Iterated Expectations with Common Beliefs

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August 2001

Abstract

This paper generalizes a result by Samet concerning iterated expectations and common priors. When a player in some state of the world is allowed to ascribe probability zero to that state, something not allowed in Samet's framework, iterated expectations may not converge, and when they do, common knowledge of their limit may not characterize a common prior. It is shown here that replacing common knowledge with common belief, convergence is still lost in general, but when it obtains, the full characterization is restored.

1 Introduction

In interactive contexts where a player forms different expectations about random variables at different states of the world, her beliefs are themselves random variables. For a given random variable, we are thus naturally led to analyze a player's infinite hierarchy of expectations of random variables, expectations of other players' expectations of random variables, and so on. Samet (1998) uses the Markovian structure of type functions to characterize existence and uniqueness of a common prior, and when a unique common prior exists, to meaningfully construct it in terms of players' expectations only. The main results in Samet (1998) are as follows: first, if for a given random variable and a given sequence of players $i_1, i_2, \ldots$ where each player appears infinitely often we could compute, at each state, i_1's expectation of it, i_2's expectation of i_1's expectation of it, and so on, then we would end up with a number whose value is common knowledge among the players in every state. Second, there is a common prior if and only if, for every random variable, the limit is the same regardless of what sequence of players we choose.

These results hold under the assumption that every player, at every state, must assign positive probability to that state. While this is not a very restrictive

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assumption, it is interesting to investigate the effects of relaxing it, and this is what this work does. In Samet (1998) it is suggested that analogous results are true once we replace the notion of common knowledge with that of common 1-belief (as defined by Monderer and Samet (1989)). Our results show that the second result indeed holds in our more general framework, while the first one (in particular, convergence of iterated expectations to values which are common knowledge at every state, even if a common prior doesn't exist) cannot be proved.

2 Common Beliefs and Common Priors

We take as given a *type space* $\langle \mathcal{I}, \Omega, (\Pi_i, t_i)_{i \in \mathcal{I}} \rangle$, where $\mathcal{I} = \{1, \dots, I\}$ is a finite set of *players*; Ω is a finite set whose elements we call *states* and whose subsets we call *events*; Π_i is player i's *information partition*, a partition of Ω whose unique element containing ω we denote by $\Pi_i(\omega)$; t_i is player i's *type function*: for each $\omega \in \Omega$, $t_i(\omega)(\cdot)$ is a probability measure on $(\Omega, 2^{\Omega})$ satisfying

$$(1) \qquad t_i(\omega)(\Pi_i(\omega)) = 1 \qquad \forall\, \omega \in \Omega,$$

$$(2) \qquad t_i(\omega)(\cdot) = t_i(\omega')(\cdot) \qquad \forall\, \omega, \omega' \in \Omega \text{ s.t. } \omega' \in \Pi_i(\omega).$$

An event E is *common knowledge*[1] at ω if it includes an event which contains ω and is a union of elements of Π_i for every i. An event E is *evident 1-belief* if $t_i(\omega)(E) = 1$ for every $i \in \mathcal{I}$ and every $\omega \in E$. Finally, an event E is *common 1-belief at* ω if there exists an evident 1-belief event F such that $\omega \in F$ and $t_i(\omega')(E) = 1$ for every $i \in \mathcal{I}$ and every $\omega' \in F$.

We denote by $\mathcal{B}$ the set of nonempty minimal evident 1-belief events. These are thus the nonempty minimal events B which are commonly 1-believed at every $\omega \in B$. These events are disjoint. In fact, if $B, C \in \mathcal{B}$ and $B \cap C \neq \varnothing$, then $t_i(\omega)(B) = t_i(\omega)(C) = 1$, hence $t_i(\omega)(B \cap C) = 1$ for every $\omega \in B \cap C$, hence by minimality of B and C, $B = C$. It follows that Ω can be uniquely partitioned as

$$(3) \qquad \Omega = B_1 \cup \dots \cup B_k \cup N,$$

for some finite number k, where $B_1, \dots, B_k \in \mathcal{B}$ and N is such that $t_i(\omega)(N) = 0$ for every $i \in \mathcal{I}$, $B \in \mathcal{B}$, $\omega \in B$. Note that N can't contain any evident 1-belief event.

A probability measure p on Ω is a *common prior*, provided

$$(4) \qquad p(\Pi_i(\omega)) t_i(\omega)(E) = p(E \cap \Pi_i(\omega))$$

for all $i \in \mathcal{I}$, $E \subset \Omega$ and $\omega \in \Omega$. Similarly, for $B \in \mathcal{B}$, a probability measure p^B on B is a *common prior on* B if for all $i \in \mathcal{I}$, $E \subset \Omega$ and $\omega \in B$,

$$(5) \qquad p^B(B \cap E \cap \Pi_i(\omega)) = p^B(B \cap \Pi_i(\omega)) t_i(\omega)(B \cap E).$$

[1] The notion of common knowledge is only used in section 3.1.

Lemma 1. *If p is a common prior, then $p(N) = 0$.*

Proof. Suppose $S := \{\omega \in N \mid p(\omega) > 0\}$ is nonempty, and let $\omega \in S$ and $i \in \mathcal{I}$. By (4) and (2) above, $t_i(\omega')(\omega) > 0$ for each $\omega' \in \Pi_i(\omega)$. But, by (3) and by definition of $\mathcal{B}$, for every $B \in \mathcal{B}$ and every $\omega' \in B$ we have $t_i(\omega')(\omega) = 0$, because $\omega \in N$. It follows that $\Pi_i(\omega) \cap B = \varnothing$ and thus, again by (3), $\Pi_i(\omega) \subset N$, which implies by (1) that $t_i(\omega)(S) = 1$. But this is true for every $i \in \mathcal{I}$, meaning that S is evident 1-belief, a contradiction. ■

Lemma 2. *If p^B is a common prior on $B \in \mathcal{B}$, then $p^B(\omega) > 0$ for each $\omega \in B$.*

Proof. Call S the support of p^B. Then, by (5), $p^B(S \cap \Pi_i(\omega)) = p^B(B \cap \Pi_i(\omega))$, hence $t_i(\omega)(S) = 1$, for every $i \in \mathcal{I}$ and every $\omega \in S$. In other words, S is a nonempty, evident 1-belief event which is contained in B, and since B is minimal for this property, we conclude that $B = S$. ■

Proposition 1. *The set of common priors is the convex hull of the set of common priors on the elements of $\mathcal{B}$.*

Proof. Suppose p^B is a common prior on $B \in \mathcal{B}$, and define the probability measure p on Ω as $p(E) = p^B(B \cap E)$ for every event E. If $\omega \in B$ we see that (4) is obviously satisfied. The same is clearly true if $\omega \notin B$ and $p^B(B \cap \Pi_i(\omega)) = 0$. Finally, if $\omega \notin B$ but $p^B(B \cap \Pi_i(\omega)) > 0$, then there exists $\omega' \in B \cap \Pi_i(\omega)$; since $\omega' \in B$, (4) holds for ω', and since $\Pi_i(\omega) = \Pi_i(\omega')$, hence $t_i(\omega)(E) = t_i(\omega')(E)$, (4) also holds for ω.

Conversely, suppose p is a common prior. Then, by Lemma 1, there exists $B \in \mathcal{B}$ such that $p(B) > 0$. Let $\omega \in B$ and $\omega' \in \Pi_i(\omega)$. Since B is evident 1-belief, $p(\omega') > 0$ clearly implies $\omega' \in B$, hence we see that $p(B \cap \Pi_i(\omega)) = p(\Pi_i(\omega))$. In other words, the probability measure p^B on B defined by $p^B(E \cap B) = p(E \cap B)/p(B)$ for every event E satisfies (5).

■

From now on we assume that $\Omega = B \cup N$, i.e. we assume that $\mathcal{B}$ contains only one element. Proposition 1 ensures this is meaningful, and together with Lemma 1 and Lemma 2, it implies that the support of any common prior must be exactly B. But, as Samet (1998) notes, defining for every player $i \in \mathcal{I}$ the stochastic matrix M_i as $M_i(\omega, \omega') = t_i(\omega)(\omega')$ for every $\omega, \omega' \in \Omega$, a probability measure p on Ω is a common prior if and only if it is a stationary probability measure of M_i, that is $pM_i = p$, for every $i \in \mathcal{I}$. We conclude by Lemma 2 that, whenever a common prior exists, B is an ergodic set of states for *each* Markov chain M_i, hence in particular that the common prior is unique.

3 Iterated Expectations

For a random variable $f : \Omega \to \mathbb{R}$ and an infinite sequence $i_1, i_2, \ldots$ in $\mathcal{I}$ where each player appears infinitely often, the corresponding *iterated expectation* is the sequence of random variables $M_{i_1} f, M_{i_2} M_{i_1} f, \ldots$.

3.1 Divergent Expectations and Common Knowledge

In our framework, the main result in Samet (1998), which enables a full characterization of existence and uniqueness of a common prior, and a meaningful construction of it (when it uniquely exists) in terms of players' beliefs, need not hold even in simple cases. As an example, let $\mathcal{I} = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\Pi_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$, and let the type functions of 1 and 2 be represented by

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The only event which is common knowledge at any state is Ω itself. However, despite the existence of many common priors (any convex combination of [1 0 0] and [0 0 1]), for a random variable $f = [f_1\ f_2\ f_3]$ such that $f_1 \neq f_3$ we see that no iterated expectation of it can converge to a value which is common knowledge at any state, because it takes both values $M_1 f$ and $M_2 f$ infinitely often, and the two are different since $f_1 \neq f_3$.

3.2 Divergent Expectations and Common 1-Belief

As a preliminary result (which is important in its own right) Samet (1998) proves that, under the assumption that $t_i(\omega)(\omega) > 0$ for every $i \in \mathcal{I}$, $\omega \in \Omega$, every iterated expectation converges to a value which is common knowledge in every state. Here an analogous claim, in terms of common 1-belief, fails to be true, as the following example shows.

Let $\mathcal{I} = \{1, 2, 3\}$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\Pi_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and $\Pi_2 = \Pi_3 = \{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}\}$, and let the type functions of 1,2 and 3 be represented by

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Note that $\mathcal{B} = \{\Omega\}$. Since $M_2 M_1 = M_2$ and $M_1 M_2 = M_1 M_3 M_2 M_1 = M_1$, given a random variable $f = [f_1\ f_2\ f_3\ f_4]$ (taken as a column vector) we see that the iterated expectation

$$M_1 f, \ M_2 M_1 f, \ M_3 M_2 M_1 f, \ M_1 M_3 M_2 M_1 f, \ \ldots$$

doesn't converge, because it takes both values $M_1 f$ and $M_2 f$ infinitely often. And since $\mathcal{B} = \{\Omega\}$, in no state it is common 1-belief that the iterated expectation converges.

3.3 Convergent Expectations

The second result in Samet (1998), according to which a common prior exists if and only if, for every random variable, every iterated expectation converges to the same value, and this value is common knowledge, *can* be proved here once we replace common knowledge with common 1-belief (with convergence obtaining almost surely with respect to the common prior).

As said earlier, there's no loss of generality in assuming $t_i(\omega)(\omega) > 0$ for all $\omega \in B$ and all $i \in \mathcal{I}$, since we already proved that otherwise a common prior doesn't exist. On the other hand this is all we need to prove our main result.

Theorem 1. *There uniquely exists a common prior p if and only if, for every random variable f, it is common 1-belief at B that every iterated expectation of f converges to pf.*

Proof. Let σ be a permutation of $\mathcal{I}$. The restriction of the Markov chain $M_{\sigma(1)} \cdots M_{\sigma(I)}$ to B is ergodic, so sufficiency follows immediately from Theorem 1' in Samet (1998). Conversely, suppose p is a common prior. Then, by Proposition 1, the restriction of p to B is the unique common prior on B, and convergence follows immediately by Theorem 1 in Samet (1998). ■

Acknowledgements

I am grateful to Prof. Dov Samet for his advice during my M.Sc. studies at University College London in 1999, when these results were first obtained, and to Prof. George Mailath for helpful comments.

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