E$ preserve ω colimits.*

Proof. They do so in cpo_E, and fcpo_E is closed under ω colimits.

3.4. The Lawson topology

Recall that in Theorem 5 we extended the payoff function u_i from the aggregate strategy space $A_1 \times \cdots \times A_n$ of the original game to the aggregate strategy space $\Gamma(A_1) \times \cdots \times \Gamma(A_n)$ of an extended game. Each $\Gamma(A_i)$ is compact and Hausdorff in the Vietoris topology, and the extension v_i of u_i is continuous in this topology. To do the same thing on cpos, we need the counterpart of the Vietoris topology on cpos: this is the Lawson topology.

Definition 27. If L is a cpo and x is in L, then $\uparrow x = \{y \in L : x \subseteq y\}$ is the set of all the properties that refine x.

Definition 28. The *Lawson topology* on a cpo L is generated by a sub-basis consisting of the Scott-open sets and the sets of the form $L \setminus \uparrow x$, x in L.

Examples.
If $L = [0,1]^k$, the Lawson and the usual metric topology are identical. If $L = \Gamma(S)$, where S is compact Hausdorff, the Lawson and the Vietoris topologies are identical.

Theorem 20. *If L is a finitely continuous cpo, its Lawson topology is compact Hausdorff and has a countable base.*

Proof. Lawson (1989, pp. 152–154); for the countable base, see Gierz (1980, p. 170).

Our ultimate objective in the rest of this section is to show that Lawson-open sets are measurable in the σ-algebra generated by Scott-open sets. To this end, we define continuous cpos and continuous lattices.

Definition 29. Let A be a cpo, x, y elements of A: x is *way below* y, $x \ll y$, if for any directed subset D of A, $y \subseteq \bigcup D$ implies $x \subseteq d$ for some d in D: any pairwise consistent collection D of properties that implies y contains a property that implies x.

Examples.
If $A = \{0,1\}$, $x \ll y$ iff $x < y$ in the usual order. If $A = \Gamma(S)$, where S is a compact Hausdorff space, then for K, L in A, $K \ll L$ iff $L \subseteq \mathrm{int}(K)$.

Definition 30. A cpo A is *continuous* if for any $y \in A$:
(a) $\{x \in A : x \ll y\}$ is directed.
(b) $y = \bigcup \{x \in A : x \ll y\}$.

In words, every element of A can be approximated by the elements way below it.

Examples.

Compact real intervals $[a, b]$, $-\infty \leqslant a$ & $b \leqslant \infty$ are continuous cpos. $\Gamma(S)$ is a continuous cpo for every compact Hausdorff space S; see Gierz (1980, p. 284); Lawson (1989, p. 138).

Definition 31. A continuous cpo A is *countably based* if there is a countable subset B of A such that for any element y of A

(a) $\{x \in B : x \ll y\}$ is directed.

(b) $y = \bigcup \{x \in B : x \ll y\}$.

Theorem 21. *Finitely continuous cpos are countably based continuous cpos.*

Proof. Gunther (1989), Theorem 22 and the preceding discussion.

Theorem 22. *If L is a finitely continuous cpo, every Lawson-open set is measurable with respect to the Borel σ-algebra determined by the Scott topology.*

Proof. It suffices to show that the sub-basic elements of the Lawson topology are Borel measurable (with respect to the Scott topology). The Scott-open sets certainly are. The sets of the form $L \setminus \uparrow x$ also are (recall that $\uparrow x = \{y \in L : x \subseteq y\}$. To see this, let $\Uparrow x = \{y \in L : x \ll y\}$. Sets of this form are Scott-open (Lawson, 1989, p. 145). In addition, if B is a countable basis of L, then $L \setminus \uparrow x = \bigcup \{L \setminus \Uparrow b : b \ll x,\ b \in B\}$. Hence, $L \setminus \uparrow x$ is measurable, as a countable union of Scott-closed sets. □

Corollary. *If L is a finitely continuous cpo, $p \in \Delta(L)$ (see Definition 24), and $u : L \to R^*$ is Lawson continuous, then the integral $\int_L u \, \mathrm{d}p$ exists.*

Proof. Rudin (1974, p. 20, Definition 1.23).

We are finally ready for the construction of universal games. We conclude this section with a crucial result of Gierz (1980, Theorem 4.7, p. 129).

Theorem 23. *If X is a compact space and L a continuous lattice endowed with the Scott topology, the set $[X, L]$ of continuous functions from X to L ordered pointwise is a continuous lattice. In particular, this holds when L is the unit interval* $[0, 1]$.

4. Universal games

Motivation for the constructions in this section was provided in Section 2 and in the Introduction.

4.1. *The category G of games*

Let $I=[-1,1]$ be endowed with its natural order, and the Scott topology.

Definition 32. A *game* $g=(A,u)$ is an n-tuple $A=\langle A_1,\dots,A_n\rangle$ of finitely continuous posets and an n-tuple $u=\langle u_1,\dots,u_n\rangle$ of Scott-continuous payoff functions $u_i:\prod(A)\to I$.

Definition 33. A morphism $\lambda:g\to g'$ of games is an n-tuple $\lambda=\langle\lambda_1,\dots,\lambda_n\rangle$ of embeddings $\lambda_i:A_i\to A_i'$ such that, for all i, $u_i\circ\prod(\lambda^*)\leqslant u_i'$ and $u_i=u_i'\circ\prod(\lambda)$ (see Fig. 9).

In words, λ embeds the strategy spaces of g into those of g' in such a way that u_i is a restriction of u_i' and if $a'\in\prod(A')$, the payoff $u_i(\prod(\lambda^*)(a'))$ associated with the approximation $\prod(\lambda^*)(a')$ of a' in g is smaller than the payoff $u_i'(a')$ associated with a' itself in g'. Note that there might be no morphisms between two given games g and g'.

To see how colimits are formed in G, let $T:\omega\to G$ be a functor, denote $T(t)=(A^t,u^t)$ and $T(t\to t+1)=f^{t,t+1}$, $\forall t\in\omega$. Let $g=(A,u)$ be a game, and $\lambda=\langle\lambda_t:t\in\omega\rangle$ a natural transformation from the functor $A^1\to A^2\to A^3\to\dots$ to the constant functor $K_A:\omega\to \mathrm{fcpo_E}$.

Theorem 24. *(g,λ) is a colimit of T iff*

(a) *(A,λ) is a colimit of the functor $A^1\to A^2\to A^3\dots$;*

(b) $u_i=\bigcup u_i^t\ \prod(\lambda_t^*)$, $\forall i=1,\dots,n$.

Proof. First, note that each u_i is Scott-continuous by Theorem 8. Secondly, the colimit in (a) exists, by Theorem 14. Thirdly $\lambda_t:T(t)\to g$ is a morphism of games because for all i, $\prod(\lambda_t^*)u_i^t\subseteq u_i$ and $u_i^t=u_i\circ\prod(\lambda_i)$. The rest comes from Coquand, Gunther and Winskel (1989, p. 137).

Corollary. *G has all ω colimits.*

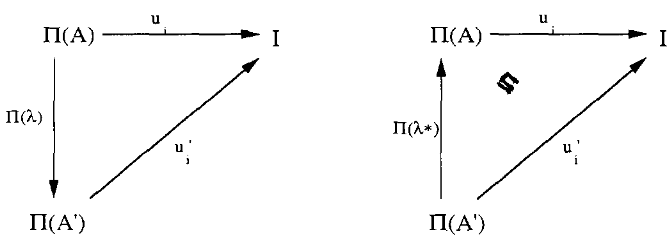

Fig. 9. Morphism of games.

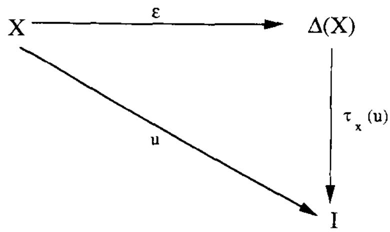

Fig. 10. Extension property.

4.2. *Extensions of preferences on probabilities*

Recall that the constructions in the Introduction required that agents have preferences over lotteries that extend their preferences on outcomes. To describe this extension, let X be a finitely continuous cpo and $u: X \to I$ a Scott-continuous function. An extension of u to $\Delta(X)$ is a Scott-continuous $\tau_X(u): \Delta(X) \to I$ that makes Fig. 10 commute, where ε assigns to each x in X the probability $\varepsilon(x)$ with unit mass on x; such an extension is given by expected utility: $\tau_X(u)(p) = \int u \, \mathrm{d}p$, $p \in \Delta(X)$. This particular extension enjoys two properties that we want τ to inherit.

Definition 34. $\tau = \langle \tau_X : X \text{ in fcpo} \rangle$ is a collection of morphisms with the following properties:

(a) for each X, $\tau_X: [X, I] \to [\Delta(X), I]$ is Scott-continuous;

(b) for each $u \in [X, I]$, for each X, $\tau_X(u) \circ \varepsilon_X = u$;

(c) for each measurable $f: Y \to X$ and $u \in [X, I]$, $\tau_X(u) \circ \Delta(f) = \tau_Y(u \circ f)$, i.e. Fig. 11 commutes.

Note that if τ_X is expected utility, property (a) is Lebesgue's monotone convergence theorem and property (c) is the change of variable formula in p. 163 of

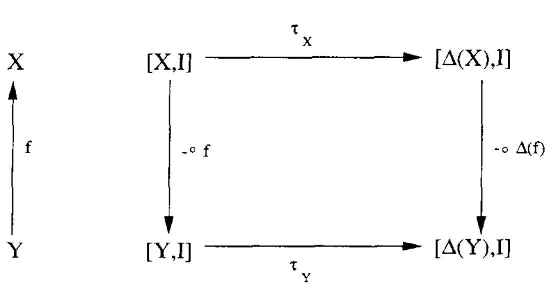

Fig. 11. Change of variable property.

Holmos (1950). To see this, note that (c), in the case of expected utility, reduces to, for all $q \in \Delta(Y)$,

$$\int_X u \, \mathrm{d}(q \circ f^{-1}) = \int_Y (u \circ f) \, \mathrm{d}q$$

(see Definition 26 for $\Delta(f)$).

4.3. *The game constructor $F: G \to G$*

We now define precisely the game constructor F that first appeared in the Introduction.

Definition 35. The functor $R_i: \mathrm{fcpo}_{\mathrm{E}}^{\mathrm{n}} \to \mathrm{fcpo}_{\mathrm{E}}$ is defined
(i) on objects by $R_i(A) = \Delta \prod (A) \times [A_i^n \to A_i]$,
(ii) on morphisms by $R_i(\lambda) = \Delta \prod (\lambda) \times (\lambda_i \circ - \circ \lambda_i^{*n})$,
where $\lambda : A \to A'$.

Definition 36. The functor $R: \mathrm{fcpo}_E^n \to \mathrm{fcpo}_E^n$ is defined by

$$R(A) = \langle R_1(A), \ldots, R_n(A) \rangle,$$

$$R(\lambda) = \langle R_1(\lambda), \ldots, R_n(\lambda) \rangle.$$

Theorem 25. *R preserves ω colimits.*

Proof. By Theorems 12, 13 and 17, the functors Δ, $\to$, X preserve ω colimits; R is a composite of these functors.

R will define F on objects; to define F on morphisms we will need to compute the probability in $\Delta \prod (A)$ induced by a vector of strategies (p, δ) in $\prod R(A)$. By Definitions 32 and 35, $p = (p_1, \ldots, p_n)$, $\delta = (\delta_1, \ldots, \delta_n)$, $p_i \in \Delta \prod (A)$, $\delta_i : A_i^n \to A_i$. Let $P_{ij} \in \Delta(a_j)$ be the marginal of P_i in A_j; the product measure $P_{1j} \times \cdots \times P_{nj}$ in $\Delta(A_j^n)$ indicates the probability with which agent j will receive recommendations on what to do from each of the n players; these recommendations are then fed into j's deviation function $\delta_j : A_i^n \to A_j$ to produce j's action. Hence, the probability measure $\xi_A^j(p, \delta) = (P_{1j} \times \cdots \times P_{nj}) \circ \delta_j^{-1}$ in $\Delta(A_j)$ computes the probability of each property of the actions open to agent j, given that (p, δ) is played. Hence, the product measure

$$\xi_A(p, \delta) = \prod \xi_A^j(p, \delta)$$

is the probability measure on the aggregate strategy space $\prod (A)$ induced by (p, δ). It is easy to show that for each A, $\xi_A : \prod R(A) \to \Delta \prod (A)$ is Scott-continuous and that $\xi : \prod R \to \Delta \prod$ is a natural transformation.

Definition 37. The *game constructor* $F: G \to G$ assigns to each game $g = (A, u)$ a new

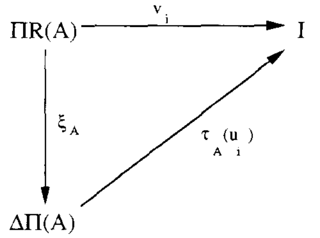

Fig. 12. Definition of game constructor.

game $F(g)=(R(A), v)$, where v is defined by Fig. 12, $\forall i$. Also, F assigns to each morphism $\lambda : g \to g'$ in G the morphism $F(\lambda)=R(\lambda)$.

It can be shown that $F(\lambda)$ is a morphism in G (Vassilakis, 1990, p. 33), and that the following holds (pp. 34–38).

Theorem 26. *F preserves ω colimits.*

This is one of the two crucial theorems that will allow us to apply Theorem 2 on the existence of fixed points. The next result is the other crucial ingredient.

Theorem 27. *For each game g, there is an embedding* $\varphi_g : g \to F(g)$. *Moreover,* $\varphi : \mathrm{Id} \to F$ *is a natural transformation* ($\mathrm{Id} : G \to G$ *is the identity functor*).

Proof. If $g=(A, u)$, then for each i, $\varphi_g^i : A_i \to R_i(A)$ is defined by $\varphi_g^i(a_i)=(\perp, a_i^c)$, where $\perp \in \Delta \prod A$ is the probability measure that assigns unit mass to the least element of $\prod(A)$ and $a_i^c : A_i^n \to A_i$ is the constant function at a_i. The rest is in Vassilakis (1990, p. 39). □

Theorem 28. *For each game g, there is a universal game X(g), unique up to isomorphism, defined as the colimit of the functor of iterations of F with respect to* $\varphi_g : g \to F(g)$ *(see Definition* 10); *X(g) is then a fixed point of F (Theorem* 2).

X can be extended to morphisms $\mu : g \to g'$ as well (Vassilakis, 1990, p. 40). The resulting functor, $X : G \to G$, satisfies the following theorem.

Theorem 29. *X preserves ω colimits.*

Proof. Lehmann and Smyth (1981, p. 119, Theorem 4.1).

This theorem is useful when the universal game constructor X is used as an ingredient in the definition of other functors whose fixed points we seek.

5. The category Σ of solution concepts

5.1. Motivation

A solution concept is an operation that assigns to each game a property of its aggregate strategy vectors. For example, the Nash solution concept is defined by the following operation: 'For each game g, find the set of fixed points of its best-reply correspondence.' There are two points worth abstracting from this example. First, to perform the operation, we always apply the same rule on each game. Secondly, from each player's point of view, a solution concept predicts the behavior of the other players in each game; in other words, it is a theory about the behavior of others. If this view is accepted, each player must be allowed to choose his or her own theory; it follows that, in principle, two different players might choose two different theories about the behavior of a third player; it is well known that this is not allowed by the Nash solution concept and its refinement. More importantly, we now have to *model* the choice of theories by agents. On the one hand, this sounds very attractive because it might lead to the development of new solution concepts. On the other hand, it generates modelling problems at least as difficult as those generated by infinite regress. For example, solution concepts have to be defined abstractly; the definition has to capture the fact that solution concepts are operations that prescribe the same rule in each game; the 'space' of solution concepts has to have the kind of properties, say compactness, that we usually need when we model choice of a point from a space of points. But the 'space' of solution concepts might be too large to even be a set, let alone a compact set, given that the category of games is too large to be a set.

It turns out that similar problems were faced by computer scientists trying to model abstractly 'polymorphic operations'. The following example clarifies the meaning of such operations, draws an analogy between them and solution concepts, and motivates their abstract definition.

Example.
Consider the operation: 'for each cpo X, and for each Scott-continuous $f: X \to X$, assign to f its least fixed point'.

Intuitively, this is a polymorphic operation because for each X and f we apply the same rule to perform the operation, namely the rule $m_X(f) = \bigcup_{n=1}^{\infty} f^n(\bot)$, where $\bot$ is the least element of X, and $m_X(f)$ is the least fixed point of f. To represent the operation abstractly, note that m_X belongs to the cpo $H(X) = [[X \to X] \to X]$. A pair (X, k), $k \in H(X)$, is a fibration of (the functor) H. Let $\mathrm{Fib}(H)$ be the category of fibrations of H. Then the operation 'take the least fixed point' can be abstractly represented by a continuous functor $\sigma: \mathrm{cpo}_E \to \mathrm{Fib}(H)$, where $\sigma(X) = (X, m_X)$. Such a functor is called a (continuous) section of H. The fact that performing the operation σ involves the application of the same rule to different cpos is captured by the fact that σ is both continuous and a functor (the appropriate diagrams commute). For more information, see Section 5.3 and Coquand, Gunter and Winskel (1989).

By analogy, then, we will model solution concepts as continuous sections of the functor $\Delta \prod PR_1$, that assigns to each game $g = \langle A, u \rangle$ the cpo $\Delta \prod (A)$ of probability measures on its aggregate strategy space. This definition has an unexpected bonus: the 'space' Σ of all solution concepts is (equivalent to) a cpo. To see why this is so, recall that the strategy spaces of each game are finitely continuous posets (Definition 22). A characterization theorem cited in Graham (1988, p. 221) and in Gunter (1989, ch. 22) shows that X is finitely continuous if and only if X is either a finite poset, or an ω colimit in cpo_E of finite posets, or a project of such a colimit (X is a project of D if there is an embedding-projection pair (f, f^*), with $f: X \to D$ and $f^*: D \to X$; then, given that f^* is onto, $X = f^*(D)$). The same result holds for games (Section 5.1). Hence, if we assume that a solution concept, in addition to preserving ω colimits, also preserves projects, we conclude that each solution concept is completely determined by its values on finite games. The collection of finite games, though, is a set (by identifying isomorphic games); hence, the collection of all possible solution concepts on finite games is a set, and therefore so is the collection of all solution concepts. This is basically the argument in Section 5.4. Modelling *already existing* solution concepts in the way sketched here is still an open problem, for reasons discussed in Sections 6 and 7.

5.2. *Obtaining games as projects of profinite games*

Motivated by the discussion in Section 5.1, we can now make precise the sense in which infinite games can be obtained by operations on finite games.

Definition 38. A cpo X is a project of a cpo D if there is an embedding-projection pair $f: X \to D$, $f^*: D \to X$.

Definition 39. A cpo D is profinite if it is an ω colimit in cpo_E of finite cpos.

Theorem 30. *A cpo X is finitely continuous if and only if it is a project of a profinite.*

Proof. Gunter (1989, Theorem 22).

We now give a characterization of profinites that will be used when we define solution concepts.

Definition 40. Let D be a cpo. A finitary projection $p: D \to D$ of D is a Scott-continuous map with finite range that satisfies $p^2 = p \leqslant \text{id}_D$. Let $M(D)$ be the set of finitary projections of D.

Theorem 31. *A cpo D is profinite if and only if $M(D)$ is countable, directed, and* $\bigsqcup M(D)=\mathrm{id}_D$ (recall that $\bigsqcup$ denotes least upper bound).

Proof. Gunter (1985, p. 47).

We can now extend Theorems 30 and 31 to games.

Definition 41. A game $g=\langle A,u\rangle$ is a project of a game $g'=\langle A',u'\rangle$ if

(a) each A_i is a project of A_i', i.e. there are embedding-projection pairs $f_i: A\rightarrow A_i'$, $f_i^*: A_i'\rightarrow A_I$ for all i;

(b) $u_i'=u_i\circ\prod(f^*)$, for all i.

Definition 42. A game is profinite if it is the ω colimit in G_E of (a chain of) finite games.

Theorem 32. *If g is a game in the category G_{E}, then g is either*

(a) *finite*; *or*

(b) *profinite*; *or*

(c) *a project of profinite.*

Proof. Let $g=\langle A,u\rangle$. By Theorem 30, each A_i is either finite, profinite or a project of a profinite. If all A_i are finite, we are in case (a). If all A_i are profinite, then there are finite cpos $A_i(t)$ such that $A_i=\mathrm{colimit}_t\, A_i(t)$. Let $u_i(t)=$ restriction of u_i on $\prod_{i=1}^n A_i(t)$, and let $g(t)=\langle A(t),u(t)\rangle$. Then $g=\mathrm{colimit}_t\, g(t)$, and we are in case (b). Finally, if all A_i are projects of profinites A_i', these are embeddings $f_i: A_i\rightarrow A_i'$. Let $u_i'=u_i\circ\prod(f^*)$ and $g'=\langle A',u'\rangle$. Then g is a project of the profinite g'.

Finally, we define for future reference a more 'symmetric' relation between games.

Definition 43. Two games g and g' are adjoint if there are Scott-continuous functions $\lambda: A\rightarrow A'$ and $\mu: A'\rightarrow A$ such that

(a) $\mu_i\circ\lambda_i\leqslant\mathrm{id}_{A_i}$, $\lambda_i\circ\mu\leqslant\mathrm{id}_{A_i'}$;

(b) $u_i'=u_i\circ\prod(\mu)$, $u_i=u_i'\circ\prod(\lambda)$.

Note that λ uniquely determines μ, and vice versa.

5.3. Fibrations and sections

Motivated by the discussion in Section 5.1, we define solution concepts as continuous, adjoint-preserving sections of the functor $H=\Delta\prod PR_1$, that assigns to each game $g=\langle A,u\rangle$ the cpo $H(g)=\Delta\prod(A)$ of probability measures on its aggregate strategy space.

Definition 44. The category of fibrations (Fib(H)), of the functor $H: G \to \mathrm{fcpo}_E$ has

objects: all pairs (g,p), $p \in H(g)$;
morphisms: $\lambda : (g,p) \to (g',p')$ iff $\lambda : g \to g'$ is a morphism in G and $H(\lambda)(p) \subseteq p'$.

Definition 45. A section of H is a functor $\sigma : G \to \mathrm{Fib}(H)$ such that
(a) $\sigma(g) = (g, p_g)$, for all g in G;
(b) $\sigma(\lambda) = \lambda$, for all morphisms λ in G.

For the purpose of the next definition, if σ is a section of H, we write $\sigma(g) = (g,p)$, $\sigma(g') = (g',p')$.

Definition 46. A section σ of H preserves adjoints if, whenever $(\lambda,\mu) : g \to g'$ is an adjunction, then $p = H(\mu)(p')$, $p' = H(\lambda)(p)$. It is easy to see that if g and g' have unique Nash equilibria and are adjoint, then these Nash equilibria satisfy the requirements of Definition 46.

Definition 47. A solution concept is a section of H that preserves adjoints and countable directed colimits.

Definition 48. A morphism of solution concepts is a natural transformation $\upsilon : \sigma \to \sigma'$ such that for all games g, $\upsilon(g) = \mathrm{id}_g$.

Unpacking the content of this definition, we can show that $\upsilon : \sigma \to \sigma'$ if and only for each game g, $p \subseteq p'$, where $\sigma(g) = (g,p)$, $\sigma'(g) = (g,p')$. Hence, the category Σ of solution concepts is a (possibly large) partial order. The next section shows that Σ is in fact small, i.e. equivalent to a cpo.

5.4. Σ is a cpo

Motivated by the discussion in Section 5.1, we can now sketch the argument that shows that Σ is a cpo.

Let FG be the subcategory of G consisting of finite games and all morphisms between them. Each strategy space of a game in FG can be identified with a finite subset of ω; the number of partial orders on each subspace is also finite: hence the set of all such strategy spaces A is countable. On the one hand, for each A, the set of all payoff functions on $\prod(A)$ has cardinality equal to I. On the other hand, for each of two finite games g and g', there are finitely many morphisms from g to g', and therefore the collection of all such morphisms has cardinality equal to I. This argument shows that the category FG is equivalent to a small subcategory S of FG, where in S all games have finite subsets of ω^n as strategy spaces. The category Σ_s of solution concepts *on* S is also small, because the set of all possible probability measures on each game in g is a subset of some finite dimensional space, and S is a set. We are now ready for the following result.

Theorem 33. *Σ is equivalent to Σ_s, and Σ_s is a cpo.*

Proof. We define two functors, res$:\Sigma\to\Sigma_s$ and ext$:\Sigma_s\to\Sigma$, and show that they form an equivalence of categories. First, res(σ) is simply σ restricted on S. Secondly, ext(σ) is defined as follows:

- If g is in S, $\mathrm{ext}(\sigma)(g)=\sigma(g)$.
- If g is a profinite, then by Theorem 31 each $M(A_i)$ is countable, directed and $\mathrm{id}_{A_i}=\bigcup M(A_i)$. For each n-tuple f in $M(A_1)\times\cdots\times M(An)$, let $g_f=\langle f(A), u|_{f(A)}\rangle$. Each g_f is finite. Let $\sigma(g_f)=(f_f, P_f)$, and let $p=\bigcup_f H(\lambda_f)(P_f)$, where $\lambda_f: g_f\to g$ embeds each g_f into g. Then define $\mathrm{ext}(\sigma)(g)=(g,p)$.
- If $(\lambda,\mu):g\to g'$ is an adjunction and if $\mathrm{ext}(\sigma)(g')=(g',p')$, then define $\mathrm{ext}(\sigma)(g)=(g,H(\mu)(p'))$. One can show, by mimicking the arguments in Coquand et al. (1989), that these definitions make sense, and that ext and res are 'inverse' to each other, i.e.

$$\mathrm{res}\,\mathrm{ext}(\sigma)=\sigma,\quad \sigma\in\Sigma_s,$$

$$\mathrm{ext}\,\mathrm{res}(\sigma)=\sigma,\quad \sigma\in\Sigma;$$

and that Σ_s is a cpo.

It is not known yet whether Σ_s is a finitely continuous cpo. We can now use our knowledge about Σ to solve equations whose variables are solution concepts.

6. Coordination-proof solution concepts

This section outlines one possible way to impose restrictions on solution concepts, namely that the prediction of a solution concept σ on each game g should be immune to successful attempts to coordinate on g, where success is also defined by the solution concept σ. The prediction of σ on the universal game $X(g)$ generated by g, call it $q=\mathrm{pr}_2\sigma(X(g))$, defines (stochastically) the successful attempts to coordinate on g. When q is projected back on g, it yields a probability measure $\gamma(g)(q)$ that is interpreted as the (stochastic) actions undertaken in g as a result of (in compliance with) the successful attempts to coordinate on g. Given that these attempts are *successful*, the actions in g will be $\gamma(g)(q)$; but the actions in g are predicted by $\sigma(g)$. Hence, it should be true that for all g

$$\sigma(g)=(g,\gamma(g)(q))$$

or

$$\sigma(g)=(g,\gamma(g)(\mathrm{pr}_2\sigma(X(g)))\equiv\psi(\sigma)(g)$$

or

$$\sigma=\psi(\sigma).$$

Note that σ appears on both sides of the equation, i.e. σ is defined as a fixed point of the functor ψ. Vassilakis (1990, p. 57) shows that ψ preserves ω colimits and that for each solution concept σ there is a solution concept $\hat{\sigma} \simeq \psi(\hat{\sigma})$ obtained as the colimit of the functor $\sigma \to \psi(\sigma) \to \psi^2(\sigma) \to \dots$; $\hat{\sigma}$ is the coordination-proof solution concept generated by σ. It can be computed pointwise by

$$\hat{\sigma}(g) = \bigcup_{t=0}^{\infty} \psi^t(\sigma)(g).$$

Unfortunately, we cannot take σ to be the Nash solution concept: even if g has a unique Nash equilibrium, $X(g)$ will in general have multiple Nash equilibria, while we restrict σ to be single-valued. The extension of this construction to multivalued solution concepts is still an open problem.

7. Open problems

This section briefly indicates how one would build on the results and techniques in the main body of this paper.

7.1. Solution concepts

The main open problem of this research is the construction of solution concepts that are coordination-proof (Section 6); are determined by their values on finite games (Section 5); are allowed to be multivalued (as is Nash); are based on mutually consistent optimization (as is Nash); and are computable. The results of Lewis (1991) suggest that such solution concepts will sometimes have to sacrifice full optimization to attain computability. The objective of the construction of such solution concepts is to take the first steps towards a theory of equilibrium institutions.

7.2. Equilibrium institutions

Recall that for each game g, a universal game $X(g)$ is a colimit of the functor of iterations $g \to F(g) \to F^2(g) \to$. Let $\lambda = \langle \lambda_t : F^t(g) \to X() \rangle$ be a colimit natural transformation. Then, by the basic lemma on p. 765 of Smyth and Plotkin (1982), there is a unique $f: X(g) \to F(X(g))$ that makes Fig. 13 diagram commute for all t

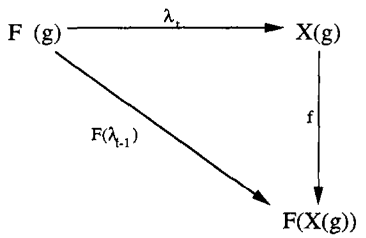

Fig. 13. Isomorphism between the universal game and its image under the game constructor.

(and f is an isomorphism). This fact implies that if a solution concept predicts p on $X(g)$, it predicts $p' = \Delta \prod PR_1(f)(p)$ on $F(X(g))$. By the definition of F, p' contains a prediction about the kind of institutions that prevail in an environment described by g; their characterization, even for the simplest games, is an open problem.

7.3. More restrictions on solution concepts

The construction sketched below owes a lot to the ideas in the cheap-talk literature, and assumes that Σ is (equivalent to) a finitely continuous poset (see the conclusion of section 6). A solution concept σ in Σ is now interpreted as a theory of agent i, predicting a property of the outcome of each game g: the idea borrowed from the cheap-talk literature is that theory σ_i should be immune to revisions caused by communication with other agents where the method is utilized to revise his theory, and the messages communicated by other agents are those predicted by σ_i itself. One way to formalize this is to assume that agents propose solution concepts to each other ('this is how I will play'; 'this is how I think others will play'). For each game $g=(A,u)$ define a new game $H(g)=(B,\upsilon)$, where $B_i = \Sigma \times [\Sigma^n \to \Sigma]$ and $\upsilon_i : \prod(B) \to I$. Each agent i proposes a solution concept σ_i in Σ, and adopts a deviation strategy $\delta_i : \Sigma^n \to \Sigma$ that maps each n-tuple of proposed solution concepts $\hat{\sigma} = (\sigma_1 \ldots \sigma_n)$ to the solution concept $\delta^i(\hat{\sigma})$ adopted by agent i. Each vector $(\hat{\sigma}, \delta)$ in $\prod(B)$ induces a probability measure $p \in \Delta \prod(A)$, defined by $p = p_1 \times \cdots \times p_n$, where p_i is the marginal on A_1 of $\delta_i(\hat{\sigma})(g)$; the payoff $\upsilon_i(\hat{\sigma}, \delta)$ associated with a strategy $(\hat{\sigma}, \delta)$ in $H(g)$ is equal to the payoff $\tau_{A_i}(u_i)(p)$ associated with p in the original game g. Finally, define a functor $Q_i : \Sigma \to \Sigma$ for each i that assigns to each solution concept $\sigma_i \in \Sigma$ a solution concept $Q_i(\sigma_i)$; $Q_i(\sigma_i)(g) = \mathrm{pr}_i \sigma_i(H(g))$ = the solution concept adopted by i in the game $H(g)$, as predicted by σ_i. We say that σ_i is communication-proof if $\sigma_i = Q_i(\sigma_i)$; the existence of such fixed points is an open problem.

7.4. Working in simpler categories

Finitely continuous posets are not easy to work with, because (a) they lack an intrinsic characterization in terms of properties of their order relation, and (b) they are not complete lattices. Finitely continuous posets that are also complete lattices, though, are exactly the continuous lattices (with a countable base). The category of continuous lattices and embeddings has all the properties needed for the constructions (see Gierz, 1980) except one: if L is a continuous lattice, $\Delta(L)$ is not necessarily a continuous lattice. A standard way of resolving such problems is by completion, i.e. by 'adding elements' to $\Delta(L)$ until it becomes a continuous lattice $\tilde{\Delta}(L)$. Such completions have been studied in different contexts (i.e. not in the case of the probability functor): see Hrbacek (1987, 1989). The construction of a completion $\tilde{\Delta}(L)$ of $\Delta(L)$ with the following properties is an open problem:

(a) $\tilde{\Delta}$ is continuous;

(b) there is an extension operation $\tau_L : [L, I] \rightarrow [\Delta(L), I]$ with the continuity and naturality properties described in Section 4.2;

(c) $\tilde{\Delta}(L)$ is the minimal completion of $\Delta(L)$ that satisfies (a) and (b);

(d) the elements in $\tilde{\Delta}(L) \setminus \Delta(L)$ are economically meaningful.

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