How vague can one be? Rational preferences without completeness or transitivity.

Paola Manzini*
Queen Mary, University of London

Marco Mariotti
Queen Mary, University of London

This version: April 2004

Abstract

What can it mean for preferences to be rational when transitivity or completenss are not assumed? In this paper we provide a framework and a set of conditions to deal with this question. We provide representation results in terms of a pair of functions, a utility function and a vagueness function.

Keywords: incomplete preferences, vagueness, sure-thing principle

*Corresponding author, Department of Economics, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom. We are grateful to Eric Danan, Yiannis Kountouris and Duncan Luce for useful comments.

1 Introduction

Standard preferences in economic modelling are complete and transitive. However, such assumptions may not be adequate at the descriptive level, especially when the alternatives are complex (for example when they carry several attributes).[1] Even at the normative level, arguments such as 'money pumps' used to justify the need for transitivity meet many objections.[2] Moreover, there has been some recent interest in incomplete preferences suggesting how an individual may be '*vague*', or 'undecided', i.e. unable to choose between alternatives, without necessarily being irrational (Mandler [7], Dubra, Maccheroni and Ok [3], Masatlioglu and Ok [8]). The two issues of transitivity and completeness are related: if preferences are incomplete (but transitive) because of cognitive limitations, the way they are completed may in principle generate cycles. For example, we may judge alternatives first on the basis of a certain criterion, say fairness, and when we encounter two alternatives that cannot be ranked by fairness we turn to a supplementary criterion, say efficiency. It is easy to see that even when the two criteria of justice and efficiency are themselves transitive, the combination of the two need not be.[3]

In this paper we propose a framework in which the question of rationality can be posed in a different way. We show that even when the two standard rationality properties of transitivity and completeness are dropped, it is still possible to draw a sharp distinction between rationality and irrationality. We view incompleteness, interpreted as cognitive vagueness, as the fundamental phenomenon, since we will show that acyclicity is implied by some standard and more basic rationality requirement. The question we would like to address is then: How vague can a rational individual be?

This question can be taken in two senses. First, it may mean: *To what extent can someone be vague*? But it can also mean: *In what way can someone be vague?* It is the second question that interests us. We find that while there is no limit to the 'amount' of vagueness a rational individual may experience (including the extreme case in which no two alternatives can be compared), there is a rather specific structure that 'rational vagueness' must obey.

[1] See for instance Tversky [12], [13] and Slovic [11].

[2] See e.g. Fishburn [5]and Mongin [9].

[3] Suppose that a is fairer than c but b cannot be ranked with either a or c in terms of fairness. If c is more efficient than b and b more efficient than a, and fairness is applied before efficiency, the resulting preferences are cyclic: c is preferred to b which is preferred to a which is preferred to c.

Let's now be more specific as to the rationality conditions we consider. Our main innovation is adding a new primitive to the standard formulation of a decision problem (consisting of a set of alternatives and a preference relation). This new primitive is a *binary operation*, which we call Keep Your Options Open (KYOO) operation. When two alternatives a and b are combined through the KYOO operation, the result is a 'higher order' alternative with the meaning that the decision between a and b is left 'open'. The possibility to KYOO is common in economic and other everyday decisions. Except in cases where there is an absolute deadline and you are exactly at the deadline, when valuing objects or states you can always not decide immediately, 'sleep over it', 'take a deep breath', and so on. Keeping the options open between several alternatives creates a new decision situation. We propose that, in order to assess the rationality of an individual's preferences over certain alternatives, we also consider his preferences over the *extended* set of alternatives, which includes such decision situations. We are able to impose sharp, powerful and easily understandable axioms on such preferences, which lead to specific characterisation results for the preferences on the basic alternatives. The first condition is a 'sure-thing' property. If you prefer a to b and c to d, then you will prefer to keep your options open between a and c rather than between b and d. This seemingly uncontroversial property guarantees acyclicity, and via a classical representation result by Bridges [1], leads to our first representation result (Theorem 3) which uses both a 'utility' and a 'vagueness' function. We argue that this representation is suited to express 'psychological' preferences - based on introspection -, as opposed to 'behavioural' ones - based on choice. Our second result (Theorem 4) specialises the vagueness function somewhat, to obtain an interval order. This is done by imposing a second axiom ('noncomparability') which states that an individual is vague if and only if he prefers to postpone a choice rather than making it. We argue that this characterises all 'behavioural' (i.e. choice-revealed) preferences, as well as a certain class of psychological preferences. The proof uses a classical theorem by Fishburn [4], as well as a mild auxiliary condition.

There are two notable contributions (Danan [2] and Luce [6]) with which our own shares the insight of adding a binary operation on the basic objects of analysis. Luce's measurement theory based on the idea of 'joint receipts' has in fact inspired our approach. The joint receipt operation has some formal features similar to the KYOO operation of this paper, but is somewhat different in other respects. We highlight the differences and similarities as they arise in the text (Remarks 1 and 2).

Danan [2] introduced a 'flexibility' operator which is very close, both formally and conceptually, to the KYOO operator. In Danan's approach, an agent can decide to 'learn then act'. This is similar to the idea of 'keeping one's options' open. At the formal level, his 'learning-then-acting' axiom and our 'noncomparability' axiom are the same. However, the focus in Danan's paper is quite different from ours. His concern is to establish a link between behavioural and psychological preferences. This is done through several properties including learning-then-acting. Our interest, on the contrary, is on the structure and representation of incomplete preferences that can be described as 'rational'.

2 Preliminaries

A binary relation B on a set A of alternatives is said to satisfy:

- *asymmetry*, if xBy implies *not* yBx;
- *acyclicity*, if $x_1Bx_2B...Bx_{n-1}Bx_n$ implies $x_n \neq x_1$;
- *transitivity*, if $xByBz$ implies xBz;
- *intervality*, if xBy and vBw implies xBw or vBy.

The following two theorems due to Bridges [1] and Fishburn [4] establish representation results[4] which will be instrumental for our own representation theorems:

Theorem 1 *(Bridges [1]) Let B be a binary relation on a countable set X. Then B satisfies acyclicity if and only if there exists a function $u : X \to \mathcal{R}$ such that, for all $x, y \in X$:*

$$xBy \Rightarrow u(x) > u(y)$$

Theorem 2 *(Fishburn [4]) Let B be a binary relation on a countable set X. Then B satisfies intervality if and only if there exist functions $u : X \to \mathcal{R}$ and $\sigma : X \to \mathcal{R}_{++}$ such that, for all $x, y \in X$:*

$$xBy \Longleftrightarrow u\left(x\right) > u\left(y\right) + \sigma\left(y\right)$$

[4]Bridges [1] also offers a simpler proof for the theorem by Fishburn reported here.

Let A be a countable set of alternatives. Let $\oplus$ denote a binary operation on A. We assume that the operation $\oplus$ can be extended recursively to pairs of the type $(a_1 \oplus ... \oplus a_n, b_1 \oplus ... \oplus b_m)$ provided that $a_i \neq b_j$ for all $i = 1, ..., n$ and $j = 1, ..., m$. Let $A^{\oplus}$ be the closure of A under $\oplus$. This operation is also assumed to be:

- commutative: $a \oplus b = b \oplus a$
- associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

So, for example, if $A = \{a, b, c\}$ then $A^{\oplus}$ can be written as $A^{\oplus} = \{a, b, c, a \oplus b, a \oplus c, b \oplus c, a \oplus b \oplus c\}$. The set $A^{\oplus}$ is clearly isomorphic to the power set 2^A, but as will be clear later, our contribution is definitely not in the vein of the literature on extending a preference relation on a set to the power set (to express preferences over 'opportunity sets').[5]

Although we are ultimately interested in characterising preferences over 'pure' outcomes in A, we consider also preferences over the set of 'mixtures' obtained by applying $\oplus$ recursively. That is, preferences are modelled as a relation $\succ$ on $A^{\oplus}$. The only assumption made throughout on $\succ$ is that it is irreflexive. We write $a \sim b$ to mean both $a \not\succ b$ and $b \not\succ a$.

The interpretation of $\oplus$ is as a 'Keep Your Options Open' (KYOO) operation: $a \oplus b$ means that both a and b are left available. The commutativity property simply means that having the option to decide between a and b is just the same thing as having the option to decide between b and a. The associativity property is a bit more subtle. It means that having the option to decide between: (i) a and (ii) having the option to decide between b and c is the same thing as having the option to decide between (iii) having the option to decide between a and b and (iv) c. This simply *defines* the notion of a higher order option. If one thinks of a sequential choice interpretation of objects such as $a \oplus (b \oplus c)$, associativity amounts to an assumption of 'consequentialism' in the following sense: two 'decision trees' are the same whenever for every path of the first tree there is a path leading to the same outcome in the second tree, and conversely. Alternatively, one may interpret the KYOO operation simply as one of 'lumping together', with no sequential structure.

[5]For example, while in that literature a 'monotonicity' axiom - stating that a set is better than at least some of its subsets - is standard and sensible, in our context this is not necessarily true: there is no reason to prefer $a \oplus b$ to either a or b (when for example $a \succ b$).

Remark 1 The associativity feature of the KYOO operation makes it similar to Luce's 'joint receipt' operation (see e.g. [6]), and dissimilar from a lottery over basic alternatives.

In the sequel, we will introduce some properties of preferences over $A^{\oplus}$ that involve the application of the KYOO operation.

3 A fundamental property

The first property that we introduce has to do with the 'mixing' of alternatives for which the individual *is* able to establish a preference:

$\succ -$**Sure Thing** For all $a, b, c, d \in A^{\oplus}$, $a \succ b, c \succ d \Rightarrow a \oplus c \succ b \oplus d$

This is for us an uncontroversial postulate of rationality at least if $a \oplus c$ and $b \oplus d$ are viewed as decision situations whose only value is their capacity to lead to a final resolution. No matter how the decision situation $b \oplus d$ is eventually resolved, the decision situation $a \oplus c$ can be resolved in a way that is strictly better. The only potential criticism we can think of, used sometimes against axioms which are formally similar, has to do with positive or negative 'complementarities'. However, given our interpretation of the KYOO operation this line of criticism does not apply, since the receipt of one of the alternatives in a mixture is *exclusive* and not joint: spillovers cannot arise.

Remark 2 This distinguishes our interpretation from Luce's 'joint receipt' approach. In this respect two alternatives combined through the KYOO operation are more similar to a gamble over those two alternatives.

For future reference, we also define a similar principle, which however appears to have a less compelling appeal than $\succ -$Sure Thing.

$\sim -$**Sure Thing** For all $a, b, c, d \in A^{\oplus}$, $a \sim b, c \sim d \Rightarrow a \oplus c \sim b \oplus d$

As we show below, the $\succ -$Sure Thing axiom on its own is powerful enough to restrict preferences dramatically:

Theorem 3 *If $\succ$ satisfies $\succ -$Sure Thing then*
(i) there exist functions $u : A \to \mathcal{R}$ and $\sigma : A \times A \to \mathcal{R}_+$ with $\sigma(a,b) = \sigma(b,a)$, such that

$$a \succ b \Leftrightarrow u(a) > u(b) + \sigma(a,b)$$

(ii) In addition, if A is finite then the function σ can be chosen as follows:

$$\sigma(a,b) = \sigma(b,a) = 0 \textit{ for all } a,b \textit{ with } a \succ b$$

and

$$\sigma(a,b) = \sigma(b,a) = 1 \textit{ for all } a,b \textit{ with } a \sim b$$

Proof. ***(i)*** First we show that $\succ -$Sure Thing implies acyclicity. Suppose in negation that $a_1 \succ a_2 \succ ... \succ a_n \succ a_1$ for some $a_1, ..., a_n \in A$. Then by successive applications of $\succ -$Sure Thing we have:

$$\begin{aligned} a_1 \oplus a_2 &\succ a_2 \oplus a_3 \\ (a_1 \oplus a_2) \oplus a_3 &\succ (a_2 \oplus a_3) \oplus a_4 \\ &\vdots \\ (...(a_1 \oplus a_2) \oplus ...a_{n-1}) \oplus a_n &\succ (...(a_2 \oplus a_3) \oplus ... \oplus a_n) \oplus a_1 \end{aligned}$$

The last relation is a contradiction in view of the irreflexivity[6] of $\succ$ and the commutativity and associativity of $\oplus$.

By Theorem 1, this implies that there exists a function $u : A \to \mathcal{R}$ such that $a \succ b \Rightarrow u(a) > u(b)$ for all $a,b \in A$. Now for all $a,b \in A$ such that $a \succ b$ choose

$$\sigma(a,b) = \sigma(b,a) = \frac{u(a) - u(b)}{2}$$

and for all $a,b \in A$ such that $a \sim b$ choose

$$\sigma(a,b) = \sigma(b,a) = |u(a) - u(b)|$$

So we have that

$$a \succ b \Rightarrow u(a) = u(b) + (u(a) - u(b)) > u(b) + \sigma(a,b)$$

proving one direction of the statement, and

$$u(a) > u(b) + \sigma(a,b) \Rightarrow \sigma(a,b) < |u(a) - u(b)|$$

[6]Note that although irreflexivity is implied by acyclicity, we have to assume it independently in order to prove acyclicity.

The last inequality and the definition of σ imply that it cannot be $a \sim b$. Since it cannot be $b \succ a$ either (given that $u(a) > u(b)$), it must be $a \succ b$. This proves the other direction of the statement.

(ii) Let

$$m := \min_{a,b \in A} \{u(a) - u(b) \,|a \succ b\}$$

Note that $m > 0$. Now for all $a, b \in A$ such that $a \succ b$ choose

$$\sigma(a,b) = \sigma(b,a) = \frac{m}{2}$$

and for all $a, b \in A$ such that $a \sim b$ choose

$$\sigma(a,b) = \sigma(b,a) = |u(a) - u(b)|$$

Proceeding as for the general case one can see that this specialisation represents preferences. Rescaling the functions u and σ finally yields the claim of the statement[7]. ■

This result has an obvious interpretation. Provided the single rationality principle of $\succ -$Sure Thing is satisfied, an individual's preferences can be modeled by means of a 'utility' function u and a 'vagueness' function σ. The individual is able to make a comparison between two alternatives if and only if his 'vagueness' is sufficiently low compared to the utility difference of the alternatives.

When the set of alternatives is finite, the representation is considerably simplified, as the vagueness function can be taken to assume only two values. A natural interpretation is that the individual can be in one of only two psychological states: 'confused' or 'not confused'.

Finally, a principle of 'optimality', at least as embodied by acyclicity, can be recovered from the $\succ -$Sure Thing axiom[8]. This is noteworthy in view of the opinion, expressed by leading scholars[9], that acyclicity of preferences is not a normatively compelling property.

[7]Note that obviously if u and σ represent preferences, then so do $v = \lambda u + \alpha$ and $\tau = \lambda\sigma$.

[8]We call this a principle of optimality bacause, at least in the finite case, acyclicity of the strict preference is sufficient to guarantee the existence of a maximal element, so that the individual can be seen as 'maximising' the utility u, albeit imperfectly due to the vagueness σ.

[9]E.g. Fishburn [5], Mongin [9].

4 'Psychological' versus 'behavioural' preferences

We view Theorem 3 as representing rational 'psychological preferences', that is those disclosed by the decision maker after accessing his own 'internal feelings': the key aspect is that the decision maker only has to express an opinion in his comparison of the alternatives. From this perspective it is entirely plausible for an agent to declare that for instance he is undecided (i.e. unable to compare) between two alternatives a and b, and yet prefer any of them to postponing a decision, that is both a and b might be preferred to their mixture $a \oplus b$.

This type of reasoning is ruled out of necessity if we look at 'choice-revealed preferences', that is if we require an agent to *choose* (rather than express a preference) between a and b. In the case of the two alternatives a and b he can either choose a, or choose b, or postpone a choice between these two alternatives by selecting their mixture $a \oplus b$: there are no other options. So *if* the mixture is selected, one must infer that the agent is unable to compare a and b, in the sense of a and b being in the relation $\sim$. Based on these considerations, in the case of *behavioural* (i.e. choice-revealed) preferences the following axiom seems compelling (in addition to $\succ$ −Sure Thing):

Noncomparability For all $a, b \in A^{\oplus}$, $a \sim b \Leftrightarrow a \oplus b \succ a$ and $a \oplus b \succ b$.

So we regard the Noncomparability axiom just as a matter of logic in the choice-revealed interpretation of preferences: being unable to compare a and b *means* preferring not to choose between a and b.

The fact that this property is not simply a matter of logic in the alternative, psychological, interpretation does not imply that it does not have a meaning in that context. In that case its correct interpretation is as a 'bias' towards keeping one's options open. To understand this, consider first the implication

$$\{a \oplus b \succ a \text{ and } a \oplus b \succ b\} \Rightarrow a \sim b \tag{1}$$

If there was a strict preference between a and b, say, $a \succ b$, it would seem plain contradictory to prefer to keep one's options open between a and b. Therefore we regard implication 1 of the axiom as an uncontroversial rationality requirement, whether or not preferences are interpreted psychologically.

The psychologically substantive part of the axiom is the converse implication

$$a \sim b \Rightarrow \{a \oplus b \succ a \text{ and } a \oplus b \succ b\} \tag{2}$$

This means that if you cannot decide between a and b, then you prefer to postpone a decision. Although this is reasonable, it is conceivable that one would regard this postponement as worse than either a or b simply because he dislikes a situation of indecision. Hence our interpretation as a *bias* in favour of keeping one's options open.

The following example illustrates the situation in which this bias may not apply with a psychological interpretation of preferences. Suppose Mr. Q is taking Miss She out on their first date, which will comprise a movie followed by dinner at a nearby restaurant. There are three movie theatres showing the chosen film, with a restaurant just round the corner. Cinema 1 is adjacent to a restaurant with both a (cheap) bar and a (more expensive) restaurant menu in distinct dining rooms. Cinemas 2 and 3 are next to a (cheap) bar and a (more expensive) restaurant, respectively. Mr. Q may have conflicting fears of, on the one hand, appearing stingy and on the other hand overdoing it in trying to impress his new date. So he may be undecided on what type of dinner is better. Choosing cinema 1 would keep Mr. Q's options open. Still, he might regard the postponement of the decision as spoiling his enjoyment of the movie. Thus, although he is psychologically indifferent between cinema 2 or 3, he prefers either one of them to keeping the matter unresolved.

To summarise, the Noncomparability axiom may or may not be violated by psychological preferences, but it is a compelling requirement in the case of behavioural preferences.

As an additional straightforward rationality requirement we introduce the following axiom:

Independence of Dominated Alternatives (IDA) For any distinct alternatives $a_1, a_2, ..., a_n \in A$, $a_1 \oplus ... \oplus a_{i-1} \oplus a_{i+1} ... \oplus a_n \sim a_1 \oplus a_2 \oplus ... \oplus a_{i-1} \oplus a_i \oplus a_{i+1} ... \oplus a_n$ if there exists some $a_j \succ a_i$.

This axiom simply states that enlarging the set of options further to a dominated alternative does not make any difference to the evaluation of the original composite alternative.

Adding Noncomparability and IDA to $\succ -$Sure Thing yields our second and main representation result. In this case we are able to derive even more structure by specifying the 'vagueness' function σ so that it only depends on one alternative:

Theorem 4 *If $\succ$ satisfies IDA, $\succ$ −Sure Thing and Noncomparability then there exist functions $u : A \to \mathcal{R}$ and $\sigma : A \to \mathcal{R}_{+}$ such that $a \succ b$ if and only if $u(a) > u(b) + \sigma(b)$*

Proof. *Step 1. For all $a, b, c, d \in A^{\oplus}$, $a \sim b$, $c \sim d \Rightarrow a \oplus c \sim b \oplus d$ ($\sim$ −Sure Thing).* Suppose $a \sim b$, $c \sim d$. By Noncomparability this implies

$$\begin{aligned} a \oplus b &\succ a \\ a \oplus b &\succ b \\ c \oplus d &\succ c \\ c \oplus d &\succ d \end{aligned}$$

Then by $\succ$ −Sure Thing

$$\begin{aligned} a \oplus b \oplus c \oplus d &\succ a \oplus c \\ a \oplus b \oplus c \oplus d &\succ b \oplus d \end{aligned}$$

At this point a second application of Noncomparability yields $a \oplus c \sim b \oplus d$, as desired.

Step 2. Intervality. Suppose $a \succ b$ and $c \succ d$. Then $a \succ d$ or $c \succ b$. Violations can occur in four cases:

1. $b \succ c$ and $d \succ a$: this generates the cycle $a \succ b \succ c \succ d \succ a$, which by the proof of Theorem 3 contradicts $\succ$ −Sure Thing.

2. $a \sim d$ and $b \succ c$: $a \sim d$ and Noncomparability imply

   $$a \oplus d \succ a$$

   Moreover by $\succ$ −Sure Thing applied to $b \succ c$ and $c \succ d$ we have

   $$b \oplus c \succ c \oplus d$$

   Applying once more $\succ$ −Sure Thing to the displayed preferences yields $a \oplus b \oplus c \oplus d \succ a \oplus c \oplus d$, which contradicts IDA.

3. $a \succ d$ and $b \sim c$: as case 2 above.

4. $a \sim d$ and $b \sim c$: $\succ$ −Sure Thing on $a \succ b$ and $c \succ d$ implies

$$a \oplus c \succ b \oplus d$$

whereas step 1 implies the contradiction

$$a \oplus c \sim b \oplus d$$

Step 3: There exist functions as in the statement. This step follows from step 2 and Theorem 2. ■

Remark 3 Although transitivity of strict preference was not assumed, the result above shows that it is a consequence of the axioms, since $u(x) > u(y) + \sigma(y)$ and $u(y) > u(z) + \sigma(z)$ imply that $u(x) > u(z) + \sigma(z)$.

5 Concluding Remarks

In this paper we have considered what it may mean for an individual's preferences to be rational even when they are not assumed to be complete or transitive. To this aim, we have enlarged the space of basic alternatives to allow preferences for 'keeping one's options open'. We have found, first, that a standard sure-thing rationality property implies acyclicity. We have then shown that the *rational* incompleteness of preferences must possess a specific structure, and may be captured by a pair of functions: a standard utility function and a 'vagueness' function.

In our first representation theorem the vagueness term depends on both alternatives. This representation fits 'psychological' preferences. In our second representation result, which adds some conditions, vagueness depends on one alternative only. At the formal level, rational incompleteness is here representable as an interval order. This representation fits behavioural preferences, as well a particular variety of psychological preferences: those which are biased in favour of keeping one's options open.

References

[1] Bridges, D. S., 1983, 'Numerical representation of intransitive preferences on a countable set', *Journal of Economic Theory,* 30: 213- 217.

[2] Danan, E., 2003, 'A Behavioral Model of Individual Welfare' mimeo EUREQua, Universite' de Paris I', mimeo.

[3] Dubra, J., F. Maccheroni and E. A. Ok 'Expected Utility without the Completeness Axiom', forthcoming in *Journal of Economic Theory.*

[4] Fishburn, P. C., 1970, *Utility Theory for Decision Making*, Wiley, New York.

[5] Fishburn, P. C., 1988 *Nonlinear Preference and Utility Theory,* John Hopkins University Press, Baltimore.

[6] Luce, D., 2000, *Utility of Gains and Losses: Measurement-Theoretical and Experimental Approaches,* Lawrence Elrbaum Associates, Mahwah NJ.

[7] Mandler, M. , 2001, 'Incomplete Preferences and Rational Intransitive Choice', mimeo, Royal Holloway, University of London.

[8] Masatlioglu, Y. and E. A. Ok, 2003, 'Rational Choice with a Status-Quo Bias', mimeo, New York University.

[9] Mongin, P. , 2000, 'Does Optimization Imply Rationality?', *Synthese*, 124:73-111.

[10] Ok, E. A., 2002, 'Utility Representation of an Incomplete Preference Relation', *Journal of Economic Theory*, 104: 429-449.

[11] Slovic, P., 1975, "Choice between equally valued alternatives", *Journal of Experimental Psychology: Human Perception and Performance*, 1: 280-287.

[12] Tversky, A., 1969, "Intransitivity of Preferences", *Psychological Review*, 76: 31-48.

[13] Tversky, A., 1972, "Elimination by Aspects: a Theory of Choice", *Psychological Review*, 79: 281-299.