Stability of Equilibria in Games with Procedurally Rational Players*

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Abstract

One approach to the modeling of bounded rationality in strategic environments is based on the dynamics of evolution and learning in games. An entirely different approach has been developed recently by Osborne and Rubinstein (1998). This latter approach is static and equilibrium based, but relies on less stringent assumptions regarding the knowledge and understanding of players than does the standard theory of Nash equilibrium. This paper formalizes Osborne and Rubinstein's dynamic interpretation of their equilibrium concept and thereby facilitates a comparison of this approach with the explicitly dynamic approach of evolutionary game theory. It turns out that the two approaches give rise to radically different static and dynamic predictions. For instance, dynamically stable equilibria can involve the playing of strictly dominated actions, and equilibria in which strictly *dominant* actions are played with probability 1 can be unstable. Sufficient conditions for the instability of equilibria are provided for symmetric and asymmetric games.

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1 Introduction

Dissatisfaction with the stringent rationality requirements of the standard equilibrium approach to behavior in strategic environments has fueled the growth of a large literature on the dynamics of evolution and learning in games.[1] The focus of this literature has been on questions such as whether trajectories converge to Nash equilibria, whether dominated actions are eliminated along convergent or nonconvergent trajectories, and whether dynamic stability can provide an effective equilibrium selection criterion. The unifying theme in this otherwise diverse body of work is that individuals are assumed to have only a limited understanding of the strategic setting in which they participate, and to adjust their behavior in accordance with some adaptive process. This process is typically assumed to be payoff monotonic: more successful actions are more likely to be adopted with greater frequency than are less successful ones.

An entirely different approach to the modeling of bounded rationality has been developed recently by Osborne and Rubinstein (1998). The approach is static and equilibrium based, but relies on less stringent assumptions regarding the knowledge and understanding of players than does the standard theory of Nash equilibrium. Their equilibrium concept, $S(1)$ equilibrium, is based on an explicit process of reasoning on the part of players and therefore corresponds to procedural rather than substantive rationality in the sense of Simon (1978). Although $S(1)$ equilibrium is a static concept, Osborne and Rubinstein interpret it as a steady state of a dynamic process of sampling. This paper formalizes their informal description of the dynamic process and thereby facilitates a comparison of this approach with the explicitly dynamic approach of evolutionary game theory. It turns out that the two approaches give rise to radically different static and dynamic predictions. For instance, dynamically stable $S(1)$ equilibria can involve the playing of strictly dominated actions, and equilibria in which strictly *dominant* actions are played with probability 1 can be unstable. The criterion of dynamic stability also yields a refinement of $S(1)$ equilibrium that is both intuitively appealing and effective in application. For instance, although it is the case that all strict Nash equilibria are also $S(1)$ equilibria, some strict Nash equilibria may be unstable with respect to the dynamics. This provides a simple basis for selection among strict

[1] Recent books which survey and extend the main results in this literature include Weibull (1995), Vega-Redondo (1996), Fudenberg and Levine (1997), Samuelson (1997) and Young (1998).

Nash equilibria in certain coordination games which, unlike the commonly used criterion of stochastic stability, relies neither on the ultra-long run nor on the presence of rare mutations.

The paper is organized as follows. Osborne and Rubinstein's informal dynamic interpretation of $S(1)$ equilibrium is formalized in Section 2. The use of dynamic stability as a method of selection among alternative $S(1)$ (and alternative strict Nash) equilibria is developed in Section 3. It is shown in Section 4 that strictly dominated actions may be played with positive probability in stable $S(1)$ equilibria, and that equilibria in which only dominant actions are played can be unstable. This can occur in symmetric games only if the number of players plus and the number of actions is at least five. Sufficient conditions for the instability of strict Nash equilibria is provided in Section 5 for symmetric games. These conditions are easy to verify and are satisfied in many commonly studied games. The case of asymmetric games with multiple player populations is examined in Section 6, and sufficient conditions for the instability of strict Nash equilibria are provided also for this case. Section 7 concludes.

2 Equilibrium and Dynamics

Consider first the case of symmetric n-player games.[2] As in Osborne and Rubinstein, let $A = \{a_1, ..., a_m\}$ represent the finite set of actions available to each player, and let $u(a_i, b)$ denote the payoff to a player of choosing the action $a_i \in A$ when the remaining $n-1$ players choose the action profile $b \in A^{n-1}$. This payoff function represents each player's ordinal preferences over the set of outcomes. Let $\alpha = (\alpha_1, ..., \alpha_m)$ be a probability distribution on A, and let $v(a_i, \alpha)$, $i = 1, ..., m$, be the random variables yielding $u(a_i, b)$ with probability $\Pr(b)$ for each $b \in A^{n-1}$. Here $\Pr(b)$ is the probability that the action profile chosen by the remaining $n-1$ players is b, assuming that each action is chosen independently subject to the distribution α on A. Finally, let $w(a_i, \alpha)$ be the probability that, when each random variable $v(x, \alpha)$ is drawn independently and exactly once, the action a_i yields the best outcome. In the case of realizations in which a_i is not unique in yielding the best outcome, the probability is weighted by the reciprocal of the total number of tied alternatives. An $S(1)$ equilibrium

[2] Just as the concept of $S(1)$ equilibrium generalizes in a straightforward manner to the case of asymmetric games, so too does the disequilibrium dynamic process introduced below. The case of asymmetric games is treated separately in Section 6 below.

is defined as a probability distribution α on the set of actions A with the property that

$$w(a_i, \alpha) = \alpha_i \text{ for every action } a_i \in A.$$

The concept of $S(1)$ equilibrium is based on the idea that players sample each action exactly once and select the action which yields the highest payoff. In the case of ties, one of the tied alternatives is picked at random, with each alternative having the same probability of being chosen. The probability with which a given action is chosen under this procedure will depend, of course, on the (mixed) strategies chosen by the remaining players during each of the sampling periods. An $S(1)$ equilibrium is a mixed strategy α with the following property: if all other players adopt this strategy throughout the sampling procedure, then the probability that action a_i is best under the sampling procedure is precisely α_i.

One interpretation of $S(1)$ equilibria that is advanced by Osborne and Rubinstein is that it is the steady state of a dynamic process involving a large population of individuals who are randomly matched to play the game. Each member of the population adopts the same action throughout her stay in the population, and the population composition changes as a result of new entrants and departures. When entering, a player samples each action once and selects that which yields the best outcome according the procedure described above. In this case, an $S(1)$ equilibrium is a distribution of actions in the incumbent population which induces the same distribution of actions in the flow of entrants.

This dynamic process may be formalized as follows. Let $\alpha(t)$ represent the distribution of actions in the population at time t. That is, the proportion of the population choosing action $a_i \in A$ is given by $\alpha_i(t)$. Let $\dot{\alpha}_i(t)$ represent the rate of change of this proportion. Under the dynamics of sampling described above, the representation in the population of an action that is "best" with a higher (lower) probability than it is currently being played should increase (decrease). This suggests the following dynamics:

$$\dot{\alpha}_i(t) = F(\alpha(t)),$$

where F is Lipschitz continuous and satisfies:[3]

$$w(a_i, \alpha(t)) > \alpha_i(t) \Leftrightarrow \dot{\alpha}_i(t) > 0,$$

$$\alpha(t_0) \in \Omega^m \Rightarrow \alpha(t) \in \Omega^m \textit{ for all } t > t_0,$$

[3] Lipschitz continuity is required to ensure that trajectories are well-defined from all initial conditions.

where

$$\Omega^m = \left\{ x \in \mathbf{R}^m \;\middle|\; x_i \geq 0 \text{ and } \sum_{i=1}^{m} x_i = 1 \right\}$$

is the set of probability distributions on A. We shall refer to any system of differential equations $\dot{\alpha}_i(t) = F(\alpha(t))$ which satisfies the above conditions as a *sampling dynamic*. The simplest possible specification of a sampling dynamic is the following:

$$\dot{\alpha}_i(t) = w(a_i, \alpha(t)) - \alpha_i(t). \tag{1}$$

This specification will be used for illustrative purposes in the numerical examples which follow, and, for convenience, will also be used in the statement and proof of all formal results.[4] It is easy to verify, however, that all results will continue to hold without modification for arbitrary sampling dynamics.

Clearly, a distribution α is a rest point of a sampling dynamic if and only if it is an $S(1)$ equilibrium. Not all rest points of the dynamics will be stable, however. In examining the question of stability, the following standard definitions (see for instance, Hirsch and Smale, 1974) will be used below.

Definition. *A* rest point α is *stable* if, for every neighborhood $U \subset \mathbf{R}^m$ containing α, there is a neighborhood $V \subset U$ such that if $\alpha(t_0) \in V \cap \Omega^m$, then $\alpha(t) \in U \cap \Omega^m$ for all $t > t_0$.

In other words, α is stable if, for every neighborhood of α, it is possible to find an open set of initial conditions from which trajectories never leave this neighborhood. A rest point α is *unstable* if it is not stable. A sufficient condition for instability is that one or more of the eigenvalues of the Jacobean, when evaluated at the rest point, has positive real part.

Definition. A rest point α is *asymptotically stable* if it is stable and if there is some neighborhood $U \subset \mathbf{R}^m$ such that all trajectories initially in $U \cap \Omega^m$ converge to α.

A sufficient condition for asymptotic stability (and hence stability) is that all eigenvalues of the Jacobean, when evaluated at the rest point, have negative real part. In the next section, it is shown that stability (and asymptotic stability) can provide a method of selection among $S(1)$ equilibria.

[4] To see that (1) leaves Ω^m invariant, note that by definition $\sum_i w(a_i, \alpha(t_0)) = 1$. Suppose $\alpha(t_0) \in \Omega^m$. Then $\sum_i \dot{\alpha}_i(t_0) = \sum_i w(a_i, \alpha(t_0)) - \sum_i \alpha_i(t_0) = 0$ so $\sum_i \alpha_i(t) = 1$ for all $t > t_0$. Hence we need only check that $\alpha_i(t) \geq 0$ for all $t > t_0$. This follows form the fact that if $\alpha_i(t) = 0$, then $\dot{\alpha}_i(t) = w(a_i, \alpha(t)) \geq 0$.

3 Equilibrium Selection

Let $S(1)$ equilibria which place probability 1 on some action be referred to as *pure*. Although the set of Nash equilibria of a game will not generally coincide with the set of $S(1)$ equilibria, every strict symmetric Nash equilibrium corresponds to a pure $S(1)$ equilibrium. Specifically, if $(a_q, ..., a_q)$ is a strict Nash equilibrium, then the mixed strategy that places probability 1 on the action a_q is a pure $S(1)$ equilibrium. Hence coordination games have multiple $S(1)$ equilibria. The following example shows that stability with respect to the dynamics (1) can provide an equilibrium refinement that eliminates some equilibria in such games.

Example 1 (Coordination). Consider the symmetric game with the following payoff matrix (payoffs correspond to the row player; by symmetry the column player's payoff is given by the transpose):

	a_1	a_2
a_1	1	x
a_2	y	0

where $y < 1$ and $x < 0$. Both pure strategies correspond to strict Nash equilibria and hence also to pure $S(1)$ equilibria. If $x < y$, then all probability distributions on the action space are $S(1)$ equilibria (see Example 2 in Osborne and Rubinstein).[5] Suppose, however, that $x > y$. Then the probability that the first action is best under the sampling procedure is

$$w(a_1, \alpha) = \alpha_1 + \alpha_1(1 - \alpha_1).$$

Hence we have

$$\dot{\alpha}_1 = \alpha_1(1 - \alpha_1) \geq 0,$$

with strict inequality holding whenever $\alpha_1 \in (0, 1)$. The equilibrium $\alpha_1 = 1$ is therefore the only stable equilibrium, and all trajectories except the one originating at the unstable equilibrium converge to it. ||

The above example is interesting because strict Nash equilibria are always stable with respect to any deterministic payoff monotonic evolutionary selection dynamics, such as the

[5] In this case, all distributions are also stable $S(1)$ equilibria, though none is asymptotically stable.

replicator dynamics (Weibull, 1995). Evolutionary approaches to the equilibrium selection problem in coordination games has therefore focused on the criterion of stochastic stability (Kandori, Mailath and Rob 1993, Young 1993). Stochastically stable states are states which occur with positive probability in models which combine payoff monotonic selection dynamics with rare mutations, as the mutation rate approaches zero.[6] The notion of stochastic stability is appropriate only for the ultra-long run, since movements across basins of attraction can become very infrequent as the mutation rate is decreased (Ellison, 1993). In contrast, the use of the dynamics (1) to select among strict Nash equilibria provides a refinement that depends neither on mutations, nor on the ultra-long run. In Section 5, easily verifiable sufficient conditions are provided which can be applied directly to the equilibrium selection problem in coordination games.

4 Strictly Dominated Strategies

One of the more striking results in Osborne and Rubinstein is the finding that strategies that are strictly dominated by pure strategies can receive positive probability in $S(1)$ equilibria. It is interesting to raise the question, therefore, of whether this can occur at *stable* $S(1)$ equilibria. It turns out that not only can $S(1)$ equilibria which place positive probability on strictly dominated strategies be dynamically stable, they can be uniquely so. In other words, the $S(1)$ equilibrium which corresponds to the strict Nash equilibrium in which the *dominant* strategy is played with probability 1 can be unstable. The "voluntary exchange" example given by Osborne and Rubinstein itself turns out to have this property.

Example 2 (Voluntary Exchange). Consider the symmetric game with the following payoff matrix:

	a_1	a_2	a_3
a_1	2	5	8
a_2	1	4	7
a_3	0	3	6

[6] Note that in example 1, the unique stable $S(1)$ equilibrium in the case $x > y$ is also the unique stochastically stable equilibrium, though this equivalence need not hold more generally.

The first action strictly dominates the other two in this game and therefore (a_1, a_1) is the unique Nash equilibrium. The conditions for an $S(1)$ equilibrium are as follows.

$$\begin{aligned}
\alpha_1 &= \alpha_1^3 + \alpha_2(1-\alpha_3)^2 + \alpha_3, \\
\alpha_2 &= \alpha_1\alpha_2(1-\alpha_3) + \alpha_3(1-\alpha_3), \\
\alpha_3 &= \alpha_1^2\alpha_2 + \alpha_3(1-\alpha_3)^2.
\end{aligned}$$

Exactly two probability distributions satisfy these conditions:

$$\begin{aligned}
\alpha &= (1, 0, 0) \\
\alpha &= (0.519, 0.277, 0.204)
\end{aligned}$$

Each of the two strictly dominated strategies are played with positive probability in the latter of these equilibria, while the former corresponds to the unique strict Nash equilibrium of the game. In order to determine which, if any of these equilibria are stable, consider the dynamics (1) applied to this game. Substitution for α_3 yields the following two-dimensional system:

$$\begin{aligned}
\dot{\alpha}_1 &= \alpha_1^3 + \alpha_2(\alpha_1+\alpha_2)^2 + (1-\alpha_1-\alpha_2) - \alpha_1, \\
\dot{\alpha}_2 &= \alpha_1\alpha_2(\alpha_1+\alpha_2) + (1-\alpha_1-\alpha_2)(\alpha_1+\alpha_2) - \alpha_2.
\end{aligned}$$

The Jacobean of the system is given by

$$\begin{pmatrix} 3\alpha_1^2 + 2\alpha_1\alpha_2 + 2\alpha_2^2 - 2 & \alpha_1^2 + 4\alpha_1\alpha_2 + 3\alpha_2^2 - 1 \\ 2\alpha_1\alpha_2 + \alpha_2^2 - 2\alpha_1 - 2\alpha_2 + 1 & \alpha_1^2 + 2\alpha_1\alpha_2 - 2\alpha_1 - 2\alpha_2 \end{pmatrix}.$$

Consider first the pure $S(1)$ equilibrium $\alpha = (1, 0, 0)$. Here the Jacobean is

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},$$

with eigenvalues 1 and -1. This equilibrium is therefore unstable. Consider next the equilibrium $\alpha = (0.519, 0.277, 0.204)$,in which strictly dominated strategies are played with positive probability. Here the Jacobean is

$$\begin{pmatrix} -0.750 & 0.074 \\ -0.228 & -1.035 \end{pmatrix},$$

with eigenvalues -0.84 and -0.95. This equilibrium is therefore locally asymptotically stable, and trajectories which are initially sufficiently close to it converge it. ||

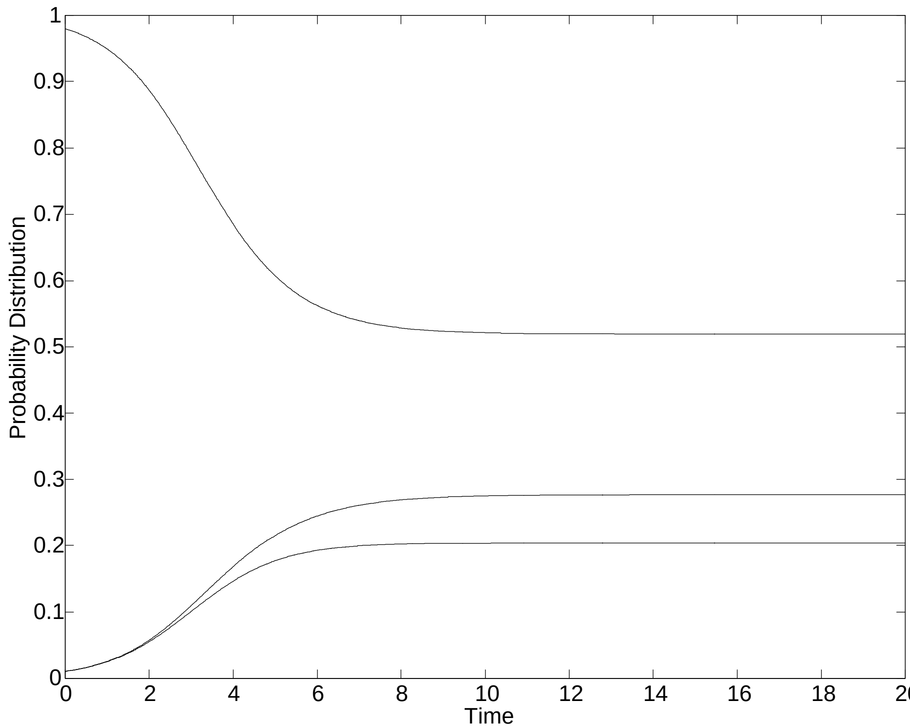

Figure 1: Convergence to the Interior $S(1)$ Equilibrium

The previous example shows that regardless of whether trajectories converge to an equilibrium, strictly dominated strategies will continue to be played indefinitely along all paths. Simulation results suggest, moreover, that the stable interior $S(1)$ equilibrium in this example attracts all trajectories except for that which originates at the unstable equilibrium. Figure 1 depicts the dynamics of the mixed strategy from the initial condition $\alpha(0) = (0.98, 0.01, 0.01)$, which is very close to the strict Nash equilibrium. Convergence to the interior equilibrium is rapid and monotonic.

The finding that strictly dominated strategies can be played in $S(1)$ equilibria of two-person games does not hold when each player has only two actions (Osborne and Rubinstein, 1998). The following example shows that such strategies can indeed be played in equilibria of games having two actions, provided that the number of players is at least three.

Example 3 (Three-Player Prisoners' Dilemma). Consider the symmetric three-person game with two actions and the following payoff matrix:

	(a_1, a_1)	(a_1, a_2)	(a_2, a_2)
a_1	1	3	5
a_2	0	2	4

The action a_1 strictly dominates a_2 and each player plays a_1 with probability 1 in the unique Nash equilibrium. Hence the mixed strategy which places probability 1 on action a_1 is a pure $S(1)$ equilibrium. There is a second $S(1)$ equilibrium, however, in which each player plays the mixed strategy $(x, 1-x)$, where

$$x = \frac{1}{6} \frac{\left(\sqrt[3]{(20+4\sqrt{29})}\right)^2 - 4 + 2\sqrt[3]{(20+4\sqrt{29})}}{\sqrt[3]{(20+4\sqrt{29})}} \approx 0.718$$

Of the two $S(1)$ equilibria, only the latter is stable. To see this, consider a mixed strategy $\alpha = (\alpha_1, \alpha_2)$. Then

$$\begin{aligned} \dot{\alpha}_1 &= w(a_1, \alpha) - \alpha_1 \\ &= \alpha_1^4 + 2\alpha_1\alpha_2\left(2\alpha_1\alpha_2 + \alpha_1^2\right) + \alpha_2^2 - \alpha_1 \\ &= \alpha_1^4 + 2\alpha_1(1-\alpha_1)\left(2\alpha_1(1-\alpha_1) + \alpha_1^2\right) + (1-\alpha_1)^2 - \alpha_1 \end{aligned}$$

It may be verified that $\dot{\alpha}_1 > 0$ for all $\alpha_1 \in [0, x)$ and $\dot{\alpha}_1 < 0$ for all $\alpha_1 \in (x, 1)$ where x is as defined above. Hence the pure $S(1)$ equilibrium $\alpha = (1, 0)$ is unstable and all trajectories except that which originates at this unstable equilibrium converge to the equilibrium in which strictly dominated strategies are played with positive probability. $\|$

The finding that strictly dominated strategies that are dominated by pure strategies can be played with positive probability at asymptotically stable $S(1)$ equilibria may be contrasted with the fact that in the standard theory of evolutionary games, payoff monotonic selection dynamics eliminate all such strategies, and even weakly payoff positive selection dynamics can never converge to a state in which strictly dominated strategies are played (Samuelson and Zhang, 1992; Weibull 1995).[7]

[7] It is possible, however, for strategies that are strictly dominated by a *mixed* strategy to survive along nonconvergent paths under payoff monotonic selection dynamics. This occurs, for instance, in a continuous-time version of Dekel and Scotchmer's (1992) example (see Björnerstedt, 1993 or Weibull, 1995). It is also possible for strategies that are strictly dominated by a pure strategy to survive along nonconvergent paths under weak payoff positive selection dynamics (Sethi, 1998).

5 Instability of Equilibria

In the examples of the previous section, $S(1)$ equilibria in which strictly dominant actions are played with probability 1 were shown to be unstable. In this section, simple and easily verifiable sufficient conditions for the instability of pure $S(1)$ equilibria are given.

Let Γ be a symmetric game with $n \geq 2$ players and $m \geq 2$ actions. Write the action space as $A = \{a_1, a_2, ...a_m\}$.

Definition. *An action profile* $(a_q, a_q, ..., a_q)$ *in a symmetric* n*–player game is* ***inferior*** *if, for every* $i \neq q$ *there exists* $j(i) \neq q$ *such that*

$$u(a_{j(i)}, a_i, a_q, ..., a_q) > u(a_q, a_q, ..., a_q).$$

It is ***twice inferior*** *if, for every action* $i \neq q$ *there exist* $j(i) \neq q$ *and* $k(i) \neq q$ *such that* $j \neq k$ *and*

$$u(a_{j(i)}, a_i, a_q, ...a_q) \geq u(a_{k(i)}, a_i, a_q, ..., a_q) > u(a_q, a_q, ..., a_q).$$

This definition states the following. A symmetric action profile $(a_q, a_q, ..., a_q)$ is inferior if, when $n-2$ of the other $n-1$ players take the action a_q while the remaining player selects $a_i \neq a_q$, there exists at least one response a_j $(j \neq q)$ by player 1 which yields an outcome that is strictly preferred by player 1 to the outcome at $(a_q, a_q, ...a_q)$. In Example 3 above, the dominant strategy equilibrium is inferior. A symmetric action profile is twice inferior if, when $n-2$ of the other $n-1$ players take the action a_q while the remaining player selects $a_i \neq a_q$, there exist at least *two* distinct responses a_j and a_k $(j, k \neq q)$ by player 1 which yield an outcome that is preferred by player 1 to the outcome at $(a_q, a_q, ...a_q)$. In Example 2 above, the dominant strategy equilibrium (a_1, a_1) is twice inferior. Note that no action profile can be twice inferior in games having only two actions.

The following result provides sufficient conditions for instability in games having at least three players.

Theorem 1. *In any symmetric game with three or more players, all inferior symmetric strict Nash equilibria (and hence all inferior pure* $S(1)$ *equilibria) are unstable under the sampling dynamics* (1)*.*

Proof. Without loss of generality, let $(a_1, a_1, ..., a_1)$ be a symmetric strict Nash equilibrium. Then the mixed strategy which places probability 1 on action a_1 is a pure $S(1)$ equilibrium. Consider a mixed strategy $\alpha = (1-\varepsilon, \varepsilon_2, \varepsilon_{3,}..., \varepsilon_m) \in \Omega^m$,where $\varepsilon \in (0,1)$. Since $\alpha \in \Omega^m$,we have $\varepsilon_i \geq 0$ for all $i \in \{2, ..., m\}$, and $\sum_2^m \varepsilon_i = \varepsilon$. Consider the following event: when a_1 is sampled, the outcome is $(a_1, a_1, ..., a_1)$; when $a_{j(2)}$ is sampled, the outcome is $(a_{j(2)}, x)$, where $x \in A^{n-1}$ contains some permutation of the actions $\{a_2, a_1, ..., a_1\}$. The probability of this event is

$$(1-\varepsilon)^{n-1} (n-1) (1-\varepsilon)^{n-2} \varepsilon_2,$$

and if it occurs, a_1 will not yield the best outcome. Next consider the following event: when a_1 is sampled, the outcome is $(a_1, a_1, ..., a_1)$; when $a_{j(2)}$ is sampled, the outcome is $(a_{j(2)}, x)$, where $x \in A^{n-1}$ does *not* contain a permutation of the actions $\{a_2, a_1, ..., a_1\}$; when $a_{j(3)}$ is sampled, the outcome is $(a_{j(3)}, y)$, where $y \in A^{n-1}$ contains some permutation of the actions $\{a_3, a_1, ..., a_1\}$. The probability of this event is (regardless of whether or not $a_{j(2)} = a_{j(3)}$) *at least*

$$(1-\varepsilon)^{n-1} \left(1-(n-1)(1-\varepsilon)^{n-2} \varepsilon_2\right) (n-1)(1-\varepsilon)^{n-2} \varepsilon_3,$$

and if it occurs, a_1 will not yield the best outcome. Note that the two events described are mutually exclusive. Reasoning in this manner, we obtain the following bound for the probability that a_1 will not yield the best outcome under the sampling procedure:

$$\begin{aligned} 1 - w(a_1, \alpha) \geq\ & (1-\varepsilon)^{n-1} (n-1) (1-\varepsilon)^{n-2} \varepsilon_2 \\ & + (1-\varepsilon)^{n-1} \left(1-(n-1)(1-\varepsilon)^{n-2} \varepsilon_2\right) (n-1)(1-\varepsilon)^{n-2} \varepsilon_3 + ... \\ & + (1-\varepsilon)^{n-1} \left(\prod_{i=2}^{m} \left(1-(n-1)(1-\varepsilon)^{n-2} \varepsilon_i\right) \right) (n-1)(1-\varepsilon)^{n-2} \varepsilon_m \end{aligned}$$

Let $o(\varepsilon^2)$ represent terms that are second order or higher in ε and/or ε_i $(\varepsilon^2, \varepsilon\varepsilon_i, \varepsilon_i\varepsilon_j, ...)$. Then the above inequality may be written as

$$w(a_1, \alpha) \leq 1 - (n-1) \sum_{i=2}^{m} \varepsilon_i + o(\varepsilon^2) \leq 1 - 2\varepsilon + o(\varepsilon^2) \tag{2}$$

since $n \geq 3$. Note that there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon < \bar{\varepsilon}$, $o(\varepsilon^2) < \varepsilon$. Hence, for all $\varepsilon < \bar{\varepsilon}$, we have

$$w(a_1, \alpha) < 1 - 2\varepsilon + \varepsilon = 1 - \varepsilon = \alpha_1. \tag{3}$$

Let $N_{\bar{\varepsilon}}$ be defined as follows

$$N_{\bar{\varepsilon}} = \{x \in \Omega^m \mid x = (1-\varepsilon, \varepsilon_2, \varepsilon_{3,...}, \varepsilon_m) \text{ with } \varepsilon < \bar{\varepsilon} \ \}.$$

Then from (1) and (3)we have $\dot{\alpha}_1 < 0$ for all $\alpha \in N_{\bar{\varepsilon}} \backslash \alpha^*$, where $\alpha^* = (1, 0, ..., 0)$. Hence all trajectories initially in $N_{\bar{\varepsilon}} \backslash \alpha^*$ eventually leave $N_{\bar{\varepsilon}}$, and α^* is unstable. □

Many experimental public goods games in which zero contributions are a dominant strategy belong to the class to which Theorem 1 applies (Ledyard, 1995). Consider, for instance, the following example.

Example 4 (Private Provision of Public Goods)**.** Each of $n \geq 3$ individuals has an endowment $e = (m-1)x$, all or part of which may be contributed to the provision of a public good in finite increments x. The action space $A = \{a_1, ..., a_m\}$ where a_i represents a contribution of $(i-1)x$ units. Let a_j represent the contribution of player j. The total contribution is $\sum_{j=1}^{n} a_j$ and the payoff to player j is $\pi_j = e - a_j + \beta \sum_{j=1}^{n} a_j$, where $1/n < \beta < 1$. Clearly a_1 is a strictly dominant action. The unique Nash equilibrium is $(a_1, ..., a_1)$, and hence the mixed strategy which places probability 1 on action a_1 is a pure $S(1)$ equilibrium. If $2\beta > 1$ the conditions for Theorem 1 are satisfied and this equilibrium is therefore unstable. ||

The example above implies that some positive contributions will be observed at any stable $S(1)$ equilibrium in this class of games, which accords with much of the experimental evidence.[8]

Theorem 1 states that inferiority is sufficient for instability of equilibria in symmetric games with three or more players. The following simple example shows that inferiority is *not* sufficient for instability in two-player games.

Example 5 (Prisoners' Dilemma). Consider the symmetric game represented by the following payoff matrix:

	a_1	a_2
a_1	1	3
a_2	0	2

[8] This conclusion is based on the premise that an individual's preference ordering over outcomes in the game depends positively on their own monetary payoff and is independent of the monetary payoffs of others. An alternative interpretation of the experimental evidence is that some individuals have preferences that are altruistic (or interdependent in other, more complex, ways.)

The action a_1 is strictly dominant and (a_1, a_1) is the unique Nash equilibrium. The mixed strategy $\alpha = (1, 0)$ which places probability 1 on action a_1 is the unique $S(1)$ equilibrium. The action profile (a_1, a_1) clearly inferior. The dynamics (1) applied to this game are as follows

$$\dot{\alpha}_1 = w(a_1, \alpha) - \alpha_1 = \alpha_1^2 + (1 - \alpha_1) - \alpha_1 = (1 - \alpha_1)^2 .$$

Hence $\dot{\alpha}_1 > 0$ for all $\alpha_1 \in [0, 1)$ and the inferior pure $S(1)$ equilibrium at $\alpha = (1, 0)$ is globally stable. $\|$

Example 5 illustrates that inferiority is not sufficient for instability when the number of players $n = 2$. Twice inferiority is, however, sufficient.

Theorem 2. *In any symmetric two-player game, all twice inferior symmetric strict Nash equilibria (and hence all twice inferior pure* $S(1)$ *equilibria) are unstable under the sampling dynamics* (1).

Proof. Without loss of generality, let $(a_1, a_1, ..., a_1)$ be a symmetric strict Nash equilibrium. Then the mixed strategy which places probability 1 on action a_1 is a pure $S(1)$ equilibrium. Consider a mixed strategy $\alpha = (1 - \varepsilon, \varepsilon_2, \varepsilon_{3,}..., \varepsilon_m) \in \Omega^m$,where $\varepsilon \in (0, 1)$. Since $\alpha \in \Omega^m$,we have $\varepsilon_i \geq 0$ for all $i \in \{2, ..., m\}$, and $\sum_2^m \varepsilon_i = \varepsilon$. Consider the following event: when a_1 is sampled, the outcome is (a_1, a_1); when $a_{j(2)}$ is sampled, the outcome is $(a_{j(2)}, a_2)$. The probability of this event is $(1 - \varepsilon)\, \varepsilon_2$, and if it occurs, a_1 will not yield the best outcome. Next consider the following event: when a_1 is sampled, the outcome is (a_1, a_1); when $a_{j(2)}$ is sampled, the outcome is *not* $(a_{j(2)}, a_2)$; when $a_{k(2)}$ is sampled, the outcome is $(a_{k(2)}, a_2)$. The probability of this event is $(1 - \varepsilon)\,(1 - \varepsilon_2)\, \varepsilon_2$, and if it occurs, a_1 will not yield the best outcome. Note that the two events described are mutually exclusive. Reasoning in this manner, we obtain the following bound for the probability that a_1 will not yield the best outcome under the sampling procedure:

$$\begin{aligned} 1 - w(a_1, \alpha) \;\; &\geq \;\; (1 - \varepsilon)\, \varepsilon_2 + (1 - \varepsilon)\,(1 - \varepsilon_2)\, \varepsilon_2 + ... \\ &\quad + (1 - \varepsilon) \prod_{i=2}^{m} (1 - \varepsilon_i)^2\, \varepsilon_m + (1 - \varepsilon) \prod_{i=2}^{m} (1 - \varepsilon_i)^2\, (1 - \varepsilon_m)\, \varepsilon_m \end{aligned}$$

Let $o(\varepsilon^2)$ represent terms that are second order or higher in ε and/or ε_i $(\varepsilon^2, \varepsilon\varepsilon_i, \varepsilon_i\varepsilon_j, ...)$. Then the above inequality may be written as

$$w(a_1, \alpha) \leq 1 - 2\sum_{i=2}^{m} \varepsilon_i + o(\varepsilon^2) = 1 - 2\varepsilon + o(\varepsilon^2).$$

This is the same as inequality (2) in the proof of Theorem 1 above, and the argument used thereafter applies unchanged to the present case. □

A profile (a_q, a_q) can be twice inferior even when a_q is a strictly dominant action, as Example 2 above illustrates. The instability of the equilibrium which places probability 1 on the strictly dominant action implies that strictly dominated actions must be played with positive probability indefinitely along all trajectories, regardless of whether or not such trajectories converge to an $S(1)$ equilibrium. The following example illustrates a further application of Theorem 2.

Example 6 (Three-Action Coordination Game). Consider the symmetric game represented by the following payoff matrix:

	a_1	a_2	a_3
a_1	2	6	6
a_2	0	7	3
a_3	1	4	8

This game has three strict Nash equilibria each of which corresponds to a pure $S(1)$ equilibrium. There are no other $S(1)$ equilibria. Of the three equilibria, the one at (a_1, a_1) is twice inferior and hence unstable by direct application of Theorem 2. ‖

The above example shows that selection among strict Nash equilibria in certain games on the basis of the dynamics (1) can be can be very easy to implement by means of the sufficient conditions identified in Theorems 1 and 2. This invites a comparison with the criterion of stochastic stability, which is currently the standard basis for selection among strict Nash equilibria. Stochastic stability is based on the limiting properties of the invariant distribution in a model of evolutionary dynamics with rare mutations, as the mutation rate get vanishingly small. It is therefore most suitable as a selection criterion applied to the very long run. In contrast, the dynamics (1) are deterministic and can converge rapidly, providing a selection criterion that applies to the short run. Stochastic stability is, however, a more powerful criterion which can distinguish among equilibria that the sampling dynamics treat as identical (as in Example 1, for the case $x < y$). Which of the two methods is more appropriate in any given context will therefore depend on the time horizon over which selection is expected to occur.

6 Multiple Populations

The analysis to this point has been based on the assumption that there is a single population from which all players are drawn. While this is a suitable assumption for symmetric games, it is not tenable in the case of asymmetric games since the action space and the potential payoff consequences of a given action generally differ across different player positions. In this case it is natural to assume that there exists a distinct population for each player position. Symmetric games too can be analyzed on the basis of multiple player populations (one for each player position). The multiple population case has different dynamic properties than does the single population case and results that hold in one case need not carry over to the other even in the case of symmetric games.

Consider, for simplicity, the case of a two-player asymmetric game with action spaces $A = \{a_1, ..., a_{m_1}\}$ and $B = \{b_1, ..., b_{m_2}\}$ respectively. Let $u_k(a_i, b_j)$ denote the payoff to Player k when Player 1 chooses the action $a_i \in A$ while Player 2 chooses $b_j \in B$. These payoff functions represent each player's ordinal preferences over the set of outcomes. Let α be a probability distribution on A and β a probability distribution on B. Let $v_1(a_i, \beta)$ be the m_1 random variables yielding $u_1(a_i, b_j)$ with probability β_j for each $b_j \in B$. Similarly, let $v_2(b_j, \alpha)$ be the m_2 random variables yielding $u_2(a_i, b_j)$ with probability α_i for each $a_i \in A$. Let $w_1(a_i, \beta)$ be the probability that, when each random variable $v_1(x, \beta)$ is drawn independently and exactly once, the action a_i yields the best outcome. As before, in the case of realizations in which a_i is not unique in yielding the best outcome, the probability is weighted by the reciprocal of the total number of tied alternatives. Finally, let $w_2(b_j, \alpha)$ be the probability that, when each random variable $v_2(y, \alpha)$ is drawn independently and exactly once, the action b_j yields the best outcome, again with the same tie-breaking convention that the probability is weighted by the reciprocal of the total number of tied alternatives.

An $S(1)$ equilibrium in this case is defined as a pair of probability distributions (α, β) on the sets of actions A and B respectively, with the property that

$$\begin{aligned} w_1(a_i, \beta) &= \alpha_i \text{ for every action } a_i \in A, \\ w_2(b_j, \alpha) &= \beta_j \text{ for every action } b_j \in B. \end{aligned}$$

The dynamics (1) can easily be generalized to cover this case as follows:

$$\dot{\alpha}_i(t) = w(a_i, \beta(t)) - \alpha_i(t), \tag{4}$$

$$\dot{\beta}_i(t) = w(b_i, \alpha(t)) - \beta_i(t). \tag{5}$$

If the number of actions available to the two players are m_1 and m_2 respectively, the above dynamics are defined for the state space $\mathbf{R}^{m_1+m_2}$. It is easily verified that under the dynamics (4–5), if $(\alpha(t_0), \beta(t_0)) \in \Omega^{m_1} \times \Omega^{m_2}$, then $(\alpha(t), \beta(t)) \in \Omega^{m_1} \times \Omega^{m_2}$ for all $t > t_0$. It is also easily seen that a state (α, β) is a rest point of these dynamics if an only if it is an $S(1)$ equilibrium. Furthermore, all of the above definitions and statements generalize in a straightforward manner to the case of three or more populations.

For symmetric games, if α is an $S(1)$ equilibrium in the single population case, then (α, α) must be an $S(1)$ equilibrium in the multiple population case. There may, however, be additional equilibria in the latter case, and the stability properties of equilibria which occur in both cases need not be identical, as the following example illustrates.

Example 7 (Hawk-Dove). Consider the game represented by the following payoff matrix:

	b_1	b_2
a_1	$(0, 0)$	$(3, 1)$
a_2	$(1, 3)$	$(2, 2)$

In the single population case there is a unique $S(1)$ equilibrium $\alpha = \left(\frac{1}{2}, \frac{1}{2}\right)$ which is globally asymptotically stable. To see this, observe that the dynamics (1) yield

$$\dot{\alpha}_1 = w(a_1, \alpha) - \alpha_1 = 1 - 2\alpha_1.$$

In the multiple population case, on the other hand, the dynamics are as follows

$$\begin{aligned} \dot{\alpha}_1 &= w(a_1, \beta) - \alpha_1 = 1 - \beta_1 - \alpha_1 \\ \dot{\beta}_1 &= w(b_1, \alpha) - \beta_1 = 1 - \alpha_1 - \beta_1 \end{aligned}$$

In this case any pair of distributions (α, β) is an $S(1)$ equilibrium provided that $\alpha_1 + \beta_1 = 1$. The single population $S(1)$ equilibrium remains an equilibrium in the multiple population case, but it is no longer asymptotically stable. ‖

Although the stability of an equilibrium need not be maintained as one moves from the single population to the multiple population case, the following result shows that *instability* is maintained.[9]

Theorem 3. *In any symmetric game, if* α^* *is a pure* $S(1)$ *equilibrium which is unstable under the single population dynamics* (1), *then* (α^*, α^*) *is unstable under the multiple population dynamics* (4–5).

Proof. Suppose that a pure $S(1)$ equilibrium $\alpha^* \in \Omega^m$ is unstable under the dynamics (1). Then, by definition, there exists a neighborhood $U \subset \mathbf{R}^m$ of α^* such that all trajectories which originate at any $\alpha(t_0) \in U \cap \Omega^m$, $\alpha(t_0) \neq \alpha^*$, must eventually leave U. Now consider the multiple population dynamics (4–5) in the neighborhood of the equilibrium (α^*, β^*), where $\beta^* = \alpha^*$. Since Γ is symmetric, any trajectory which satisfies initial conditions $\alpha(t_0) = \beta(t_0)$ must satisfy $\alpha(t) = \beta(t)$ for all $t > t_0$. Moreover, the time path of $\alpha(t)$ will be identical under (1) to the time paths of $\alpha(t)$ and $\beta(t)$ under (4–5), provided that initial conditions $\alpha(t_0)$ in the former case are identical to the initial conditions $\alpha(t_0)$ and $\beta(t_0)$ in the latter case. Hence, all trajectories from initial conditions $(\alpha(t_0), \beta(t_0))$ satisfying $\alpha(t_0) = \beta(t_0) \in U \cap \Omega^m$ will eventually leave $U \times U$ under (4–5). Since every neighborhood of (α^*, β^*) contains some points (α, β) such that $\alpha = \beta$, there can be no neighborhood W of (α^*, β^*) such that trajectories initially in W remain in $U \times U$. Hence (α^*, β^*) is unstable. □

Theorem 3 implies, in particular, that dominant strategy equilibria which are unstable under the single population dynamics (as in Examples 2 and 3 above) remain unstable under the multiple population dynamics. In such cases, strictly dominated strategies will be played with positive probability along all trajectories, regardless of whether or not they converge.

Turning, finally, to the case of genuinely asymmetric games with multiple player populations, the following definition extends the notion of inferiority to the case in which the action spaces of the two players need not be the same.

[9] Although Theorem 3 is stated and proved for the two population case, the generalization to multiple populations is straightforward.

Definition. *An action profile* (a_q, b_r) *in a two-player game is* ***inferior for player 1*** *if, for every* $i \neq r$ *there exists* $j(i) \neq q$ *such that*

$$u(a_{j(i)}, b_i) > u(a_q, b_r).$$

It is ***twice inferior for player 1*** *if, for every action* $i \neq r$ *there exist* $j(i) \neq q$ *and* $k(i) \neq q$ *such that* $j \neq k$ *and*

$$u(a_{j(i)}, b_i) \geq u(a_{k(i)}, b_i) > u(a_q, b_r).$$

(Twice) inferiority for player 2 is defined analogously. An action profile (a_q, b_r) *is (twice) inferior if it is (twice) inferior for both players.*

This definition is consistent with that given for symmetric games above (for symmetric games the two definitions are identical.) The following result provides sufficient conditions for the instability of $S(1)$ equilibria in asymmetric games with multiple player populations.

Theorem 4. *In any two-player game, all twice inferior strict Nash equilibria (and hence all twice inferior pure* $S(1)$ *equilibria) are unstable under the dynamics* (4–5)*.*

Proof. Without loss of generality, let (a_1, b_1) be a strict Nash equilibrium. Then the pair (α^*, β^*) where $\alpha^* = \beta^* = (1, 0)$ is a pure $S(1)$ equilibrium. Consider the pair of mixed strategies $\alpha = (1-\delta, \delta) \in \Omega^2$ and $\beta = (1-\varepsilon, \varepsilon) \in \Omega^2$, where $\varepsilon, \delta \in (0, 1)$. Using the same reasoning as in the proof of Theorem 2, the following inequalities may be obtained

$$\begin{aligned} w(a_1, \beta) &\leq 1 - 2\varepsilon + o(\varepsilon^2), \\ w(b_1, \alpha) &\leq 1 - 2\delta + o(\delta^2) \end{aligned}$$

Hence there exist $\bar{\delta}$, $\bar{\varepsilon} > 0$ such that for all $\delta < \bar{\delta}$ and $\varepsilon < \bar{\varepsilon}$, the following holds

$$w(a_1, \beta) < 1 - \varepsilon \tag{6}$$

$$w(b_1, \alpha) < 1 - \delta \tag{7}$$

Let $\eta = \min\{\bar{\delta}, \bar{\varepsilon}\}$ and let N_η be defined as follows

$$N_\eta = \left\{ (\alpha, \beta) \in \Omega^2 \times \Omega^2 \;\middle|\; (\alpha, \beta) = ((1-\delta, \delta), (1-\varepsilon, \varepsilon)) \text{ with } \delta, \varepsilon < \eta \;\right\}.$$

Then from (4–5) and (6–7), the following holds for all $(\alpha, \beta) \in N_\eta$:

$$\begin{aligned} \dot{\alpha}_1 &= w(a_1, \beta) - \alpha_1 < (1-\varepsilon) - (1-\delta) = \delta - \varepsilon, \\ \dot{\beta}_1 &= w(b_1, \alpha) - \beta_1 < (1-\delta) - (1-\varepsilon) = \varepsilon - \delta. \end{aligned}$$

Hence $\dot{\alpha}_1 + \dot{\beta}_1 < 0$ for all $(\alpha, \beta) \in N_\eta \backslash (\alpha^*, \beta^*)$, and all trajectories initially in $N_\eta \backslash (\alpha^*, \beta^*)$ eventually leave N_η. Therefore (α^*, β^*) is unstable. □

Theorem 4 provides sufficient conditions for the instability of strict Nash equilibria (and hence pure $S(1)$ equilibria) in two-player asymmetric games. As in the case of symmetric games, these conditions can be satisfied at equilibria in which strictly dominant actions are played with probability 1. Hence the basic conclusion arising from the analysis of symmetric games remains unchanged: strictly dominated strategies may be played with positive probability along all trajectories even in asymmetric games with multiple player populations. As might be conjectured, the following asymmetric game analogue of Theorem 1 also holds: in games with three or more players, all inferior strict Nash equilibria (and hence pure $S(1)$ equilibria) are unstable under the multiple population dynamics. The proof of this claim follows the same logic as those of Theorems 1 and 4, but requires considerable additional notation, and is therefore omitted.

7 Conclusions

This paper has explored the dynamic implications of the sampling procedure that underlies Osborne and Rubinstein's equilibrium concept for games with procedurally rational players. Dynamic stability can serve as a criterion for selection among multiple $S(1)$ equilibria. Furthermore, since there is a correspondence between strict Nash equilibria and pure $S(1)$ equilibria, this criterion can also be used to address the standard (Nash) equilibrium selection problem. More significantly, the theory of $S(1)$ equilibria yields predictions that differ starkly from those based on the standard theory of evolutionary games. This occurs because the sampling dynamics do not generally satisfy the condition of payoff monotonicity that underlies most work in evolutionary game theory. Which of the two approaches is more suitable in particular applications will depend, naturally, on which of the dynamic specifications more accurately describes individual learning and adjustment in the environment being studied.

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